Quantum teleportation with infinite reference-frame uncertainty

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We present two schemes for quantum teleportation between parties whose local reference frames are misaligned by the action of a compact Lie group G. These schemes require no prior alignment of reference frames and are unaffected by arbitrary changes in reference-frame alignment during execution, suiting them to situations of rapid reference-frame drift. Our *tight* scheme yields improved purity compared to standard teleportation, in some cases substantially—this includes the case of qubit teleportation under arbitrary SU(2) reference-frame uncertainty—while communicating no information about either party's reference-frame alignment at any time. Our *perfect* scheme performs perfect teleportation, but does communicate some reference-frame information. The mathematical foundation of these schemes is a unitary error basis permuted up to a phase by the conjugation action of a finite subgroup of G.

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I. OVERVIEW

A. Motivation

A shared reference frame is an important implicit assumption underlying the correct execution of many multiparty quantum protocols [1–7]. As quantum technologies move into space [8–10] and into handheld devices [11–13], scenarios where this assumption is violated are naturally encountered. This problem has already received considerable attention in the case of ground-to-satellite quantum key distribution [10,14,15]; there is also a smaller body of work on quantum teleportation without a shared reference frame [16–18], a subject which is increasingly important as quantum repeaters [19] and ground-to-satellite quantum teleportation [8] become experimentally viable.

Prior alignment of reference frames [1,20–23] may become impractical in the case of time-varying misalignment, or where the parties are far apart; prior alignment also involves communication of reference-frame information, which may be cryptographically sensitive [4,24,25]. Another approach involves the use of decoherence-free subspaces [26]; because this requires larger Hilbert spaces, practical implementation can be nontrivial, although experimental solutions have been developed for optical systems [27].

B. Our approach

We use a classical channel whose configurations are interpreted with respect to the local reference frame, such as might be used for prior alignment. Indeed, such a channel could be used to align frames by observing how a preagreed configuration transmitted by Alice is perceived by Bob. However, this does *not* occur in our schemes; in particular, our schemes work when rapidly varying reference-frame alignment renders prior alignment impossible, and our tight scheme in fact communicates no information about either party's frame configuration at any time. Rather, in our schemes, Alice communicates *the measurement result itself* using this channel. If the parties' frames are not aligned, Bob will perform correction operations with respect to his own frame; these may not correspond to the measurement Alice performed, causing error. In our approach, however, the misalignment also causes errors in transmission of the measurement result; Bob may receive a different index to that sent by Alice. These errors are correlated, and our key idea is to construct schemes where they cancel out.

C. Equivariant unitary error bases

A standard teleportation protocol can be described mathematically in terms of a *unitary error basis* (UEB) [28], a basis of unitary operators on a Hilbert space $\mathbb{C}^d$ which are orthogonal under the trace inner product. Let G be a finite reference-frame transformation group; we define a UEB to be *G-equivariant* when its elements are permuted up to a phase under conjugation by $\rho(g)$ for any $g \in G$, where $\rho : G \to U(d)$ is the representation of G on Bob's system [1].

Equivariant UEBs are the mathematical foundation of our teleportation schemes. In previous work we exhaustively classified these for qubit systems [29, Theorem 4.1]; they exist precisely when the image of the composite homomorphism $G \xrightarrow{\rho} \mathrm{U}(2) \xrightarrow{\tau} \mathrm{SO}(3)$ is isomorphic to 1, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, D_2, D_3, D_4, A_4, or S_4, where τ is the obvious projection. We also provided constructions in higher dimension, and a method for proving nonexistence in some cases.

D. Tight scheme

For any finite subgroup $H \subseteq G$ admitting an H-equivariant UEB, we construct a *tight* teleportation scheme immune to

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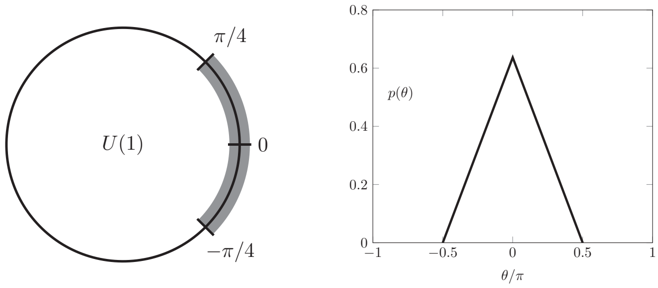

FIG. 1. Effective channel for a conventional protocol with uniform U(1) reference-frame uncertainty is a uniform average over the channels induced by all misalignments $\theta \in [-\pi, \pi)$. The cyclic subgroup $\mathbb{Z}_4 \subset \mathrm{U}(1)$ possesses an equivariant UEB, allowing our tight scheme to "quotient out" $\mathbb{Z}_4$ reference-frame uncertainty. Roughly, this reduces uncertainty to the region $\theta \in (-\pi/4, \pi/4)$ highlighted in the left subfigure; more precisely, the average over all misalignments is now weighted by $p(\theta)$, shown in the right subfigure.

reference-frame errors arising from H. When $H = G$, the protocol allows error-free teleportation. When G is larger than H, the protocol roughly allows us to "quotient" by the subgroup H, restricting the error to a *fundamental domain* for H in G. (See Fig. 1.) This can result in significant improvements in channel purity [30] compared to conventional teleportation, even for infinite compact Lie groups. For $G = \mathrm{SU}(2)$, for example, corresponding to arbitrary reference-frame uncertainty for a qubit system, standard teleportation yields an average channel purity of 0.21; with our tight scheme for the subgroup $\mathrm{BOct} \subset \mathrm{SU}(2)$, where BOct is the binary octahedral group, we obtain a channel purity of 0.44 ± 0.03, more than double that for standard teleportation. The results are shown in Table I.

The tight scheme additionally possesses the following desirable properties.

(i) *Dynamical robustness (DR).* It is unaffected by arbitrary changes in reference-frame alignment during transmission of the measurement result, provided Bob's frame alignment remains approximately constant between his receipt of the measurement result and his performance of the unitary correction.

(ii) *Minimal entanglement (ME).* The parties only require a d-dimensional maximally entangled resource state.

(iii) *Minimal communication (MC).* Only 2 dits of classical information are communicated from Alice to Bob.

(iv) *No reference-frame leakage (NL).* No information about either party's reference-frame alignment at any time is communicated. (This property is of cryptographic significance [4,24,25].)

TABLE I. Qubit teleportation using a matched channel for U(1) and SU(2) reference-frame uncertainty. The numbers shown are the purities of the effective quantum channels.

Transformation group	Conventional purity	New tight scheme purity
U(1)	0.59	0.65 (matched channel)
SU(2)	0.21	0.32 ± 0.02 (matched channel)
		0.44 ± 0.03 (rod channel)

E. Perfect scheme

The tight scheme yields an improvement in the quality of the channel. Our perfect scheme, on the other hand, performs *perfect* teleportation, up to a global phase, while retaining properties (DR) and (ME) and without communicating full information about Alice's frame configuration at the time of measurement. To achieve this, additional reference-frame information is transmitted by Alice in the same package as the measurement result, reducing reference-frame uncertainty exactly to the finite group H, for which perfect teleportation is possible. Our techniques allow us to "fold" the measurement result in with the reference-frame information, obviating the need to communicate it through a separate channel and, importantly, maintaining the novel (DR) property.

F. Related work

Chiribella *et al.* [16] argued that, when the reference transformation group is a continuous compact Lie group, there is no teleportation procedure yielding perfect state transfer. They did not consider transmission of the measurement result in a reference-frame-dependent manner, and their no-go theorem therefore does not apply to our results.

Some other approaches for finite G can be found in the literature. These rely on a variety of techniques: using additional preshared entanglement [16], sharing additional entanglement during the protocol [4], and transmitting more complex resources [1, Sec. V A]. None of these share the (DR) property, and they all require additional resources and additional quantum operations.

G. Outlook

Work has been done on reference-frame-independent quantum key distribution between handheld devices sharing an optical link [11–13]; such devices seem an obvious application for our perfect scheme for U(1) uncertainty. There may also be cryptographic applications for these results, as it has been noted that a private shared reference frame may be used as a secret key [4,24,25], and our tight scheme does not leak reference-frame information.

II. EXAMPLES

We begin with two illustrative examples.

A. Example 1: Phase reference-frame uncertainty

1. Physical setup

Alice and Bob share an optical link along a line of sight; through this link they can perform quantum or classical communication, mediated by individual photons or beams of classical light. Alice transfers one-half of a polarization-entangled pair of photons to Bob through the optical link, which can be used to teleport the state σ of a qubit in her possession. However, they do not share a Cartesian frame defining the x- and y-polarization axes in the plane perpendicular to the axis of the link. Due to frame misalignment, Bob's description of the polarization state of the transmitted photon may differ from Alice's [14].

