Incremental $\mathcal{L}_2$-gain stability of piecewise-affine systems with piecewise-polynomial storage functions

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Abstract

This paper concerns the incremental $\mathcal{L}_2$-gain stability of piecewise-affine (PWA) systems. We propose sufficient conditions derived from dissipativity theory to compute an upper bound on the incremental $\mathcal{L}_2$-gain. This is achieved by constructing piecewise-polynomial storage functions through the use of sum of squares (SOS) relaxations. The constraints are expressed as linear matrix inequalities (LMIs), which can be solved numerically in an efficient way. The proposed conditions are verified to be less conservative than previous results found in the literature by means of a numerical example.

Key words: Incremental stability; incremental gain; nonlinear systems; piecewise-affine systems; dissipativity; linear matrix inequalities.

1 Introduction

The concept of incremental stability of a dynamical system concerns the behavior of each possible trajectory with respect to the others, i.e. whether they tend to converge to one another. There exists in the literature a variety of definitions concerning incremental stability, both from the input-output and state-space points of view. Concerning the former, Zames introduced the *maximum incremental amplification* [35] and used it to establish conditions for the stability of feedback loops [36,37]. This notion was later proposed as part of a framework to tackle robust performance analysis of nonlinear systems [9]. With respect to the latter, we may cite incremental asymptotic stability and incremental input-to-state stability [1], as well as convergence [26] and contraction [20], among some other variants. All of these definitions share the fact that an incremental notion of stability ensures stronger properties on the behavior of the system than its non-incremental counterpart. Among these, we may cite the existence of a unique asymptotically stable constant (resp. T-periodic) trajectory in response to a constant (resp. T-periodic) input, the asymptotic independence of initial conditions and the unicity of the steady state [1,26,20,8]. The aforementioned properties make incremental stability a suitable tool to deal with tracking and synchronization problems, as well as observer design.

Necessary and sufficient conditions for incremental $\mathcal{L}_2$-gain stability based on the celebrated dissipativity framework [34] are proposed in [29]. The analysis amounts to searching for a solution of a Hamilton-Jacobi-Bellman inequality [15], a problem of infinite dimension involving a partial differential inequality (PDI). Although numerical procedures to find approximate solutions to the PDI exist [16], the analysis may become intractable for complex nonlinear systems. A different approach is to search for relaxed sufficient conditions to compute an upper bound on the incremental $\mathcal{L}_2$-gain. In [10], the notion of *quadratic incremental stability* is introduced, and the analysis is conducted by embedding the dynamics of the time-varying linearizations of the system in a linear parameter-varying (LPV) model with polytopic description. The drawback of performing an analysis based on relaxed sufficient conditions comes in the form of conservatism, and to try to cope with that we shall focus the analysis on piecewise-affine (PWA) systems.

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The interest in nonlinear systems described by piecewise-affine functions is not new (see e.g. [17] for a historical review). This may be credited to two concurring factors: 1) PWA functions allow the description of a wide range of nonlinearities appearing in applied control theory – such as saturations, relays and dead zones – as well as the approximation of a broad class of nonlinear functions; 2) PWA systems remain quite similar to LTI systems, so that some of the results from linear control theory can be efficiently transposed, notably with respect to the possibility of recasting the analysis as an optimization problem constrained by linear matrix inequalities (LMIs). Johansson and Rantzer pioneered the analysis of piecewise-affine systems by introducing piecewise-quadratic Lyapunov functions [18]. In this sense, the analysis becomes local as to each region corresponds a different quadratic function. This was made possible via application of $\mathcal{S}$-procedure techniques [2], and the approach has been shown to provide less conservative results than those obtained with single (common) quadratic Lyapunov functions. Several extensions have been subsequently proposed, including stabilization and computation of an upper bound on the $\mathcal{L}_2$-gain, among others [12,14,28].

The study of incremental stability properties of PWA systems has already been addressed in the literature. In the context of convergent systems, [25] casts the analysis of PWA systems with continuous and discontinuous right-hand side as a search for a quadratic Lyapunov-like function. Romanchuk and Smith considered the incremental $\mathcal{L}_2$-gain stability of PWA systems, and proposed conditions to construct a quadratic storage function [30]. In common between both approaches is the proposal of LMI constraints and the restriction to quadratic functions. Morinaga et al. [21] propose piecewise-quadratic storage functions for the analysis of PWA systems; however, the same authors report being unsuccessful in reducing the conservatism.

Based on dissipativity arguments, we propose sufficient conditions to construct storage functions possessing a piecewise-polynomial structure. Using Sum-of-squares (SOS) relaxations, we obtain sufficient conditions expressed as LMI-constrained optimization problems that can be efficiently solved. The use of SOS techniques to construct Lyapunov and storage functions is well documented in the control literature, see e.g. [3,13,11] and references therein. The construction of piecewise-polynomial functions for PWA systems has been considered in [27]. This paper extends the preliminary results presented in [32], where piecewise-quadratic storage functions were considered. We show that our results are more general than those in [30,21,32], and thus potentially less conservative.

The paper is organized as follows. Section 2 presents the definition of incremental stability adopted in this paper, along with the related functional problem allowing its assessment. Subsequently we present conditions for the incremental analysis of PWA systems with piecewise-polynomial storage functions (Section 3). A numerical example illustrates the results in Section 4, with conclusions drawn in Section 5.

Notation

We denote by $\|\cdot\|$ the Euclidean norm for vectors or the corresponding induced norm for matrices. The real half line $[0, +\infty)$ is denoted by $\mathbb{R}_+$. The space of $n \times n$ symmetric matrices is denoted $\mathbb{S}^n$. For a vector $v = (v_1, \ldots, v_n)$, $v \succ 0$ (resp. $v \succeq 0$) is equivalent to the componentwise inequality $v_i > 0$ (resp. $v_i \geq 0$), $\forall i \in \{1, \ldots, n\}$. For a matrix $A \in \mathbb{R}^{n\times n}$, $A \succ 0$ (resp. $A \succeq 0$) denotes that A is positive definite (resp. semidefinite). The symbol $\bullet$ replaces the corresponding symmetric block in a block-symmetric matrix. The columnwise concatenation of two matrices A and B of compatible dimensions, denoted by col, is such that $\mathrm{col}(A, B) = \begin{bmatrix} A^\top & B^\top \end{bmatrix}^\top$. The $n \times n$ identity matrix is denoted by I_n.

