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TEST VECTORS FOR RANKIN–SELBERG L -FUNCTIONS

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ABSTRACT. We study the local zeta integrals attached to a pair of generic representations (π, τ) of $\mathrm{GL}_n \times \mathrm{GL}_m$, $n > m$, over a p -adic field. Through a process of unipotent averaging we produce a pair of corresponding Whittaker functions whose zeta integral is non-zero, and we express this integral in terms of the Langlands parameters of π and τ . In many cases, these Whittaker functions also serve as a test vector for the associated Rankin–Selberg (local) L -function.

1. INTRODUCTION

Let F be a non-archimedean local field with ring of integers \mathfrak{o} and residue field of cardinality q . For $m < n$, let π and τ be irreducible admissible representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively. We fix an additive character ψ of F with conductor \mathfrak{o} and assume that π and τ are generic relative to ψ .

Recall that the local *zeta integral* $\Psi(s; W, W')$ is defined by

$$(1.1) \quad \Psi(s; W, W') = \int_{U_m(F) \backslash \mathrm{GL}_m(F)} W \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W'(h) \|\det h\|^{s - \frac{n-m}{2}} dh,$$

where $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\tau, \psi^{-1})$ are Whittaker functions in the corresponding Whittaker spaces, and U_m is the group of unipotent matrices. It converges for $\Re(s) \gg 1$, and the collection of such zeta integrals spans a fractional ideal $\mathbb{C}[q^s, q^{-s}]L(s, \pi \boxtimes \tau)$ of the ring $\mathbb{C}[q^s, q^{-s}]$. We may choose the generator to satisfy $1/L(s, \pi \boxtimes \tau) \in \mathbb{C}[q^{-s}]$ and $\lim_{s \rightarrow \infty} L(s, \pi \boxtimes \tau) = 1$, and this gives the local Rankin–Selberg factor attached to the pair (π, τ) in [5, §2.7].

In particular, if we define a map

$$\mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\tau, \psi^{-1}) \longrightarrow \mathbb{C}(q^{-s})$$

via

$$W \otimes W' \mapsto \Psi(s; W, W'),$$

then there is an element in $\mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\tau, \psi^{-1})$ that maps to $L(s, \pi \boxtimes \tau)$. However, *a priori* this element need not be a pure tensor. In this paper, we produce a pure tensor $W \otimes W'$ for which the associated zeta integral is explicitly computable and non-zero. The precise result that we prove is the following.

Theorem 1.1. *Let $\{\alpha_i\}_{i=1}^n$ and $\{\gamma_j\}_{j=1}^m$ denote the Langlands parameters of π and τ , respectively, and let $L(s, \pi \times \tau)$ be the naive Rankin–Selberg L -factor defined by*

$$L(s, \pi \times \tau) = \prod_{i=1}^n \prod_{j=1}^m (1 - \alpha_i \gamma_j q^{-s})^{-1}.$$

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Then there is a pair $(W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tau, \psi^{-1})$, described explicitly in §3, such that

$$\Psi(s; W, W') = L(s, \pi \times \tau).$$

When $\Psi(s; W, W') = L(s, \pi \boxtimes \tau)$, the pair (W, W') is called a *test vector* for (π, τ) . Hence the theorem produces a test vector whenever $L(s, \pi \times \tau) = L(s, \pi \boxtimes \tau)$ —for instance, if either π or τ is unramified or if $L(s, \pi \boxtimes \tau) = 1$. In general, one has $L(s, \pi \times \tau) = P(q^{-s})L(s, \pi \boxtimes \tau)$ for a non-zero polynomial $P \in \mathbb{C}[X]$ (see Lemma 2.1).