The reference-frame transformation group here is the two-dimensional rotation group U(1). If $\theta \in [0, 2\pi)$ is the angle of a clockwise rotation of the 2D Cartesian frame, U(1) acts as follows on the polarization state:

$$\theta \mapsto \rho(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{1}$$

Here the vector acted on by the matrix is $(v_L, v_R)^T$, where v_L is the left and v_R the right circular polarization coefficient. The transformation $g(t) \in$ U(1), which relates Alice and Bob's frames at time t is unknown, and may vary non-negligibly on time scales shorter than the message transmission time between the parties, rendering prior alignment impossible.

2. *Conventional scheme*

Alice creates a polarization-entangled photon pair

$$\eta = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

She communicates one photon to Bob through the optical link, and measures the other, together with the state σ, in the maximally entangled orthonormal basis $|\phi_i\rangle = (\mathbb{1} \otimes U_i^T)|\eta\rangle$, where U_i are the Pauli matrices:

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

She communicates the result to Bob through an ordinary classical channel, who applies the correction U_i to his half of the entangled state. Should both parties' reference frames be aligned, Bob's system will finish in the state σ; this is because the Pauli matrices form a *unitary error basis* (UEB), a structure we will define later.

However, if Bob's frame is related to Alice's by a nontrivial transformation $g \in$ U(1), then from the perspective of Alice's frame, Bob will not perform the intended correction U_i, but rather the conjugated unitary [31]

$$\rho(g)^\dagger U_i \rho(g). \tag{3}$$

The transformation g is unknown, so we must average over the whole of U(1) to find the effective channel, yielding the following expression:

$$\mathcal{T}_i(\sigma) = \int_{\mathrm{U}(1)} dg[\rho(g)^\dagger U_i \rho(g) U_i^\dagger](\sigma). \tag{4}$$

Here dg is the Haar measure on U(1), and we have used the notation $[X](\sigma)$ for the conjugation $X\sigma X^\dagger$. Averaging over the four equiprobable measurement results, we find (Appendix C 2) that the effective channel for a conventional scheme has the following effect on an input density matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b/2 \\ c/2 & d \end{pmatrix}.$$

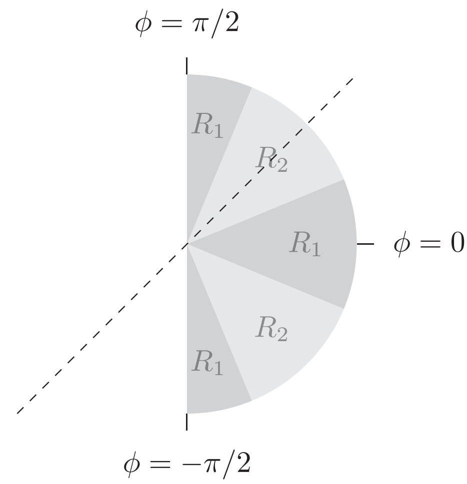

FIG. 2. Regions R_1 and R_2. The polarization axis of a beam of light linearly polarized at angle $\theta = \pi/4$ is shown in the figure.

3. *Tight scheme*

Alice measures as before, but now transmits her measurement result using a beam of polarized classical light sent along the optical link, according to the following prescription. If she measures zero or 3, she transmits a beam of clockwise or counterclockwise circularly polarized light, respectively; since the direction of circular polarization is preserved under reference-frame transformations, Bob will receive the measurement result as it was sent. If she measures 1 or 2, she sends the measurement result encoded in the polarization axis of a beam of linearly polarized light, which is chosen using the regions in Fig. 2: if she measures 1 or 2, she sends the light linearly polarized along an axis selected uniformly at random from the region R_1 or R_2, respectively. Bob then observes the polarization direction of the light he receives respective to his own frame and decodes in the inverse manner, performing the correction as before. The rationale behind this choice of encoding will be made clear in Sec. III.

This scheme is *tight*. In particular, we highlight two of the properties listed in Sec. I as follows.

(i) (NL). To an observer outside Alice's laboratory, the information she communicates is uniformly random. This follows from the fact that her measurement outcomes are equiprobable, and given the measurement outcome i all polarization directions in the corresponding region are equiprobable. Therefore, nothing can be deduced from her transmission about her reference-frame orientation.

(ii) (MC). There are four messages Bob can receive: left or right circularly polarized light, or light linear polarized through an axis in the region E_1 or E_2. All four messages are equiprobable. He therefore obtains precisely two bits of classical information.

We will see (Appendix C 2) that the effective channel—averaging over Alice's equiprobable measurement results—has the following action on an input density matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b\left(\frac{2}{\pi^2} + \frac{1}{2}\right) \\ c\left(\frac{2}{\pi^2} + \frac{1}{2}\right) & d \end{pmatrix}. \tag{5}$$

The quality of the channel has increased, despite the fact that no reference-frame information has been transmitted. In particular, the final state is now asymmetric even when Alice measures 1 or 2.

4. Perfect scheme

For perfect teleportation, Alice need not transmit full information about the frame in which she measured, as shown by the following scheme. If Alice measures zero or 3, she transmits a beam of left or right circularly polarized light, respectively. If she measures 1 or 2, she transmits linearly polarized light with polar angle zero or $\pi/4$, respectively. If Bob receives circularly polarized light, he decodes as before. If he receives linearly polarized light in the region E_1 with respect to his own frame, he rotates his frame actively or passively so that the light is polarized along the axis with polar angle zero in his frame, and performs the correction U_1. If the polarization direction is in the region E_2, he rotates his frame actively or passively so that the light is polarized along the axis with polar angle $\pi/4$ in his frame, and performs the correction U_2. We will see (Proposition III.5) that this procedure results in perfect teleportation. However, the reference-frame information communicated by this protocol is only sufficient to reduce reference-frame uncertainty to a finite subgroup $\mathbb{Z}_4$.

B. Example 2: Spatial reference-frame uncertainty

1. Physical setup

Alice and Bob are spatially separated; their qubits are spin-$\frac{1}{2}$ particles. Alice plans to teleport a state σ to Bob. They each possess half of the following maximally entangled pair [32]:

$$|\eta\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

However, the Cartesian frame according to which Alice's x-, y-, and z-spin axes are defined is related to Bob's by some unknown three-dimensional rotation. The reference-frame transformation group is SU(2), which acts on a qubit Hilbert space H by its standard matrix representation $\rho : \mathrm{SU}(2) \to B(H)$. Again, the transformation $g(t) \in \mathrm{SU}(2)$ which relates Alice's and Bob's frames at time t is unknown, and may vary on time scales shorter than the message transmission time between the parties.

2. Conventional scheme

Alice and Bob use the entangled state $|\eta\rangle$ to attempt a standard teleportation protocol [33], again based on the Pauli matrices (2). Alice measures the state σ together with her entangled qubit in the maximally entangled orthonormal basis $|\phi_i\rangle = [\mathbb{1} \otimes -i(U_i U_2)^T]\,|\eta\rangle$ [34] and communicates the measurement result to Bob through an ordinary classical channel; Bob then applies the correction U_i. We must average over all misalignments in SU(2) to find the effective channel. For measurement result i we obtain the following expression:

$$\mathcal{T}_i(\sigma) = \int_{\mathrm{SO}(3)} dg[\rho(g)^\dagger U_i \rho(g) U_i^\dagger](\sigma). \tag{6}$$

Here dg is the Haar measure on SO(3). Averaging over the four equiprobable measurement results, we find (Appendix C 3) that the effective channel purity is approximately 0.21.

TABLE II. Tight encoding scheme for the rod channel. Alice chooses the precise orientation of the rod uniformly at random from the set of all orientations satisfying the intersection condition.

Measurement result	Classical transmission
0	Featureless sphere
1	Rod oriented along any axis intersecting the 1-faces
2	Rod oriented along any axis intersecting the 2-faces
3	Rod oriented along any axis intersecting the 3-faces

3. Tight scheme

Alice considers a cube centered at the origin of her frame, oriented so that the x, y, and z axes form normal vectors to its faces; we call the faces intersected by the x, y, and z axes the 1-, 2-, and 3-faces, respectively. She measures in the basis $\{|\phi_i\rangle\}$, and transmits her measurement result using the encoding scheme given in Table II, and illustrated in Fig. 3, which we summarize as follows. If Alice receives measurement result zero, she sends a spherically symmetric object (in other words, a sphere) to Bob. Otherwise, if she receives measurement result $n \in \{1, 2, 3\}$, she prepares a rigid rod in an arbitrary orientation in space, centered at the origin of her frame, such that it intersects the n-faces of the cube. She then sends this object to Bob by parallel transport.

When Bob receives the object from Alice, he performs the reverse of Alice's encoding scheme. If he receives the spherically symmetric object he performs correction U_0. If he receives a rod, he moves it by parallel transport to his origin, and observes which faces of the cube it intersects. Bob's cube will of course in general be oriented differently to Alice's, and so he may observe a different intersection than that encoded by Alice. Having observed an intersection with the n-faces, he then performs correction U_n.

In Appendix C 3 we numerically calculate the purity of the effective channel as 0.44 ± 0.03, approximately double the value for a conventional scheme.

This scheme is *tight*, possessing in particular the (NL) and (MC) properties, for exactly the same reasons as the previous example.

