Region of attraction analysis with Integral Quadratic Constraints ⋆

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Abstract

A general framework is presented to estimate the Region of Attraction of attracting equilibrium points. The system is described by a feedback connection of a nonlinear (polynomial) system and a bounded operator. The input/output behavior of the operator is characterized using an Integral Quadratic Constraint. This allows to analyze generic problems including, for example, hard-nonlinearities and different classes of uncertainties, adding to the state of practice in the field which is typically limited to polynomial vector fields. The IQC description is also nonrestrictive, with the main result given for both hard and soft factorizations. Optimization algorithms based on Sum of Squares techniques are then proposed, with the aim to enlarge the inner estimates of the ROA. Numerical examples are provided to show the applicability of the approaches. These include a saturated plant where bounds on the states are exploited to refine the sector description, and a case study with parametric uncertainties for which the conservativeness of the results is reduced by using soft IQCs.

Key words: Region of attraction, Integral quadratic constraints, Nonlinear uncertain systems, Local analysis, Dissipation Inequality.

1 Introduction

Stability guarantees are often valid only locally for nonlinear systems, and for this reason the notion of Region of Attraction (ROA) has been proposed (Khalil 1996). The ROA of an equilibrium point x^* is the set of all the initial conditions from which the trajectories of the system converge to x^* as time goes to infinity. This paper proposes a new framework for the analysis of the ROA for generic uncertain systems. In full generality, the problem considered in this article consists of the feedback interconnection of a system G with polynomial vector field and a bounded causal operator Δ. Motivation for this kind of description stems from the objective to compute ROA of systems which are affected by generic nonlinearities (in addition to the polynomial ones) and/or uncertainties.

The Integral Quadratic Constraint (IQC) (Megretski and Rantzer 1997) paradigm, building on work by Yakubovich (Yakubovich 1971), is particularly suited to address the aforementioned operator Δ, because it characterizes a broad class of nonlinearities, and allows to refine the description of the uncertainties by specifying their nature. The setup provided by the feedback interconnection G-Δ is thus believed to be quite general and to adequately cover a large class of nonlinear systems encountered in applications.

The time domain interpretation of IQCs is instrumental to prove the main results of the paper. In particular, the connection between dissipation inequality and IQC is exploited to provide guarantees of local stability. One of the known issues is that the dissipation inequality requires the IQC to be *hard* in the sense that the integral constraints must hold over all finite times (Seiler 2015). This is not immediate because the frequency domain IQC only guarantees an equivalent counter part in the time domain as an infinite-horizon integral constraint (*soft* IQC) (Megretski and Rantzer 1997). Recent studies have proposed lower bounds on the finite-horizon integral constraint when only the soft property holds (Seiler 2015, Seiler 2018, Fetzer *et al.* 2018). In (Fetzer *et al.* 2018), this was provided as a convex constraint on the IQC multiplier, and it will be employed in this work.

The exact ROA is often difficult to compute either numerically or analytically (Genesio *et al.* 1985), therefore algorithms have been proposed to numerically calculate inner estimates of the ROA. The state of practice in the field focuses on determining Lyapunov function level sets, which are contractive and invariant and thus

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are subsets of the ROA (Chakraborty *et al.* 2011*b*, Valmorbida *et al.* 2009). Non-Lyapunov methods have also been studied to reduce the conservatism typically associated with the characterization of ROA as contractive level sets (Valmorbida and Anderson 2017, Henrion and Korda 2014). All the approaches above share the common feature that are either only applicable to polynomial vector fields or rely on Sum Of Squares (SOS) techniques. As a result, a limitation holds on the types of nonlinearities that can be considered. An example of a more general approach is the so-called *Zubov's method* (Zubov 1964), which is based on a converse Lyapunov theorem and requires to solve a partial differential equation, but this makes it difficult to be employed for practical cases. Relaxed versions of this result have also been proposed, for example in (Vannelli and Vidyasagar 1985) where the concept of maximal Lyapunov function was introduced. Possible extensions of Lyapunov (Topcu *et al.* 2010, Chesi 2004) and non-Lyapunov (Iannelli *et al.* 2019) methods to the case of uncertain systems have also been proposed recently. In general, a major drawback of the approaches employed to deal with uncertain systems is that they do not exploit specific properties of the uncertainties (e.g. linear time invariant, real constant). This inherently leads to conservative outcomes because the results must hold for a larger set of uncertainties than the one actually affecting the system.

The contribution of this article is therefore to propose a general and flexible framework for local stability analysis of nonlinear uncertain systems. The problem is formulated by defining an augmented plant which comprises the polynomial part of the vector field G as well as the Linear Time Invariant (LTI) system Ψ provided by the state-space factorization of the IQC. Based on this problem setup, Section 3 establishes certificates for the domain of attraction with both hard and soft factorizations. Specifically, the ROA is formulated as the level set of a polynomial function of generic degree (which is not necessarily a Lyapunov function for the system). Since the fictitious plant and the sought function are polynomial, the problem can be solved numerically via Sum of Squares (SOS) techniques, allowing it to be recast as a set of semidefinite programs (Parrilo 2003).

Numerical examples of polynomial systems affected by hard nonlinearities (i.e. actuator saturation) and real parametric uncertainties illustrate the application of the approaches in Section 4. In the former case, the bounds on the states inherently given by ROA are employed to provide a less conservative expression for the sector IQC. As for the case with parametric uncertainties, the favourable feature of this framework of allowing to refine the description of the uncertainties is showcased.

The work here, extending preliminary results in (Iannelli *et al.* 2018*b*) which only considered the case of hard IQC, is related to recent studies (Seiler 2015, Pfifer and Seiler 2015, Fetzer *et al.* 2018, Seiler 2018) which focused on the reconciliation between Lyapunov function methods and multiplier theory. In particular, the distinction between hard and soft IQCs plays a crucial role in the estimation of the domain of attraction. In view of this, the active area of research concerned with finding less conservative bounds for soft IQCs (Seiler 2015, Fetzer *et al.* 2018, Seiler 2018) can have a big impact on the estimation of local stability regions with the framework presented here. This paper is also connected to the work in (Chakraborty *et al.* 2010), where the time domain interpretation of IQC (hard IQCs only were considered) was exploited for performance analysis of polynomial systems subject to hard nonlinearities.

2 Background

2.1 Notation

$\mathbb{RL}_\infty$ denotes the set of rational functions with real coefficients that are proper and have no poles on the imaginary axis. $\mathbb{RH}_\infty$ is the subset of functions in $\mathbb{RL}_\infty$ that are analytic in the closed right half of the complex plane. $\mathbb{RL}_\infty^{m\times n}$ and $\mathbb{RH}_\infty^{m\times n}$ denote the sets of $m \times n$ matrices whose elements are in $\mathbb{RL}_\infty$ and $\mathbb{RH}_\infty$ respectively. Vertical concatenation of two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ is denoted by $[x; y] \in \mathbb{R}^{n+m}$. For a matrix $M \in \mathbb{C}^{m\times n}$, $M^\top$ and M^* denotes respectively the transpose and the complex conjugate transpose. The para-Hermitian conjugate of $G \in \mathbb{RL}_\infty^{m\times n}$, denoted as $G^\sim$, is defined by $G^\sim(s) := G(-\bar{s})^*$, where $\bar{s}$ is the complex conjugate of s. $\mathcal{L}_2^n$ is the space of all square integrable functions $v : [0,\infty) \to \mathbb{R}^n$, i.e. satisfying $\|v\|_2 < \infty$ where $||v||_2 := \left(\int_0^\infty v(t)^\top v(t)dt \geq 0\right)^{1/2}$. $\mathbb{R}[x]$ indicates the set of all polynomials $r : \mathbb{R}^n \to \mathbb{R}$ in n variables, and $\partial(r)$ indicates the degree of r. Given a scalar $c > 0$, the level set of r is defined as $\Omega_{r,c} = \{x \in \mathbb{R}^n : r(x) \leq c\}$. A polynomial g is said to be a Sum of Squares if there exists a finite set of polynomials $g_1, ..., g_k$ such that $g = \sum_{i=1}^k g_i^2$. The set of SOS polynomials in x will be denoted by $\Sigma[x]$.