$\mathcal{L}_2^q(\mathbb{R}_+)$ is the space of square integrable $\mathbb{R}^q$-valued functions defined on $\mathbb{R}_+$, and the associated norm is defined by $\|f\|_2 = (\int_0^\infty \|f(t)\|^2 dt)^{1/2}$. The *extended space* of locally square integrable functions is denoted by $\mathcal{L}_{2e}^q(\mathbb{R}_+)$.

2 Incremental stability

2.1 Piecewise-affine systems

Let $\Sigma : \mathcal{L}_{2e}^{n_w}(\mathbb{R}_+) \to \mathcal{L}_{2e}^{n_z}(\mathbb{R}_+)$ be a piecewise-affine dynamical system with a state space representation given by

$$z = \Sigma(w) \begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i w(t) \\ z(t) = C_i x(t) + c_i + D w(t) \\ x(0) = x_0 \end{cases} \quad \text{for } x(t) \in X_i \tag{1}$$

where $x(t) \in X = \mathbb{R}^n$ is the state at time t, $w \in \mathcal{L}_{2e}^{n_w}(\mathbb{R}_+)$ is the input taking values in $\mathbb{R}^{n_w}$, and $z \in \mathcal{L}_{2e}^{n_z}(\mathbb{R}_+)$ is the output taking values in $\mathbb{R}^{n_z}$. The regions X_i, for $i \in \mathcal{I} := \{1, \ldots, N\}$, are closed convex polyhedral sets defined by $X_i = \{x \in X \mid G_i x + g_i \succeq 0\}$ with non-empty and pairwise disjoint interiors such that $\bigcup_{i\in\mathcal{I}} X_i = X$. Then, $\{X_i\}_{i\in\mathcal{I}}$ constitutes a finite partition of X. From the geometry of X_i, the intersection $X_i \cap X_j$ between two different regions is always contained in a hyperplane, i.e. $X_i \cap X_j \subseteq \{x \in X \mid E_{ij} x + e_{ij} = 0\}$.

The continuous functions f_{PWA} and h_{PWA}, defined by $f_{\mathrm{PWA}}(x, w) := A_i x + a_i + B_i w$ and $h_{\mathrm{PWA}}(x, w) := C_i x + c_i + D w$, for $x \in X_i$, are such that $f_{\mathrm{PWA}}(0, 0) = 0$ and $h_{\mathrm{PWA}}(0, 0) = 0$, so that the origin is an equilibrium point of Σ associated to the null input with zero output. This means that, for any $i \in \mathcal{I}$, $0 \in X_i$ implies $a_i = 0$

and $c_i = 0$. Continuity of the vector field f_{PWA} ensures that no sliding modes occur at the intersection between regions [17].

The goal of this paper is to provide sufficient conditions to assess incremental $\mathcal{L}_2$-gain stability of piecewise-affine systems of the form (1). In the remainder of this section we shall formally define this performance measure and present the general approach for its assessment.

2.2 Incremental $\mathcal{L}_2$-gain and dissipativity

We begin by recalling the definition of $\mathcal{L}_2$-gain stability of nonlinear systems.

Definition 1 ($\mathcal{L}_2$-gain stability) *The system* (1) *is said to be $\mathcal{L}_2$-gain stable if there exists $\gamma \geq 0$ such that for all $w \in \mathcal{L}_2^{n_w}(\mathbb{R}_+)$ we have*

$$\int_0^\infty \|z(t)\|^2 \, dt \leq \gamma^2 \int_0^\infty \|w(t)\|^2 \, dt \tag{2}$$

for $z = \Sigma(w)$ with $x_0 = 0$. We define the $\mathcal{L}_2$-gain of Σ as the smallest γ for which (2) *holds.*

We now proceed to define the incremental $\mathcal{L}_2$-gain of a dynamical system. This notion is also understood as the Lipschitz continuity of the operator Σ [33].

Definition 2 (Incremental $\mathcal{L}_2$-gain stability) *The system* (1) *is said to be incrementally $\mathcal{L}_2$-gain stable if it is $\mathcal{L}_2$-gain stable and there exists $\eta \geq 0$ such that for all w, $\tilde{w} \in \mathcal{L}_2^{n_w}(\mathbb{R}_+)$ we have*

$$\int_0^\infty \|z(t) - \tilde{z}(t)\|^2 \, dt \leq \eta^2 \int_0^\infty \|w(t) - \tilde{w}(t)\|^2 \, dt \tag{3}$$

for $z = \Sigma(w)$ and $\tilde{z} = \Sigma(\tilde{w})$. We define the incremental $\mathcal{L}_2$-gain of Σ as the smallest η for which (3) *holds.*

We note that, given the assumptions on functions f_{PWA} and h_{PWA}, boundedness of the $\mathcal{L}_2$-gain is implied by incremental $\mathcal{L}_2$-gain stability. We now introduce the framework of dissipative systems [34], a standard approach when studying input-output properties such as boundedness and passivity. The following recalls the main concepts needed.

Definition 3 (Dissipative system) *A dynamical system Σ is said to be dissipative with respect to the* supply rate *$s : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \to \mathbb{R}$ if there exists a nonnegative function $S : X \to \mathbb{R}_+$, called the* storage function*, such that for all $t_1, t_0 \in \mathbb{R}_+$, $t_1 \geq t_0$, and $w \in \mathcal{L}_{2e}^{n_w}(\mathbb{R}_+)$,*

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t)) \, dt \tag{4}$$

where $x(t_1) = \phi(t_1, t_0, x(t_0), w)$ and $z = \Sigma(w)$.

The incremental $\mathcal{L}_2$-gain stability of system (1) can be assessed via dissipativity analysis of a fictitious augmented system $\overline{\Sigma} : \mathcal{L}_{2e}^{n_w}(\mathbb{R}_+) \times \mathcal{L}_{2e}^{n_w}(\mathbb{R}_+) \to \mathcal{L}_{2e}^{n_z}(\mathbb{R}_+)$ given by

$$\overline{z} = \overline{\Sigma}(\overline{w}) \begin{cases} \dot{\overline{x}}(t) = \overline{A}_{ij}\overline{x}(t) + \overline{B}_{ij}\overline{w}(t) \\ \overline{z}(t) = \overline{C}_{ij}\overline{x}(t) + \overline{D}\overline{w}(t) \\ \overline{x}(0) = \mathrm{col}(x_0, x_0, 1) \end{cases} \quad \text{for } \overline{x}(t) \in X_{ij} \tag{5}$$

where $\overline{x} = \mathrm{col}(x, \tilde{x}, 1)$, $\overline{w} = \mathrm{col}(w, \tilde{w})$, and

$$\overline{A}_{ij} = \begin{bmatrix} A_i & 0 & a_i \\ 0 & A_j & a_j \\ 0 & 0 & 0 \end{bmatrix} \qquad \overline{B}_{ij} = \begin{bmatrix} B_i & 0 \\ 0 & B_j \\ 0 & 0 \end{bmatrix}$$
$$\overline{C}_{ij} = \begin{bmatrix} C_i & -C_j & c_i - c_j \end{bmatrix} \qquad \overline{D} = \begin{bmatrix} D & -D \end{bmatrix}. \tag{6}$$

We note that $\overline{\Sigma}(\overline{w}) = \Sigma(w) - \Sigma(\tilde{w})$.