4. Perfect scheme

Again, transmission of a full reference frame is unnecessary for perfect teleportation. We call the following family of unitary matrices the *tetrahedral* qubit unitary error basis [29]:

$$V_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/3} \end{pmatrix}, \quad V_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2}e^{2\pi i/3} \\ \sqrt{2} & e^{5\pi i/3} \end{pmatrix},$$
$$V_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2}e^{4\pi i/3} \\ \sqrt{2}e^{4\pi i/3} & e^{5\pi i/3} \end{pmatrix}, \tag{7}$$
$$V_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2}e^{2\pi i/3} & e^{5\pi i/3} \end{pmatrix}.$$

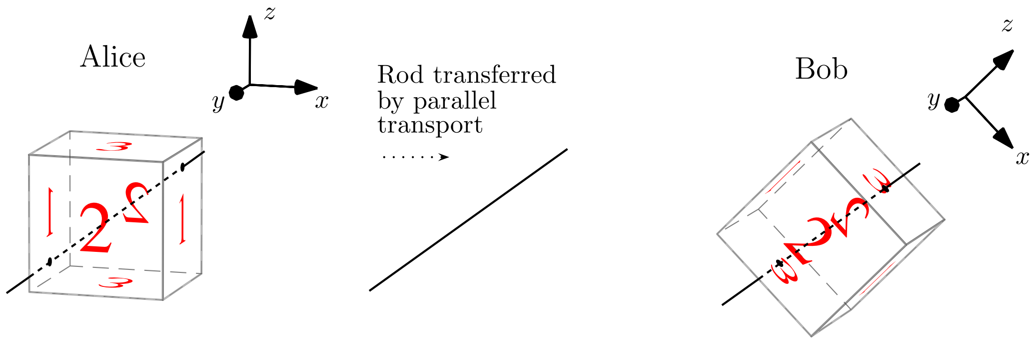

FIG. 3. Tight encoding scheme for the rod channel. Alice measures 1, chooses at random an orientation of the rod which intersects the 1-faces of the cube in her frame, and communicates the rod to Bob by parallel transport along a straight path. In Bob's frame, related to Alice's by a $\pi/4$ rotation around the y axis, the rod intersects the 3-faces; he therefore performs the correction U_3.

Let Tet $\subset$ SO(3) be the subgroup preserving a regular tetrahedron centered at the origin with vertices:

$$v_0 = \hat{z}, \quad v_1 = \frac{1}{3}(\sqrt{8}\hat{x} - \hat{z}), \quad v_2 = \frac{1}{3}(-\sqrt{2}\hat{x} + 2\sqrt{3}\hat{y} - \hat{z}),$$
$$v_3 = \frac{1}{3}(-\sqrt{2}\hat{x} - 2\sqrt{3}\hat{y} - \hat{z}).$$

We identify the elements of Tet $\cong$ SO(3) with the permutation they induce on these vertices.

Alice again measures in the basis $\{|\phi_i\rangle\}$, where $|\phi_i\rangle = [\mathbb{1} \otimes -i(V_i U_2)^T]\,|\eta\rangle$. To perform the classical communication, Alice uses a completely asymmetric classical object whose orientation exactly determines a frame of reference. In order to transmit the measurement result i, she aligns the asymmetric object so that the frame determined by its orientation matches her own Cartesian frame. She then rotates the object by an element $r^A \in$ Tet, according to the prescription in Table III, and sends it to Bob/ Bob observes the orientation of the object according to his own Cartesian frame, and realigns his frame (actively or passively) by the smallest possible angle so that the rotation r^B taking his frame onto that determined by the orientation of the asymmetric object is in Tet. He then uses Table III to decide which measurement result j to correct for, and performs—in his own frame—the correction V_j.

While this procedure only reduces reference-frame uncertainty to the *binary tetrahedral* subgroup of SU(2), it will be shown in Proposition III.5 that it results in perfect teleportation. As before, it possesses the (DR) and (ME) properties, but violates (MC) and (NL).

III. THEORY

We now explain the theory behind the examples in Sec. II.

TABLE III. Type C encoding scheme for the matched channel.

Measurement result	Alice's rotation r^A	Bob's observation r^B
0	()	() or (234) or (243)
1	(132)	(142) or (132) or (12)(34)
2	(123)	(13)(24) or (123) or (143)
3	(134)	(134) or (124) or (14)(23)

A. Equivariant unitary error bases

We first recall the notion of a unitary error basis.

Definition III.1. A *unitary error basis* (UEB) for a d-dimensional Hilbert space V is a basis of d^2 unitary matrices $\{U_i\}_{i\in I}$ in $B(V)$ (where $I = \{1, \dots, d^2\}$ is the index set), which is orthonormal under the Hilbert-Schmidt inner product:

$$\langle U_i | U_j\rangle := \frac{1}{d}\mathrm{Tr}(U_i^\dagger U_j) = \delta_{ij}. \tag{8}$$

Theorem III.1 ([28, Theorem 1]). A teleportation protocol satisfying the (ME) property corresponds to a choice of unitary error basis for V, along with any other unitary matrix X.

Under this correspondence, the shared entangled state η is the maximally entangled state $\sum_i |i\rangle \otimes X\,|i\rangle$ for a chosen orthonormal basis $\{|0\rangle, |1\rangle, \dots\}$ and some unitary X. (Any bipartite maximally entangled pure state is of this form.) Alice measures in the maximally entangled orthonormal basis $\{|\phi_i\rangle\}_{i\in I}$, where

$$|\phi_x\rangle = \sum_i |i\rangle \otimes (U_x X)^T\,|i\rangle. \tag{9}$$

Bob's correction for measurement outcome x is U_x.

We now consider the effect of reference-frame misalignment on such a procedure. Let G be a compact Lie group of reference-frame transformations, with unitary representation $\rho : G \to B(V)$ on Bob's system; here and throughout we assume uniform reference-frame uncertainty, where the probability measure over G is the Haar measure dg. We assume that the maximally entangled state $|\eta\rangle \in V \otimes V$ is invariant up to a phase under changes in frame, so that the entanglement is not itself degraded by reference-frame uncertainty [35]. We work in Alice's frame. In this frame, Alice performs the measurement correctly and sends the result i, but Bob performs the correction $\rho(g)^\dagger U_i \rho(g)$ [36]. Since $g \in G$ is unknown, the effective channel when Alice measures i is

$$\sigma_i' = \int_G dg[\rho(g)^\dagger U_i \rho(g) U_i^\dagger](\sigma). \tag{10}$$

For finite G, we can use an *equivariant* UEB together with a classical channel carrying a G action to perform perfect reference-frame-independent teleportation [37].

Definition III.2. Let a finite group H act on a Hilbert space V of dimension d by the representation $\rho : H \to B(V)$. We say that a unitary error basis $\{U_i\}_{i\in I}$ for V is H *equivariant* when the right conjugation action of H permutes the elements of $\{U_i\}_{i\in I}$ up to a phase. Explicitly,

$$\rho(h)^\dagger U_i \rho(h) = \alpha(i,h) U_{\sigma(i,h)} \quad \forall\, h \in H, i \in I,$$

where $\sigma : I \times H \to I$ is a right action of H on the index set $I = \{0, \dots, d^2-1\}$, and $\alpha : I \times H \to \mathrm{U}(1)$ is some phase.

Proposition III.1. ([29, Theorem 2.7]). Let H be a finite group of reference-frame transformations with an equivariant unitary error basis $\{U_i\}_{i\in I}$ and corresponding right action $\sigma : I \times H \to I$. Let Alice communicate the measurement results using a channel whose set of messages I carries the inverse left action $\sigma^{-1} : H \times I \to I$. Then the teleportation protocol with data $\{U_i\}_{i\in I}$ will function perfectly for all $h_{AB} \in H$.

Proof. In Alice's frame, for measurement result i and any misalignment $h \in H$, Bob will perform the correction $\rho(h)^\dagger U_{\sigma^{-1}(h,i)} \rho(h) \sim U_{\sigma(\sigma^{-1}(h,i),h)} = U_i$. ■

Here we consider actions of general (i.e., possibly infinite) compact Lie groups G, for which equivariant UEBs generally do not exist. Our approach here is to identify a finite subgroup $H \subset G$ such that there exists an equivariant UEB for H under the restricted representation. We then choose an encoding of the measurement result in the classical channel which carries the inverse action in the sense of Proposition III.1, allowing us to "quotient" the space of possible misalignments G by the subgroup H.

Remark III.1. If the representation of G on the system to be teleported is not faithful, we can consider the natural faithful representation of the *reduced reference-frame transformation group* $\tilde{G} := G/\mathrm{Ker}(\rho)$. In Sec. II A, for instance, the reduced transformation group was $\mathrm{U}(1)/\mathbb{Z}_2 \cong \mathrm{U}(1)$, because the representation (1) obeys $\rho(2\theta) = \rho(\theta)$. For the faithful action, we can use the results about existence of equivariant UEBs from [29]. We cannot simply assume that G acts faithfully, though, since when constructing a compatible classical channel it will be necessary to consider the physical rather than the reduced transformation group.

