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ON A SHARP POINCARÉ-TYPE INEQUALITY ON THE 2-SPHERE AND ITS APPLICATION IN MICROMAGNETICS*

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Abstract. The main aim of this note is to prove a sharp Poincaré-type inequality for vector-valued functions on $\mathbb{S}^2$ that naturally emerges in the context of micromagnetics of spherical thin films.

Key words. Poincaré inequality, vector spherical harmonics, magnetic skyrmions

AMS subject classifications. 35A23, 35R45, 49R05, 49S05, 82D40

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1. Introduction.

The Poincaré-type inequalities are a crucial tool in analysis, as they provide a relation between the norms of a function and its gradient. As such they are deeply relevant in analytic models appearing in geometry, physics, and biology. Such models often exhibit different qualitative behaviors for various ranges of parameters and therefore sharply estimating the Poincaré constant is fundamental for a proper understanding of a model.

The Poincaré-type inequalities always involve some constraints on the target of the function in order to eliminate the constants, which are not seen by the gradient part. The most commonly used ones, for scalar-valued functions, involve either local restrictions (zero values on the boundary of the domain) or nonlocal ones (zero mean). The optimal constant strongly depends on the type of constraint imposed and provides a piece of significant geometric information about the problem under consideration [4, 18, 12].

There exists an enormous body of literature about Poincaré-type inequalities for *scalar-valued* functions but virtually nothing about *vector-valued* ones despite their use in many physical contexts. The last four decades have witnessed an extraordinary interest in manifold-valued function spaces but Poincaré inequalities naturally relevant in this context have not been explored much. The various constraints on the range

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of the vector-valued function, motivated by physical or geometrical considerations reduce the degrees of freedom allowed on the function and generate natural questions concerning the optimal constants. Such questions require special approaches, going beyond what is available in the scalar case.

We are interested in proving a sharp Poincaré-type inequality for vector-valued functions on the 2-sphere $\mathbb{S}^2 := \left\{\xi \in \mathbb{R}^3 : |\xi| = 1\right\}$ and using this result to obtain nontrivial information about magnetization behavior inside thin spherical shells. Topological magnetic structures arising in nonflat geometries attract a lot of interest due to their potential in the application to magnetic devices [17]. Thin spherical shells are one of the simplest examples where an interplay between topology, geometry, and curvature of the underlying space results in nontrivial magnetic structures [16].

The magnetization distribution $\boldsymbol{u} \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ in thin spherical shells can be found by minimizing the following reduced micromagnetic energy [8, 13]:

$$\mathcal{F}_\kappa(\boldsymbol{u}) = \int_{\mathbb{S}^2} \left|\nabla^*_\xi \boldsymbol{u}(\xi)\right|^2 \mathrm{d}\xi + \kappa \int_{\mathbb{S}^2} (\boldsymbol{u}(\xi) \cdot \boldsymbol{n}(\xi))^2 \mathrm{d}\xi, \tag{1}$$

where $\boldsymbol{n}(\xi) := \xi$ is the normal field to the unit sphere and $\kappa \in \mathbb{R}$ is an effective anisotropy parameter. Here, we have denoted by $\nabla^* : H^1(\mathbb{S}^2, \mathbb{R}^3) \to L^2(\mathbb{S}^2, \mathbb{R}^3)$ the tangential gradient on $\mathbb{S}^2$.

The existence of minimizers can be easily obtained using direct methods of the calculus of variations and nonuniqueness of minimizers follows due to the invariance of the energy $\mathcal{F}_\kappa$ under the orthogonal group. An exact characterization of the minimizers in this problem is a nontrivial task and so far has been carried out only numerically [16]. However, sometimes it is enough to obtain a meaningful lower bound on the energy in order to gain some information of the ground states. This lower bound is typically obtained by relaxing the constraint $\boldsymbol{u} \in \mathbb{S}^2$ to the following weaker constraint:

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} |\boldsymbol{u}(\xi)|^2 \mathrm{d}\xi = 1. \tag{2}$$

This kind of relaxation, which physically corresponds to a passage from classical physics to a probabilistic quantum mechanics perspective, has been proved to be useful in obtaining nontrivial lower bounds of the ground state micromagnetic energy (see, e.g., [5]). Mathematically, replacing a constraint $\boldsymbol{u} \in \mathbb{S}^2$ with (2) puts us in a realm of Poincaré-type inequalities, where in many cases the relaxed problem can be solved exactly and the dependence of the minimizers on the geometrical and physical properties of the model made explicit. Sometimes this relaxation turns out to be helpful to obtain sufficient conditions for minimizers to have specific geometric structures (see, e.g., [5]).

We note that the constraint $|\boldsymbol{u}|^2 = 1$ a.e. on $\mathbb{S}^2$ is equivalent to the following two energy constraints in terms of the L^2 and L^4 norms:

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} |\boldsymbol{u}(\xi)|^2 \mathrm{d}\xi = 1 \quad \text{and} \quad \frac{1}{4\pi} \int_{\mathbb{S}^2} |\boldsymbol{u}(\xi)|^4 \mathrm{d}\xi = 1. \tag{3}$$

This observation follows from the Cauchy–Schwarz inequality

$$4\pi = (|\boldsymbol{u}|^2, 1)_{L^2(\mathbb{S}^2,\mathbb{R}^3)} \leqslant \||\boldsymbol{u}|^2\|_{L^2(\mathbb{S}^2,\mathbb{R}^3)} \|1\|_{L^2(\mathbb{S}^2)} = 4\pi, \tag{4}$$

where equality holds when $|\boldsymbol{u}|^2$ is a constant. Therefore our relaxed problem is the one obtained by removing the L^4 constraint.

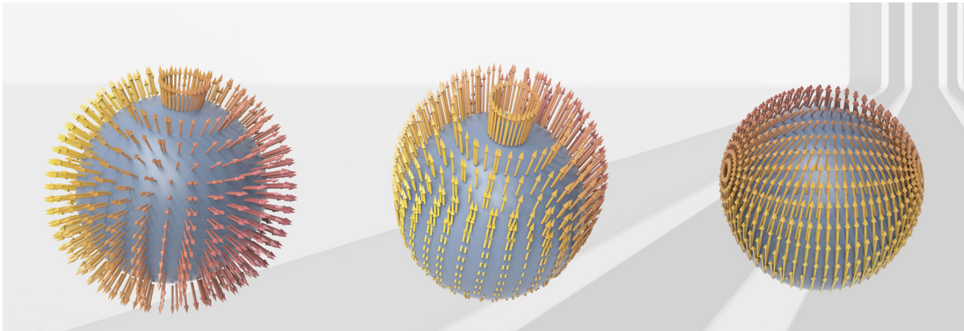

FIG. 1. *Examples of vector fields for which the equality sign is attained in the Poincaré inequality* (5). *Left,* $\kappa = -8$*; center,* $\kappa = -4$*; right,* $\kappa = 6$.

Main results. Our results include the precise characterization of the minimal value and global minimizers of the energy functional $\mathcal{F}_\kappa$, defined in (1), on the space of $H^1(\mathbb{S}^2, \mathbb{R}^3)$ vector fields satisfying the relaxed constraint (2). In particular, we prove the following Poincaré-type inequality.