2.2 Problem statement

Consider an autonomous nonlinear system of the form:

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field. The vector $x^* \in \mathbb{R}^n$ is called a *fixed* or *equilibrium* point of Eq. (1) if $f(x^*) = 0$. Let $\phi(t, x_0)$ denote the solution of Eq. (1) at time t with initial condition x_0. The ROA associated with x^* is defined as:

$$\mathcal{R} := \left\{x_0 \in \mathbb{R}^n : \lim_{t\to\infty} \phi(t, x_0) = x^*\right\} \tag{2}$$

For nonlinear systems it generally holds $\mathcal{R} \subseteq \mathbb{R}^n$.

Computing the exact ROA for nonlinear systems is difficult (Genesio *et al.* 1985). A standard approach to calculate inner estimates consists of finding Lyapunov function level sets.

Lemma 1 *(Khalil 1996) Let $\mathcal{D} \subset \mathbb{R}^n$ and let $x^* \in \mathcal{D}$. If there exists a 1-time continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that:*

$$\begin{aligned}
&V(x^*) = 0 \quad \text{and} \quad V(x) > 0 && \forall x \in \mathcal{D}\backslash x^* \\
&\nabla V(x) f(x) < 0 && \forall x \in \mathcal{D}\backslash x^* \\
&\Omega_{V,\gamma} = \{x \in \mathbb{R}^n : V(x) \leq \gamma\} && \text{is bounded and} \\
&\Omega_{V,\gamma} \subseteq \mathcal{D}
\end{aligned} \tag{3}$$

then $\Omega_{V,\gamma} \in \mathcal{R}$.

If f is a polynomial vector field, a function V satisfying the conditions in Lemma 1 can be determined by means of Sums of Squares (SOS) techniques by exploiting their connection with convex optimization (Parrilo 2003).

First, recall that $g \in \Sigma[x]$ if and only if there exists $Q{=}Q^T \succeq 0$ such that $g = z^T\ Q\ z$, where z gathers the monomials of g of degree less than or equal to $\partial(g)/2$. This problem can be recast as a semidefinite program and there are freely available toolboxes to solve this in an efficient manner. In this work, the library SOSOPT (Balas *et al.* n.d.) is used. Lemma 2 reports an application of the Positivstellensatz (P-satz) Theorem, which allows to recast the set containment conditions (3) as SOS constraints.

Lemma 2 *Given* $h, f_0, ..., f_r \in \mathbb{R}[x]$, *the following set containment holds:*

$$\{x : h(x) = 0, f_1(x) \geq 0, ..., f_r(x) \geq 0\} \subseteq \{x : f_0(x) \geq 0\} \tag{4}$$

if there exist polynomials $p \in \mathbb{R}[x]$ *and* $s_1, ..., s_r \in \Sigma[x]$ *such that:*

$$p(x)h(x) - \sum_{i=1}^{r} s_i(x)f_i(x) + f_0(x) \in \Sigma[x] \tag{5}$$

2.3 Integral Quadratic Constraints

IQCs provide a unified framework to assess the robustness of uncertain, nonlinear systems (Megretski and Rantzer 1997). The basic idea is to describe the generic nonlinear uncertain operator Δ by means of IQCs on its input v and output w channels.
Let $\Pi \in \mathbb{RL}_{\infty}^{(n_v+n_w)\times(n_v+n_w)} : j\mathbb{R} \to \mathbb{C}^{(n_v+n_w)\times(n_v+n_w)}$ be a measurable Hermitian-valued function, commonly named multiplier. It is said that the two signals $v \in \mathcal{L}_2^{n_v}$ and $w \in \mathcal{L}_2^{n_w}$ satisfy the IQC defined by Π if:

$$\int_{-\infty}^{+\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{6}$$

where $\hat{v}$ and $\hat{w}$ indicate the Fourier transforms of the corresponding signals. A bounded and causal operator Δ is said to satisfy the frequency domain IQC defined by Π if the signals v and $w = \Delta(v)$ satisfy Eq. (6) for all $v \in \mathcal{L}_2^{n_v}$. We will denote this by writing $\Delta \in$ IQC (Π).

A library of IQCs exists for various types of uncertainties and nonlinearities as summarized in (Megretski and Rantzer 1997, Veenman *et al.* 2016), many of them conveniently derived in the frequency domain. However, the dissipativity framework, used here to provide local stability guarantees, is inherently formulated in time domain, hence it is useful to connect frequency and time domain IQCs (Seiler 2015, Fetzer *et al.* 2018). Let $\Pi \in \mathbb{RL}_{\infty}^{(n_v+n_w)\times(n_v+n_w)}$, and (Ψ, M) be a (non-unique) factorization of $\Pi = \Psi^{\sim}\ M\ \Psi$, where $M = M^{\top} \in \mathbb{R}^{n_z\times n_z}$ and $\Psi \in \mathbb{RH}_{\infty}^{n_z\times(n_v+n_w)}$ is constructed from pre-selected basis transfer functions. Note that M is typically sign indefinite. By Parseval's theorem (Zhou *et al.* 1996), substituting the proposed factorization of Π in Eq. (6), the following holds:

$$\int_{0}^{\infty} z(t)^{\top} M z(t) dt \geq 0 \tag{7}$$

where z is the output of the LTI system Ψ with state matrices A_{Ψ}, $B_{\Psi} = [B_{\Psi 1}\ B_{\Psi 2}]$, C_{Ψ}, and $D_{\Psi} = [D_{\Psi 1}\ D_{\Psi 2}]$:

$$\begin{aligned} \dot{x}_{\Psi} &= A_{\Psi}x_{\Psi} + B_{\Psi 1}v + B_{\Psi 2}w, \quad x_{\Psi}(0) = 0 \\ z &= C_{\Psi}x_{\Psi} + D_{\Psi 1}v + D_{\Psi 2}w \end{aligned} \tag{8}$$

It is stressed here that an important distinction holds for time domain IQCs (Seiler 2015). Namely, a bounded causal operator $\Delta : \mathcal{L}_2^{n_v} \to \mathcal{L}_2^{n_w}$ satisfies the time domain *soft* IQC defined by (Ψ, M), denoted by $\Delta \in$ SoftIQC (Ψ, M), if the inequality in Eq. (7) holds for all $v \in \mathcal{L}_2^{n_v}$ and $w = \Delta(v)$. On the other hand, Δ satisfies the time domain *hard* IQC defined by (Ψ, M), denoted by $\Delta \in$ HardIQC (Ψ, M), if the following inequality holds for all $v \in \mathcal{L}_2^{n_v}$, $w = \Delta(v)$ and for all $T \geq 0$:

$$\int_{0}^{T} z(t)^{\top} M z(t) dt \geq 0 \tag{9}$$

While by Parseval's theorem $\Delta \in$ SoftIQC $(\Psi, M) \to \Delta \in$ IQC (Π) and $\Delta \in$ HardIQC $(\Psi, M) \to \Delta \in$ IQC (Π), the converse is only true in the former case. In fact, $\Delta \in$ IQC does not, in general, imply the existence of a factorization (Ψ, M) for which $\Delta \in$ HardIQC (Ψ, M), the hard/soft property being not inherent to the multiplier Π but dependent on the factorization (Ψ, M).