Example III.1. (i) The UEB in both the tight and perfect schemes for U(1) (Sec. II A) is the set of Pauli matrices, which is equivariant for the subgroup $\mathbb{Z}_4 < \mathrm{U}(1)$ of the reduced transformation group. A generator of $\mathbb{Z}_4$ acts as the swap (12) on the index set of the UEB under conjugation.

(ii) In the tight scheme for SU(2) (Sec. II B) the Pauli UEB is equivariant for the binary octahedral subgroup BOct $\subset$ SU(2) preserving the cube.

(iii) In the perfect scheme for SU(2) (Sec. II B) the tetrahedral UEB is equivariant for the binary tetrahedral subgroup BTet $\subset$ SU(2) preserving the tetrahedron.

B. Compatible encoding of classical information

We now consider the other component of the scheme, a classical channel carrying an action of the reference-frame transformation group. The spaces of readings of all the classical channels we consider in this work carry a smooth manifold structure with normalized measure dx, and all actions are smooth and measure preserving.

Definition III.3. We say that a classical channel *communicates unspeakable information* [23], or is an *unspeakable channel*, if its space of readings C carries a nontrivial action of the reference-frame transformation group G.

We call a channel whose space of readings carries a trivial G action a *speakable channel*.

Throughout this paper we make the simplifying assumption that there is no channel noise, apart from that arising from frame misalignment. A classical channel is therefore fully described by its space of readings and the G action on that space; for this reason we conflate the channel with its space of readings, using the same letter C for both. Since we have chosen the convention that the effect of a change of reference frame on the states of a quantum system corresponds to a left action of the transformation group (see Appendix A), the action of G on the classical channel will be a left action.

Example III.2. (i) For the tight and perfect schemes in Sec. II A, the space of readings was the linear polarization direction of the light beam. As a smooth manifold, this is the real projective line $\mathbb{RP}^1$; it carries a nonfaithful smooth action of U(1) with kernel $\mathbb{Z}_2$ (since a π rotation does not change the polarization direction).

(ii) For the tight scheme in Sec. II B, the space of readings was the space of possible orientations of a rod. As a manifold, this is the real projective plane $\mathbb{RP}^2$, carrying the obvious smooth action of SU(2).

(iii) For the perfect scheme in Sec. II B, the space of readings was the space of possible orientations of a completely asymmetric object. As a manifold, this is the Stiefel (frame) manifold $V_2(\mathbb{R}^3) \cong \mathrm{SO}(3)$, carrying the obvious smooth action of SU(2).

We now specify a framework for encoding of measurement values in such a channel.

Definition III.4 (Encoding scheme). Let C be an unspeakable channel and I be a finite set of values to be sent through it. An *encoding scheme* for I is as follows.

(i) A set of open subsets $\{E_i \subset C \mid i \in I\}$, the *encoding subsets*, where E_i are disjoint open sets.

(ii) A set of open subsets $\{D_i \subset C \mid i \in I\}$, the *decoding subsets*, where D_i are disjoint open sets which cover C up to a set of measure zero.

The encoding subset E_i is the set of all possible readings Alice can send in order to transmit the value $i \in I$. The decoding subset D_i is the set of all possible readings upon receipt of which Bob will record the value $i \in I$.

Recalling Proposition III.1, the success of our protocol depends on encoding schemes which are compatible with the right action of H on the index set of the UEB.

Definition III.5 (Compatible channel). Let C be an unspeakable channel for a finite group H. Let $\sigma : I \times H \to I$ be a right action of H on an index set I. We say that an encoding scheme for I is *compatible with* σ if (i) the decoding subsets $\{D_i\}_{i\in I}$ and the encoding subsets $\{E_i\}_{i\in I}$ are each permuted under the action of H on C, inducing left actions $\tau_D, \tau_E : H \times I \to I$, and (ii) the left actions $\tau_D, \tau_E : H \times I \to I$ are equal and inverse to the action $\sigma : I \times H \to I$ of H on I. That is, for all $i \in I$,

$$\tau_D(i,-) = \tau_E(i,-) = \sigma^{-1}(i,-).$$

In other words, given a right action of a finite reference-frame transformation group on the UEB index set, a compatible encoding scheme transmits the indices through the classical channel with the inverse left action.

Example III.3. (i) In Sec. II A, the encoding and decoding subsets for the tight scheme are the same, namely the regions R_1 and R_2 (Fig. 2). In the physical (unfaithful) representation, the Pauli UEB is equivariant for the subgroup $\mathbb{Z}_8 < \mathrm{U}(1)$, where a generator of $\mathbb{Z}_8$ acts as the swap (12). Compatibly, the regions R_1 and R_2 are swapped under the action of a generator of $\mathbb{Z}_8$. For the perfect scheme, the encoding subsets are singletons, namely the polar angles zero and $\pi/4$; the decoding subsets are the regions R_1 and R_2.

(ii) In the tight scheme of Sec. II B, the encoding and decoding subsets are the same: $D_i = E_i$ is the subset of orientations of the rod through the i-faces of the cube. The indices of the Pauli UEB are permuted inversely to the labels on the cube's faces under the conjugation action of BOct.

(iii) In the perfect scheme of Sec. II B, the encoding subsets E_i are singletons, namely the orientations given by rotating the object according to Table III. The decoding subsets are Voronoi cells around these orientations [38]. The indices of the tetrahedral UEB are permuted inversely to the encoding and decoding subsets under the conjugation action of BTet.

Construction of compatible encoding schemes

We now provide a general construction of a compatible encoding scheme for any transitive action $\sigma : I \times H \to I$ of a finite subgroup of G. Since all actions split into transitive actions on the orbits, this loses no generality, since we can communicate the orbit index using speakable communication. For the construction, we need an unspeakable classical channel of the following type. Recall that an action is *free* if all stabilizers are trivial and *transitive* if it possesses only one orbit.

Definition III.6. Let G be the reference-frame transformation group, with representation ρ on the system to be teleported. Let C be an unspeakable classical channel, carrying the action $\alpha : G \times C \to C$. We say C is *matched to* ρ if $\mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\alpha)$, and the reduced action $\tilde{G} \times C \to C$, where $\tilde{G} = G/\mathrm{Ker}(\rho)$ is the reduced transformation group, is free and transitive.

Example III.4. (i) In Sec. II A the kernel of the representation ρ is $\mathbb{Z}_2$, generated by the rotation through an angle π. Likewise, the kernel of the action of U(1) on polarization directions is U(2). The reduced group $G/\mathrm{Ker}(\rho)$ corresponds to the rotations $\theta \in (-\pi/2, \pi/2]$, which clearly act freely and transitively on the polarization directions.

(ii) The channel for the perfect scheme in Sec. II B, where a completely asymmetric classical object was transmitted, is a matched channel for the representation of SO(3). Here the kernel of ρ is trivial, and the action of SO(3) on the set of orientations is clearly free and transitive.

The readings of a matched channel C can be identified with elements of the reduced transformation group $\tilde{G}$, by choosing an "identity" reading $[e] \in C$ based on their own reference-frame configuration. All other readings in C are then identified uniquely by $[g] := g \cdot [e]$, for any $g \in \tilde{G}$.

Example III.5. (i) For the channel of Sec. II A, the channel reads $[e]$ when the polarization axis is the x axis of the observer.

(ii) For the perfect scheme of Sec. II B, the channel reads $[e]$ when the frame defined by the asymmetric object is aligned with the Cartesian frame of the observer.

In general, Alice and Bob will have different labelings of the channel, given that their reference frames are oriented differently. We write $[g]_A, [g]_B$ for the reading associated to $g \in G$ by Alice and Bob, respectively.

Proposition III.2. If Bob's frame is related to Alice's by a transformation $g_{AB} \in G$, then their labelings are related as follows:

$$[g]_A = \left[g g_{AB}^{-1}\right]_B. \tag{11}$$

We now construct the compatible encoding scheme. We recall the following characterization of transitive actions.

Lemma III.1. Let H be a finite group. Any transitive right H-set is isomorphic to a right coset space $L\backslash H$ for a subgroup $L \subset H$ under the right action $(Lh_2) \cdot h_1 = Lh_2h_1$.

Our construction divides the matched channel C up into regions $\{R_h \subset C \mid h \in H\}$, which are permuted by reference-frame transformations in H according to the inverse left action $h_2 \cdot R_{h_1} = R_{h_1 h_2^{-1}}$. We then identify these regions to obtain the desired transitive action. To define the R_h, we choose a fundamental domain for the finite subgroup $H \subset \tilde{G}$.

Definition III.7. A *fundamental domain* for a finite subgroup $H \subset G$ is an open subset $F \subset G$ containing the identity such that the H-translates Fh have empty intersection and cover G up to a set of measure zero [39].

Example III.6. In the example of Sec. II A, the rotations through an angle $\theta \in (-\pi/8, \pi/8)$ are a fundamental domain for $\mathbb{Z}_4 \subset \tilde{G}$.