THEOREM 1 (Poincaré inequality on $\mathbb{S}^2$). *Let* $\kappa \in \mathbb{R}$. *For every* $\boldsymbol{u} \in H^1(\mathbb{S}^2, \mathbb{R}^3)$ *the following inequality holds:*

$$\int_{\mathbb{S}^2} \left|\nabla_\xi^* \boldsymbol{u}(\xi)\right|^2 \mathrm{d}\xi + \kappa \int_{\mathbb{S}^2} (\boldsymbol{u}(\xi) \cdot \boldsymbol{n}(\xi))^2 \mathrm{d}\xi \;\geqslant \gamma(\kappa) \int_{\mathbb{S}^2} |\boldsymbol{u}(\xi)|^2 \mathrm{d}\xi \tag{5}$$

with

$$\gamma(\kappa) := \begin{cases} \kappa + 2 & \text{if } \kappa \leqslant -4, \\ \frac{1}{2}((\kappa+6) - \sqrt{\kappa^2 + 4\kappa + 36}) & \text{if } \kappa > -4\,. \end{cases} \tag{6}$$

For any $\kappa \in \mathbb{R}$ *the equality in* (5) *holds* if and only if *the function* $\boldsymbol{u}$ *has the following form in terms of vector spherical harmonics* (*see section* 2*, Definition* 1)*:*

$$\boldsymbol{u}(\xi) = c_0 \boldsymbol{y}_{0,0}^{(1)}(\xi) + \sum_{j=-1}^{1} \sigma_j \boldsymbol{y}_{1,j}^{(1)}(\xi) + \tau_j \boldsymbol{y}_{1,j}^{(2)}(\xi), \tag{7}$$

where coefficients $c_0, (\sigma_j, \tau_j)_{|j|\leqslant 1}$ *are defined as follows:*

- *if* $\kappa < -4$*, then* $c_0 = \pm\sqrt{4\pi}$*,* $\sigma_j = \tau_j = 0$ *for* $|j| \leqslant 1$*;*
- *if* $\kappa > -4$*, then*
(8)
$$c_0 = 0, \quad \tau_j = \frac{-2\sqrt{2}}{(\gamma(\kappa) - 2)}\sigma_j \quad \forall |j| \leqslant 1, \quad \sum_{|j|\leqslant 1} \sigma_j^2 = 2\pi \frac{-(\kappa+2) + \sqrt{\kappa^2 + 4\kappa + 36}}{\sqrt{\kappa^2 + 4\kappa + 36}};$$

- *if* $\kappa = -4$*, then*

$$\tau_j = \frac{\sqrt{2}}{2}\sigma_j \quad \forall |j| \leqslant 1, \quad 2c_0^2 + 3\sum_{|j|\leqslant 1} \sigma_j^2 = 8\pi. \tag{9}$$

We recall that the superscripts (1) and (2) in (7) correspond to maps which are, respectively, normal and tangent to $\mathbb{S}^2$ at ξ (see section 2, Definition 1). In particular, since $\sqrt{4\pi}\boldsymbol{y}_{0,0}^{(1)}(\xi) = \boldsymbol{n}(\xi)$, we discover that for $\kappa \leqslant -4$ the relaxed minimization problem admits $\mathbb{S}^2$-valued minimizers. Thus, surprisingly, as a byproduct of Theorem 1, we obtain the following characterization of micromagnetic ground states in thin spherical shells.

THEOREM 2 (micromagnetic ground states in thin spherical shells). *For every $\kappa \in \mathbb{R}$, the normal vector fields $\pm\boldsymbol{n}(\xi)$ are stationary points of the micromagnetic energy functional $\mathcal{F}_\kappa$ given by* (1) *on the space $H^1(\mathbb{S}^2, \mathbb{S}^2)$. Moreover, they are strict local minimizers for every $\kappa < 0$ and are unstable for $\kappa > 0$. If $\kappa \leqslant -4$, the normal vector fields $\pm\boldsymbol{n}(\xi)$ are the only global minimizers of $\mathcal{F}_\kappa$.*

Remark 1.1. Although the inequality (5) holds for any $\kappa \in \mathbb{R}$, it is sometimes more convenient to restate it in the standard form where both the term on the right side and the term on the left side are nonnegative. Therefore when $\kappa \geqslant 0$ we can use (5) and if $\kappa < 0$ we note that $|\boldsymbol{u}(\xi) \times \boldsymbol{n}(\xi)|^2 - |\boldsymbol{u}(\xi)|^2 = -(\boldsymbol{u}(\xi) \cdot \boldsymbol{n}(\xi))^2$ and rewrite relation (5) as

$$\int_{\mathbb{S}^2} \left|\nabla_\xi^* \boldsymbol{u}(\xi)\right|^2 \mathrm{d}\xi + |\kappa| \int_{\mathbb{S}^2} |\boldsymbol{u}(\xi) \times \boldsymbol{n}(\xi)|^2 \mathrm{d}\xi \;\geqslant\; (|\kappa| + \gamma(\kappa)) \int_{\mathbb{S}^2} |\boldsymbol{u}(\xi)|^2 \mathrm{d}\xi \tag{10}$$

with $|\kappa| \geqslant |\kappa| + \gamma(\kappa) \geqslant 0$ and the tangential part of the vector field appearing on the left-hand side.

Examples of vector fields for which the equality sign is attained in (5) are depicted in Figure 1. Plots of the best constants $\kappa \in \mathbb{R} \mapsto \gamma(k)$ and $\kappa \in \mathbb{R} \mapsto \gamma(k) + |\kappa|$ for $\kappa > 0$ and $\kappa < 0$, respectively, are given in Figure 2. We note that for $\kappa < -4$ the minimizing configurations are normal vector fields, for $\kappa \gg 1$ the tangential configurations are favored, and for the critical case $\kappa = -4$ various minimizing states may coexist.

Remark 1.2. Note that the maximum value of $\gamma(\kappa)$ (see Figure 2) is reached at $\kappa = +\infty$, where $\gamma(+\infty) = 2$. It follows that for purely *tangential* vector fields one has the Poincaré inequality

$$\frac{1}{2} \int_{\mathbb{S}^2} \left|\nabla_\xi^* \boldsymbol{u}(\xi)\right|^2 \mathrm{d}\xi \;\geqslant \int_{\mathbb{S}^2} |\boldsymbol{u}(\xi)|^2 \mathrm{d}\xi. \tag{11}$$

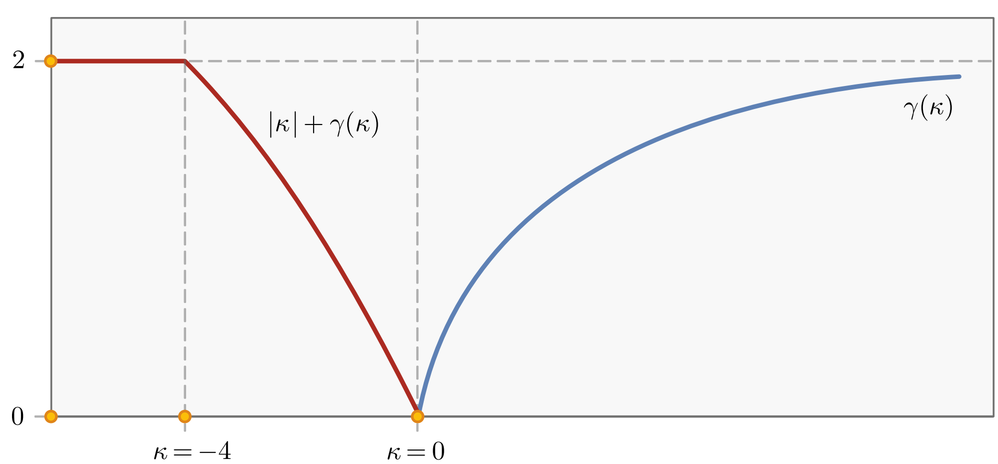

FIG. 2. *The values of the best constants in the Poincaré inequalities* (5) *and* (10) *for $\kappa > 0$ and $\kappa < 0$, respectively.*

The inequality (11) is sharp as equality is achieved, for instance, by a vector field $\boldsymbol{u}(\xi) = \pm\sqrt{4\pi}\boldsymbol{y}^{(2)}_{1,0}(\xi)$. In fact, one can characterize all vector fields delivering an optimal Poincaré constant by taking the limit for $\kappa \to +\infty$ of the coefficients τ_j in (8).

Remark 1.3. We note that Theorem 2 implies that the minimizers of micromagnetic energy don't have full radial symmetry in the case $\kappa > 0$. It follows from the fact that the only radially symmetric vector fields are $\pm\boldsymbol{n}(\xi)$ and these are unstable for $\kappa > 0$.

Remark 1.4. It is worth noting that, in the language of modern physics, the two ground states $\pm\boldsymbol{n}$ carry a different *skyrmion number* (or topological charge). Indeed, since $\deg(\pm\boldsymbol{n}) = \pm 1$, by the Hopf theorem [14], these two configurations cannot be homotopically mapped one into the other and are, therefore, topologically protected against external perturbations and thermal fluctuations. These considerations make the two ground states $\pm\boldsymbol{n}$ promising in view of novel spintronic devices [9, 10].

We also want to point out a correspondence between our Theorem 2 and Brown's fundamental theorem on fine ferromagnetic particles [5, 7, 2], as Theorem 2 implies an existence of a critical value $\kappa_0 < 0$ below which the only ground states are $\pm\boldsymbol{n}(\xi)$.