Example

To exemplify this aspect, let us consider a real constant uncertain parameter $\delta_u \in \mathbb{R}$ satisfying $|\delta_u| \leq k$. A possible frequency domain multiplier Π_{δ_u} is:

$$\Pi_{\delta_u} = \begin{bmatrix} k^2 X(j\omega) & Y(j\omega) \\ Y^*(j\omega) & -X(j\omega) \end{bmatrix} \tag{10}$$

with $X = X^* \geq 0$ and $Y = -Y^*$ bounded functions of ω (Megretski and Rantzer 1997). A possible time domain factorization for Π_{δ_u} is:

$$\Psi_{\delta_u} = \begin{bmatrix} kH_{\nu} & 0 \\ 0 & H_{\nu} \end{bmatrix}; \qquad M_{DG} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\top} & -M_{11} \end{bmatrix} \tag{11}$$

where $H_{\nu} \in \mathbb{RH}_{\infty}^{N_d}$ is a column vector of basis functions (typically chosen as low-pass filters, i.e. $H_{\nu} := [1; \frac{1}{s+a_1}; ...; \frac{1}{s+a_{\nu}}]$), and $M_{11}{=}M_{11}^{\top}$, $M_{12}{=}{-}M_{12}^{\top}$ are decision matrices constrained to satisfy $H_{\nu}^{\sim}(j\omega)M_{11}H_{\nu}(j\omega){\geq}0$. This factorization is a general expression for the so called *D-G* scalings (Veenman and Scherer 2014, Veenman *et al.* 2016), which specifies that δ_u is a constant and real parameter (Zhou *et al.* 1996). It holds that $\delta_u \in$ SoftIQC $(\Psi_{\delta_u}, M_{DG})$.

The special case of (10) with $Y \equiv 0$ corresponds to the frequency-domain IQC for a dynamic, norm-bounded LTI system, i.e. $\delta_u \in \mathbb{RH}_\infty$ and $\|\delta_u\|_\infty \le k$. For this case, a valid time-domain factorization, known as D scalings, is given by (Ψ_{δ_u}, M_D), with M_D obtained from M_{DG} by setting $M_{12} \equiv 0$. It holds that $\delta_u \in$ HardIQC (Ψ_{δ_u}, M_D) (Balakrishnan 2002). Note that (Ψ_{δ_u}, M_D) holds the (stronger) hard property, but it does not fully capture the nature of δ_u as real uncertainty, thus it can possibly lead to more conservative results than those obtained with $(\Psi_{\delta_u}, M_{DG})$.

Remark 1 *Most IQCs require the perturbation to map zero input to zero output, i.e. $v = 0$ maps to $w = \Delta(v) = 0$. Specifically, if $v = 0$ then the IQC in (6) simplifies to $\int_{-\infty}^{+\infty} \hat{w}(j\omega)^* \Pi_{22}(j\omega) \hat{w}(j\omega) d\omega \ge 0$. It is typical that $\Pi_{22}(j\omega) < 0$ for all ω and hence this requires $w = 0$. As a consequence, if Δ has internal dynamics, e.g. LTI uncertainties, then it is assumed to have zero initial condition. This assumption was used in (Balakrishnan 2002), and can be interpreted as the absence of initial stored energy in Δ. The instance of energy stored in the IQC has recently been addressed in (Pfifer and Seiler 2015, Fetzer* et al. *2018) and could allow to relax this assumption, which is not deemed overly restrictive in principle.*

3 Region of Attraction estimation with IQC

In this section a general framework to estimate the ROA of attracting equilibria is formulated based on SOS and IQCs. First, the problem setup is detailed, and then local stability certificates for the cases of hard and soft IQCs are stated. Algorithms based on SOS are finally proposed to numerically solve the problem.

3.1 Problem setup

The proposed framework aims to analyze the local stability of autonomous nonlinear systems of the form:

$$\dot{x} = f(x, w) \tag{12a}$$
$$v = h(x, w) \tag{12b}$$
$$w = \Delta(v) \tag{12c}$$

where f and h are polynomial functions of x and w (defining the plant G), and Δ is a generic bounded operator (gathering nonlinearities and uncertainties for which an IQC description holds). The prototype of systems considered by this work thus consists of the interconnection G-Δ (standard in robust control) where G is polynomial but not Δ in general, making therefore the combined system non-polynomial.

In the rest of this work it will be assumed for simplicity that the equilibrium point x^* is not a function of the uncertainties in Δ. This hypothesis is largely established in the literature (Chesi 2004, Topcu and Packard 2009, Anderson and Papachristodoulou 2017), although strategies to overcome this limitation have been proposed (Aylward *et al.* 2008, Iannelli *et al.* 2018*a*). Without loss of generality, it will also be assumed $x^* = 0$.

Starting from the generic description given in (12), the first step consists in defining the augmented plant sketched in Fig. 1. The feedback interconnection comprises the subsystems G (defined by (12a)-(12b)), Δ (12c), and Ψ (8).

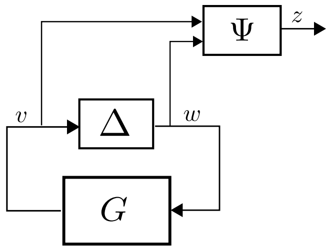

Fig. 1. Augmented plant for ROA analysis.

Introducing the vector $\tilde{x} = [x; x_\Psi]$ gathering the states of the analyzed system x and of the LTI system Ψ, the plant can be reorganized as follows:

$$\begin{aligned} \dot{\tilde{x}} &= F(\tilde{x}, w) \\ z &= H(\tilde{x}, w) \end{aligned} \tag{13}$$

where $F, H : \mathbb{R}^{n_{\tilde{x}}+n_w} \to \mathbb{R}^{n_{\tilde{x}}}$ are polynomial maps depending on both G and Ψ. It is stressed that this manipulation of (12) does not make any assumption on Δ except the existence of a factorization Ψ for the associated multiplier Π.

3.2 Region of Attraction certificates with Hard IQCs

The proposed estimation of invariant subsets of the ROA for the system in Eq. (12) when Δ has an hard IQC factorization is based on the following theorem.

Theorem 1 *Let F be the polynomial vector field defined in Eq. (13) and $\Delta : \mathcal{L}_2^{n_v} \to \mathcal{L}_2^{n_w}$ be a bounded, causal operator. Further assume:*

(1) $\Delta \in$ HardIQC (Ψ, M)
(2) There exist a smooth function $V\colon \mathbb{R}^{n_{\tilde{x}}} \to \mathbb{R}$ and positive scalars ϵ_x and ϵ_w such that:

$$V(0) = 0 \quad \text{and} \quad V(\tilde{x}) > 0 \qquad \forall \tilde{x} \backslash \{0\} \tag{14a}$$
$$\nabla V(\tilde{x}) F(\tilde{x}, w) + z^\top M z + \epsilon_x \tilde{x}^\top \tilde{x} + \epsilon_w w^\top w < 0 \qquad \forall \tilde{x} \in \Omega_{V,\gamma} \backslash \{0\}, \quad \forall w \in \mathbb{R}^{n_w} \tag{14b}$$

where z is the output of the polynomial map H (13). Then the intersection of $\Omega_{V,\gamma}$ with the hyperplane $x_\Psi = 0$ is an inner estimate of the ROA of (12).