Definition III.8. Fix a subgroup $H \subset G$, and a fundamental domain F for H in G. Then the regions $\{R_h \mid h \in H\}$ are defined as

$$R_h := \{[fh] \mid f \in F\}.$$

Lemma III.2. Let Bob's reference-frame configuration be related to Alice's by a transformation $h_{AB} \in H$. Then

$$(R_h)_A = \left(R_{hh_{AB}^{-1}}\right)_B.$$

Proof. Immediate from (11). ■

We can now construct a compatible encoding scheme for the transitive action $L\backslash H$ by grouping regions R_h into cosets. Let $c_i \in H$ be right coset representatives for L in H.

Definition III.9. The *tight matched scheme* for σ is defined as

$$D_i = \bigsqcup_{l \in L} R_{lc_i}, \quad E_i = D_i.$$

The *perfect matched scheme* is defined as

$$D_i = \bigsqcup_{l \in L} R_{lc_i}, \quad E_i = \left\{\bigsqcup_{l \in L} [lc_i]\right\}.$$

The reason for the nomenclature will become apparent in the next section.

C. Teleportation schemes

We now specify and prove correctness for our teleportation schemes. Throughout this section, let $H \subset G$ be a finite subgroup, let $\{U_i\}_{i\in I}$ be an equivariant UEB for H, let $\sigma : I \times H \to I$ be the corresponding right action of H on the index set of the UEB, let $I_k \subset I$ be the orbits in I under σ, where k is some index for the orbits, and let $\sigma_k : I_k \times H \to I_k$ be the corresponding (transitive) restricted actions.

1. Tight scheme

Procedure III.1 (Tight teleportation scheme). Let C be an unspeakable channel for G (and therefore also for H), and let $(D_i^k, E_i^k)_{i\in I}$ be encoding schemes for I_k on C compatible with $\sigma_k : I_k \times H \to I_k$ and such that, for each k, the decoding regions are the same as the encoding regions, that is, $D_i^k = E_i^k$ for all i, k.

Alice measures in the basis $\{|\phi_i\rangle\}_{i\in I}$ (9) as in a standard teleportation protocol, and obtains the result $i \in I_k$. The result is transmitted as follows.

(1) Alice transmits the orbit label k through a speakable channel.

(2) Alice sends a reading x chosen uniformly at random from the region E_i^k.

(3) Bob receives $g \cdot x \in D_j^k$ and performs the correction U_j.

Here g is the reference-frame transformation taking Alice's frame at the time of measurement onto Bob's frame at the time of receipt.

We now derive an explicit expression for the effective channel obtained using Procedure III.22. Recall that, for operators $M, \sigma \in B(H)$, we write $[M](\sigma)$ for $M\sigma M^\dagger$.

Theorem III.2 (Effective channel for Procedure III.1). Suppose that Alice measures some result $i \in I_k$, where $D_k^i = E_k^i$ for all $i \in I_k$. Then the channel induced by Procedure III.1 is

$$\mathcal{T}_k(\sigma) = \frac{|I_k|}{\mu_C(E_0^k)}[\rho(c_i)] \circ \int_G (dg\, p(g)\, [\rho(g)^\dagger U_0 \rho(g) U_0^\dagger] \circ [\rho(c_i)^\dagger]\, (\sigma)). \tag{12}$$

Here $0 \in I_k$ is some fixed element of the orbit; the normalizing factor $\mu_C(E_0^k)$ is the measure of E_0^k in C; $p(g) = \int_{E_0^k \subset C} dx\, \mathbb{1}_{D_0^k}(g \cdot x)$, where $\mathbb{1}_{D_0^k}$ is a continuous approximation to the indicator function for $D_0^k \subset C$; and $\{c_i\}_{i\in I_k}$, $c_i \in H$ are such that $c_i \cdot E_0^k = E_i^k$.

Proof. The proof is somewhat technical, so it has been placed in Appendix B. ■

Proposition III.3. Procedure III.1 satisfies (MC), (NL), (ME), and (DR).

Proof. (NL) Alice has an equal probability of measuring any $i \in I_k$, and chooses a reading with uniform probability from the subsets $\{E_i^k = D_k^i\}_{i\in I}$, which have equal measure and cover the space of readings up to a set of measure zero. The message therefore communicates no information about Alice's frame configuration, since without prior knowledge of the reading Alice sent, nothing can be learned from the reading that is received.

(MC) The only useful information Bob learns from the message he receives is which of his decoding subsets $\{D_i^k\}_{i\in I_k}$ the reading he receives lies in; there are $\sum_k |I_k| = |I| = d^2$ possible messages, which are equiprobable. In total, therefore, he receives two dits of unspeakable classical information.

(ME) Obvious.

(DR) In Alice's frame, reference-frame misalignment affects Bob's reading of the transmitted measurement result, and his unitary correction. Provided that his frame configuration remains approximately constant between these steps, the effective channel (12) is unaffected by arbitrary changes in reference-frame alignment throughout the rest of the procedure. ■

2. Perfect scheme

Procedure III.2 (Perfect scheme). Let C be an unspeakable channel for G (and therefore also for H), and let $(D_i^k, E_i^k)_{i\in I}$ be encoding schemes for I_k compatible with $\sigma_k : I_k \times H \to I_k$, and where $E_i^k = X_i^k$, where $X_i^k \subset D_i^k$ is a finite set of readings in C, and moreover H acts transitively on $\sqcup_i X_i^k$.

Alice measures in the basis $\{|\phi_i\rangle\}_{i\in I}$ (9) as in a standard teleportation protocol and obtains the result $i \in I_k$. The result is transmitted as follows.

(1) Alice transmits the orbit label k through a speakable channel.

(2) Alice sends a reading $x_i^k \in X_i^k$ chosen uniformly at random.

(3) Bob receives $y = g \cdot x_i^k \in g \cdot X_i^k = X_j^k \subset D_j^k$ and performs the correction $\rho(r_j(y)) U_j \rho(r_j(y))^\dagger$, where $r_j(y) \in G$ is any element such that $r_j(y) \cdot x_j^k = y$ for some $x_j^k \in X_j^k$.

In other words, Bob realigns his frame (actively or passively) so that the reading he receives is $x_j^k \in X_j^k$, and then performs the correction U_j. Here g is the reference-frame transformation taking Alice's frame at the time of measurement onto Bob's frame at the time of receipt.

Proposition III.4 (Effective channel for Procedure III.25). Suppose that Alice measures some result $i \in I_k$. Then the quantum channel induced by Procedure III.2 is as follows:

$$\mathcal{T}_i(\sigma) = \int_{Stab_G(x_i)} ds[\rho(s)^\dagger U_i \rho(s) U_i^\dagger](\sigma). \tag{13}$$

Here ds is the Haar measure on $Stab_G(x_i^k)$.

Proof. Alice measures $i \in I_k$ and communicates x_i^k to Bob, who receives $y \in D_j$, where $y = g \cdot x_i^k = (r_j(y) h_{ij} s) \cdot x_i^k$ for $h_{ij} \in H$ such that $h_{ij} \cdot x_i^k = x_j^k$ (this exists because H acts transitively on $\sqcup_i X_i^k$) and some $s \in \mathrm{Stab}_G(x_i^k)$.

The distribution over $\mathrm{Stab}_G(x_i^k)$ is uniform. We therefore have the following expression for the effective channel:

$$\begin{aligned}\mathcal{T}_k(\rho) &= \int_{\mathrm{Stab}_G(x_i^k)} ds[\rho(r_j(y)h_{ij}s)^\dagger \rho(r_j(y)) U_j \rho(r_j(y))^\dagger \\ &\quad \times \rho(r_j(y)h_{ij}s) U_i^\dagger](\sigma) \\ &= \int_{\mathrm{Stab}_G(x_i^k)} ds[\rho(h_{ij}s)^\dagger U_j \rho(h_{ij}s) U_i^\dagger](\sigma) \\ &= \int_{\mathrm{Stab}_G(x_i^k)} ds[\rho(s)^\dagger U_i \rho(s) U_i^\dagger](\sigma).\end{aligned}$$

At each step, we used the fact ρ is a representation. For the final equality, we used equivariance of the unitary error basis. ■

In particular, this produces perfect teleportation for matched channels.

Proposition III.5. Procedure III.2 with the perfect encoding scheme on a matched channel (Definition III.9) results in perfect teleportation.

Proof. The stabilizer of any reading is trivial, since the action is free. ■

The perfect scheme also possesses the (ME) and (DR) properties, for exactly the same reasons as the tight scheme.

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APPENDIX A: REFERENCE-FRAME TRANSFORMATION RULES

In this Appendix we briefly summarize the effect of reference-frame transformations on measurements and operations. Let $\mathcal{F}$ be the space of reference-frame configurations. Let V be the d-dimensional Hilbert space of a system whose states are described according to a reference frame. The Hilbert space carries a unitary representation $\rho : G \to B(V)$, which encodes how states transform upon a change of reference frame: a state with vector $|\psi\rangle$ in reference frame $f \in \mathcal{F}$ will have vector $\rho(g)\,|\psi\rangle$ in reference frame $g \cdot f$. Let $g_{AB} \in G$ be the reference-frame transformation taking Alice's frame $f_A \in \mathcal{F}$ onto Bob's frame $f_B \in \mathcal{F}$; that is, $f_B = g_{AB} \cdot f_A$. We then have the following expressions.