The regions X_{ij} are defined as $X_{ij} = \{\overline{x} = \mathrm{col}(x, \tilde{x}, 1) \mid x \in X_i \text{ and } \tilde{x} \in X_j\}$. Each region X_{ij} is described by $X_{ij} = \{\overline{x} \in \overline{X} \times \{1\} \mid \overline{G}_{ij}\overline{x} \succeq 0\}$ where $\overline{X} = X \times X$ and $\overline{G}_{ij} \in \mathbb{R}^{l_{ij} \times (2n+1)}$ is given by

$$\overline{G}_{ij} = \begin{bmatrix} G_i & 0 & g_i \\ 0 & G_j & g_j \end{bmatrix}. \tag{7}$$

Analogously to the state partition $\{X_i\}_{i \in \mathcal{I}}$ of system Σ, the intersection between any two regions X_{ij} and X_{kl} of $\overline{\Sigma}$ is either empty or contained in the hyperplane given by

$$X_{ij} \cap X_{kl} \subseteq \left\{ \overline{x} \in \overline{X} \times \{1\} \mid \overline{E}_{ijkl}\overline{x} = 0 \right\}, \tag{8}$$

for some row vector $\overline{E}_{ijkl}$.

The following theorem is an important result concerning incremental $\mathcal{L}_2$-gain stability and dissipativity of the augmented system. For a proof and details, the reader is referred to [31, Theorem 2.18].

Theorem 1 *Let Σ be the piecewise-affine system defined in* (1)*. Then Σ is incrementally $\mathcal{L}_2$-gain stable with an incremental $\mathcal{L}_2$-gain smaller than or equal to η, if the augmented system $\overline{\Sigma}$ is dissipative with respect to the supply rate $\overline{s} : \mathbb{R}^{n_w} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \to \mathbb{R}$ given by*

$$\overline{s}(w, \tilde{w}, \overline{z}) = \eta^2 \|w - \tilde{w}\|^2 - \|\overline{z}\|^2 \tag{9}$$

and if there exists a storage function $\overline{S} : \overline{X} \to \mathbb{R}_+$ such that $\overline{S}(x, x) = 0$, $\forall x \in X$.

This theorem is the starting point for providing sufficient conditions to compute an upper bound on the incremental $\mathcal{L}_2$-gain of PWA systems. This has already been considered in [30], where the search for a quadratic storage function of the form $\overline{S}(x, \tilde{x}) = (x - \tilde{x})^T P (x - \tilde{x})$ is expressed as an LMI-constrained optimization problem. In the course of this paper we shall propose less conservative results based on the search for piecewise-polynomial storage functions of x and $\tilde{x}$.

3 Incremental analysis of piecewise-affine systems: SOS techniques

In this section we consider the use of polynomial storage functions for the assessment of incremental $\mathcal{L}_2$-gain stability. We begin by recalling some concepts about polynomials and sum-of-squares representation, and present the main results in Section 3.2.

3.1 Polynomials and convex optimization

A monomial is a function $\upsilon : \mathbb{R}^n \to \mathbb{R}$ such that $\upsilon(\xi) = c\xi^a$, where $c \in \mathbb{R}$ is a coefficient and $a \in \mathbb{N}^n$ is a multi-index, i.e. if $a = (a_1, \ldots, a_n)$, then $\xi^a = \xi_1^{a_1} \cdots \xi_n^{a_n}$. The degree of υ is given by $|a| = \sum_{i=1}^n a_i$. A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is a finite sum of monomials with finite degree. The degree of the polynomial is the largest degree of its monomials. In what follows, $\mathbb{R}[\xi]$ denotes the ring of polynomials in $\xi \in \mathbb{R}^n$ with coefficients in $\mathbb{R}$.

We are interested in constructing nonnegative polynomials to be used as storage functions. It can be shown that, in general, testing global nonnegativity of polynomials is NP-hard, see e.g. [23,24]. For this reason, we turn our attention to a special class of polynomials, namely those that can be represented as *sums of squares*. The next definition is adapted from [3,22].

Definition 4 (Sum of squares polynomials) *For $\xi \in \mathbb{R}^n$, the polynomial $p \in \mathbb{R}[\xi]$ is a sum of squares (SOS) if there exist some polynomials p_i, $i = 1, \ldots, M$ such that*

$$p(\xi) = \sum_{i=1}^{M} p_i^2(\xi). \tag{10}$$

In this case we say that $p \in SOS[\xi]$.

It is clear that SOS polynomials are nonnegative. It can be shown that, in the general case, not all nonnegative polynomials are SOS (see e.g. [3, Theorem 2]). However, even if the existence of an SOS decomposition is not equivalent to nonnegativity, this representation is quite important, as the test of whether or not a polynomial is SOS can be cast into a convex optimization problem, namely the feasibility of LMI constraints. To see this, let $\chi_d(\xi)$ denote a vector containing all monomials in $\xi \in \mathbb{R}^n$ of degree less than or equal to d. This vector takes values in $\mathbb{R}^{\varrho(n,d)}$, where

$$\varrho(n, d) = \binom{n+d}{d}. \tag{11}$$

Then, a polynomial p of degree less than or equal to d can be written as

$$p(\xi) = \mathcal{O}^{\mathsf{T}} \chi_d(\xi) \tag{12}$$

for some $\mathcal{O} \in \mathbb{R}^{\varrho(n,d)}$, and a polynomial p of degree less than or equal to $2d$ can be written as

$$p(\xi) = \chi_d(\xi)^{\mathsf{T}} \mathcal{P} \chi_d(\xi) \tag{13}$$

for some $\mathcal{P} \in \mathbb{S}^{\varrho(n,d)}$, i.e. a $\varrho(n, d) \times \varrho(n, d)$ symmetric matrix. In what follows, we drop the dependence of χ_d on ξ to ease the notation. Due to the interdependence among the different elements of χ_d, the representation (13) is not unique. Let us define the set

$$\mathscr{Q}(n, d) := \{Q \in \mathbb{S}^{\varrho(n,d)} \mid \chi_d^{\mathsf{T}} Q \chi_d = 0, \forall \xi \in \mathbb{R}^n\}. \tag{14}$$