In the following, in section 2, we define suitable vector spherical harmonics. Afterward, in section 3, by means of these vector spherical harmonics, we recast the minimization problem for $\mathcal{F}_\kappa$ as a constrained minimization problem on a suitable space of sequences. Then, in section 4, by proper use of the Euler–Lagrange equations in sequence space, we derive necessary minimality conditions which allow us to reduce the infinite dimensional problem to a finite dimensional one. Finally, arguments based on the method of Lagrange multipliers complete the proof of Theorem 1 and afterward of Theorem 2.

2. Notation and setup. Vector spherical harmonics.

In this section, we define a natural basis and characterize vector spherical harmonics on the unit sphere $\mathbb{S}^2$; see [11]. Every point $\xi \in \mathbb{S}^2$ can be expressed via the polar coordinates parametrization

$$\sigma(\varphi, t) = \left(\sqrt{1-t^2}\cos\varphi, \sqrt{1-t^2}\sin\varphi, t\right), \tag{12}$$

where $\varphi \in [0, 2\pi)$ is the longitude, $t = \cos\theta \in [-1, 1]$ is the polar distance, and $\theta \in [0, \pi]$ is the latitude.

We can define the surface gradient operator ∇^*_ξ for a.e. $\xi \in \mathbb{S}^2$ in the following way:

$$\nabla^*_\xi = \varepsilon^\varphi \frac{1}{\sqrt{1-t^2}}\partial_\varphi + \varepsilon^t\sqrt{1-t^2}\partial_t, \tag{13}$$

where $\varepsilon^\varphi(\varphi, t) := (-\sin\varphi, \cos\varphi, 0)$, $\varepsilon^t(\varphi, t) := \left(-t\cos\varphi, -t\sin\varphi, \sqrt{1-t^2}\right)$. For any $u \in C^2(\mathbb{S}^2, \mathbb{R})$, the Laplace–Beltrami operator is defined as

$$\Delta^*_\xi u(\xi) := \nabla^*_\xi \cdot \nabla^*_\xi u(\xi). \tag{14}$$

NOTATION 2.1. *We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0$ the set of nonnegative integers. For every $n \in \mathbb{N}$ we set $\mathbb{N}_n := \{1, 2, \ldots, n\}$ and $\mathbb{Z}_n := \{0, \pm 1, \ldots, \pm n\}$, and for every $N \in \mathbb{N}_0$ we introduce the set $J_N \subseteq \mathbb{N}_0 \times \mathbb{Z}$ consisting of all pairs $(n, j) \in \mathbb{N}_0 \times \mathbb{Z}$ such that $n \leqslant N$ and $|j| \leqslant n$. We set $J := J_\infty$.*

Vector spherical harmonics are an extension of the scalar spherical harmonics to square-integrable vector fields on the sphere; in fact, they can be introduced in terms of the scalar spherical harmonics and their derivatives. Motivated by different physical problems, various sets of vector spherical harmonics have been introduced in the literature. The system that best fits our purposes is the one introduced in [3] and obtained from the splitting of vector fields into a radial and a tangential component. We have the following definition (see [11]).

DEFINITION 1. *The vector spherical harmonics* $\boldsymbol{y}_{n,j}^{(1)}$, $\boldsymbol{y}_{n,j}^{(2)}$, *and* $\boldsymbol{y}_{n,j}^{(3)}$ *of degree* n *and order* j, *with* $(n,j) \in J$, *are defined by*

$$\boldsymbol{y}_{n,j}^{(1)}(\xi) := Y_{n,j}(\xi)\boldsymbol{n}(\xi), \quad \boldsymbol{y}_{n,j}^{(2)}(\xi) := \frac{1}{\sqrt{n_*}}\nabla_\xi^* Y_{n,j}(\xi), \quad \boldsymbol{y}_{n,j}^{(3)}(\xi) := \frac{1}{\sqrt{n_*}}\xi \wedge \nabla_\xi^* Y_{n,j}(\xi), \tag{15}$$

where $n_* := n(n+1)$. *Here, for every* $(n,j) \in J$, *the function* $Y_{n,j}$ *is the real-valued scalar spherical harmonics of degree* n *and order* j, *defined by*

$$Y_{n,j}(\xi) := \begin{cases} \sqrt{2}X_{n,|j|}(t)\cos(j\varphi) & \text{if } -n \leqslant j < 0, \\ X_{n,0}(t) & \text{if } j = 0, \\ \sqrt{2}X_{n,j}(t)\sin(j\varphi) & \text{if } 0 < j \leqslant n, \end{cases} \tag{16}$$

where for every $t \in [-1,1]$ *and every* $0 \leqslant j \leqslant n$

$$X_{n,j}(t) = (-1)^j \sqrt{\left(\frac{2n+1}{4\pi}\right)\frac{(n-j)!}{(n+j)!}}P_{n,j}(t), \tag{17}$$

and $P_{n,j}$ *is the associate Legendre polynomial given by* $P_{n,j}(t) := \frac{1}{2^n n!}(1-t^2)^{j/2}\partial_t^{n+j}$ $(t^2-1)^n$.

It is well-known (cf. [3, 15]) that the system $(Y_{n,j})_{(n,j)\in J}$ so defined is a complete orthonormal system for $L^2\left(\mathbb{S}^2, \mathbb{R}\right)$, consisting of eigenfunctions of the Laplace–Beltrami operator. Precisely, for every $n \in \mathbb{N}_0$ we have $-\Delta_\xi^* Y_{n,j} = n_* Y_{n,j}$ with $n_* := n(n+1)$. Not so widely known seems to be that the system of vector spherical harmonics is complete in $L^2(\mathbb{S}^2,\mathbb{R}^3)$ and forms an orthonormal system (cf. [11]). Therefore, any vector field $\boldsymbol{u} \in L^2(\mathbb{S}^2,\mathbb{R}^3)$ can be represented by its Fourier series:

$$\sum_{i\in\mathbb{N}_3}\sum_{(n,j)\in J} \hat{u}_i(n,j)\boldsymbol{y}_{n,j}^{(i)} = \boldsymbol{u} \quad \text{in } L^2(\mathbb{S}^2,\mathbb{R}^3) \tag{18}$$

with the Fourier coefficients $\hat{u}_i$ being given by $\hat{u}_i(n,j) := (\boldsymbol{u}, \boldsymbol{y}_{n,j}^{(i)})_{L^2(\mathbb{S}^2,\mathbb{R}^3)}$.

As the minimizers of our problem will be fully characterized in terms of the first vector spherical harmonics, it is worth writing down their explicit expressions. By the relation $\boldsymbol{y}_{n,j}^{(1)}(\xi) := Y_{n,j}(\xi)\boldsymbol{n}(\xi)$ we get, for $n = 0$, that

$$\boldsymbol{y}_{0,0}^{(1)}(\xi) = \frac{1}{\sqrt{4\pi}}\boldsymbol{n}(\xi). \tag{19}$$

For $n = 1$, we get

$$\mu^{(1)}\boldsymbol{y}_{1,-1}^{(1)}(\xi) = \sin\theta\cos\varphi\,\boldsymbol{n}(\xi), \tag{20}$$

$$\mu^{(1)}\boldsymbol{y}_{1,0}^{(1)}(\xi) = \cos\theta\,\boldsymbol{n}(\xi), \tag{21}$$

$$\mu^{(1)}\boldsymbol{y}_{1,1}^{(1)}(\xi) = \sin\theta\sin\varphi\,\boldsymbol{n}(\xi) \tag{22}$$

with $\mu^{(1)} := \sqrt{4\pi/3}$. Also, by the relation $\boldsymbol{y}^{(2)}_{n,j}(\xi) := \frac{1}{\sqrt{n_*}}\nabla^*_\xi Y_{n,j}(\xi)$, we obtain, for $n = 1$, the following identities:

$$\mu^{(2)}\boldsymbol{y}^{(2)}_{1,-1}(\xi) = \cos\theta\cos\varphi\,\boldsymbol{\tau}_\theta(\xi) - \sin\varphi\,\boldsymbol{\tau}_\varphi(\xi), \tag{23}$$

$$\mu^{(2)}\boldsymbol{y}^{(2)}_{1,0}(\xi) = -\sin\theta\,\boldsymbol{\tau}_\theta(\xi), \tag{24}$$

$$\mu^{(2)}\boldsymbol{y}^{(2)}_{1,1}(\xi) = \cos\theta\sin\varphi\,\boldsymbol{\tau}_\theta(\xi) + \cos\varphi\,\boldsymbol{\tau}_\varphi(\xi) \tag{25}$$

with $\mu^{(2)} := \sqrt{8\pi/3}$, $\boldsymbol{\tau}_\theta(\xi) := (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$, and $\boldsymbol{\tau}_\varphi(\xi) := (-\sin\varphi, \cos\varphi, 0)$. Note that the tangent vectors $\boldsymbol{\tau}_\theta$ and $\boldsymbol{\tau}_\varphi$ have unit norms. The previous expressions turn out to be extremely useful to obtain both a qualitative and a quantitative comprehension of the energy landscape as in Figure 1.