Proof. The theorem assumes that Eq. (14b) holds only over the set $\Omega_{V,\gamma}$. Hence, the proof must ensure first that all the trajectories originating in $\Omega_{V,\gamma}$ remain within for all finite time. Assume there exists a $T_1 > 0$ such that $\tilde{x}(T_1) \notin \Omega_{V,\gamma}$, and define $T_2 := \inf_{\tilde{x}(T) \notin \Omega_{V,\gamma}} T$. Since F and H are polynomial maps, solutions of Eq. (13) are continuous, thus $\tilde{x}(T_2)$ is on the boundary of $\Omega_{V,\gamma}$ and $\tilde{x}(t) \in \Omega_{V,\gamma} \forall t \in [0, T_2]$. Therefore, it is possible to integrate the inequality (14b) in this range:

$$\begin{aligned} &V(\tilde{x}(T_2)) - V(\tilde{x}(0)) + \int_0^{T_2} z^\top M z + \\ &+ \epsilon_x \int_0^{T_2} \tilde{x}^\top \tilde{x} \, dt + \epsilon_w \int_0^{T_2} w^\top w \, dt < 0 \end{aligned} \tag{15}$$

Since by hypothesis $\Delta \in$ HardIQC (Ψ, M) and $V(\tilde{x}(0)) \leqslant \gamma$, it thus holds:

$$\gamma = V(\tilde{x}(T_2)) < \gamma \tag{16}$$

This is contradictory and hence the assumption that $\exists T_1 > 0$ such that $\tilde{x}(T_1) \notin \Omega_{V,\gamma}$ can not hold. Thus $\tilde{x}(0) \in \Omega_{V,\gamma}$ implies $\tilde{x}(t) \in \Omega_{V,\gamma}$ for all finite time (invariance of the level set). Note that this proof by contradiction is needed otherwise there is no guarantee that the trajectories stay in the set (and thus that the inequality (14b), leveraged in the rest of the proof, holds).

Next, it is required to prove that the equilibrium point is attractive. Let us consider Eq. (15) with the integrals performed in a generic interval $[0, T]$. From $\Delta \in$ HardIQC (Ψ, M) and $V(\tilde{x}(T)) \geq 0$, it follows that:

$$\epsilon_x \int_0^T \tilde{x}^\top \tilde{x}\, dt + \epsilon_w \int_0^T w^\top w\, dt < V(\tilde{x}(0)) \tag{17}$$

Let $T \to \infty$ to see that $\tilde{x} \in \mathcal{L}_2^{n_{\tilde{x}}}$ and $w \in \mathcal{L}_2^{n_w}$.
Let us now define $y(\tilde{x}, w) = [\tilde{x}; w]$ and $\mathcal{D}_y = \{y(\tilde{x}, w) : \tilde{x} \in \Omega_{V,\gamma}, w \in \mathbb{R}^{n_w}\}$. The vector field F is a polynomial function of $\tilde{x}$ and w. Therefore, F is locally Lipschitz (Khalil 1996):

$$\|F(y_2) - F(y_1)\| \leq L\|y_2 - y_1\| \quad \forall y_1, y_2 \in \mathcal{D}_y \tag{18}$$

with L a real constant. In particular, for $y_1 = [0; 0]$ and a generic y_2, it holds that:

$$\|F(y_2)\| \leq L\|y_2\| \tag{19}$$

Eq. (19) is valid for a generic y_2 in $\mathcal{D}_y$, hence the subscript will be omitted. It follows directly from Eq. (19) that:

$$\dot{\tilde{x}}^T \dot{\tilde{x}} = \|F(y)\|^2 \leq L^2\|y\|^2 = L^2 \left[\tilde{x}^T \tilde{x} + w^T w\right] \tag{20}$$

By integrating both sides from 0 to ∞, it holds for any admissible trajectory:

$$\begin{gathered} \int_0^\infty \dot{\tilde{x}}^T \dot{\tilde{x}}\, dt \leq L^2 \int_0^\infty \left[\tilde{x}^T \tilde{x} + w^T w\right]\, dt \\ ||\dot{\tilde{x}}||_2^2 \leq L^2 \left[||\tilde{x}||_2^2 + ||w||_2^2\right] \end{gathered} \tag{21}$$

Since $\tilde{x} \in \mathcal{L}_2^{n_{\tilde{x}}}$ and $w \in \mathcal{L}_2^{n_w}$, it follows from (21) that $\dot{\tilde{x}} \in \mathcal{L}_2^{n_{\tilde{x}}}$. Note also that an identical argument applied to the map h from Eq. (12b) allows to conclude that $v \in \mathcal{L}_2^{n_v}$, which guarantees that the signals in the loop belong to the space of signals requested from IQCs. Finally, $(\dot{\tilde{x}}, \tilde{x}) \in \mathcal{L}_2$ implies that $\tilde{x} \to 0$ as $T \to \infty$ (Desoer and Vidyasagar 1975). Therefore, all the trajectories originated by initial conditions in $\Omega_{V,\gamma}$ stay in the set and eventually converge to the equilibrium point. That is, $\Omega_{V,\gamma}$ is a subset of the ROA of the system in Eq. (13).
Note finally that by definition of Ψ (8) $x_\Psi(0)$=0, i.e. the initial condition for the states x_Ψ always lies on the hyperplane x_Ψ=0. Thus its intersection with $\Omega_{V,\gamma}$ provides an inner estimate of the ROA of the original system (12). □

Remark 2 *Note that V is not a Lyapunov function of the system in Eq. (13). In fact, it is possible for $\dot{V}$ to be nonnegative at some points in time. This is a consequence of the term $z^\top Mz$ which in general only provides integral constraints. When the IQC defines a pointwise-in-time constraint (i.e. $z^\top Mz \geq 0 \;\; \forall t$), then V is a Lyapunov function of the system. This is the case for example of the sector bound multiplier used in Sec. 4.1.*

Remark 3 *It is common practice to tackle the nonlinear stability problem of systems subject to polynomial nonlinearities with Lyapunov techniques, whereas the study of systems subject to hard nonlinearities (and uncertainties) is addressed with multipliers-based techniques. The proposed result allows to consider the asymptotic stability problem (Khalil 1996) of systems generically described by Eq. (12) within a unified framework. To determine whether or not an equilibrium point x^* is asymptotically stable (without determining its ROA) it suffices indeed to satisfy Theorem 1 in any domain $\mathcal{D} \subset \mathbb{R}^n$ containing x^*.*

A subtle aspect of the proposed framework is that it is not automatically guaranteed that $v \in \mathcal{L}_2^{n_v}$ (this is required for application of the IQC framework, Eq. 6) because v is here the solution of the polynomial vector field G (unlike standard IQC problems where G is an LTI system). However, if the conditions of Theorem 1 are satisfied, then the proof demonstrates that all the signals in the loop belong to $\mathcal{L}_2$. As a result, the conditions of the theorem also guarantees that v is in the space of signals required by IQCs. By contraposition, any plant G that does not generate solutions in $\mathcal{L}_2$ will not satisfy the conditions of Theorem 1 and, consequently, cannot be studied with the proposed method.

3.3 An SOS-algorithm for ROA estimates with hard IQCs

Theorem 1 is used here to compute inner estimates of the ROA of the original system (12). By restricting the attention to the class of polynomial functions V, SOS optimization can be exploited to enforce the set containment constraints in Eq. (14). The following program is first proposed.

Program 1

$$\max_{s_1 \in \Sigma[\tilde{x}, w];\, V \in \mathbb{R}[\tilde{x}]} \gamma$$

$$V \in \Sigma[\tilde{x}] \tag{22a}$$

$$-(\nabla V f + z^\top M z + L_\epsilon) - s_1(\gamma - V) \in \Sigma[\tilde{x}, w] \tag{22b}$$

where $L_\epsilon = \epsilon[\tilde{x}; w]^\top [\tilde{x}; w]$ with ϵ small real number on the order of 10^{-6}. These constraints are sufficient conditions for (14). Indeed, (22a) ensures positivity of V (14a), whereas (22b) is, by direct application of Lemma 2, a sufficient condition for the set containment (14b).
Note that in Program 1 there are bilinear terms featuring s_1, γ and V. If V is held fixed then the bilinearity only appears in the terms $s_1\gamma$. In this case, since one of the two terms in the bilinearity is the objective function, the problem is quasiconvex (Seiler and Balas 2010) and the global optimum can be computed via bisection. However, the term $s_1 V$ makes the above program nonconvex and this is handled by means of iterative schemes.