Then, $\mathscr{Q}(n, d)$ is the null space of the map that associates to every matrix $Q \in \mathbb{S}^{\varrho(n,d)}$ a polynomial $\chi_d^{\mathsf{T}} Q \chi_d$ in $\mathbb{R}[\xi]$. Let $\{Q_\ell^{n,d}\}_{\ell=1,\ldots,\iota(n,d)}$ be a basis of $\mathscr{Q}(n, d)$, where $\iota(n, d)$ is given by

$$\iota(n, d) = \frac{1}{2} \varrho(n, d) \left(\varrho(n, d) + 1\right) - \varrho(n, 2d). \tag{15}$$

We call $Q_\ell^{n,d}$ the slack matrices associated with the representation of polynomials of degree d in $\xi \in \mathbb{R}^n$. The first term on the right-hand side of (15) represents the number of independent terms in a symmetric matrix belonging to $\mathbb{S}^{\varrho(n,d)}$, and the second is the number of distinct monomials in the polynomial representation $\chi_d^{\mathsf{T}} Q \chi_d$, for some $Q \in \mathbb{S}^{\varrho(n,d)}$. Then, $\iota(n, d)$ represents the number of redundant terms in the representation $\chi_d^{\mathsf{T}} Q \chi_d$. A method to construct a basis $\{Q_\ell^{n,d}\}_{\ell=1,\ldots,\iota(n,d)}$ is given in [5, Table 4]. Finally, $Q^{n,d}(\tau)$ denotes a linear parametrization of $\mathscr{Q}(n, d)$, i.e. $Q^{n,d}(\tau) = \sum_{\ell=1}^{\iota(n,d)} \tau_\ell Q_\ell^{n,d}$, for $\tau \in \mathbb{R}^{\iota(n,d)}$. Then, the following result may be stated [3,23].

Lemma 1 *Let $p \in \mathbb{R}[\xi]$ be a polynomial of degree $2d$ in $\xi \in \mathbb{R}^n$ and let $\mathcal{P} \in \mathbb{S}^{\varrho(n,d)}$ be such that $p(\xi) = \chi_d^{\mathsf{T}} \mathcal{P} \chi_d$. Then, $p \in SOS[\xi]$ if and only if there exists $\tau \in \mathbb{R}^{\iota(n,d)}$ such that*

$$\mathcal{P} + Q^{n,d}(\tau) \succeq 0. \tag{16}$$

Condition (16) is an LMI feasibility problem over τ, and hence testing whether a polynomial is SOS can be done by solving a convex optimization problem.

As we have seen in the previous section, in order to be able to analyse piecewise-affine systems we need to use the $\mathcal{S}$-procedure to go from the constrained inequalities for every region to LMIs. Using polynomial functions, the approach remains the same, but we are able to consider a more flexible application of the $\mathcal{S}$-procedure using a key result in real algebraic geometry: the *Positivstellensatz* (see [3,23,24] for its statement and details). It provides a way to certify whether a given set, defined by polynomial equations and inequalities, is empty, and can be used as a test for constrained positivity of polynomials. In this sense, it can be relaxed to provide a generalization of the $\mathcal{S}$-procedure, as it should become clear after the following lemma (adapted from [3,22]).

Lemma 2 *The polynomial function $f_0 \in \mathbb{R}[\xi]$ is nonnegative for all ξ such that $f_k(\xi) \geq 0$, where $f_k \in \mathbb{R}[\xi]$, $k = 1, \ldots, M$, if there exist polynomials $g_k \in SOS[\xi]$ such that*

$$f_0 - \sum_{k=1}^{M} g_k f_k \in SOS[\xi], \quad \forall \xi \in \mathbb{R}^n. \tag{17}$$

From (17), it is clear why Lemma 2 can be seen as a generalization of the $\mathcal{S}$-procedure, since by taking g_k to be nonnegative scalars and f_k to be quadratic functions, we recover the classic result.

3.2 Analysis with piecewise-polynomial functions

We now consider a candidate storage function in the form of a continuous piecewise-polynomial function composed of polynomials of degree $2d$ given by:

$$\overline{S}(x, \tilde{x}) = \chi_d(\overline{x})^\mathsf{T} \mathcal{P}_{ij} \chi_d(\overline{x}), \quad \text{for } \overline{x} \in X_{ij}, \tag{18}$$

where $\chi_d(\overline{x})$ is a vector of monomials in $\overline{x}$, defined in (5), of degree less than or equal to d. As usual, the dependence on $\overline{x}$ is dropped in what follows.

We aim to rewrite the dissipativity inequality in Theorem 1 into quadratic inequalities that we can verify via LMI optimization. In the case of polynomial functions, we shall obtain quadratic inequalities on the vector of monomials χ_d. In order to consider dissipativity properties, we need to be able to take the inputs into account. This means that we need to devise a way of producing a quadratic function that leads to an LMI containing the vector of monomials χ_d as well as some vector containing the inputs.