Proposition A.1. A state with vector $|\psi\rangle$ in Bob's frame has vector $\rho(g)^\dagger\,|\psi\rangle$ in Alice's frame. An linear map with matrix $M : V \to V$ in Bob's frame has matrix $\rho(g)^\dagger M \rho(g)$ in Alice's frame. A general operation $\Phi : L(V) \to L(V)$ in Bob's frame is the operation $[\rho(g)^\dagger] \circ \Phi \circ [\rho(g)]$ in Alice's frame.

Proof. By definition a state described in Alice's frame as $|\psi\rangle$ will be described in Bob's frame as $\rho(g)\,|\psi\rangle$; the first equation follows immediately.

For the linear maps, consider that a linear map is defined by its matrix elements in some orthonormal basis. Bob performs the operation with matrix elements M_{ij} in his frame; that is, he performs the operation M_B such that $\langle i_B|\,M_B\,|j_B\rangle = M_{ij}$. Now note that $|i_B\rangle = \rho(g)^\dagger\,|i_A\rangle$, so $M_{ij} = \langle i_B|\,M_B\,|j_B\rangle = \langle i_A|\,\rho(g)M_B\rho(g)^\dagger\,|j_A\rangle$. In Alice's frame, therefore, Bob has performed the operation M_B such that $\rho(g)M_B\rho(g)^\dagger = M_A$; this operation is therefore related to M_A by $M_A = \rho(g)^\dagger M_B \rho(g)$. The same argument can be extended to general operations by considering the Kraus maps. ■

APPENDIX B: PROOF OF THEOREM III.2

We now provide the postponed proof of this theorem.

Theorem B.1 (Effective channel for a general encoding scheme). Suppose that Alice measures some result $i \in I_k$, where $D_k^i = E_k^i$ for all $i \in I_k$. Then the channel induced by Procedure III.1 is as follows:

$$\mathcal{T}_k(\rho) = \frac{|I_k|}{\mu_C(E_0^k)}[\pi(c_i)] \circ \int_G (dg\, p(g)[\pi(g)^\dagger U_0 \pi(g) U_0^\dagger]) \circ [\pi(c_i)^\dagger](\rho). \tag{B1}$$

Here $0 \in I_k$ is any element of the orbit, the normalizing factor $\mu_C(E_0^k)$ is the measure of E_0^k in C, $p(g) = \int_{E_0^k \subset C} dx\, \mathbb{1}_{D_0^k}(g \cdot x)$, where $\mathbb{1}_{D_0^k}$ is a continuous approximation to the indicator function for $D_0^k \subset C$, and $\{c_i\}_{i\in I_k}$, $c_i \in H$ are such that $c_i \cdot E_0^k = E_i^k$.

Proof. We define $U(x) = U_j \mid x \in D_j^k$. Then, in Alice's frame, Bob's correction will be

$$\pi(g_{AB})^\dagger U(g_{AB} \cdot x)\pi(g_{AB}),$$

where $x \in E_i^k$ is the direction sent by Alice. Since both $g_{AB} \in G$ and $x \in E_i^k$ are unknown and uniformly distributed, we must average over both. When Alice measures $i \in I_k$, the channel is as follows for input state σ:

$$\mathcal{T}_i^k(\sigma) = \frac{1}{\mu_C(E_i^k)} \int_{G\times C} dg\, dx\, \mathbb{1}_{E_i^k}(x) \times [\rho(g)^\dagger U(g\cdot x)\rho(g) U_i^\dagger](\sigma). \tag{B2}$$

Here $\mathbb{1}_{E_i^k}$ is a continuous approximation to the indicator function for the region $E_i \subset C$.

First we show that $\mathcal{T}_i^k = [\rho(c_i)] \circ \mathcal{T}_0^k \circ [\rho(c_i)^\dagger]$; that is, every measurement result in a given orbit produces a similar channel. Indeed, since the product measure $dg\, d\phi$ is invariant under the left G action $g_1 \cdot (g_2, x) = (g_2 g_1^{-1}, g_1 \cdot x)$ on $G \times C$, we can make the change of variables $(g, x) \mapsto (g c_i^{-1}, c_i \cdot x)$:

$$\begin{aligned}\mathcal{T}_i^k(\sigma) &= \frac{1}{\mu_C(E_i^k)} \int_{G\times C} dg\, dx\, \mathbb{1}_{E_i^k}(c_i \cdot x) \\ &\quad\times [\rho(c_i)\rho(g)^\dagger U(g\cdot x)\rho(g)\rho(c_i)^\dagger U_i^\dagger \rho(c_i)\rho(c_i)^\dagger](\sigma) \\ &= \frac{1}{\mu_C(E_0^k)}[\rho(c_i)] \circ \int_{G\times C} dg\, dx\, \mathbb{1}_{E_0^k}(x) \\ &\quad\times [\rho(g)^\dagger U(g\cdot x)\rho(g) U_1^\dagger] \circ [\rho(c_i)^\dagger](\sigma) \\ &= [\rho(c_i)] \circ \mathcal{T}_0^k \circ [\rho(c_i)^\dagger].\end{aligned}$$

To obtain the first equality we changed variables and used the fact that ρ is a representation. For the second equality we used $\mathbb{1}_{E_i^k}(c_i \cdot x) = \mathbb{1}_{E_0^k}$, linearity, and the fact that the action of G on C is measure preserving. We can therefore restrict our attention to the channel where Alice measures the index $0 \in I_k$.

We will now express the integral for the channel T_0^k as a sum over integrals where Bob performs a definite correction. The action $\nu : (g, x) \mapsto g \cdot x$ is continuous; it follows that the preimages of the open sets D_i^k under ν are open and therefore measurable. That the open sets $\nu^{-1}(D_i^k)$ cover $G \times C$ up to a set of measure zero follows immediately from the fact that the D_i^k cover C up to a set of measure zero and ν is a submersion. We may therefore split the domain of integration over the $\nu^{-1}(D_i^k)$:

$$\mathcal{T}_0^k(\sigma) = \frac{1}{\mu_C(E_0^k)} \sum_{i\in I_k} \int_{G\times C} dg\, dx\, \mathbb{1}_{E_0^k}(x)\mathbb{1}_{D_i^k}(g\cdot x) \times [\rho(g)^\dagger U_i \rho(g) U_0^\dagger](\sigma).$$

Now we observe that the integrals over $\nu^{-1}(D_i^k)$ are identical for all $i \in I_k$:

$$\mathcal{T}_0^k(\sigma) = \frac{1}{\mu_C(E_0^k)} \sum_{i\in I_k} \int_{G\times C} dg\, dx\, \mathbb{1}_{E_0^k}(x)\mathbb{1}_{D_i^k}(g\cdot x) \times \left[\rho\left(c_i^{-1}g\right)^\dagger U_0 \pi\left(c_i^{-1}g\right)U_0\right](\sigma)$$
$$= \frac{|I_k|}{\mu_C(E_0^k)} \int_{G\times C} dg\, dx\, \mathbb{1}_{E_0^k}(x)\mathbb{1}_{D_0^k}(g\cdot x) \times [\rho(g)^\dagger U_0 \rho(g) U_0](\sigma).$$

The first equality uses that $U_i = \rho(c_i)U_0\rho(c_i)^\dagger$; in the second we performed the change of variables $(g, x) \mapsto (c_i g, x)$ and noted that $\mathbb{1}_{D_i^k}[(c_i g)\cdot x] = \mathbb{1}_{D_0^k}(g\cdot x)$, since $D_i^k = E_i^k$ for all i, k. By Fubini's theorem this may be evaluated as an iterated integral, where x is integrated over first:

$$\mathcal{T}_0^k(\sigma) = \frac{|I_k|}{\mu_C(E_0^k)} \int_G dg \int_C dx\, \mathbb{1}_{E_0^k}(x)\mathbb{1}_{D_0^k}(g\cdot x) \times [\rho(g)^\dagger U_0 \rho(g) U_0](\sigma).$$

This produces a weighting for $g \in G$ which is precisely the measure in C of the set $D_0^k \cap (g\cdot E_0^k)$. The result follows. ■

APPENDIX C: CALCULATIONS

In this appendix we derive the numerical results presented in Table I.

1. Map purity and its calculation

The measure we use to evaluate the success of the protocol is the *map purity* [40–42]. Recall that the *Choi-Jamiołkowski (CJ)* state $\rho_\mathcal{T}$ of a channel $\mathcal{T}$ on a Hilbert space of dimension d is

$$\rho_\mathcal{T} = \tfrac{1}{2}(\mathbb{1}\otimes\mathcal{T})(\omega),$$

where ω is the density matrix of the state $\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle$. (For calculations, recall that the density matrix of the CJ state can be obtained by "reshuffling" the entries of the superoperator matrix of the channel [40].)