Remark 2.1. Throughout the paper, we use summations which formally involve also $\boldsymbol{y}^{(2)}_{0,0} = \boldsymbol{y}^{(3)}_{0,0} = 0$, with the understanding that $\hat{u}_2(0,0) = \hat{u}_3(0,0) = 0$. Indeed, although these vectors are not officially present in the orthonormal system of vector spherical harmonics, such a convention allows us to express the Fourier series representation of $\boldsymbol{u}$ in the compact form $\sum_{i\in\mathbb{N}_3}\sum_{(n,j)\in J}\hat{u}_i(n,j)\boldsymbol{y}^{(i)}_{n,j}$.

3. Representation of the energy in a space of sequences. In this section we are going to rewrite the energy (1) in terms of sequences using Fourier representation (18). According to the representation formula (18), every vector field $\boldsymbol{u} \in H^1(\mathbb{S}^2, \mathbb{R}^3)$ can be expressed in the form

$$\boldsymbol{u} = \sum_{i\in\mathbb{N}_3}\sum_{(n,j)\in J}\hat{u}_i(n,j)\boldsymbol{y}^{(i)}_{n,j} \quad \text{in } L^2(\mathbb{S}^2,\mathbb{R}^3) \tag{26}$$

with the Fourier coefficients $\hat{u}_i$ being given by $\hat{u}_i(n,j) := (\boldsymbol{u}, \boldsymbol{y}^{(i)}_{n,j})_{L^2(\mathbb{S}^2,\mathbb{R}^3)}$. Also, if $\boldsymbol{u}$ is a smooth vector field, we have $\|\nabla^*_\xi\boldsymbol{u}\|^2_{L^2(\mathbb{S}^2,\mathbb{R}^3)} = (-\Delta^*_\xi\boldsymbol{u}, \boldsymbol{u})_{L^2(\mathbb{S}^2,\mathbb{R}^3)}$. Hence, by making use of the relations (cf. [11, p. 237])

$$-\Delta^*\boldsymbol{y}^{(1)}_{n,j} = (n_* + 2)\,\boldsymbol{y}^{(1)}_{n,j} - 2\sqrt{n_*}\boldsymbol{y}^{(2)}_{n,j}, \tag{27}$$

$$-\Delta^*\boldsymbol{y}^{(2)}_{n,j} = n_*\boldsymbol{y}^{(2)}_{n,j} - 2\sqrt{n_*}\boldsymbol{y}^{(1)}_{n,j}, \tag{28}$$

$$-\Delta^*\boldsymbol{y}^{(3)}_{n,j} = n_*\boldsymbol{y}^{(3)}_{n,j}, \tag{29}$$

where $n_* := n(n+1)$, we infer that for every $\boldsymbol{u} \in C^\infty(\mathbb{S}^2,\mathbb{R}^3)$

$$-\Delta^*_\xi\boldsymbol{u}(\xi) = \sum_{(n,j)\in J}\hat{u}_1(-\Delta^*_\xi\boldsymbol{y}^{(1)}) + \hat{u}_2(-\Delta^*_\xi\boldsymbol{y}^{(2)}) + \hat{u}_3(-\Delta^*_\xi\boldsymbol{y}^{(3)}) \tag{30}$$

$$= \sum_{(n,j)\in J}\left((n_*+2)\,\hat{u}_1 - 2\sqrt{n_*}\hat{u}_2\right)\boldsymbol{y}^{(1)}_{n,j} + \left(n_*\hat{u}_2 - 2\sqrt{n_*}\hat{u}_1\right)\boldsymbol{y}^{(2)}_{n,j} + n_*\hat{u}_3\boldsymbol{y}^{(3)}_{n,j} \tag{31}$$

with the understanding that $\hat{u}_2(0,0) = \hat{u}_3(0,0) = 0$ and $\hat{u}_1 = \hat{u}_1(n,j)$, $\hat{u}_2 = \hat{u}_2(n,j)$, and $\hat{u}_3 = \hat{u}_3(n,j)$. Thus, for every $\boldsymbol{u} \in C^\infty(\mathbb{S}^2,\mathbb{R}^3)$,

$$\int_{\mathbb{S}^2}\left|\nabla^*_\xi\boldsymbol{u}(\xi)\right|^2 \mathrm{d}\xi = \sum_{(n,j)\in J}(n_*+2)\,\hat{u}_1^2 - 4\sqrt{n_*}\hat{u}_1\hat{u}_2 + n_*\hat{u}_2^2 + n_*\hat{u}_3^2, \tag{32}$$

and, by density, the same relation holds for every $\boldsymbol{u} \in H^1\left(\mathbb{S}^2, \mathbb{R}^3\right)$. Also, a straightforward calculation shows that

$$\int_{\mathbb{S}^2} (\boldsymbol{u}(\xi) \cdot \boldsymbol{n}(\xi))^2 \mathrm{d}\xi = \sum_{(n,j)\in J} \hat{u}_1^2(n,j). \tag{33}$$

Therefore, the surface energy (1), in the sequence space, reads as the functional

$$\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = \sum_{(n,j)\in J} (n_* - 2 + \kappa)\, \hat{u}_1^2 + \left(2\hat{u}_1 - \sqrt{n_*}\hat{u}_2\right)^2 + n_* \hat{u}_3^2. \tag{34}$$

Denoting by $\ell_2(J)$ the classical Hilbert space of square-summable sequences endowed with the inner product $\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}} \rangle := \sum_{(n,j)\in J} \hat{u}_1\hat{v}_1 + \hat{u}_2\hat{v}_2 + \hat{u}_3\hat{v}_3$, the natural domain of $\mathcal{G}_\kappa$ is the subspace $\ell_2'(J)$ of $\ell_2(J)$ consisting of those sequences in $\hat{\boldsymbol{u}} \in \ell_2(J)$ such that $\sqrt{n_*}\hat{\boldsymbol{u}} \in \ell_2(J)$. In $\ell_2'(J)$ the constraint (2) reads as

$$\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \rangle = \sum_{(n,j)\in J} \hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_3^2 = \int_{\mathbb{S}^2} |\boldsymbol{u}(\xi)|^2 \mathrm{d}\xi = 4\pi\,. \tag{35}$$

As before, in the previous relations, to shorten notation, we avoid explicitly writing the dependence of $\hat{u}_1, \hat{u}_2, \hat{u}_3$ from (j,n).

4. Proof of the Poincaré inequality (Theorem 1). In this section, we are going to prove the main result of this note—Theorem 1. Without loss of generality, we will focus on the case $\kappa \neq 0$, because for $\kappa = 0$ the only minimizers are the constant vector fields with unit modulus. Instead of working with the original continuous formulation (1), we introduce the equivalent formulation in terms of sequences,

$$\min_{\hat{\boldsymbol{u}}\in\ell_2'(J)} \mathcal{G}_\kappa(\hat{\boldsymbol{u}}) \quad \text{subject to} \quad \frac{1}{4\pi}\|\hat{\boldsymbol{u}}\|^2_{\ell_2(J)} = 1\,, \tag{36}$$

and provide a complete characterization of the minimizers of (36).

We split the proof into several steps and first prove the following useful lemma.