Expression of the storage function Following the approach in [6], we define $\overline{w}_\chi := \overline{w} \otimes \chi_{d-1}$, where $\overline{w} = \operatorname{col}(w, \tilde{w})$ and $\otimes$ is the Kronecker product. The vector $\overline{\chi}_{\overline{w}} := \operatorname{col}(\chi_d, \overline{w}_\chi)$ is of dimension $\varrho_w(2n, d, 2n_w)$, where ϱ_w is defined as

$$\varrho_w(n, d, n_w) := \varrho(n, d) + n_w \varrho(n, d-1). \tag{19}$$

In order to obtain quadratic inequalities on χ_d and $\overline{w}_\chi$, we shall rewrite the dynamics of the augmented system in terms of these variables. For this, let us consider matrices $\mathcal{A}_{ij} \in \mathbb{R}^{\varrho(2n,d)\times\varrho(2n,d)}$ and $\mathcal{B}_{ij} \in \mathbb{R}^{\varrho(2n,d)\times\varrho_w(2n,d,2n_w)}$ implicitly defined by

$$\dot{\chi}_d = \frac{\partial \chi_d}{\partial \overline{x}} (\overline{A}_{ij}\overline{x} + \overline{B}_{ij}\overline{w}) =: \mathcal{A}_{ij}\chi_d + \mathcal{B}_{ij}\overline{w}_\chi, \text{ for } \overline{x} \in X_{ij}. \tag{20}$$

Consider the polynomial (18). Its derivative can then be written as

$$\begin{aligned}\dot{\overline{S}} &= 2\chi_d^\mathsf{T}\mathcal{P}_{ij}\dot{\chi}_d = 2\chi_d^\mathsf{T}\mathcal{P}_{ij}(\mathcal{A}_{ij}\chi_d + \mathcal{B}_{ij}\overline{w}_\chi) \\ &= \overline{\chi}_{\overline{w}}^\mathsf{T} \begin{bmatrix} \mathcal{A}_{ij}^\mathsf{T}\mathcal{P}_{ij} + \mathcal{P}_{ij}\mathcal{A}_{ij} & \mathcal{P}_{ij}\mathcal{B}_{ij} \\ \bullet & 0 \end{bmatrix} \overline{\chi}_{\overline{w}}, \quad \text{for } \overline{x} \in X_{ij}.\end{aligned} \tag{21}$$

We obtain a quadratic function on the vector $\overline{\chi}_{\overline{w}}$. As it happened with the vector of monomials χ_d, the quadratic representation of a polynomial with respect to the vector $\chi_{\overline{w}}$ is not unique. Let us define the set

$$\mathscr{R}(n, d, n_w) := \left\{ R \in \mathbb{S}^{\varrho_w(n,d,n_w)} \;\middle|\; \begin{array}{c} \chi_w^\mathsf{T} R \chi_w = 0, \\ \text{with } \chi_w = \operatorname{col}(\chi_d(x), w_\chi), \\ \forall x \in \mathbb{R}^n, \forall w \in \mathbb{R}^{n_w} \end{array} \right\}. \tag{22}$$

Let $\{R_\ell^{n,d,n_w}\}_{\ell=1,\ldots,\iota_w(n,d,n_w)}$ be a basis of $\mathscr{R}(n, d, n_w)$, where $\iota_w(n, d, n_w)$ is the number of slack matrices R_ℓ^{n,d,n_w}, and is given by [5]:

$$\begin{aligned}\iota_w(n, d, n_w) = \frac{1}{2}\varrho_w(n, d, n_w)(\varrho_w(n, d, n_w) + 1) \\ - \Big(\varrho(n, 2d) + n_w\varrho(n, 2d-1) \\ + \frac{n_w(n_w+1)}{2}\varrho(n, 2d-2)\Big).\end{aligned} \tag{23}$$

Finally, let $R^{n,d,n_w}(\tau)$ be a linear parametrization of the set $\mathscr{R}(n, d, n_w)$, i.e. $R^{n,d,n_w}(\tau) = \sum_{\ell=1}^{\iota_w(n,d,n_w)} \tau_\ell R_\ell^{n,d,n_w}$, for $\tau \in \mathbb{R}^{\iota_w(n,d,n_w)}$. By doing this, we have that a sufficient condition to ensure the nonpositivity of $\dot{\overline{S}}$ is the existence of $\mathcal{P}_{ij} \in \mathbb{S}^{\varrho(2n,d)}$ and $\mu_{ij} \in \mathbb{R}^{\iota_w(n,d,n_w)}$ such that

$$\begin{bmatrix} \mathcal{A}_{ij}^\mathsf{T}\mathcal{P}_{ij} + \mathcal{P}_{ij}\mathcal{A}_{ij} & \mathcal{P}_{ij}\mathcal{B}_{ij} \\ \bullet & 0 \end{bmatrix} + R^{2n,d,2n_w}(\mu_{ij}) \preceq 0. \tag{24}$$

Supply rate In order to assess dissipativity, we also need to rewrite the supply rate (9) as a quadratic function on $\overline{\chi}_{\overline{w}}$. Proceeding similarly to the previous discussion, let us define matrices $\mathcal{C}_{ij} \in \mathbb{R}^{n_z \times \varrho(2n,d)}$ and $\mathcal{D} \in \mathbb{R}^{n_z \times \varrho_w(2n,d,2n_w)}$ such that

$$\overline{z} = \overline{C}_{ij}\overline{x} + \overline{D}\overline{w} =: \mathcal{C}_{ij}\chi_d + \mathcal{D}\overline{w}_\chi. \tag{25}$$

and also the matrix $M_\eta \in \mathbb{S}^{\varrho_w(2n,d,2n_w)}$ such that

$$\eta^2\|w - \tilde{w}\|^2 =: \overline{w}_\chi^\mathsf{T} M_\eta \overline{w}_\chi. \tag{26}$$

In this way, the supply rate (9) can be written as the quadratic function

$$\overline{s}(w, \tilde{w}, \overline{z}) = \overline{\chi}_{\overline{w}}^\mathsf{T} \begin{bmatrix} -\mathcal{C}_{ij}^\mathsf{T}\mathcal{C}_{ij} & -\mathcal{C}_{ij}^\mathsf{T}\mathcal{D} \\ \bullet & M_\eta - \mathcal{D}^\mathsf{T}\mathcal{D} \end{bmatrix} \overline{\chi}_{\overline{w}}. \tag{27}$$

Regional description and $\mathcal{S}$-procedure Let us define some notation concerning the use of the extended $\mathcal{S}$-procedure as stated in Lemma 2. In our case, $f_0(\overline{x}) \geq 0$ denotes the polynomial inequality that we are trying to satisfy. Then, the constraint functions f_i are given in each region by each hyperplane defining the augmented region X_{ij}, i.e. each row of the constraint $\overline{G}_{ij}\overline{x} \succeq 0$. Let $\overline{G}_{ij,k}$ denote the k-th row of $\overline{G}_{ij}$, and let us define $\mathcal{T}_{ij} \in \mathbb{S}^{\varrho(2n,d)}$ as the matrix such that

$$g_{ij,1}(\overline{x})\overline{G}_{ij,1}\overline{x} + \cdots + g_{ij,l_{ij}}(\overline{x})\overline{G}_{ij,l_{ij}}\overline{x} =: \chi_d^\mathsf{T}\mathcal{T}_{ij}\chi_d. \tag{28}$$

