



Najnudel, J., Wu, C. T., & Yen, J. Y. (2020). On moments of Brownian functionals and their interpretation in terms of random walks. *Statistics and Probability Letters*, 161, Article 108724.
<https://doi.org/10.1016/j.spl.2020.108724>

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[10.1016/j.spl.2020.108724](https://doi.org/10.1016/j.spl.2020.108724)

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ON MOMENTS OF BROWNIAN FUNCTIONALS AND THEIR INTERPRETATION IN TERMS OF RANDOM WALKS

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ABSTRACT. An identity in law, proven by Yor [3, 7], involves integrals of quadratic functionals of the Brownian motion. The corresponding equalities of moments bring in equalities of multiple integrals of joint moments of the Brownian motion taken at different times. In the present paper, we compute these moments, and deduce an interpretation of some of these computations in terms of combinatorial sums indexed by different paths of random walks.

1. INTRODUCTION

Let $(B_t, t \geq 0)$ denote a 1-dimensional Brownian motion starting from 0. Let $0 \leq a \leq b$. Consider two continuous functions $f, g : [a, b] \rightarrow \mathbb{R}_+$, with f decreasing, and g increasing. It was shown in [3, 7] that the following identity between quadratic functionals of Brownian motion holds in law:

$$\int_a^b -df(x)B_{g(x)}^2 + f(b)B_{g(b)}^2 \stackrel{(\text{law})}{=} g(a)B_{f(a)}^2 + \int_a^b dg(x)B_{f(x)}^2. \quad (1)$$

To simplify matters, we assume that both f and g are C^1 , and that $f(b) = g(a) = 0$, that is:

$$\int_a^b -f'(x)dx B_{g(x)}^2 \stackrel{(\text{law})}{=} \int_a^b g'(x)dx B_{f(x)}^2. \quad (2)$$

Note that (2) is proved in [6] using the stochastic Fubini argument.

The tails of the random variables involved here have exponential decay, and hence these random variables are characterized by their moments. The identity in law described above is then equivalent to the following equality, for $n \geq 1$:

$$\mathbb{E} \left[\left(\int_a^b -f'(x)dx B_{g(x)}^2 \right)^n \right] = \mathbb{E} \left[\left(\int_a^b dx g'(x) B_{f(x)}^2 \right)^n \right]. \quad (3)$$

Now, let $L_n \equiv L_n(f, g)$ and $R_n \equiv R_n(f, g)$ be the left-hand side and the right-hand side of (3), respectively. It can be shown that

Date: January 20, 2020.

Key words and phrases. Brownian quadratic functionals, distributional integration by parts. MSC: 60J65, 60G15.

We thank M. Yor who initiated our motivation for this project. J.-Y. Yen is grateful to the Academia Sinica, Institute of Mathematics (Taipei, Taiwan) for their hospitality and support during some extended visits.

$$\left. \begin{aligned} \frac{L_n}{n!} &= \int_a^b |f'(y_1)| dy_1 \int_{y_1}^b |f'(y_2)| dy_2 \cdots \int_{y_{n-1}}^b |f'(y_n)| dy_n \beta_n(g(y_1), \dots, g(y_n)) \\ \text{and} \\ \frac{R_n}{n!} &= \int_a^b g'(x_1) dx_1 \int_{x_1}^{x_2} g'(x_2) dx_2 \cdots \int_{x_{n-1}}^{x_n} g'(x_n) dx_n \beta_n(f(x_1), \dots, f(x_n)) \end{aligned} \right\} \quad (4)$$

where

$$\beta_n(u_1, \dots, u_n) = \mathbb{E}[B_{u_1}^2 B_{u_2}^2 \cdots B_{u_n}^2], \quad 0 \leq u_1 \leq u_2 \leq \cdots \leq u_n. \quad (5)$$