The overview of our method is as follows. Let ξ^0 (resp. φ^0) denote the “essential vector” in the space of π (resp. τ), and let $W_{\xi^0} \in \mathcal{W}(\pi, \psi)$ (resp. $W_{\varphi^0} \in \mathcal{W}(\tau, \psi^{-1})$) be the associated essential Whittaker functions, as described in detail in §2. When τ is unramified, it follows from [10, Corollary 3.3] that

$$(1.2) \quad L(s, \pi \boxtimes \tau) = \int_{U_m(F) \backslash \mathrm{GL}_m(F)} W_{\xi^0} \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(h) \|\det h\|^{s - \frac{n-m}{2}} dh$$

for a suitable normalization of the measure on $U_m(F) \backslash \mathrm{GL}_m(F)$. When $m = n - 1$, the above equality is part of the characterization of the essential vector in [4, 6]; the fact that it holds for any $m < n$ is the result of a concrete realization of essential functions in [10]. On the other hand, if τ is ramified then the local integral in (1.2) vanishes. Through a process of unipotent averaging (see (3.1) below), we modify W_{ξ^0} to obtain a Whittaker function $W \in \mathcal{W}(\pi, \psi)$ such that the resulting zeta integral $\Psi(s; W, W_{\varphi^0})$ equals $cL(s, \pi \times \tau)$ for a non-zero number $c \in \mathbb{C}$, depending on the conductor of τ , its central character ω_τ , and ψ . Setting $W' = c^{-1}W_{\varphi^0}$, we obtain the required pair (W, W') .

We mention some related results in the literature. First, if π and τ are discrete series representations, then the existence of a test vector (W, W') was shown in [9], but the Whittaker function W' there is taken to be in a larger space, namely the Whittaker space associated to the standard module of τ . Second, the so-called *local Birch lemma*, arising in the context of \mathfrak{p} -adic interpolation of special values of twisted Rankin–Selberg (global) L -functions, is also related. It concerns evaluation of a local integral in the special case that π is unramified and τ is the twist of an unramified representation by a character with non-trivial conductor; see [8, Proposition 3.1] and [7, Theorem 2.1]. The approach in [8] is similar to ours in that it also uses a process of unipotent averaging in order to modify the Whittaker function on the larger general linear group. We can of course apply Theorem 1.1 to their setup: Since $L(s, \pi \times \tau) = 1$ in this case, the pair (W, W_{φ^0}) described above has the property that $\Psi(s; W, W_{\varphi^0})$ is an explicit constant (independent of s).

Finally, note that one can obtain a global version of Theorem 1.1 by combining the test vectors at all (finite) places. In work in progress, we study the analogous question over an archimedean local field.

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2. PRELIMINARIES

Let \mathfrak{p} be the unique maximal ideal in \mathfrak{o} . We fix a generator ϖ of \mathfrak{p} with absolute value $\|\varpi\| = q^{-1}$. Let $v : F^\times \rightarrow \mathbb{Z}$ denote the valuation map, and extend it to fractional ideals in the usual way. For any $n > 1$, let $B_n = \mathrm{T}_n U_n$ be the Borel subgroup of GL_n consisting of upper triangular matrices; let $P'_n \supset B_n$ be the standard parabolic subgroup of type

$(n-1, 1)$ with Levi decomposition $P'_n = M_n N_n$. Then $M_n \cong \mathrm{GL}_{n-1} \times \mathrm{GL}_1$ and

$$N_n = \left\{ \begin{pmatrix} I_{n-1} & * \\ & 1 \end{pmatrix} \right\}.$$

Also, we write Z_n to denote the center consisting of scalar matrices and $A_n \subset T_n$ to denote the subtorus consisting of diagonal matrices with lower-right entry 1.

If R is any F -algebra and H is any algebraic F -group, we write $H(R)$ to denote the corresponding group of R -points. Let $P_n(R) \subset P'_n(R)$ denote the mirabolic subgroup consisting of matrices whose last row is of the form $(0, \dots, 0, 1)$, i.e.,

$$P_n(R) = \left\{ \begin{pmatrix} h & y \\ & 1 \end{pmatrix} : h \in \mathrm{GL}_{n-1}(R), y \in R^{n-1} \right\} \cong \mathrm{GL}_{n-1}(R) \ltimes N_n(R).$$

The character

$$u \mapsto \psi \left(\sum_{i=1}^{n-1} u_{i,i+1} \right) \quad \text{for } u \in U_n(F)$$

defines a *generic character* of $U_n(F)$, and by abuse of notation we continue to denote this character by ψ . Further, for any algebraic subgroup $V \subseteq U_n$, ψ defines a character of $V(F)$ via restriction. In particular, we may consider the character $\psi|_{N_n(F)}$; its stabilizer in $M_n(F)$ is then $P_{n-1}(F)$, where we regard P_{n-1} as a subgroup of M_n via $h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix}$.