Since $\overline{G}_{ij,k}\overline{x}$ is an affine function of $\overline{x}$, we may choose polynomials $g_{ij,k}$ of degree up to $2d-1$. Let us also define $\mathcal{G}_{ij,k} \in \mathbb{S}^{\varrho(2n,d)}$ as the matrix such that

$$g_{ij,k}(\overline{x}) =: \chi_d^\mathsf{T}\mathcal{G}_{ij,k}\chi_d. \tag{29}$$

Then, if $f_0(\overline{x}) = \chi_d^\mathsf{T} F_0 \chi_d$, the conditions of Lemma 2 become

$$\begin{cases} F_0 + Q^{2n,d}(\tau) - \mathcal{T}_{ij} \succeq 0 \\ \mathcal{G}_{ij,k} + Q^{2n,d}(\nu_{ij,k}) \succeq 0, \quad \text{for } k = 1, \ldots, l_{ij}. \end{cases} \tag{30}$$

Condition $\overline{S}(x,x) = 0$ As we have seen in Theorem 1, the storage function must be such that $\overline{S}(x,x) = 0$, for every $x \in X$. In order to ensure this, let $\lambda(\overline{x}) := \chi_d(\ell(\overline{x}))$, where $\ell(\overline{x}) = \mathrm{col}(x - \tilde{x}, x + \tilde{x})$, and let $T \in \mathbb{R}^{\varrho(2n,d)\times\varrho(2n,d)}$ be such that

$$\chi_d(\overline{x}) = T\lambda(\overline{x}). \tag{31}$$

Let us define $\ell_0(x) := \mathrm{col}(0, 2x)$, i.e. the case when $x = \tilde{x}$, and then $\lambda_0(x) := \chi_d(\ell_0(x))$. If $\overline{S}(x,\tilde{x}) = \chi_d^\mathsf{T}\mathcal{P}_{ij}\chi_d$, the constraint $\overline{S}(x,x) = 0$ for every $x \in X$ then means that $\lambda_0(x)^\mathsf{T} T^\mathsf{T} \mathcal{P}_{ii} T \lambda_0(x) = 0$, for all $\lambda_0(x)$ generated by $x \in X$, and every $i \in \mathcal{I}$. Let $\mathcal{Z} \in \mathbb{R}^{\varrho(2n,d)\times\varrho(2n,d)}$ be a matrix such that

$$\lambda_0(x) = \mathcal{Z}\lambda(\overline{x}). \tag{32}$$

Then, $\mathcal{Z}$ generates all $\lambda(\overline{x})$ with $x = \tilde{x}$. Let Z be a matrix whose columns constitute an orthogonal basis of range($\mathcal{Z}$). Then, to ensure that $\overline{S}(x,x) = 0$, for every $x \in \mathbb{R}^n$, we must have that $Z^\mathsf{T} T^\mathsf{T} \mathcal{P}_{ii} T Z = 0$, for every $i \in \mathcal{I}$. Matrices T and Z can be obtained based solely on the dimensions n of the state space and on the degree $2d$ of the polynomial.

Continuity of the storage function To ensure continuity of (18), we need to lift the description of the hyperplanes between each pair of regions, given by $\overline{E}_{ijkl}\overline{x} = 0$, to the vector of monomials χ_d. Namely, we want to find $\mathcal{E}_{ijkl}$ such that $\overline{E}_{ijkl}\overline{x} = 0$ implies $\mathcal{E}_{ijkl}\chi_d = 0$. This matrix can be obtained by extending the constraint $\overline{E}_{ijkl}\overline{x} = 0$ with the multiplication of a vector of monomials of reduced order, i.e. $\mathcal{E}_{ijkl}$ is the unique matrix implicitly defined by:

$$\chi_{d-1}\overline{E}_{ijkl}\overline{x} =: \mathcal{E}_{ijkl}\chi_d = 0, \tag{33}$$

where $\mathcal{E}_{ijkl} \in \mathbb{R}^{\varrho(2n,d-1)\times\varrho(2n,d)}$. Then, continuity of the piecewise-polynomial function can be ensured by enforcing the constraint

$$\mathcal{P}_{ij} = \mathcal{P}_{kl} + L_{ijkl}\mathcal{E}_{ijkl} + \mathcal{E}_{ijkl}^\mathsf{T} L_{ijkl}^\mathsf{T} + Q^{2n,d}(\tau) \tag{34}$$

with $L_{ijkl} \in \mathbb{R}^{\varrho(2n,d)\times\varrho(2n,d-1)}$, and where we introduce $Q^{2n,d}(\tau)$ to take into account the non-uniqueness of the polynomial representation. This can be easily checked by left and right multiplying by χ_d^T and χ_d, respectively.

Theorem 2 *Let $Q^{2n,d}$ and $R^{2n,d,2n_w}$ be linear parametrizations of fixed bases of $\mathscr{Q}(2n,d)$ and $\mathscr{R}(2n,d,2n_w)$, defined respectively in* (14) *and* (22)*. Let the matrix $T \in \mathbb{R}^{\varrho(2n,d)\times\varrho(2n,d)}$ be such that* (31) *is satisfied. Finally, let Z be a matrix whose columns span the range of the matrix $\mathcal{Z}$, which is defined such that* (32) *is satisfied. If there exist matrices $\mathcal{P}_{ij} \in \mathbb{S}^{\varrho(2n,d)}$, as well as $\mathcal{T}_{ij,r} \in \mathbb{S}^{\varrho(2n,d)}$ and $\mathcal{G}_{ij,r,k} \in \mathbb{S}^{\varrho(2n,d)}$ defined respectively by* (28) *and* (29) *for $r \in \{1,2\}$ and $k \in \{1,\ldots,l_{ij}\}$, vectors $\tau_{ij} \in \mathbb{R}^{\iota(2n,d)}$ and $\nu_{ij,r,k} \in \mathbb{R}^{\iota(2n,d)}$, for $r \in \{1,2\}$ and $k \in \{1,\ldots,l_{ij}\}$, $\mu_{ij} \in \mathbb{R}^{\iota_w(2n,d,2n_w)}$ and $\vartheta_{ijkl} \in \mathbb{R}^{\iota(2n,d)}$, a scalar $\eta > 0$ and a matrix M_η, as defined in* (26)*, and matrices $L_{ijkl} \in \mathbb{R}^{\varrho(2n,d)\times\varrho(2n,d-1)}$*