Definition C.1. The *map purity* $P(\mathcal{T})$ of a channel $\mathcal{T}$ on a Hilbert space of dimension d is the normalized purity of its CJ state; that is,

$$P(\mathcal{T}) := 1 - \frac{S(\rho_\mathcal{T})}{\ln(d^2)} = 1 + \frac{\mathrm{Tr}[\rho_\mathcal{T}\ln(\rho_\mathcal{T})]}{\ln(d^2)}. \quad \text{(C1)}$$

For numerical optimization we will additionally use the linear map purity.

Definition C.2. The *linear map purity* $P^L(\mathcal{T})$ of a channel $\mathcal{T}$ on a Hilbert space of dimension d is defined as the linear purity of its CJ state; that is,

$$P^\mathrm{L}(\mathcal{T}) = \mathrm{Tr}\left(\rho_\mathcal{T}^2\right).$$

The map purity in the qubit case, which we consider in our examples, is very similar to minimum purity over pure state inputs [41].

By (10), the channels we consider are of the following sort.

Definition C.3. A *random unitary channel* is a channel of the form

$$\sigma \mapsto \int_X dx [U(x)](\sigma)$$

for some label space and probability measure (X, dx), where each $U(x)$ is a unitary matrix.

In particular, our random unitary channels are

$$\sigma \mapsto \sum_i \int_G dg\, p(i)q(g)[U(i, g)](\sigma),$$

where $U(i, g)$ are the unitaries, the label space is $I \times G$, and the probability measure on the label space is $p(i)dg$; this is the product of the probability $p(i)$ of measurement result i (which is uniform), and the Haar measure dg over the group G of reference-frame misalignments. A little straightforward algebra yields the following useful expression for the linear map purity of these channels.

Proposition C.1 (Linear map purity of a random unitary channel). Let $\mathcal{T}$ be a random unitary channel on a Hilbert space of dimension d. Let the random unitaries be indexed by a discrete index $I = \{0, \dots, n-1\}$ with probability distribution $p(i)$ and a continuous index $g \in G$ with probability measure dg. Then

$$P^L(\mathcal{T}) = \frac{1}{d^2}\sum_{i,j=0}^{n-1}\int_{G\times G} p(i)p(j)dg\,dg'|\mathrm{Tr}[U(i,g)^\dagger U(j,g')]|^2. \quad \text{(C2)}$$

We now consider teleportation of quantum systems carrying fundamental representations of the reference-frame transformation groups U(1) and SU(2). For each of these groups, we first find the UEB, which optimizes the linear map purity of the quantum channel resulting from a conventional protocol (10), and then calculate the map purity of the quantum channel arising from that UEB, obtaining the numbers in the second column of Table I. We then calculate the map purity for certain of our tight schemes, obtaining the numbers in the third column of that table.

2. Calculations for U(1)

Here we consider the case $G = \mathrm{U}(1)$, where the group of reference-frame transformations acts on the qubit state as follows:

$$\theta \mapsto \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}. \quad \text{(C3)}$$

a. Conventional scheme

We begin by finding the UEB which optimizes linear map purity for a conventional protocol. A general qubit UEB may be expressed as $U\mathcal{E}V$, where U, V are arbitrary unitary matrices and $\mathcal{E} = \{X_0, X_1, X_2, X_3\}$ is the Pauli UEB (2). Since we ignore global phase, we need only consider unitaries up to their induced rotation of the Bloch sphere. Let $R_{\hat{n}}(\theta)$ be a Bloch sphere rotation through an angle θ around the $\hat{x}$ axis, let X_i be a Pauli rotation (that is, a rotation through an angle π around the x, y, or z axis), and let $\hat{x}, \hat{y}$ be two unit vectors which correspond to the choice of UEB. Then the equiprobable unitaries are as follows:

$$U_{ig} = gV^\dagger X_i U^\dagger g^\dagger U X_i V \quad \text{(C4)}$$

$$\sim V g V^\dagger X_i U^\dagger g^\dagger U X_i \quad \text{(C5)}$$

$$= R_{\hat{x}}(\theta) R_{X_i(\hat{y})}(-\theta). \quad \text{(C6)}$$

We write $\sim$ to indicate that replacing unitaries (C4) with unitaries (C5) will yield a channel with the same purity, because of cyclicity of the trace in (C2). The second equality follows by the fact that conjugating a rotation $R_{\hat{x}}(\theta)$ by another rotation Q gives $QR_{\hat{x}}(\theta)Q^{-1} = R_{Q(\hat{x})}(\theta)$. By Proposition C.4 we therefore have the following expression for the effective channel:

$$P(\mathcal{T}) = \frac{1}{256\pi^2} \sum_{i,j} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \times \left|\mathrm{Tr}\left[R_{X_j(\hat{y})}(-\theta_2) R_{X_i(\hat{y})}(\theta_1) R_{\hat{x}}(\theta_2 - \theta_1)\right]\right|^2. \quad \text{(C7)}$$

Here the choice of UEB corresponds to a choice of two unit vectors $(\hat{x}, \hat{y})$ or equivalently a choice of angles $(\psi_{\hat{x}}, \psi_{\hat{y}}, \phi_{\hat{x}}, \phi_{\hat{y}}) \in [0, \pi]^2 \times [0, 2\pi]^2$. The factor in front of the integral is a product of the normalization factors for the parametrization of U(1) and the $1/4$ probabilities for measurement results i and j. The simplicity of the integral allows us to numerically evaluate it for given $\hat{x}, \hat{y}$ with negligible error. We performed Nelder-Mead maximization over $\hat{x}, \hat{y}$ and found optimality of the Pauli UEB, corresponding to angles (0,0,0,0). The normalized map purity for this UEB is

$$1 + \frac{1}{\ln(4)}[0.75 \ln(0.75) + 0.25 \ln(0.25)] \simeq 0.59.$$

b. Tight scheme

We must choose a finite subgroup $H \subset \mathrm{U}(1)$ for which an equivariant UEB exists. In [29] the largest such subgroup was shown to be $H \simeq \mathbb{Z}_4$, with a two-parameter family of equivariant UEBs:

$$U_0 = R_{\hat{z}}(\theta - \pi), \quad U_1 = R_{\hat{z}}(\phi) X R_{\hat{z}}(-\phi),$$
$$U_2 = R_{\hat{z}}(\phi) Y R_{\hat{z}}(-\phi), \quad U_3 = R_{\hat{z}}(\theta).$$

The Pauli UEB is the member of this family with parameters $\theta = \pi, \phi = 0$. The tight reference-frame encoding scheme for this family of UEBs was given in Fig. 2.

We use Theorem III.2 to calculate the superoperator for the effective channel. Because the group is Abelian, conjugation by $\pi(c_i)$ is irrelevant, so the channel will be identical for measurements 1 and 2. For a similar reason we need only consider the Pauli UEB, since all UEBs in the family yield identical channels. It is easy to derive an analytic expression for $p(\theta)$:

$$p(\theta) = \left|\frac{(\theta + \pi/2)}{\pi} - \left\lfloor \frac{1}{2} + \frac{(\theta + \pi/2)}{\pi} \right\rfloor\right|. \quad \text{(C8)}$$

The effective channel when Alice measures 1 is

$$4 \int_{-\pi}^{\pi} d\theta \, p(\theta) \, [\rho(\theta)^\dagger U_1 \rho(\theta) U_1^\dagger](\sigma), \quad \text{(C9)}$$

and the channel for result 2 is similar. Since measurement results zero and 3 yield perfect teleportation, we obtain the action (5) of the effective channel on input density matrices. The normalized map purity for this effective channel is

$$1 + 0.5 \ln(0.5) \simeq 0.65.$$

3. Calculations for SU(2)

We now consider the case $G = \mathrm{SU}(2)$, acting on a qubit state by its defining representation.

a. Conventional scheme

We have a channel of the form (C2), which involves integration over SU(2). In order to obtain a parametrization and measure for the integral, we use the isomorphism between SU(2) and the unit quaternions. These quaternions, being diffeomorphic to the 3-sphere S^3, may be parametrized by hyperspherical coordinates $(\theta, \psi, \phi) \in D$, where $D = [0, \pi] \times [0, \pi] \times [0, 2\pi]$. This parametrization is inherited by SU(2), along with the Haar measure $d\Omega$ on S^3, as follows:

$$g(\theta, \psi, \phi) = \begin{pmatrix} \cos(\theta) + i \sin(\theta)\sin(\psi)\sin(\phi) & \left[\cos(\psi) + i\cos(\phi)\sin(\psi)\right]\sin(\theta) \\ -\left[\cos(\psi) - i\cos(\phi)\sin(\psi)\right]\sin(\theta) & \cos(\theta) - i\sin(\phi)\sin(\psi)\sin(\theta) \end{pmatrix},$$

$$d\Omega = \frac{1}{2\pi^2}\sin^2(\theta)\sin(\psi)\, d\theta\, d\psi\, d\phi.$$

We consider the integrand. Expanding the UEB elements in the form $U\mathcal{E}V$, where U, V are arbitrary unitary matrices and $\mathcal{E} = \{X_0, X_1, X_2, X_3\}$ is the Pauli UEB, we see that the unitaries of the channel will be, for all $Y \in \mathrm{SU}(2)$ and $i \in I = \{1, \dots, 4\}$,

$$U(Y, i) = YV^\dagger X_i^\dagger U^\dagger Y^\dagger U X_i V \sim VYV^\dagger X_i^\dagger U^\dagger Y^\dagger U X_i,$$

where the equivalence is again a consequence of the cyclicity of the trace in (C2). We therefore obtain the following equation for the map purity:

$$P(\mathcal{T}) = \frac{1}{32} \int_{D\times D} d\Omega_1 d\Omega_2 |\mathrm{Tr}[X_i Y_1 X_i \widetilde{U} Y_1^\dagger Y_2 \widetilde{U}^\dagger X_j Y_2^\dagger X_j]|^2. \quad \text{(C10)}$$

Here we performed a change of variables from Y_i to $\widetilde{Y}_i = VY_iV^\dagger$, using the invariance of the measure; we omit the tilde on the new variable. We also wrote $\widetilde{U} := VU$; note that this is the only significant element in our choice of UEB.