LEMMA 1. *For any $\kappa \in \mathbb{R}$, the following upper bound on the energy* (34) *holds:*

$$\min \mathcal{G}_\kappa \leqslant \min\left\{2\pi\left((\kappa+6) - \sqrt{\kappa^2+4\kappa+36}\right), 4\pi(2+\kappa)\right\} < 8\pi, \tag{37}$$

where $\min \mathcal{G}_\kappa$ *refers to the minimization problem* (36). *Moreover, if* $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \ell_2'(J)$ *is a minimizer for* $\mathcal{G}_\kappa$, *then*

(i) *the coefficients* $\hat{u}_3(n,j) = 0$ *for any* $(n,j) \in J$,

(ii) *if* $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) < 4\pi(2+\kappa)$, *then* $\hat{u}_1(0,0) = 0$,

(iii) *the coefficients* $\hat{\boldsymbol{u}}(n,j) = 0$ *for any* $n \geqslant 2$ *and all* $|j| \leqslant n$.

Proof. We provide a simple test function $\hat{\boldsymbol{u}}_*(n,j)$ by setting all its terms to 0 except $\hat{u}_1(1,1)$ and $\hat{u}_2(1,1)$. Therefore the minimum value of $\mathcal{G}_\kappa$ is less than the minimum of $\alpha_\kappa(x,y) = (\kappa+4)x^2 - 4\sqrt{2}xy + 2y^2$ under constraint $x^2+y^2 = 4\pi$. By studying the minima of $(\alpha_\kappa \circ \gamma)(t)$ with $\gamma(t) = \sqrt{4\pi}(\cos t, \sin t)$, it is easily seen that

$$\min_{(x,y)\in\sqrt{4\pi}\mathbb{S}^1} \alpha_\kappa(x,y) = 2\pi\left((\kappa+6) - \sqrt{\kappa^2+4\kappa+36}\right). \tag{38}$$

Note that $\kappa^2+4\kappa+36 > 0$ for every $\kappa \in \mathbb{R}$ and moreover $2\pi\left((\kappa+6) - \sqrt{\kappa^2+4\kappa+36}\right) < 8\pi$ for every $\kappa \in \mathbb{R}$; therefore

$$\min \mathcal{G}_\kappa(\hat{\boldsymbol{u}}) \leqslant 2\pi\left((\kappa+6) - \sqrt{\kappa^2+4\kappa+36}\right) < 8\pi \quad \forall \kappa \in \mathbb{R}. \tag{39}$$

Next, we provide another test function $\hat{\boldsymbol{u}}^*(n,j)$ by setting all its terms to 0 except $\hat{u}_1(0,0)$ and $\hat{u}_1(1,1)$. Therefore the minimum of $\mathcal{G}_\kappa$ is less than the minimum of $\beta_\kappa(x,y) = (\kappa+2)x^2 + (\kappa+4)y^2$ on $\sqrt{4\pi}\mathbb{S}^1$. By studying the minima of $(\beta_\kappa \circ \gamma)(t)$ with $\gamma(t) = \sqrt{4\pi}(\cos t, \sin t)$, it is easily seen that

$$\min_{\sigma\in\sqrt{4\pi}\mathbb{S}^1} \beta_\kappa(\sigma) = 4\pi(2+\kappa). \tag{40}$$

Therefore, for every $\kappa \in \mathbb{R}$, relation (37) holds.

i) We compute the first variation of $\mathcal{G}_\kappa$ around the generic point $\hat{\boldsymbol{u}} \in \ell_2'(J)$ to obtain the following Euler–Lagrange equations:

$$\sum_{(j,n)\in J} (n_* + 2 + \kappa)\, \hat{u}_1\hat{v}_1 - 2\sqrt{n_*}(\hat{u}_1\hat{v}_2 + \hat{v}_1\hat{u}_2) + n_*(\hat{u}_2\hat{v}_2 + \hat{u}_3\hat{v}_3) = \lambda(\hat{\boldsymbol{u}}) \cdot \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}} \rangle \tag{41}$$

with $\lambda(\hat{\boldsymbol{u}}) \in \mathbb{R}$ the Lagrange multiplier coming from the constraint (35). Plugging $\hat{\boldsymbol{v}} := \hat{\boldsymbol{u}}$ and taking into account (35), we obtain $\lambda(\hat{\boldsymbol{u}}) = \frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})$. Thus, the Euler–Lagrange equation reads as

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}} \rangle = \sum_{(j,n)\in J} (n_* + 2 + \kappa)\, \hat{u}_1\hat{v}_1 - 2\sqrt{n_*}(\hat{u}_1\hat{v}_2 + \hat{v}_1\hat{u}_2) + n_*(\hat{u}_2\hat{v}_2 + \hat{u}_3\hat{v}_3) \tag{42}$$

for every $\hat{\boldsymbol{v}} \in \ell_2'(J)$.

We test (42) against the sequence $\hat{\boldsymbol{v}} := (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ with $\hat{v}_1 = \hat{v}_2 = 0$ and $\hat{v}_3 = \hat{e}_{(n,j)}$, with $\hat{e}_{n,j}$ denoting the sequence $(n',j') \in J \mapsto \hat{e}_{n,j}(n',j') \in \mathbb{R}$ such that $\hat{e}_{n,j}(n,j) = 1$ and $\hat{e}_{n,j}(n',j') = 0$ if $(n',j') \neq (n,j)$. We get that

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_3(n,j) = n_*\hat{u}_3(n,j) \tag{43}$$

for any $n \geqslant 1$ and any $|j| \leqslant n$. Thus, for $n \geqslant 1$ we have $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi n_* \geqslant 8\pi$ whenever $\hat{u}_3(n,j) \neq 0$. Since the minimum of energy is strictly less than 8π we necessarily have $\hat{u}_3(n,j) = 0$ for any $n \geqslant 1$. This proves the assertion.

ii) We now evaluate (42) on $\hat{\boldsymbol{v}} := (\hat{v}_1, \hat{v}_2, \hat{v}_3)$, first with $\hat{v}_2 = \hat{v}_3 = 0$, and $\hat{v}_1 = \hat{e}_{(n,j)}$, then on $\hat{v}_2 = \hat{e}_{(n,j)}$, $\hat{v}_3 = 0$, and $\hat{v}_1 = 0$. We get the following two relations:

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_1(n,j) = (n_* + 2 + \kappa)\, \hat{u}_1(n,j) - 2\sqrt{n_*}\hat{u}_2(n,j), \tag{44}$$

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_2(n,j) = -2\sqrt{n_*}\hat{u}_1(n,j) + n_*\hat{u}_2(n,j). \tag{45}$$

For $n = 0$, relation (44) gives $\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_1(0,0) = 4\pi(2+\kappa)\hat{u}_1(0,0)$ so that if $\hat{\boldsymbol{u}}$ is a minimizer and $\hat{u}_1(0,0) \neq 0$, the minimum energy agrees with the limiting value $4\pi(2+\kappa)$. Therefore, if the minimal energy is strictly less than $4\pi(2+\kappa)$, then necessarily $\hat{u}_1(0,0) = 0$. This proves the statement.

iii) If $\hat{\boldsymbol{u}}$ is a minimizer of $\mathcal{G}_\kappa$, then for $n \geqslant 1$, using (44) and (45), we have that $\hat{u}_1(n,j) = 0$ if and only if $\hat{u}_2(n,j) = 0$. Equivalently, for any $n \geqslant 1$, $\hat{u}_1(n,j)\hat{u}_2(n,j) = 0$ implies $\hat{u}_1(n,j) = 0$ and $\hat{u}_2(n,j) = 0$.