Namely, a three-steps algorithm, inspired by the V-s iteration scheme from (Chakraborty *et al.* 2011*b*), is proposed with the aim to enlarge the inner estimates of the ROA via solution of a sequence of convex programs.

Algorithm 1 *(Hard-IQCs)*
Outputs: *the level sets* $\Omega_{V,\gamma}$ *and* $\Omega_{p,\beta}$ *(both inner estimates of the ROA).*
Inputs: *a shape function p; a polynomial* V^0 *satisfying (3) for some* γ.

(i) γ*-Step: solve through bisection on* γ:

$$\max_{s_1 \in \Sigma[\tilde{x},w], M} \gamma$$
$$-\left(\nabla V^0 F + z^\top M z + L_\epsilon\right) - s_1(\gamma - V^0) \in \Sigma[\tilde{x}, w]$$

set $\bar{\gamma} \leftarrow \gamma$ *and* $\bar{s}_1 \leftarrow s_1$.

(ii) β*-Step: maximize the size of* $\Omega_{p,\beta}$ *through bisection on* β *such that* $\Omega_{p,\beta} \subseteq \Omega_{V^0,\bar{\gamma}}$:

$$\max_{s_2 \in \Sigma[\tilde{x}]} \beta$$
$$(\bar{\gamma} - V^0) - s_2(\beta - p) \in \Sigma[\tilde{x}]$$

set $\bar{\beta} \leftarrow \beta$ *and* $\bar{s}_2 \leftarrow s_2$.

(iii) V*-Step: compute a new shape for the level set by solving over* V, M:

$$V - \epsilon \tilde{x}^\top \tilde{x} \in \Sigma[\tilde{x}, w]$$
$$-\left(\nabla V F + z^\top M z + L_\epsilon\right) - \bar{s}_1(\bar{\gamma} - V) \in \Sigma[\tilde{x}, w]$$
$$(\bar{\gamma} - V) - \bar{s}_2(\bar{\beta} - p) \in \Sigma[\tilde{x}]$$

set $V^0 \leftarrow V$ *and go to* γ*-Step.*

Remark 4 *The positive polynomial* L_ϵ *is required by the perturbation argument used in Theorem 1 (see Eq. 14b). In fact, this term is also adopted in other algorithms using SOS-based schemes (Topcu and Packard 2009, Topcu* et al. *2010, Chakraborty* et al. *2011*b*) because it provides at the end of every iteration a solution satisfying the constraints with some margin, increasing in this way the possibility of achieving feasibility in the subsequent iterative step. With the same spirit it has been used here for the constraint on V (where there is no requirement in principle from Theorem 1).*

A possible option for the shape function p is the ellipsoid $p(x) = x^\top N x$ (with $N \in \mathbb{R}^{n \times n}$, $N = N^\top > 0$) specified by the user based on important directions in the state space. An alternative scheme, which is similar to Algorithm 1 but does not entail using p, is discussed in (Iannelli *et al.* 2018*b*). As for the initialization of V, if the equilibrium is asymptotically stable then the linearization of the dynamics about the origin can be used to compute a Lyapunov function that serves as a candidate input V^0. Alternatively this can be selected as discussed for p (note that its influence on the results is less than that of p because the input V^0 is only used in the first iteration of the algorithm). Note finally that M is a matrix of optimization variables, as it is the case in standard IQC analysis (Veenman *et al.* 2016). In general M will be subject to constraints in order to represent a valid IQC. These can be typically recast as SOS constraints and thus added to the optimizations. Examples of this instances will be discussed in Sec. 3.4 and Sec. 4.2.

3.4 Region of Attraction certificates with Soft IQCs

Theorem 1 assumes that $\Delta \in \text{HardIQC}\,(\Psi, M)$. As commented before, once a suitable multiplier Π is selected for the considered uncertainty or nonlinearity (i.e. $\Delta \in \text{IQC}\,(\Pi)$), its factorization can, in general, hold only the *weaker* property $\Delta \in \text{SoftIQC}\,(\Psi, M)$. Indeed it is often desirable to enrich the description of Δ by using different multipliers Π, and for some of them only a soft factorization might hold (e.g. the less conservative *D-G* factorization (11) discussed in Sec. 2.3). Therefore, it is deemed important to provide guarantees of local stability also for the cases that do not satisfy condition 1 of Theorem 1.

In order to make use of the dissipation inequality argument exploited in Theorem 1, it is important to have at least a bound on the finite-horizon integral when no hard factorization is available. To this end, a recent result is recalled next. As a preliminary, let us denote by $\text{KYP}(A, B, C, D, M)$ the following constraint on a matrix $Y = Y^\top$:

$$\begin{bmatrix} A^\top Y + YA & YB \\ B^\top Y & 0 \end{bmatrix} + \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} < 0 \qquad (23)$$

Let us also partition the frequency domain multiplier Π conformably with the dimensions of v and w as $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\sim} & \Pi_{22} \end{bmatrix}$. Then, the result proposed in (Fetzer *et al.* 2018) (building on a previous finding from (Seiler 2015)) provides the desired bound.

Lemma 3 *(Fetzer* et al. *2018) Let* $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v + n_w)}$ *and* $M = M^\top \in \mathbb{R}^{n_z \times n_z}$ *be given and define* $\Pi = \Psi^\sim M \Psi$. *If* $\Pi_{22} < 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\}$ *then:*

- $D_{\Psi,2}^T M D_{\Psi,2} < 0$ *and there exists a solution* $Y_{22} = Y_{22}^\top$ *to* $KYP(A_\Psi, B_{\Psi 2}, C_\Psi, D_{\Psi 2}, M)$.
- *If* $\Delta \in \text{SoftIQC}\,(\Psi, M)$ *then for all* $T \geq 0$, $v \in \mathcal{L}_2^{n_v}$ *and* $w = \Delta(v)$:

$$\int_0^T z(t)^\top M z(t) dt \geq -x_\Psi(T)^\top Y_{22} x_\Psi(T) \qquad (24)$$

for any Y_{22} *satisfying* $KYP(A_\Psi, B_{\Psi 2}, C_\Psi, D_{\Psi 2}, M)$

Note that this result, even though potentially more conservative than others proposed in the literature (Seiler 2015, Seiler 2018), is particularly attractive because it relates the multiplier (Ψ, M) and the bound Y_{22} via a KYP constraint, which is a convex Linear Matrix Inequality (LMI) on M and Y_{22}. By making use of Lemma 3, Theorem 2 is proposed to address the estimation of ROA with soft IQCs.