such that

$$\begin{cases} \mathcal{P}_{ij} + Q^{2n,d}(\tau_{ij}) - \mathcal{T}_{ij,1} \succeq 0 \\ \left[\begin{array}{c:c} \mathcal{A}_{ij}^{\mathsf{T}}\mathcal{P}_{ij} + \mathcal{P}_{ij}\mathcal{A}_{ij} & \\ +\mathcal{C}_{ij}^{\mathsf{T}}\mathcal{C}_{ij} + \mathcal{T}_{ij,2} & \mathcal{P}_{ij}\mathcal{B}_{ij} + \mathcal{C}_{ij}^{\mathsf{T}}\mathcal{D} \\ \hdashline \bullet & \mathcal{D}^{\mathsf{T}}\mathcal{D} - M_\eta \end{array}\right] + R^{2n,d,2n_w}(\mu_{ij}) \preceq 0 \\ \begin{cases} \mathcal{G}_{ij,1,k} + Q^{2n,d}(\nu_{ij,1,k}) \succeq 0 \\ \mathcal{G}_{ij,2,k} + Q^{2n,d}(\nu_{ij,2,k}) \succeq 0, \end{cases} \quad \textit{for } k = 1, \ldots, l_{ij} \end{cases}$$

$$\textit{for } (i,j) \in \mathcal{I}^2 \tag{35}$$

$$Z^{\mathsf{T}}T^{\mathsf{T}}\mathcal{P}_{ii}TZ = 0 \textit{ for } i \in \mathcal{I} \tag{36}$$

$$\mathcal{P}_{ij} = \mathcal{P}_{kl} + L_{ijkl}\mathcal{E}_{ijkl} + \mathcal{E}_{ijkl}^{\mathsf{T}}L_{ijkl}^{\mathsf{T}} + Q^{2n,d}(\vartheta_{ijkl})$$
$$\textit{for } (i,j),(k,l) \mid X_{ij} \cap X_{kl} \neq \varnothing \tag{37}$$

are satisfied, then

- *(i) the piecewise-affine system* (1) *is incrementally* $\mathcal{L}_2$*-gain stable, with an incremental* $\mathcal{L}_2$*-gain less than or equal to* η*;*
- *(ii) the augmented system* (6) *is dissipative with respect to the supply rate* (9)*, and* $\overline{S}$ *given by* (18) *is a storage function.*

PROOF. According to Theorem 1, the incremental $\mathcal{L}_2$-gain of (1) is less than or equal to η if the augmented system (5) is dissipative with respect to the supply rate (9). We will show that the LMIs (35) and the algebraic constraints (36) and (37) allow the construction of a continuous nonnegative piecewise-quadratic storage function $\overline{S}$ of structure given by (18) such that the above condition is met.

Continuity - We first show that $\overline{S}$ is a continuous function of $\overline{x}$. This is clearly the case inside every cell, so we just need to show continuity on the boundaries. From (8), $\overline{E}_{ijkl}\overline{x} = 0$ for all $\overline{x} \in X_{ij} \cap X_{kl}$, which implies $\mathcal{E}_{ijkl}\chi_d = 0$ through (33). Then, (37) implies that $\chi_d^{\mathsf{T}}\mathcal{P}_{ij}\chi_d = \chi_d^{\mathsf{T}}\mathcal{P}_{kl}\chi_d$ for $\overline{x} \in X_{ij} \cap X_{kl}$ and hence that $\overline{S}$ is continuous.

Nonnegativity - We now show that $\overline{S}$ is a nonnegative function. According to the notation defined in (28)–(29), the first and third inequalities in (35) ensure that the conditions in Lemma 2 are satisfied, and thus that $\chi_d^{\mathsf{T}}\mathcal{P}_{ij}\chi_d \geq 0$ for all $\overline{x} \in X_{ij}$. The first inequality in (35), post and pre multiplied respectively by χ_d and χ_d^{T}, implies that $\chi_d^{\mathsf{T}}(\mathcal{P}_{ij} + Q^{2n,d}(\tau_{ij}))\chi_d - \chi_d^{\mathsf{T}}\mathcal{T}_{ij,1}\chi_d \geq 0$. Imposing this inequality for every $(i,j) \in \mathcal{I}^2$ ensures that

$$\overline{S}(x,\tilde{x}) \geq 0, \quad \forall x, \tilde{x} \in X. \tag{38}$$

Dissipation inequality - We now show that the storage function respects the dissipation constraint (4). Using the same arguments as before, the last inequality in (36), post and pre multiplied by $\overline{\chi}_{\overline{w}}$ and $\overline{\chi}_{\overline{w}}^{\mathsf{T}}$, implies that

$$\begin{aligned} &\chi_d^{\mathsf{T}}\mathcal{P}_{ij}(\mathcal{A}_{ij}\chi_d + \mathcal{B}_{ij}\overline{w}_\chi) + (\mathcal{A}_{ij}\chi_d + \mathcal{B}_{ij}\overline{w}_\chi)^{\mathsf{T}}\mathcal{P}_{ij}\chi_d \\ &+(\mathcal{C}_{ij}\chi_d + \mathcal{D}\overline{w}_\chi)^{\mathsf{T}}(\mathcal{C}_{ij}\chi_d + \mathcal{D}\overline{w}_\chi) - \overline{w}_\chi^{\mathsf{T}}M_\eta\overline{w}_\chi \leq 0 \end{aligned} \tag{39}$$

for all $\overline{w} \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_w}$ and all $\overline{x} \in X_{ij}$. Let t_a and t_b be two time instants such that the state trajectory of system (5) remains in X_{ij} on the interval $[t_a, t_b]$. By noticing that $\dot{\chi}_d = \mathcal{A}_{ij}\chi_d + \mathcal{B}_{ij}\overline{w}_\chi$, and integrating from t_a to t_b along a trajectory of (5), we have

$$\begin{aligned} &\chi_d(\overline{x}(t_b))^{\mathsf{T}}\mathcal{P}_{ij}\chi_d(\overline{x}(t_b)) - \chi_d(\overline{x}(t_a))^{\mathsf{T}}\mathcal{P}_{ij}\chi_d(\overline{x}(t_a)) \\ &+\int_{t_a}^{t_b}\|z(\tau) - \tilde{z}(\tau)\|^2 d\tau - \eta^2\int_{t_a}^{t_b}\|w(\tau) - \tilde{w}(\tau)\|^2\, d\tau \leq 0. \end{aligned} \tag{40}$$