We now focus on the indices $n \geqslant 1$ and, using the above observation, rewrite relations (44) and (45) into the form

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_1(n,j)\hat{u}_2(n,j) = (n_* + 2 + \kappa)\, \hat{u}_1(n,j)\hat{u}_2(n,j) - 2\sqrt{n_*}\hat{u}_2^2(n,j), \tag{46}$$

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_2(n,j)\hat{u}_1(n,j) = -2\sqrt{n_*}\hat{u}_1^2(n,j) + n_*\hat{u}_2(n,j)\hat{u}_1(n,j). \tag{47}$$

If for some $n \geqslant 1$ the product $\hat{u}_1(n,j)\hat{u}_2(n,j)$ is negative, then from (46) and (47) we get

$$\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi \left[(n_* + 2 + \kappa) - 2\sqrt{n_*} \frac{\hat{u}_2^2(n,j)}{\hat{u}_1(n,j)\hat{u}_2(n,j)} \right] > 4\pi(\kappa+2), \tag{48}$$

$$\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi \left[n_* - 2\sqrt{n_*} \frac{\hat{u}_1^2(n,j)}{\hat{u}_1(n,j)\hat{u}_2(n,j)} \right] > 8\pi \tag{49}$$

and $\hat{\boldsymbol{u}}$ is not a minimizer as a consequence of (37). Thus, if $\hat{\boldsymbol{u}}$ is a minimizer of $\mathcal{G}_\kappa$, then

$$\operatorname{sign}(\hat{u}_1(n,j)) = \operatorname{sign}(\hat{u}_2(n,j)) \quad \text{for any } n \geqslant 1. \tag{50}$$

Hence, from (44) and (45) we infer

$$\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi \left[(n_* + 2 + \kappa) - 2\sqrt{n_*} \frac{|\hat{u}_2(n,j)|}{|\hat{u}_1(n,j)|} \right], \tag{51}$$

$$\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi \left[n_* - 2\sqrt{n_*} \frac{|\hat{u}_1(n,j)|}{|\hat{u}_2(n,j)|} \right]. \tag{52}$$

Imposing the condition $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) \leqslant 4\pi(\kappa+2)$ in (51) and the condition $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) < 8\pi$ in (52) we get that if $\hat{\boldsymbol{u}}$ is a minimizer, then necessarily $(n_* - 2)\,|\hat{u}_2(n,j)| < 4|\hat{u}_2(n,j)|$, but this cannot be the case for $n \geqslant 2$. Therefore, necessarily $\hat{u}_1(n,j) = \hat{u}_2(n,j) = 0$ for any $n \geqslant 2$. This concludes the proof. ∎

Combining the results stated in Lemma 1, we can reduce the infinite dimensional minimization problem for $\mathcal{G}_\kappa$ to a finite dimensional one. Precisely, we have the following proposition.

PROPOSITION 1. *The minimization problem for $\mathcal{G}_\kappa$, subject to the constraint* (35), *reduces to the minimization of the function $g_\kappa : \sqrt{4\pi}\mathbb{S}^6 \to \mathbb{R}^+$ in the variables $\sigma := (\hat{u}_1(0,0), \hat{u}_1(1,j), \hat{u}_2(1,j))_{|j|\leqslant 1}$, given by*

$$g_\kappa(\sigma) = (\kappa+2)\hat{u}_1^2(0,0) + \sum_{j=-1}^{1} \kappa \hat{u}_1^2(1,j) + \left(2\hat{u}_1(1,j) - \sqrt{2}\hat{u}_2(1,j)\right)^2. \tag{53}$$

Precisely, any minimizer $\hat{\boldsymbol{u}}_\star = (\hat{u}_1(n,j), \hat{u}_2(n,j), \hat{u}_3(n,j))_{(n,j)\in J}$ of $\mathcal{G}_\kappa$ has all the terms zero except for those presented in σ, and coming fom minimizing g_κ. Specifically, the following complete characterization of the energy landscape holds:

- *If $\kappa < -4$, the minimum value of the energy is given by $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}_\star) = 4\pi(\kappa+2)$ and, in this case, $\hat{u}_1(0,0)$ is the only nonzero variable. Therefore, necessarily $\hat{u}_1(0,0) = \pm\sqrt{4\pi}$.*
- *If $\kappa > -4$ the minimum value of the energy is given by $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}_\star) = 4\pi\gamma_+(\kappa)$ with $\gamma_+(\kappa) := \frac{1}{2}\left((\kappa+6) - \sqrt{\kappa^2+4\kappa+36}\right)$. In this case, necessarily $\hat{u}_1(0,0) = 0$ and*

$$\hat{u}_2(1,j) = \frac{-2\sqrt{2}}{(\gamma_+(\kappa) - 2)}\hat{u}_1(1,j) \quad \forall |j| \leqslant 1. \tag{54}$$

The minimum value is reached on any vector $\hat{\sigma} = (\hat{u}_1(1,j))_{|j|\leqslant 1}$ such that

$$|\hat{\sigma}|^2 = 2\pi \frac{-(\kappa+2) + \sqrt{\kappa^2+4\kappa+36}}{\sqrt{\kappa^2+4\kappa+36}}. \tag{55}$$

- *If $\kappa = -4$, the minimum value of the energy is given by $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}_\star) = -8\pi$ and it is reached on any vector σ such that* (54) *holds and $2\hat{u}_1^2(0,0) + 3|\hat{\sigma}|^2 = 8\pi$.*

Remark 4.1. The limiting value $\kappa = -4$ represents a special case in which different topological states may coexist. Indeed, for $|\hat{\sigma}| = 0$ we recover the solutions $\hat{u}_1(0,0) := \pm\sqrt{4\pi}$ formally arising as the limit for $\kappa \to -4^-$ of the family of minimization problems for g_κ. Similarly, for $\hat{u}_1(0,0) = 0$, we recover the minimal solutions arising as the limit for $\kappa \to -4^+$ of the family of minimization problems for g_κ.

Proof. According to Lemma 1, the Euler–Lagrange equations (42) can be simplified to read, for every $\hat{\boldsymbol{v}} \in \ell_2'(J)$, as

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\langle\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}\rangle = \sum_{(n,j)\in J_1} (n_* + 2 + \kappa)\,\hat{u}_1\hat{v}_1 - 2\sqrt{n_*}(\hat{u}_1\hat{v}_2 + \hat{v}_1\hat{u}_2) + n_*(\hat{u}_2\hat{v}_2 + \hat{u}_3\hat{v}_3). \tag{56}$$

Taking, in order, $\hat{\boldsymbol{v}} = (\hat{e}_{0,0}, 0, 0)$, $\hat{\boldsymbol{v}} = (\hat{e}_{1,j}, 0, 0)$, $\hat{\boldsymbol{v}} = (0, \hat{e}_{1,j}, 0)$, we get that if $\hat{\boldsymbol{u}}$ is a minimizer, then

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_1(0,0) = (2+\kappa)\hat{u}_1(0,0), \tag{57}$$

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_1(1,j) = (4+\kappa)\hat{u}_1(1,j) - 2\sqrt{2}\hat{u}_2(1,j), \tag{58}$$

$$\frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})\hat{u}_2(1,j) = -2\sqrt{2}\hat{u}_1(1,j) + 2\hat{u}_2(1,j). \tag{59}$$

From (57) and Lemma 1 we immediately obtain that if $\hat{u}_1(0,0) \neq 0$, then $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi(2+\kappa)$. On the other hand, from (59), setting $G_\kappa := \frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}})$ and noting that $G_\kappa < 2$, we obtain

$$\hat{u}_2(1,j) = \frac{-2\sqrt{2}}{(G_\kappa - 2)}\hat{u}_1(1,j). \tag{60}$$

Substituting this last expression into (58) we obtain $(G_\kappa - 2)(G_\kappa - (4+\kappa))\hat{u}_1(1,j) = 8\hat{u}_1(1,j)$, and this, together with (60), implies that if $\hat{u}_1(1,j) \neq 0$ for some $|j| \leqslant 1$, then $\hat{u}_2(1,j)$ is different from zero too, and $(G - (4+\kappa))(G-2) = 8$, that is,

$$\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi\gamma_+(\kappa), \quad \gamma_+(\kappa) := \frac{1}{2}\left((\kappa+6) - \sqrt{\kappa^2 + 4\kappa + 36}\right). \tag{61}$$

We have proved the following implication:

$$(\exists |j| \leqslant 1 \quad \hat{u}_1(1,j) \neq 0 \quad \textbf{or} \quad \hat{u}_2(1,j) \neq 0) \quad \Longrightarrow \quad \mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi\gamma_+(\kappa).$$

Therefore, if $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) \neq 4\pi\gamma_+(\kappa)$, then necessarily

$$\hat{u}_1(1,j) = \hat{u}_2(1,j) = 0 \quad \forall |j| \leqslant 1.$$

Since $\gamma_+(\kappa) \leqslant (\kappa+2)$ if, and only if, $\kappa \geqslant -4$, by (37) we infer that for $\kappa < -4$ we have $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) < 4\pi\gamma_+(\kappa)$ and $\hat{u}_1(1,j) = \hat{u}_2(1,j) = 0\ \forall |j| \leqslant 1$. Since the variables in σ must be in $\sqrt{4\pi}\mathbb{S}^6$ this means that $\hat{u}_1(0,0)$ is the only variable different from zero and therefore necessarily equal to $\pm\sqrt{4\pi}$.