An irreducible representation (π, V_π) of $\mathrm{GL}_n(F)$ is said to be *generic* if

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}(V_\pi, \mathrm{Ind}_{U_n(F)}^{\mathrm{GL}_n(F)} \psi) \neq 0.$$

By Frobenius reciprocity, this means that there is a non-zero linear form $\lambda : V_\pi \rightarrow \mathbb{C}$ satisfying $\lambda(\pi(u)v) = \psi(u)\lambda(v)$ for $v \in V_\pi, u \in U_n(F)$. It is known (see [3]) that for a generic π the space of such linear functionals, or equivalently the space $\mathrm{Hom}_{\mathrm{GL}_n(F)}(V_\pi, \mathrm{Ind}_{U_n(F)}^{\mathrm{GL}_n(F)} \psi)$, is of dimension 1. Let $\mathcal{W}(\pi, \psi)$ denote the Whittaker model of π , viz. the space of functions W_v on $\mathrm{GL}_n(F)$ defined by $W_v(g) = \lambda(\pi(g)v)$ for $v \in V_\pi$. Then $\mathcal{W}(\pi, \psi)$ is independent of the choice of λ , and for $u \in U_n(F), g \in \mathrm{GL}_n(F)$,

$$\begin{aligned} W_v(ug) &= \psi(u)W(g), \\ W_v(g) &= W_{\pi(g)v}(I_n). \end{aligned}$$

We will consider certain compact open subgroups of $\mathrm{GL}_n(F)$; namely, for any integer $f \geq 0$, set

$$\begin{aligned} K_1(\mathfrak{p}^f) &= \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) : g \equiv \begin{pmatrix} & * \\ & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}^f} \right\}, \\ K_0(\mathfrak{p}^f) &= \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) : g \equiv \begin{pmatrix} & * \\ & \vdots \\ 0 & \dots & 0 & * \end{pmatrix} \pmod{\mathfrak{p}^f} \right\}, \end{aligned}$$

so that $K_1(\mathfrak{p}^f)$ is a normal subgroup of $K_0(\mathfrak{p}^f)$, with quotient $K_0(\mathfrak{p}^f)/K_1(\mathfrak{p}^f) \cong (\mathfrak{o}/\mathfrak{p}^f)^\times$.

Next we introduce our choice of measures. For $n \geq 1$ we normalize the Haar measure on $\mathrm{GL}_n(F)$ and $\mathrm{GL}_n(\mathfrak{o})$ so that $\mathrm{vol}(\mathrm{GL}_n(\mathfrak{o})) = 1$, and we fix the Haar measure on $U_n(F)$ for which $\mathrm{vol}(U_n(F) \cap \mathrm{GL}_n(\mathfrak{o})) = 1$. From these, we obtain a right-invariant measure on $U_n(F) \backslash \mathrm{GL}_n(F)$. We may make this explicit using the Iwasawa decomposition. For instance, let dx be the Haar measure on F such that \mathfrak{o} has unit volume, and let $d^\times x$ be the multiplicative measure on F^\times such that $\mathrm{vol}(\mathfrak{o}^\times) = 1$, i.e., $d^\times x = \frac{q}{q-1} \frac{dx}{\|x\|}$. Let dz and da be the corresponding measures on the center $Z_n(F) \cong F^\times$ and the subtorus $A_n(F) \cong$

2.1. Rankin–Selberg L -functions. In this subsection alone we drop the assumption that $m < n$ and allow (m, n) to be an arbitrary pair of positive integers. For π and τ irreducible, admissible, generic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively, let $L(s, \pi \boxtimes \tau)$ be as defined in [5]. When $m < n$, $L(s, \pi \boxtimes \tau)$ is defined as in the introduction. For $m > n$, one defines $L(s, \pi \boxtimes \tau) = L(s, \tau \boxtimes \pi)$. For $m = n$, the defining local integrals are different and involve a Schwartz function on F^n ; see loc. cit.