Theorem 2 *Let F be the polynomial vector field defined in Eq. (13) and* $\Delta : \mathcal{L}_2^{n_v} \rightarrow \mathcal{L}_2^{n_w}$ *be a bounded, causal operator. Further assume:*

(1) $\Delta \in SoftIQC(\Psi, M)$

(2) There exists a $Y_{22}{=}Y_{22}^\top$ *that satisfies* $KYP(A_\Psi, B_{\Psi 2}, C_\Psi, D_{\Psi 2}, M)$

(3) There exist smooth functions $V : \mathbb{R}^{n_{\tilde{x}}} \to \mathbb{R}$ *and* $\tilde{V} = V - x_\Psi^\top Y_{22} x_\Psi$*, and positive scalars* ϵ_x *and* ϵ_w *such that:*

$$\tilde{V}(0) = 0 \quad \text{and} \quad \tilde{V}(\tilde{x}) > 0 \qquad \forall \tilde{x} \backslash \{0\} \tag{25a}$$

$$\nabla V(\tilde{x}) F(\tilde{x}, w) + z^\top M z + \epsilon_x \tilde{x}^\top \tilde{x} + \epsilon_w w^\top w < 0 \qquad \forall \tilde{x} \in \Omega_{\tilde{V},\gamma} \backslash \{0\}, \quad \forall w \in \mathbb{R}^{n_w} \tag{25b}$$

Then the intersection of $\Omega_{V,\gamma}$ *with the hyperplane* $x_\Psi = 0$ *is an inner estimate of the ROA of (12).*

Proof. Integrating the inequality in Eq. (25b) in the interval $[0, T]$, it follows:

$$V(\tilde{x}(T)) - V(\tilde{x}(0)) + \int_0^T z^\top M z + + \epsilon_x \int_0^T \tilde{x}^\top \tilde{x} \, dt + \epsilon_w \int_0^T w^\top w \, dt < 0 \tag{26}$$

Contrary to the case in the proof of Theorem 1, since $\Delta \in$ SoftIQC (Ψ, M), the finite-horizon integral involving the IQC term is not necessarily positive here. According to the result in Lemma 3, the following lower bound on the soft IQC (Ψ, M) is valid:

$$\int_0^T z^\top M z dt \geq -x_\Psi(T)^\top Y_{22} x_\Psi(T) \tag{27}$$

for any Y_{22} satisfying KYP$(A_\Psi, B_{\Psi 2}, C_\Psi, D_{\Psi 2}, M)$. Thus, by making use of this lower bound in Eq. (26), it holds:

$$V(\tilde{x}(T)) - V(\tilde{x}(0)) - x_\Psi(T)^\top Y_{22} x_\Psi(T) + + \epsilon_x \int_0^T \tilde{x}^\top \tilde{x} \, dt + \epsilon_w \int_0^T w^\top w \, dt < 0 \tag{28}$$

Define $\tilde{V}(\tilde{x}) = V(\tilde{x}) - x_\Psi^\top Y_{22} x_\Psi$. Since $x_\Psi(0) = 0$ (recall the definition of Ψ in Eq. (8)), the following holds directly from Eq. (28):

$$\tilde{V}(\tilde{x}(T)) - \tilde{V}(\tilde{x}(0)) + \epsilon_x \int_0^T \tilde{x}^\top \tilde{x} \, dt + \epsilon_w \int_0^T w^\top w \, dt \leq 0 \tag{29}$$

Note that (29) represents a formally equivalent expression of Eq. (15), with the crucial difference that $\tilde{V}$ is now the level set function. Specifically, the same proof by contradiction of Theorem 1 can be used to prove the invariance of the set $\Omega_{\tilde{V},\gamma}$. Moreover, due to the positivity of $\tilde{V}$, it holds again that $\tilde{x} \in \mathcal{L}_2^{n_{\tilde{x}}}$ and $w \in \mathcal{L}_2^{n_w}$. Therefore, the same arguments apply to prove that the equilibrium point is attracting in the invariant set. It can then be concluded that the level set $\Omega_{\tilde{V},\gamma}$ is a subset of the ROA of the system in Eq. (13). Thus, the intersection of $\Omega_{\tilde{V},\gamma}$ with the hyperplane $x_\Psi = 0$ (equivalently, the intersection of $\Omega_{V,\gamma}$ with $x_\Psi = 0$) is a subset of the ROA of Eq. (12). □

Remark 5 *The key step in proving Theorem 2 is the finite-horizon bound on the soft IQC. In this case this is specified as an LMI constraint and thus can be easily incorporated in convex optimization algorithms. However, as it has been discussed in Section 3.3, the numerical estimation of ROA via SOS proposed here leads to bilinear terms. In view of this, other (less conservative) bounds involving non-convex bilinear matrix inequalities could be similarly considered.*

3.5 An SOS-algorithm for ROA estimates with soft IQCs

The algorithm discussed in this section allows to find inner estimates of the ROA when uncertainties and nonlinearities are possibly described by soft IQCs. In order to do this, a result allowing condition 2 in Theorem 2 to be enforced as an SOS constraint is first provided with the following Lemma.

Lemma 4 *The constraint* $KYP(A_\Psi, B_{\Psi 2}, C_\Psi, D_{\Psi 2}, M)$ *on* Y_{22} *holds if and only if there exists a function* $V_\Psi = x_\Psi^\top Y_{22} x_\Psi : \mathbb{R}^{n_{x_\Psi}} \to \mathbb{R}$ *such that:*

$$\nabla V_\Psi(x_\Psi) f_\Psi(x_\Psi, w) + z^\top M z < 0 \tag{30}$$

$$\text{with} \quad f_\Psi = A_\Psi x_\Psi + B_{\Psi 2} w, \quad z = C_\Psi x_\Psi + D_{\Psi 2} w \tag{31}$$

Proof. It immediately follows by left and right multiplying the LMI (23) by $[x_\Psi; w]^\top$ and $[x_\Psi; w]$ respectively. □

The algorithm for the case of soft IQC is then stated.

Algorithm 2 *(Soft-IQCs)*
Outputs: *the level sets* $\Omega_{\tilde{V},\gamma}$ *and* $\Omega_{p,\beta}$ *(both inner estimates of the ROA).*
Inputs: *a shape function* p*; a polynomial* V^0 *satisfying (3) for some* γ*;* M^0*;* Y_{22}^0*.*

(i) γ*-Step: set* $\tilde{V}^0 \leftarrow V^0 - x_\Psi^\top Y_{22}^0 x_\Psi$ *and solve through bisection on* γ*:*

$$\max_{s_1 \in \Sigma[\tilde{x}, w]} \gamma$$
$$-(\nabla V^0 F + z^\top M^0 z + L_\epsilon) - s_1(\gamma - \tilde{V}^0) \in \Sigma[\tilde{x}, w]$$

set $\bar{\gamma} \leftarrow \gamma$ *and* $\bar{s}_1 \leftarrow s_1$*.*

(ii) β*-Step: maximize the size of* $\Omega_{p,\beta}$ *through bisection on* β *such that* $\Omega_{p,\beta} \subseteq \Omega_{\tilde{V}^0,\bar{\gamma}}$*:*

$$\max_{s_2 \in \Sigma[\tilde{x}]} \beta$$
$$(\bar{\gamma} - \tilde{V}^0) - s_2(\beta - p) \in \Sigma[\tilde{x}]$$

set $\bar{\beta} \leftarrow \beta$ *and* $\bar{s}_2 \leftarrow s_2$*.*

(iii) V*-Step: compute a new shape for the level set by solving over* V, M, Y_{22}*:*

$$\tilde{V} - \epsilon\tilde{x}^\top\tilde{x} \in \Sigma[\tilde{x}, w];$$
$$-(\nabla VF + z^\top Mz + L_\epsilon) - \bar{s}_1(\bar{\gamma} - \tilde{V}) \in \Sigma[\tilde{x}, w]$$
$$(\bar{\gamma} - \tilde{V}) - \bar{s}_2(\bar{\beta} - p) \in \Sigma[\tilde{x}]$$
$$-(\nabla V_\Psi f_\Psi + z^\top Mz) \in \Sigma[\tilde{x}, w]$$

set $V^0 \leftarrow V$, $M^0 \leftarrow M$, $Y_{22}^0 \leftarrow Y_{22}$, *and go to* γ-*Step.*

where V_Ψ, f_Ψ are defined in Eqs. (30-31). It is worth remarking that in the proposed algorithm only the V-step is affected by the KYP constraint and, since this only features quadratic forms (30), enforcing it with as an SOS constraint is lossless. In fact, in the γ-step only a maximization of the *size* (given by the scalar γ) of the level set is performed, but its shape (given by $\tilde{V}$) is held fixed to a given value $\tilde{V}^0$. Therefore, by keeping in the γ-Step the multipliers M fixed to the value optimized at the previous iteration, the KYP constraint is automatically satisfied. In this regard, note that the algorithm needs an initialization for M and Y_{22}. There are not formal guidelines for their selection, and in this work the γ-Step at the first iteration is initialized with $Y_{22}^0 = 0$ and it optimizes over M– that is, it is applied as in the case of hard IQCs (Algorithm 1). This potentially leads to an error on $\bar{\gamma}$ which, however, only serves in the first iteration to perform the β-step and V-step. The latter then computes a level set $\Omega_{\tilde{V},\gamma}$ fulfilling all the prescribed constraints, and thus a valid ROA estimate of the system is achieved. This will in turn provide the sought initializations for the next iteration and so forth.