On the other hand, from (57) we immediately obtain that if $\hat{u}_1(0,0) \neq 0$, then $\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = 4\pi(2+\kappa)$, which, in turn, implies $\kappa \leqslant -4$. Therefore, if $\kappa > -4$, then necessarily $\hat{u}_1(0,0) = 0$ and, due to the constraint, at least one of the $\hat{u}_1(1,j)$ is different from zero. Thus, $G_\kappa := \frac{1}{4\pi}\mathcal{G}_\kappa(\hat{\boldsymbol{u}}) = \gamma_+(\kappa)$. This observation, in combination with (60), implies that for $\kappa > -4$ the problem trivializes to the minimization of

$$g_\kappa(\hat\sigma) = \left(\frac{\kappa(\gamma_+(\kappa)-2)^2 + 4\gamma_+^2(\kappa)}{(\gamma_+(\kappa)-2)^2}\right)|\hat\sigma|^2, \quad \hat\sigma := (\hat u_1(1,j))_{|j|\leqslant 1}, \tag{62}$$

subject to the constraint $|\hat\sigma|^2 = 4\pi(\gamma_+(\kappa)-2)^2/((\gamma_+(\kappa)-2)^2+8)$. This leads to the already computed minimal value $g_\kappa(\hat\sigma) = \gamma_+(\kappa)$ reached on any vector $\hat\sigma = (\hat u_1(1,j))_{|j|\leqslant 1}$ such that (55) holds.

Finally, for $\kappa = -4$, we have $\gamma_+(-4) = -2$, and again by (60), the problem trivializes to the minimization of

$$g_\kappa(\sigma) = -2\hat u_1^2(0,0) - 3|\hat\sigma|^2, \quad \sigma := (\hat u_1(0,0), \hat\sigma), \tag{63}$$

subject to the constraint $2\hat u_1^2(0,0) + 3|\hat\sigma|^2 = 8\pi$. This leads to the minimal value $g_\kappa(\sigma) = -8\pi$ reached on any vector $\sigma := (\hat u_1(0,0), \hat\sigma)$ such that $2\hat u_1^2(0,0) + 3|\hat\sigma|^2 = 8\pi$. $\square$

Finalizing the proof of Theorem 1. Going back to the minimization problem (1), (2) for the energy functional $\mathcal{F}_\kappa$, the results of Proposition 1 immediately translate into the context of Theorem 1 via the Fourier isomorphism that maps $\mathcal{F}_\kappa$ into $\mathcal{G}_\kappa$. It is therefore sufficient to apply the results to $\mathcal{F}_\kappa(\tilde{\boldsymbol{u}})$ with $\tilde{\boldsymbol{u}} := \sqrt{4\pi}\boldsymbol{u}/\|\boldsymbol{u}\|_{L^2(\mathbb{S}^2,\mathbb{R}^3)}$.

Proof of Theorem 2. Due to the saturation constraint $|\boldsymbol{u}(\xi)|^2 = 1$ for a.e. $\xi \in \mathbb{S}^2$, the Euler–Lagrange equations for $\mathcal{F}_\kappa$ reads, in strong form, as

$$\boldsymbol{u}(\xi) \times (-\Delta_\xi^* \boldsymbol{u}(\xi) + \kappa(\boldsymbol{u}(\xi)\cdot\boldsymbol{n}(\xi))\boldsymbol{n}(\xi)) = 0 \quad \forall \xi \in \mathbb{S}^2. \tag{64}$$

Since $-\Delta_\xi^*\boldsymbol{n}(\xi) = 2\boldsymbol{n}(\xi)$, the vector fields $\boldsymbol{u}_\pm(\xi) := \pm\boldsymbol{n}(\xi)$ satisfy (64) and, therefore, are stationary points of $\mathcal{F}_\kappa$.

Next, consider the second order variation $\mathcal{F}''_\kappa(\boldsymbol{u},\cdot)$ of $\mathcal{F}_\kappa$ at $\boldsymbol{u} \in H^1(\mathbb{S}^2,\mathbb{S}^2)$, which reads, for every $\boldsymbol{v} \in H^1(\mathbb{S}^2,\mathbb{R}^3)$ such that $\boldsymbol{u}(\xi)\cdot\boldsymbol{v}(\xi) = 0$ for a.e. in $\mathbb{S}^2$, as

$$\mathcal{F}''_\kappa(\boldsymbol{u},\boldsymbol{v}) = \int_{\mathbb{S}^2} |\nabla_\xi^*\boldsymbol{v}|^2 - |\nabla_\xi^*\boldsymbol{u}|^2|\boldsymbol{v}|^2 \mathrm{d}\xi + \kappa\int_{\mathbb{S}^2} (\boldsymbol{v}\cdot\boldsymbol{n})^2 - (\boldsymbol{u}\cdot\boldsymbol{n})^2|\boldsymbol{v}|^2\mathrm{d}\xi. \tag{65}$$

In particular, for $\boldsymbol{u}(\xi) := \pm\boldsymbol{n}(\xi)$, noting that $|\nabla_\xi^*\boldsymbol{n}(\xi)|^2 = 2$, we get

$$\mathcal{F}''_\kappa(\pm\boldsymbol{n},\boldsymbol{v}) = \int_{\mathbb{S}^2} |\nabla_\xi^*\boldsymbol{v}|^2 - (\kappa+2)|\boldsymbol{v}|^2\mathrm{d}\xi. \tag{66}$$

To show instability of $\boldsymbol{u}(\xi) := \pm\boldsymbol{n}(\xi)$ for $\kappa > 0$ we use a test function $\boldsymbol{v}(\xi) = \sqrt{4\pi}\boldsymbol{y}_{1,0}^{(2)}(\xi)$ from Remark 1.2 and obtain negativity of the second variation (66), which implies instability of $\boldsymbol{u}(\xi) := \pm\boldsymbol{n}(\xi)$.

Now, we concentrate on the case $\kappa < 0$ and first prove uniform local stability of $\boldsymbol{u}(\xi) := \pm\boldsymbol{n}(\xi)$, namely, there exists $\alpha(\kappa) > 0$ such that $\forall \boldsymbol{v} \in H^1(\mathbb{S}^2,\mathbb{R}^3)$ with $\boldsymbol{u}(\xi)\cdot\boldsymbol{v}(\xi) = 0$ for a.e. in $\mathbb{S}^2$ the following holds:

$$\mathcal{F}''_\kappa(\pm\boldsymbol{n},\boldsymbol{v}) \geqslant \alpha(\kappa)\int_{\mathbb{S}^2} |\nabla_\xi^*\boldsymbol{v}|^2 + |\boldsymbol{v}|^2\mathrm{d}\xi. \tag{67}$$

Using (66) and the Poincaré inequality (11) we obtain that $\forall\kappa < 0$

$$\mathcal{F}''_\kappa(\pm\boldsymbol{n},\boldsymbol{v}) \geqslant |\kappa|\int_{\mathbb{S}^2} |\boldsymbol{v}|^2\mathrm{d}\xi. \tag{68}$$

For $|\kappa| \geq 2$ it follows that

$$\mathcal{F}_\kappa''(\pm \boldsymbol{n}, \boldsymbol{v}) \geqslant \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{v}|^2 \mathrm{d}\xi \tag{69}$$

and summing up (68) and (69) we obtain

$$\mathcal{F}_\kappa''(\pm \boldsymbol{n}, \boldsymbol{v}) \geqslant \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{v}|^2 + |\boldsymbol{v}|^2 \mathrm{d}\xi. \tag{70}$$

For $|\kappa| < 2$ we define $\beta(\kappa) = \frac{1}{2}(2 - |\kappa|)$ $(0 < \beta(\kappa) < 1)$ and using (11) obtain

$$\begin{aligned} \mathcal{F}_\kappa''(\pm \boldsymbol{n}, \boldsymbol{v}) &= (1 - \beta(\kappa)) \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{v}|^2 \mathrm{d}\xi + \beta(\kappa) \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{v}|^2 - 2|\boldsymbol{v}|^2 \mathrm{d}\xi \\ &\geqslant (1 - \beta(\kappa)) \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{v}|^2 \mathrm{d}\xi. \end{aligned} \tag{71}$$

Summing up (71) and (68) we deduce

$$\mathcal{F}_\kappa''(\pm \boldsymbol{n}, \boldsymbol{v}) \geqslant \alpha(\kappa) \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{v}|^2 + |\boldsymbol{v}|^2 \mathrm{d}\xi, \tag{72}$$

where $\alpha(\kappa) = \frac{1}{2} \min\{(1 - \beta(\kappa)), |\kappa|\}$. Therefore (67) holds.