Next we elaborate on the definition of the naive Rankin–Selberg L -factor, $L(s, \pi \times \tau)$, introduced in Theorem 1.1. By definition, the L -function $L(s, \pi)$ is of the form $P_\pi(q^{-s})^{-1}$, where $P_\pi \in \mathbb{C}[X]$ has degree at most n and satisfies $P_\pi(0) = 1$. We may then find n complex numbers $\{\alpha_i\}_{i=1}^n$ (allowing some of them to be zero) satisfying

$$L(s, \pi) = \prod_{i=1}^n (1 - \alpha_i q^{-s})^{-1}.$$

We call the set $\{\alpha_i\}$ the *Langlands parameters* of π ; if π is spherical, they agree with the usual Satake parameters. Let $\{\gamma_j\}_{j=1}^m$ be the Langlands parameters of τ , and set

$$(2.5) \quad L(s, \pi \times \tau) = \prod_{i=1}^n \prod_{j=1}^m (1 - \alpha_i \gamma_j q^{-s})^{-1}.$$

$L(s, \pi \times \tau) = L(s, \pi \boxtimes \tau)$ if π or τ is spherical.

In the following lemma we describe the connection between $L(s, \pi \times \tau)$ and $L(s, \pi \boxtimes \tau)$. To that end, we first recall the classification of irreducible admissible representations of $\mathrm{GL}_n(F)$. Let \mathcal{A}_n denote the set of equivalence classes of such representations, and put $\mathcal{A} = \bigcup \mathcal{A}_n$. The essentially square-integrable representations of $\mathrm{GL}_n(F)$ have been classified by Bernstein and Zelevinsky, and they are as follows. If σ is an essentially square-integrable representation of $\mathrm{GL}_n(F)$, then there is a divisor $a \mid n$ and a supercuspidal representation η of $\mathrm{GL}_a(F)$ such that if $b = \frac{n}{a}$ and Q is the standard (upper) parabolic subgroup of $\mathrm{GL}_n(F)$ of type (a, \dots, a) , then σ can be realized as the unique quotient of the (normalized) induced representation

$$\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_n(F)}(\eta, \eta \|\cdot\|, \dots, \eta \|\cdot\|^{b-1}).$$

The integer a and the class of η are uniquely determined by σ . In short, σ is parametrized by b and η , and we denote this by $\sigma = \sigma_b(\eta)$; further, σ is square-integrable (also called “discrete series”) if and only if the representation $\eta \|\cdot\|^{\frac{b-1}{2}}$ of $\mathrm{GL}_a(F)$ is unitary.

Now, let P be an upper parabolic subgroup of $\mathrm{GL}_n(F)$ of type (n_1, \dots, n_r) . For each $i = 1, \dots, r$, let τ_i^0 be a discrete series representation of $\mathrm{GL}_{n_i}(F)$. Let (s_1, \dots, s_r) be a sequence of real numbers satisfying $s_1 \geq \dots \geq s_r$, and put $\tau_i = \tau_i^0 \otimes \|\cdot\|^{s_i}$ (an essentially square-integrable representation). Then the induced representation

$$\xi = \mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL}_n(F)}(\tau_1 \otimes \dots \otimes \tau_r)$$

is said to be a representation of $\mathrm{GL}_n(F)$ of Langlands type. If $\tau \in \mathcal{A}_n$, then it is well known that it is uniquely representable as the quotient of an induced representation of Langlands type. We write $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$ to denote this realization of τ . Thus one obtains a sum operation on the set \mathcal{A} [5, §9.5]. It follows easily from the definition that $L(s, \pi \times \tau)$ is bi-additive, i.e.

$$\begin{aligned} L(s, \pi \times (\tau \boxplus \tau')) &= L(s, \pi \times \tau) L(s, \pi \times \tau') \\ L(s, (\pi \boxplus \pi') \times \tau) &= L(s, \pi \times \tau) L(s, \pi' \times \tau) \end{aligned}$$

for all $\pi, \pi', \tau, \tau' \in \mathcal{A}$. The local factor $L(s, \pi \boxtimes \tau)$ is also bi-additive in the above sense, by [5, §9.5, Theorem].