4 Numerical examples

This section provides two numerical examples to illustrate the application of the proposed framework.

4.1 Closed-loop short period GTM aircraft

The closed-loop short period motion of the NASA's Generic Transport Model (GTM) can be approximated as a 2 states polynomial system (Chakraborty *et al.* 2011*a*):

$$\begin{aligned}
\dot{\alpha} &= -1.492\alpha^3 + 4.239\alpha^2 + 0.2402\alpha\delta_e + 0.003063\alpha q - \\
&+ 0.0649\delta_e^2 + 0.006226q^2 - 3.236\alpha - 0.3166\delta_e + 0.9227q; \\
\dot{q} &= -7.228\alpha^3 + 18.36\alpha^2 + 41.5\alpha\delta_e - 45.34\alpha - 59.99\delta_e + \\
&- 4.372q + 1.103q^3; \\
\delta_{e_{CMD}} &= Kq; \quad K = 4\frac{\pi}{180};
\end{aligned} \tag{32}$$

where α is the angle of attack, q is the pitch rate, δ is the elevator deflection. The GTM steady-state solution consists of a locally stable equilibrium point at the origin, i.e. $x^* = 0$. Previous studies focused on Region of Attraction analysis of the Open (OL) and Closed-Loop (CL) system (Chakraborty *et al.* 2011*a*), and worst-case $\mathcal{L}_2$ gain analysis with saturated rate of δ (Chakraborty *et al.* 2010). In this work, δ is assumed to be subject to actuator magnitude saturation, that is:

$$\delta = \begin{cases} sgn(\delta_{CMD})\,\delta^{sat}; & |\delta_{CMD}| > \delta^{sat} \\ \delta_{CMD}; & |\delta_{CMD}| \le \delta^{sat} \end{cases} \tag{33}$$

where δ^{sat} is the saturation level. A characterization of the saturation by means of IQCs holding as finite-horizon time domain constraints (i.e. hard IQCs) is discussed next. For example, regarding the saturation as a memoryless, bounded, nonlinearity within the sector $[\sigma,\ \eta]$, the *sector* multiplier Π_S enforces this property:

$$\Pi_S = \begin{bmatrix} -2\sigma\eta & \sigma+\eta \\ \sigma+\eta & -2 \end{bmatrix}; \quad \Psi_S = I_2; \quad M_S = \lambda_S \Pi_S; \tag{34}$$

with $\lambda_S > 0$ decision scalar.

If in addition the saturation is considered a monotonic and odd function, then a slope restriction in the sector $[\sigma_1,\ \eta_1]$ holds. This property leads to the Zames-Falb IQC Π_{ZF} (Heath and Wills 2005, Megretski and Rantzer 1997). By parametrizing the dynamic part of Π_{ZF} with low pass filters (one low pass filter with frequency 1 $\frac{rad}{s}$ is used here) (Veenman *et al.* 2016), a J-spectral factorization $J_{sf}(\Pi_{ZF})$ can be performed with the algorithm from (Seiler 2015), which in turn provides an hard IQC factorization (Ψ_{ZF}, M_{ZF}).

A common choice in IQC analysis of saturated systems is to consider the sector $[0, 1]$ for both Π_S and Π_{ZF}. However, these IQCs include the OL system (i.e. δ=0) as a particular case and thus the estimated ROA cannot be larger than the corresponding one. For these reasons, a relaxation of the sector IQC is proposed, exploiting the fact that the ROA inherently provides a bound on the values of the states of the system. The premise, based on the notion of local IQCs (Summers and Packard 2010), is sketched in Fig. 2, showing the relationship between commanded (δ_{CMD}) and saturated (δ) input. On the horizontal axis it is highlighted $\delta_{CMD}^{max} = Kq_{max}$, where q_{max} denotes the largest value for which q belongs to the region of attraction. It is then apparent that the lower bound σ depends on q_{max} and the saturation level, specifically it holds $\sigma = \frac{\delta^{sat}}{K}\frac{1}{q_{max}}$.

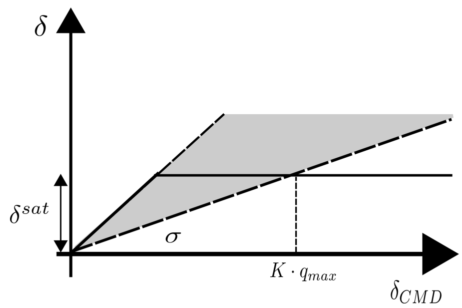

Fig. 2. Relaxed sector constraint exploiting bounds on the states.

A strategy is discussed next to include this relaxation in the algorithms presented in Section 3. This has connections with previous works (Hu *et al.* 2006, da Silva and Tarbouriech 2005) that considered a similar problem, but allows for polynomial plants G (32) and level set functions. The idea is to determine q_{max} at the end of each iteration (i.e. after the V-step), and based on that update the lower bound σ of the sector multiplier. The expression of Π_S at iteration $n+1$ is thus given by:

$$\Pi_S|^{n+1} = \Pi_S|_{\left[\sigma=\frac{\delta^{sat}}{K}\frac{1}{q_{max}|^n};\eta=1\right]} \tag{35}$$

where the value $q_{max}|^n$ can be computed at the end of iteration n with the following SOS program:

$$\begin{aligned} q_{max}|^n = \max_{s_{f_+},s_{f_-}\in\Sigma[\tilde{x}]} q_{max} \\ q + q_{max} - s_{f_+}(\gamma|^n - V|^n) \in \Sigma[\tilde{x}] \\ -q + q_{max} - s_{f_-}(\gamma|^n - V|^n) \in \Sigma[\tilde{x}] \end{aligned} \tag{36}$$

This program is an application of Lemma 2 and guarantees that $-\,q_{max}|^n \leq q|^n \leq q_{max}|^n$.
This strategy has the desired property that the sector employed at iteration $n+1$ (function of $q_{max}|^n$) is always consistent with the ROA computed at the same iteration. This results from the fact that the computed ROA is non-decreasing throughout the iterations, therefore $q_{max}|^{n+1} \geq q_{max}|^n$. In (Iannelli *et al.* 2018*b*) an alternative strategy where the lower bound of the sector is integrated in the program as optimized variable is also discussed.

Fig. 3 displays inner estimates of the ROA of the saturated GTM model obtained with Algorithm 1 and the discussed relaxation strategy for three levels of saturation (δ^{sat}=[0.05, 0.1, 0.2] rad). The open loop OL and unsaturated closed-loop CL cases are also reported for comparison. A quartic level set (i.e $\partial(V) = 4$) is considered, and the shape matrix $N = diag(8.16, 1.31)$ from (Chakraborty *et al.* 2011*a*) is employed to define the shape function p.