In order to prove strict local minimality, without loss of generality, we only consider the case $\boldsymbol{u}(\xi) = \boldsymbol{n}(\xi)$. We fix $\boldsymbol{\psi} \in H^1(\mathbb{S}^2, \mathbb{R}^3)$ such that $|\boldsymbol{n} + \boldsymbol{\psi}|^2 = 1$ and write down the difference of the energies

$$\mathcal{F}_\kappa(\boldsymbol{n} + \boldsymbol{\psi}) - \mathcal{F}_\kappa(\boldsymbol{n}) = \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{\psi}|^2 + \kappa(\boldsymbol{\psi} \cdot \boldsymbol{n})^2 + 2(\kappa + 2)(\boldsymbol{\psi} \cdot \boldsymbol{n}) \mathrm{d}\xi, \tag{73}$$

where we use integration by parts and equation $-\Delta_\xi^* \boldsymbol{n}(\xi) = 2\boldsymbol{n}(\xi)$. Now we define a component of $\boldsymbol{\psi}$ orthogonal to $\boldsymbol{n}$ as

$$\boldsymbol{\phi} = \boldsymbol{\psi} - \boldsymbol{n}(\boldsymbol{\psi} \cdot \boldsymbol{n}) \tag{74}$$

and write $\boldsymbol{\psi}$ as a sum

$$\boldsymbol{\psi} = \boldsymbol{\phi} + \boldsymbol{n}(\boldsymbol{\psi} \cdot \boldsymbol{n}). \tag{75}$$

A simple computation yields

$$\int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{\psi}|^2 = \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{\phi}|^2 + |\nabla_\xi^* (\boldsymbol{\psi} \cdot \boldsymbol{n})|^2 + (\boldsymbol{\psi} \cdot \boldsymbol{n})^2 |\nabla_\xi^* \boldsymbol{n}|^2 + 4(\boldsymbol{\psi} \cdot \boldsymbol{n}) \nabla_\xi^* \boldsymbol{\phi} : \nabla_\xi^* \boldsymbol{n} \mathrm{d}\xi. \tag{76}$$

Recalling that $|\nabla_\xi^* \boldsymbol{n}|^2 = 2$ we obtain

$$\begin{aligned} \mathcal{F}_\kappa(\boldsymbol{n} + \boldsymbol{\psi}) - \mathcal{F}_\kappa(\boldsymbol{n}) = \int_{\mathbb{S}^2} &|\nabla_\xi^* \boldsymbol{\phi}|^2 + |\nabla_\xi^* (\boldsymbol{\psi} \cdot \boldsymbol{n})|^2 + 4(\boldsymbol{\psi} \cdot \boldsymbol{n}) \nabla_\xi^* \boldsymbol{\phi} : \nabla_\xi^* \boldsymbol{n} \\ &+ (2 + \kappa)(\boldsymbol{\psi} \cdot \boldsymbol{n})^2 + 2(\kappa + 2)(\boldsymbol{\psi} \cdot \boldsymbol{n}) \mathrm{d}\xi. \end{aligned} \tag{77}$$

Now, we observe that $|\boldsymbol{\psi}|^2 = -2(\boldsymbol{\psi} \cdot \boldsymbol{n})$ and using relation (75) together with $\boldsymbol{n} \cdot \boldsymbol{\phi} = 0$ we obtain $|\boldsymbol{\psi}|^2 = |\boldsymbol{\phi}|^2 + \frac{1}{4}|\boldsymbol{\psi}|^4$. A straightforward calculation yields

(78)
$$\mathcal{F}_\kappa(\boldsymbol{n}+\boldsymbol{\psi}) - \mathcal{F}_\kappa(\boldsymbol{n}) = \int_{\mathbb{S}^2} |\nabla_\xi^* \boldsymbol{\phi}|^2 - (\kappa+2)|\boldsymbol{\phi}|^2 \mathrm{d}\xi + \int_{\mathbb{S}^2} |\nabla_\xi^*(\boldsymbol{\psi}\cdot\boldsymbol{n})|^2 - 2|\boldsymbol{\psi}|^2 \nabla_\xi^*\boldsymbol{\phi} : \nabla_\xi^* \boldsymbol{n} \mathrm{d}\xi.$$

The first integral in (78) is exactly the second variation of $\mathcal{F}_\kappa$ along direction $\boldsymbol{\phi}$ and can be controlled using (67). We can estimate the second integral by

$$\int_{\mathbb{S}^2} |\nabla_\xi^*(\boldsymbol{\psi}\cdot\boldsymbol{n})|^2 - 2|\boldsymbol{\psi}|^2 \nabla_\xi^*\boldsymbol{\phi} : \nabla_\xi^* \boldsymbol{n} \mathrm{d}\xi \geqslant -\int_{\mathbb{S}^2} \frac{1}{\delta}|\boldsymbol{\psi}|^4 + 2\delta |\nabla_\xi^*\boldsymbol{\phi}|^2 \mathrm{d}\xi, \tag{79}$$

where $\delta > 0$ is some fixed number. Therefore using $|\boldsymbol{\psi}|^2 = |\boldsymbol{\phi}|^2 + \frac{1}{4}|\boldsymbol{\psi}|^4$ we have

$$\begin{aligned} \mathcal{F}_\kappa(\boldsymbol{n}+\boldsymbol{\psi}) - \mathcal{F}_\kappa(\boldsymbol{n}) &\geqslant (\alpha(\kappa) - 2\delta) \int_{\mathbb{S}^2} |\nabla_\xi^*\boldsymbol{\phi}|^2 \mathrm{d}\xi \\ &\quad + \alpha(\kappa) \int_{\mathbb{S}^2} |\boldsymbol{\psi}|^2 \mathrm{d}\xi - \left(\frac{1}{\delta} + \frac{\alpha(\kappa)}{4}\right) \int_{\mathbb{S}^2} |\boldsymbol{\psi}|^4 \mathrm{d}\xi. \end{aligned} \tag{80}$$

Choosing $\delta = \alpha(\kappa)/2$ and using the L^2-Riemannian Gagliardo–Nirenberg inequality on $\mathbb{S}^2$ [6, Theorem 1.1] (see also [1, Theorem 3.70])

$$\|\boldsymbol{\psi}\|_{L^4}^4 \leqslant C \|\boldsymbol{\psi}\|_{H^1}^2 \|\boldsymbol{\psi}\|_{L^2}^2 \tag{81}$$

we obtain

$$\mathcal{F}_\kappa(\boldsymbol{n}+\boldsymbol{\psi}) - \mathcal{F}_\kappa(\boldsymbol{n}) \geqslant (\alpha(\kappa) - \chi(\kappa)\|\boldsymbol{\psi}\|_{H^1}^2) \int_{\mathbb{S}^2} |\boldsymbol{\psi}|^2 \mathrm{d}\xi, \tag{82}$$

where $\chi(\kappa) = C(\frac{2}{\alpha(\kappa)} + \frac{\alpha(\kappa)}{4}) > 0$. Taking $\varepsilon_0^2 = \frac{\alpha(\kappa)}{\chi(\kappa)}$ we obtain that

$$\mathcal{F}_\kappa(\boldsymbol{n}+\boldsymbol{\psi}) - \mathcal{F}_\kappa(\boldsymbol{n}) > 0 \tag{83}$$

$\forall \boldsymbol{\psi} \in H^1(\mathbb{S}^2;\mathbb{R}^3)$ ($\boldsymbol{\psi} \neq 0$) such that $\|\boldsymbol{\psi}\|_{H^1} < \varepsilon_0$ and $|\boldsymbol{n}+\boldsymbol{\psi}|^2 = 1$. This proves strict local minimality of $\boldsymbol{n}$.

Finally, for $\kappa \leqslant -4$, the global minimality of $\pm\boldsymbol{n}(\xi)$ is clear from Theorem 1 and the fact that $\mathcal{F}_\kappa$ is constrained to $H^1(\mathbb{S}^2, \mathbb{S}^2)$.

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