Lemma 2.1. *Let m and n be positive integers, and consider $\pi \in \mathcal{A}_n$, $\tau \in \mathcal{A}_m$. Then*

$$L(s, \pi \times \tau) = P(q^{-s})L(s, \pi \boxtimes \tau)$$

for a polynomial $P \in \mathbb{C}[X]$ (depending on π and τ) satisfying $P(0) = 1$.

Proof. Since π and τ are sums of essentially square-integrable representations and $L(s, \pi \times \tau)$ and $L(s, \pi \boxtimes \tau)$ are both additive with respect to \boxplus , it suffices to prove the lemma for a pair (π, τ) of essentially square-integrable representations. In particular, assume $\pi = \sigma_b(\eta)$ as above.

We proceed by induction on m . If $m = 1$ then $\tau = \chi$ is a quasi-character of F^\times and $L(s, \pi \boxtimes \tau) = L(s, \pi \otimes \chi)$, where $\pi \otimes \chi$ is the representation of $\mathrm{GL}_n(F)$ defined by $g \mapsto \pi(g)\chi(\det g)$. If χ is unramified, then

$$L(s, \pi \otimes \chi) = L(s, \pi \times \chi),$$

and consequently $P = 1$. On the other hand, if χ is ramified then $L(s, \pi \times \chi) = 1$, and the assertion follows since $L(s, \pi \otimes \chi)^{-1}$ is a polynomial in q^{-s} .

We now assume $m > 1$ and τ is an essentially square-integrable representation of $\mathrm{GL}_m(F)$, say $\tau = \sigma_{b'}(\eta')$, where $\eta' \in \mathcal{A}_{a'}$ is supercuspidal and $a'b' = m$. Then the standard L -factor $L(s, \tau)$ is given by $L(s, \tau) = L(s + b' - 1, \eta')$ [5]. Therefore, $L(s, \tau) = 1$ unless $a' = 1$ and $\eta' = \chi$ is an unramified quasi-character of F^\times . On the other hand, if $L(s, \tau) = 1$, then $L(s, \pi \times \tau) = 1$ and the assertion of the lemma follows. Hence we may assume $\tau = \sigma_m(\chi)$ for an unramified quasi-character χ of F^\times , in which case

$$(2.6) \quad \begin{aligned} L(s, \pi \times \tau) &= L(s, \pi \otimes \chi \|\cdot\|^{m-1}) = L(s, \sigma_b(\eta) \otimes \chi \|\cdot\|^{m-1}) \\ &= L(s + m - 1 + b - 1, \eta \otimes \chi). \end{aligned}$$

On the other hand, it follows from [5, §8.2, Theorem] that

$$(2.7) \quad L(s, \pi \boxtimes \tau) = \begin{cases} \prod_{j=0}^{m-1} L(s + j + b - 1, \eta \otimes \chi) & \text{if } m \leq n, \\ \prod_{i=0}^{b-1} L(s + m - 1 + i, \eta \otimes \chi) & \text{if } m > n. \end{cases}$$

From (2.6) and (2.7), one sees that the ratio $\frac{L(s, \pi \times \tau)}{L(s, \pi \boxtimes \tau)}$ is a polynomial in q^{-s} , thus proving the lemma. \square

Corollary 2.2. *If $L(s, \pi \boxtimes \tau) = 1$ then either $L(s, \pi) = 1$ or $L(s, \tau) = 1$.*

Proof. If $L(s, \pi \boxtimes \tau) = 1$ then Lemma 2.1 implies that $L(s, \pi \times \tau)$ is a polynomial in q^{-s} , and hence must be 1. This in turn implies the conclusion. \square