There are only three relevant angle variables in the choice of UEB, corresponding to a choice of a single unitary $\widetilde{U} := VU$. We performed random sampling of 100 angle triples; none of these UEBs outperformed the Pauli matrices, whose normalized map purity is

$$1 - \frac{1}{2\,\ln(4)}\left[\ln\left(\frac{1}{2}\right) + \ln\left(\frac{1}{6}\right)\right] \simeq 0.21.$$

b. Tight scheme with rod channel

The action on the rod channel considered in Sec. II B can be most easily expressed using the inner product-preserving isomorphism of SU(2) spaces

$$\begin{aligned} S^2 \subset \mathbb{R}^3 &\xrightarrow{\alpha} B(\mathbb{C}^2), \\ (n_x, n_y, n_z) &\mapsto \frac{I + (n_x, n_y, n_z)\cdot(X, Y, Z)}{2}, \end{aligned} \tag{C11}$$

where I, X, Y, and Z are the Pauli matrices, S^2 carries the obvious quotient left action of SU(2), and $B(\mathbb{C}^2)$ carries the left action of SU(2) by conjugation. The encoding and decoding regions are then made up of Voronoi cells for the cardinal points under the metric derived from the Hilbert-Schmidt inner product.

Using the above identification, we calculated $p(g)$ and the integral (6) using Monte Carlo integration with rejection sampling [43], took the average over the four measurement results, and found normalized map purity 0.44 ± 0.03.

c. Tight scheme with reference-frame channel

Again, we choose the largest possible subgroup $H \subset$ SU(2) for which an equivariant UEB exists; in previous work [29] this was shown to be $H \simeq$ BOct, where BOct is the binary octahedral group, which has order 48. The Pauli UEB is, up to phase, the unique UEB equivariant for this subgroup.

We choose the encoding and decoding regions to be Voronoi cells for the elements of BOct $<$ SU(2) under the Frobenius distance function.

We evaluated the integral in Theorem III.2 using Monte Carlo integration with rejection sampling, took the average over the four measurement results, and calculated the normalized map purity of the effective channel as 0.32 ± 0.02.

[1] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Rev. Mod. Phys. **79**, 555 (2007).
[2] I. Marvian and R. W. Spekkens, New J. Phys. **15**, 033001 (2013).
[3] I. Marvian and R. W. Spekkens, Phys. Rev. A **90**, 014102 (2014).
[4] A. Kitaev, D. Mayers, and J. Preskill, Phys. Rev. A **69**, 052326 (2004).
[5] S. van Enk, J. Mod. Opt. **48**, 2049 (2001).
[6] D. Gottesman and I. L. Chuang, Nature (London) **402**, 390 (1999).
[7] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).
[8] J.-G. Ren, P. Xu, H.-L. Yong, L. Zhang, S.-K. Liao, J. Yin, W.-Y. Liu, W.-Q. Cai, M. Yang, L. Li *et al.*, Nature (London) **549**, 70 (2017).
[9] J. Yin, Y. Cao, Y.-H. Li, S.-K. Liao, L. Zhang, J.-G. Ren, W.-Q. Cai, W.-Y. Liu, B. Li *et al.*, Science **356**, 1140 (2017).
[10] L. Bacsardi, IEEE Commun. Mag. **51**, 50 (2013).
[11] J. Wabnig, D. Bitauld, H. W. Li, A. Laing, J. L. O'Brien, and A. O. Niskanen, New J. Phys. **15**, 073001 (2013).
[12] J. L. Duligall, M. S. Godfrey, K. A. Harrison, W. J. Munro, and J. G. Rarity, New J. Phys. **8**, 249 (2006).
[13] J. L. Duligall, M. S. Godfrey, A. M. Lynch, W. J. Munro, K. J. Harrison, and J. G. Rarity, in *CLEO Europe and IQEC 2007 Conference Digest* (Optical Society of America, Washington, DC, 2007).
[14] A. Laing, V. Scarani, J. G. Rarity, and J. L. O'Brien, Phys. Rev. A **82**, 012304 (2010).
[15] W.-Y. Liang, S. Wang, H.-W. Li, Z.-Q. Yin, W. Chen, Y. Yao, J.-Z. Huang, G.-C. Guo, and Z.-F. Han, Sci. Rep. **4**, 3617 (2014).
[16] G. Chiribella, V. Giovannetti, L. Maccone, and P. Perinotti, Phys. Rev. A **86**, 010304(R) (2012).
[17] U. Marzolino and A. Buchleitner, Phys. Rev. A **91**, 032316 (2015).
[18] U. Marzolino and A. Buchleitner, Proc. R. Soc. London A **472**, 20150621 (2016).
[19] S. Muralidharan, L. Li, J. Kim, N. Lütkenhaus, M. D. Lukin, and L. Jiang, Sci. Rep. **6**, 20463 (2016).
[20] M. Skotiniotis and G. Gour, New J. Phys. **14**, 073022 (2012).
[21] T. Islam, L. Magnin, B. Sorg, and S. Wehner, New J. Phys. **16**, 063040 (2014).
[22] T. Islam and S. Wehner, New J. Phys. **18**, 033018 (2016).
[23] A. Peres and P. F. Scudo, in *Quantum Theory: Reconsideration of Foundations*, edited by A. Khrennikov (Vaxjo University Press, Vaxjo, 2002).
[24] L. M. Ioannou and M. Mosca, Theor. Comput. Sci. **560**, 33 (2014).
[25] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Phys. Rev. A **70**, 032307 (2004).
[26] D. A. Lidar and K. Birgitta Whaley, Decoherence-free subspaces and subsystems, in *Irreversible Quantum Dynamics*, edited by F. Benatti and R. Floreanini (Springer, Berlin, 2003), pp. 83–120.
[27] V. D'Ambrosio, E. Nagali, S. P. Walborn, L. Aolita, S. Slussarenko, L. Marrucci, and F. Sciarrino, Nat. Commun. **3**, 961 (2012).
[28] R. F. Werner, J. Phys. A: Math. Gen. **34**, 7081 (2001).
[29] D. Verdon and J. Vicary, Phys. Rev. A **98**, 012306 (2018).
[30] We define this as the purity of the Choi-Jamiołkowski state associated to the quantum channel induced by the protocol, where we take a convex sum over all frames $g \in G$ weighted by the Haar measure. Where figures are computed by numerical methods we give an error range in the reported figure.

[31] For a proof, see Appendix A.
[32] Note that the entangled state is invariant under changes in reference frame, so both parties' frames may shift arbitrarily following its creation without affecting the quality of the entangled resource.
[33] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[34] The $-i$ and the U_2 here correspond to the choice of maximally entangled state; see the discussion following Theorem III.1.
[35] The existence of such states is treated in an Appendix of our earlier work [29].
[36] For a proof, see Appendix A.
[37] The existence of equivariant UEBs is treated in [29], with a complete classification for qubit systems.
[38] Y. Yan and G. Chirikjian, in *2013 IEEE 52nd Annual Conference on Decision and Control (CDC)* (IEEE, New York, 2013), pp. 1137–1143.
[39] Note that we are trying to approximately limit the domain of possible reference-frame transformations to F. It is therefore sensible to pick F so that all the readings in it are as close to the identity as possible under some metric. To make this precise one can use *Voronoi cells* [38].
[40] W. Roga, Z. Puchała, Ł. Rudnicki, and K. Życzkowski, Phys. Rev. A **87**, 032308 (2013).
[41] W. Roga, K. Życzkowski, and M. Fannes, Int. J. Quantum Inf. **09**, 1031 (2011).
[42] M. Ziman, Phys. Rev. A **78**, 032118 (2008).
[43] W. Reiher, Biom. Z. **8**, 209 (1966).