We note that the first terms in (40) represent the storage function (18). Let us consider a trajectory $\overline{x}(t)$, $\forall t \in [t_0, t_1]$, with $t_0 \geq 0$. The time t_1 can be decomposed as $t_1 = t_1 - t_{in,q} + \sum_{k=0}^{q-1}(t_{out,k} - t_{in,k})$, with $t_{out,k} = t_{in,k+1}$ and $t_{in,0} = t_0$, so that during each time interval $[t_{in,k}, t_{out,k}]$ the trajectory stays in a given region. Then, replacing t_a by $t_{in,k}$ and t_b by $t_{out,k}$ in (40), adding up to q for every region X_{ij} crossed, and using the continuity of $\overline{S}$ yields

$$\begin{aligned} &\overline{S}(x(t_1), \tilde{x}(t_1)) - \overline{S}(x(t_0), \tilde{x}(t_0)) \\ &+ \int_{t_0}^{t_1}\|z(\tau) - \tilde{z}(\tau)\|^2 d\tau - \eta^2\int_{t_0}^{t_1}\|w(\tau) - \tilde{w}(\tau)\|^2\, d\tau \leq 0. \end{aligned} \tag{41}$$

From (4), this shows that $\overline{S}$ is a storage function such that the augmented system $\overline{\Sigma}$ is dissipative with respect to the supply rate (9). As we have discussed before stating the theorem, the algebraic constraint (36) ensures that $\overline{S}(x,x) = 0$, for every $x \in X$. The piecewise-polynomial function $\overline{S}$ in (18) is then a storage function satisfying the conditions in Theorem 1. This in turn implies that Σ has an incremental $\mathcal{L}_2$-gain less than or equal to η, which concludes the proof. □

The inequalities in (35) are LMIs on the variables $\mathcal{P}_{ij}$ (storage function), $\mathcal{T}_{ij,r}$ and $\mathcal{G}_{ij,r,k}$ ($\mathcal{S}$-procedure), τ_{ij}, μ_{ij} and $\nu_{ij,r,k}$ (slack matrices) and η (incremental $\mathcal{L}_2$-gain). These can be efficiently solved using semidefinite programming solvers. However, not all of these solvers can handle algebraic constraints. To deal with this, we may eliminate (36) and (37) by performing a change of variables (see e.g. [7, Section 4.3.1]). In this way, we obtain an equivalent optimization problem on the new variables, but without equality constraints.

It might be possible to show that the conditions in Theorem 2 become necessary for some sufficiently large d, in the same line as in [4], and with the additional assumption on reachability of the state space X from x_0. However, increasing d quickly renders the problem unmanageable from a numerical point of view, reason why this route is not pursued in the present paper.

4 Numerical example

Here we consider an example to illustrate the application of the techniques presented in the last section.

Example 1 *Let us consider the bimodal system given by*

$$\begin{aligned}\dot{x}(t) &= \begin{cases} A_1 x(t) + Bw(t) & \text{for } x_1 \le 0 \\ A_2 x(t) + Bw(t) & \text{for } x_1 > 0 \end{cases} \\ z(t) &= Cx(t)\end{aligned} \tag{42}$$

with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -11 & -2 \end{bmatrix} \tag{43}$$

and $B = \mathrm{col}(0,1)$, $C = [1\,0]$. *This system can also be represented as the interconnection of the LTI system*

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -0.5 & -2 & 1 \\ \hline 1 & 0 & 0 \end{array}\right] \tag{44}$$

with the nonlinearity φ given by

$$\varphi(q) = \begin{cases} 0.5q, & \text{for } q \le 0 \\ 10.5q, & \text{for } q > 0 \end{cases} \tag{45}$$

through a negative feedback. We are interested in computing, if possible, an upper bound on the incremental $\mathcal{L}_2$-gain of this system. For this system, no common quadratic or piecewise-quadratic storage function could be found using the methods in [30] and [21], respectively. It remains to check whether we can use Theorem 2 to construct piecewise-polynomial storage functions. For this, we choose to construct polynomial functions of degree 4, i.e. with d in (18) *equal to 2. Since the model has 2 states and 1 input, the augmented system is a 4-state system with 2 inputs, which leads to a vector χ_d of monomials of size* $\varrho(4,2) = 15$ *and* $\varrho_w(4,2,2) = 25$. *This leads to basis of slack matrices for the sets* $\mathscr{Q}(4,2)$ *and* $\mathscr{R}(4,2,2)$ *of size* $\iota(4,2) = 50$ *and* $\iota_w(4,2,2) = 140$, *respectively. The LMIs are then solved using the parser YALMIP [19] together with Matlab®. The computation took 27.378 seconds on an Intel® Core i7 3.4 GHz, and successfully yield a piecewise-polynomial storage function ensuring an upper bound on the incremental $\mathcal{L}_2$-gain of* $\eta = 6.6778$.

5 Conclusion

In this paper we have presented new results for the assessment of incremental $\mathcal{L}_2$-gain stability of piecewise-affine systems. We propose a method of analysis using piecewise-polynomial functions using sum-of-squares techniques. With the help of a numerical example, we have shown that this approach can be successfully applied for the analysis of incremental properties of piecewise-affine systems. To the best of our knowledge, these are the first results allowing the assessment of incremental stability of PWA systems taking advantage of their regional description. We are then able to construct storage functions that are more general than single quadratic ones. In this sense, we have gone beyond the results of Romanchuk [30] and Morinaga et al. [21], which fail to provide a conclusive outcome to the analysis of system (42), for example. Additional results concerning incremental $\mathcal{L}_2$-gain stability and also internal incremental stability can be found in [31].

The method proposed in this paper is based on sufficient conditions for incremental stability, and hence no conclusion can be reached when the LMIs in Theorem 2 are unfeasible. This means that no bound on the incremental $\mathcal{L}_2$-gain can be found, but also that no guarantees on the qualitative behavior of the system (such as unicity of the steady state and independence of initial conditions) can be obtained. It should also be noted that, as seen in Example 1, computing the polynomial functions can involve solving LMI optimization problems of increased complexity. For this reason, it is recommended to consider first polynomials of low order, and then to increase it if necessary. For now, this seems to be the price to pay for reduced conservatism in the incremental analysis of piecewise-affine systems, when there is a need to go beyond the results obtained with quadratic functions.

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