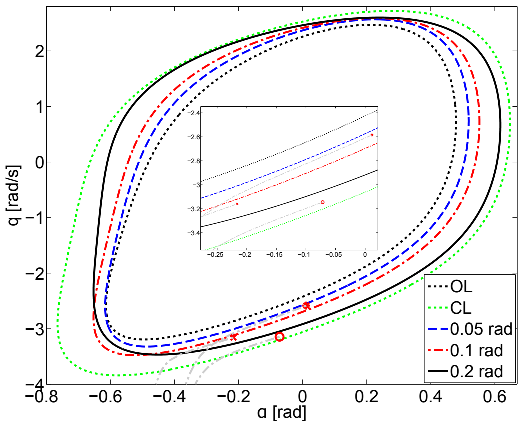

Fig. 3. Estimates of the ROA for different saturation levels.

It can be observed that the three curves obtained with the proposed approach are larger than the OL curve. Moreover, as the value of δ^{sat} is increased, the corresponding curves get closer to the unsaturated closed-loop, as expected. Fig. 3 also shows three markers (asterisk for δ^{sat}=0.05 rad, cross for δ^{sat}=0.1 rad, and circle for δ^{sat}=0.2 rad) corresponding to initial conditions for which the dynamics was found unstable, and the relative escaping trajectories in dashed-dotted lines. The small gap between markers and corresponding estimates of ROA suggests that the effect of saturation is well predicted by the proposed approach.

4.2 Van der Pol oscillator

The Van der Pol (VdP) oscillator is a polynomial nonlinear system with 2 states. In here the case with an uncertain scalar parameter $\delta_1 \in [-1, 1]$ is studied:

$$\begin{aligned} \dot{x}_1 &= -x_2(1 + 0.2\delta_1) \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned} \tag{37}$$

The VdP steady-state solutions are an unstable limit cycle and a locally stable equilibrium point (coinciding with the origin for all the values of the uncertainty, as it was assumed in Section 3.1).

The estimation of the ROA is performed with the multipliers (Ψ_{δ_u}, M_D) and $(\Psi_{\delta_u}, M_{DG})$ discussed in Sec. 2.3. The D-G scalings multipliers (11), providing soft IQCs, require the additional constraint $H_\nu^\sim M_{11} H_\nu > 0$ to be a valid factorization. This can be enforced without conservatism by finding a matrix Y such that KYP$(A_{H_\nu}, B_{H_\nu}, C_{H_\nu}, D_{H_\nu}, M_{11})$ holds (23). This in turn can be expressed by Lemma 4 as an inequality involving the optimization variables, and thus it is appended as additional SOS constraint in Algorithm 2.

Fig. 4 presents the results for $\partial(V) = 6$. The shape function $p = 0.378x_1^2 + 0.278x_2^2 - 0.274x_1x_2$ from (Topcu and Packard 2009) is employed. H_ν is parameterized with $\nu = 1$, i.e. $H_\nu = [1, \frac{1}{s+a_1}]$ where a_1=1 $\frac{rad}{s}$. This value is close to the frequency of the limit cycle, and yielded an improvement in the estimation of the ROA compared to other parameterizations. The limit cycles of the system, enclosing the region of attraction of the origin and thus providing an upper bound on the estimations, is reported for eight values of δ_1 ($ROA(\delta_1)$).

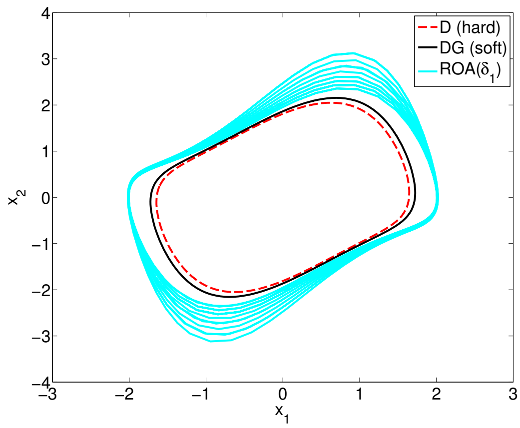

Fig. 4. Estimates of the ROA: hard versus soft IQCs.

The results show that the inner estimates of the ROA lie close to the smallest in size of the LCOs. It can also be noted that the estimation obtained with the D-G scalings (specifying the nature of the uncertainty as real parameter) is the largest. This stresses once more the favorable effect of enriching the description of the uncertainties/nonlinearities with an appropriate selection of the IQCs. In addition, it motivates the importance of providing the main result of the paper also for the case of soft IQCs (Theorem 2).

4.3 General concluding remarks

The numerical examples have showcased some of the properties of the new proposed framework for ROA analysis. Before briefly commenting them, the main assumptions underlying the analyses are recapped here. System (12) has a locally stable equilibrium point x^*, and it is well-posed (hence guaranteeing continuity of the resulting trajectories) for initial conditions in the region of attraction of x^*. For the sake of simplicity, x^* is assumed to be independent of Δ, but extensions of the methods proposed in (Aylward *et al.* 2008, Iannelli *et al.* 2018*a*) to the present formulation can be employed for cases in which this hypothesis does not hold.

The presented approach allows to capture the effect of different kind of uncertainties on the local stability of the analyzed system. One of the advantages of this method is indeed the possibility to refine the description of the uncertainty by adding various classes of multipliers. Different from other solutions proposed in the literature (Topcu and Packard 2009, Topcu *et al.* 2010, Anderson and Papachristodoulou 2017, Iannelli *et al.* 2019), this framework exploits the nature of the uncertainties (e.g. real constant, linear time varying) to reduce the conservatism by specializing the IQC description. In addition to that, a further refinement is allowed by the parameterization of the dynamic multipliers (e.g. increasing ν for the D-G scalings). In practice, a trade-off between computational time and accuracy will arise, and system-dependent investigations are required to assess the benefits of a more sophisticated description.

Another favourable feature is that the estimate of the ROA is given by means of parameter-independent level sets, i.e. V is a function of $\tilde{x}$ only. The fact that $\Omega_{V,\gamma}$ does not depend on the uncertainty set Δ avoids the computation of the intersection of the parameterised estimates, resulting in a more accurate and easier to visualise outcome. Nonetheless, the parameter-independent option is known to lead to more conservative estimations because a single function is used to certify the set containment properties over the entire uncertainty set (Topcu *et al.* 2010). The solution proposed in this work can be interpreted as a trade-off in that regard, since the function V indirectly depends on the uncertainties via the IQC states x_Ψ (for dynamic IQCs). This has no effect on the interpretation of the results because it always holds $x_\Psi(0) = 0$, thus the analyst will only look at the intersection of $\Omega_{V,\gamma}$ with the hyperplane $x_\Psi = 0$.

The main drawback of this approach is the presence of the states x_Ψ associated with dynamic multipliers, which determines an increase in the run time. Note however that there are a number of static multipliers which allow to specify features of the operator Δ (e.g. sector, norm bound, time varying real scalar) without affecting the size of $\tilde{x}$. Another aspect worth noting is that the computation of $\Omega_{V,\gamma}$ relies on non-convex programs, hence convergence to local minima and sensitivity to user-specified parameters are possible issues. Future works can consider efficient convex formulations of this problem (Henrion and Korda 2014).

5 Conclusions

This paper presents a new framework for region of attraction analysis of systems affected by generic nonlinearities and uncertainties. Non-polynomial nonlinearities and uncertainties are described by means of Integral Quadratic Constraints, which are allowed to have both hard and soft factorizations. The main results of the article give sufficient conditions to determine inner estimates of the ROA of attracting fixed points for both types of factorization. For the soft IQC case, a recently proposed convex lower bound on the hard IQC is employed. One of the features of the results is that the invariant sets are not level sets of a Lyapunov function.

Based on Sum of Squares techniques, iterative algorithms that allow to enlarge the provable invariant and attractive sets are formulated, and then applied to two case studies. A computational strategy to combine the estimation of ROA with the definition of the sector multiplier is discussed, together with the effect of refining the description of real parametric uncertainties by means of more sophisticated IQCs. The results show the prowess of the proposed framework in analyzing the local stability of generic uncertain nonlinear systems.

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