3. THE MAIN CALCULATION

Recall that ξ^0 and φ^0 are the essential vectors of π and τ , respectively. Here we construct a pair $(W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tau, \psi^{-1})$ as in Theorem 1.1. Let \mathfrak{n} , \mathfrak{q} and \mathfrak{c} denote the conductors of π , τ and ω_τ , respectively. If τ is an unramified representation of $\mathrm{GL}_m(F)$, then by (1.2) we have

$$\Psi(s; W_{\xi^0}, W_{\varphi^0}) = L(s, \pi \boxtimes \tau) = L(s, \pi \times \tau).$$

Thus, in this case we can take $(W, W') = (W_{\xi^0}, W_{\varphi^0})$.

Let us assume from now on that τ is ramified, meaning $v(\mathfrak{q}) > 0$. Since $\mathfrak{c} \supseteq \mathfrak{q}$, we have $v(\mathfrak{c}) \leq v(\mathfrak{q})$. Consider $\beta = (\beta_1, \dots, \beta_m) \in F^m$, with $\beta_i \in \mathfrak{q}^{-1}$ for $i = 1, \dots, m$, and

let $u(\beta)$ denote the $n \times n$ matrix with 1s on the diagonal and β^t embedded above the diagonal in the $(m+1)^{\text{st}}$ column. Let ξ_β^0 denote the vector $\xi_\beta^0 = \pi(u(\beta))\xi^0$, and define

$$(3.1) \quad \bar{\xi} = \frac{1}{[\mathfrak{o} : \mathfrak{q}]^{m-1}} \sum_{(\beta_1, \dots, \beta_{m-1}) \in (\mathfrak{q}^{-1}/\mathfrak{o})^{m-1}} \xi_{(\beta_1, \dots, \beta_{m-1}, \varpi^{-v(c)})}^0.$$

(When $m = 1$ we understand there to be one summand, so that $\bar{\xi} = \xi_{(\varpi^{-v(c)})}^0$.) We will now calculate $\Psi(s; W_{\bar{\xi}}, W_{\varphi^0})$, which by linearity equals

$$\frac{1}{[\mathfrak{o} : \mathfrak{q}]^{m-1}} \sum_{(\beta_1, \dots, \beta_{m-1}) \in (\mathfrak{q}^{-1}/\mathfrak{o})^{m-1}} \Psi(s; W_{\xi_{(\beta_1, \dots, \beta_{m-1}, \varpi^{-v(c)})}^0}, W_{\varphi^0}).$$

Put $K = \text{GL}_m(\mathfrak{o})$. By (2.2), for fixed $\beta = (\beta_1, \dots, \beta_m)$, we have

$$\begin{aligned} \Psi(s; W_{\xi_\beta^0}, W_{\varphi^0}) &= \int_{Z_m(F) \times A_m(F) \times K} \omega_\tau(z) W_{\xi^0} \left(\begin{pmatrix} z a k & \\ & I_{n-m} \end{pmatrix} u(\beta) \right) W_{\varphi^0}(a k) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|\det(z a)\|^{s - \frac{n-m}{2}} dz da dk \\ &= \int_{F \times A_m(F) \times K} \omega_\tau(z) \left(\sum_{j=1}^m \psi(z k_j \beta_j) \right) W_{\xi^0} \begin{pmatrix} z a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a k) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|z^m \det a\|^{s - \frac{n-m}{2}} d^\times z da dk, \end{aligned}$$

where (k_1, \dots, k_m) is the bottom row of the matrix k . Here we have used the fact that the function $h \mapsto W_{\xi^0} \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix}$, $h \in \text{GL}_m(F)$, is right K -invariant. Now, performing the average over $\beta_j \in \mathfrak{q}^{-1}/\mathfrak{o}$ for each $j < m$, we see that $\Psi(s; W_{\bar{\xi}}, W_{\varphi^0})$ equals

$$\begin{aligned} &\int_{\left\{ \begin{array}{l} F \times A_m(F) \times K \\ z k_j \in \mathfrak{q} \forall j < m \end{array} \right\}} \omega_\tau(z) \psi(z k_m \varpi^{-v(c)}) W_{\xi^0} \begin{pmatrix} z a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a k) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|z^m \det a\|^{s - \frac{n-m}{2}} d^\times z da dk. \end{aligned}$$

By (2.4) we have $W_{\xi^0}(t(a_1, \dots, a_{n-1})) = 0$ unless $a_1, \dots, a_{n-1} \in \mathfrak{o}$. In view of (2.1), it follows that the integrand vanishes unless z is integral.

Note that

$$z k_j \in \mathfrak{q} \forall j < m \iff k \in K_0(z^{-1} \mathfrak{q} \cap \mathfrak{o}).$$

For $r \in \mathbb{Z}_{\geq 0}$, put

$$\begin{aligned} \Psi_r &= \int_{A_m(F) \times K_0(\mathfrak{q} \mathfrak{p}^{-r} \cap \mathfrak{o})} \mathcal{G}(\omega_\tau, \psi, \varpi^{r-v(c)} k_m) W_{\xi^0} \begin{pmatrix} \varpi^r a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a k) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|\det a\|^{s - \frac{n-m}{2}} da dk, \end{aligned}$$

where $\mathcal{G}(\omega_\tau, \psi, y)$ denotes the Gauss sum

$$\mathcal{G}(\omega_\tau, \psi, y) = \int_{\mathfrak{o}^\times} \omega_\tau(z) \psi(y z) d^\times z.$$

Then we have

$$\Psi(s; W_{\bar{\xi}}, W_{\varphi^0}) = \sum_{r \geq 0} \omega_\tau(\varpi^r) q^{-rm(s - \frac{n-m}{2})} \Psi_r.$$

Suppose that ω_τ is ramified, so that $v(\mathfrak{c}) > 0$. Then $\mathcal{G}(\omega_\tau, \psi, \varpi^{r-v(\mathfrak{c})}k_m)$ vanishes unless $v(\varpi^{r-v(\mathfrak{c})}k_m) = -v(\mathfrak{c})$, which implies $v(k_m) = -r$. Since k_m is integral, it follows that $r = 0$ is the only contributing term to $\Psi(s; W_{\bar{\xi}}, W_{\varphi^0})$, so that

$$\Psi(s; W_{\bar{\xi}}, W_{\varphi^0}) = \int_{A_m(F) \times K_0(\mathfrak{q})} \mathcal{G}(\omega_\tau, \psi, \varpi^{-v(\mathfrak{c})}k_m) W_{\xi^0} \begin{pmatrix} a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(ak) \cdot \delta_{B_m}(a)^{-1} \|\det a\|^{s-\frac{n-m}{2}} da dk.$$

Moreover, since $k_m \in \mathfrak{o}^\times$, we have $\mathcal{G}(\omega_\tau, \psi, \varpi^{-v(\mathfrak{c})}k_m) = \omega_\tau(k_m)^{-1} \mathcal{G}(\omega_\tau, \psi, \varpi^{-v(\mathfrak{c})})$, and thus

$$(3.2) \quad \Psi(s; W_{\bar{\xi}}, W_{\varphi^0}) = c \int_{A_m(F)} W_{\xi^0} \begin{pmatrix} a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a) \delta_{B_m}(a)^{-1} \|\det a\|^{s-\frac{n-m}{2}} da,$$

where

$$c = \frac{\mathcal{G}(\omega_\tau, \psi, \varpi^{-v(\mathfrak{c})})}{[\mathrm{GL}_m(\mathfrak{o}) : K_0(\mathfrak{q})]} \neq 0.$$

Suppose now that ω_τ is unramified, so that $\mathfrak{c} = \mathfrak{o}$ and $m > 1$. Then $\mathcal{G}(\omega_\tau, \psi, \varpi^{r-v(\mathfrak{c})}k_m) = 1$ for all r . If $r > 0$ then $K_0(\mathfrak{q}\mathfrak{p}^{-r} \cap \mathfrak{o}) \supsetneq K_0(\mathfrak{q})$; since the conductor of τ is \mathfrak{q} , it follows from [4, Theorem 5.1] that

$$\int_{K_0(\mathfrak{q}\mathfrak{p}^{-r} \cap \mathfrak{o})} W_{\varphi^0}(ak) dk = 0,$$

which in turn implies that $\Psi_r = 0$. Hence only the $r = 0$ term contributes, and again we arrive at (3.2).

It remains only to identify the integral over $A_m(F)$.

Lemma 3.1. *When τ is ramified, we have*

$$\int_{A_m(F)} W_{\xi^0} \begin{pmatrix} a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a) \delta_{B_m}(a)^{-1} \|\det a\|^{s-\frac{n-m}{2}} da = L(s, \pi \times \tau).$$

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $(\gamma_1, \dots, \gamma_m)$ denote the Langlands parameters of π and τ , respectively. Since τ is ramified, we may take $\gamma_m = 0$. We set $\gamma_{m+1} = \dots = \gamma_n = 0$ and write $\gamma = (\gamma_1, \dots, \gamma_n)$.

Writing $a = t(a_1, \dots, a_{m-1})$ as in (2.1), by (2.4) we see that $W_{\varphi^0}(a)$ vanishes unless each a_i is integral. Setting

$$\lambda_i = \sum_{1 \leq j \leq m-i} v(a_j) \quad \text{for } i = 1, \dots, n,$$

the integral in question may be written as

$$\sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_{m-1} \\ \lambda_m = \dots = \lambda_n = 0}} W_{\xi^0}(d_n(\lambda)) W_{\varphi^0}(d_m(\lambda)) \delta_{B_m}(d_m(\lambda))^{-1} \|\det d_m(\lambda)\|^{s-\frac{n-m}{2}},$$

where

$$d_n(\lambda) = \mathrm{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_n}) \quad \text{and} \quad d_m(\lambda) = \mathrm{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_m}).$$

On the other hand, by [11, Theorem 4.1], we have

$$W_{\xi^0}(d_n(\lambda)) = \delta_{B_n}(d_n(\lambda))^{\frac{1}{2}} s_\lambda(\alpha) \quad \text{and} \quad W_{\varphi^0}(d_m(\lambda)) = \delta_{B_m}(d_m(\lambda))^{\frac{1}{2}} s_\lambda(\gamma),$$

where s_λ denotes the Schur polynomial

$$s_\lambda(X_1, \dots, X_n) = \frac{\det(X_j^{\lambda_i+n-i})_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (X_i - X_j)}$$

(see [1, Theorem 38.1]). Thus the integral becomes

$$\begin{aligned} & \sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_{m-1} \\ \lambda_m = \dots = \lambda_n = 0}} s_\lambda(\alpha) s_\lambda(\gamma) \left(\frac{\delta_{B_n}(d_n(\lambda))}{\delta_{B_m}(d_m(\lambda))} \right)^{\frac{1}{2}} \|\det d_m(\lambda)\|^{s - \frac{n-m}{2}} \\ &= \sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_n}} s_\lambda(\alpha) s_\lambda(\gamma) \|\det d_m(\lambda)\|^s, \end{aligned}$$

where the last line follows from (2.3) and the fact that $\gamma_m = \dots = \gamma_n = 0$.

By the Cauchy identity [1, Theorem 43.3], for sufficiently small $x \in \mathbb{C}$, we have

$$\prod_{i=1}^n \prod_{j=1}^n (1 - x \alpha_i \gamma_j)^{-1} = \sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_n}} s_\lambda(x\alpha) s_\lambda(\gamma) = \sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_n}} s_\lambda(\alpha) s_\lambda(\gamma) x^{\lambda_1 + \dots + \lambda_n}.$$

Therefore,

$$\sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_n}} s_\lambda(\alpha) s_\lambda(\gamma) \|\det d_m(\lambda)\|^s = \prod_{i=1}^n \prod_{j=1}^n (1 - \alpha_i \gamma_j q^{-s})^{-1} = L(s, \pi \times \tau).$$

□

Finally, we take $W = W_{\bar{\xi}}$ and $W' = c^{-1}W_{\varphi_0}$ to conclude the proof of Theorem 1.1.

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