A variant of these equalities is obtained when we begin with:

$$\int_a^b -f'(x) dx (B_{g(x)}^2 - g(x)) \stackrel{(\text{law})}{=} \int_a^b g'(x) dx (B_{f(x)}^2 - f(x)) \quad (6)$$

which is clearly equivalent to (2). When considering the n^{th} moments \bar{L}_n and \bar{R}_n of both sides of (6), there is a formula similar to (4) for \bar{L}_n and \bar{R}_n , where β_n is replaced by:

$$\bar{\beta}_n(u_1, \dots, u_n) = \mathbb{E}[M_{u_1} M_{u_2} \cdots M_{u_n}], \quad 0 \leq u_1 \leq u_2 \leq \cdots \leq u_n, \quad (7)$$

with $M_u \equiv B_u^2 - u = 2 \int_0^u B_v dB_v$. The main point of the paper is the study of the joint moments of β_n and $\bar{\beta}_n$.

Now, we denote more generally:

$$b_n(\underline{u}_n; \underline{e}_n) \equiv b_n(u_1, \dots, u_n; e_1, \dots, e_n) := \mathbb{E}[B_{u_1}^{e_1} \cdots B_{u_n}^{e_n}] \quad (8)$$

and

$$m_n(\underline{u}_n; \underline{e}_n) \equiv m_n(u_1, \dots, u_n; e_1, \dots, e_n) := \mathbb{E}[M_{u_1}^{e_1} \cdots M_{u_n}^{e_n}] \quad (9)$$

where e_1, \dots, e_n are integers. Note that since $M_u = B_u^2 - u$, the m_k 's may be expressed in terms of the b_j 's. In addition, the martingale properties of B_t and M_t imply

$$b_n(u_1, \dots, u_n; 1, 1, \dots, 1) = b_{n-1}(u_1, \dots, u_{n-1}; 1, 1, \dots, 1, 2), \quad (10)$$

and

$$\bar{\beta}_n(u_1, \dots, u_n) = m_{n-1}(u_1, \dots, u_{n-1}; 1, 1, \dots, 1, 2). \quad (11)$$

In this paper, we utilize the classical Isserlis' theorem [2] (also known as Wick's theorem [5]) to express b_n , and hence m_n . We also interpret our results in terms of random walks.

2. COMPUTATIONS OF MOMENTS AND THEIR RANDOM WALK INTERPRETATION

2.1. We first compute the moment

$$b_{2\ell}(u_1, \dots, u_{2\ell}; 1, \dots, 1) = \mathbb{E}[B_{u_1} \cdots B_{u_{2\ell}}]$$

where

$$0 = u_0 \leq u_1 \leq \cdots \leq u_{2\ell}.$$

Since $(B_{u_1}, \dots, B_{u_{2\ell}})$ is a centered Gaussian vector, we have the following expression involving the pairings of $\{1, \dots, 2\ell\}$ due to a classical result by Isserlis:

$$\begin{aligned} b_{2\ell}(u_1, \dots, u_{2\ell}; 1, \dots, 1) &= \mathbb{E}[B_{u_1} B_{u_2} \dots B_{u_{2\ell}}] \\ &= \sum_{E_1 \cup E_2 \cup \dots \cup E_\ell = \{1, 2, \dots, 2\ell\}, |E_1| = \dots = |E_\ell| = 2, \min(E_1) < \dots < \min(E_\ell)} \prod_{j=1}^{\ell} \mathbb{E}[B_{u_{m_j}} B_{u_{n_j}}] \end{aligned}$$

where $E_j = \{m_j, n_j\}$. Therefore,

$$\begin{aligned} b_{2\ell}(u_1, \dots, u_{2\ell}; 1, \dots, 1) &= \mathbb{E}[B_{u_1} B_{u_2} \dots B_{u_{2\ell}}] \\ &= \sum_{E_1 \cup E_2 \cup \dots \cup E_\ell = \{1, 2, \dots, 2\ell\}, |E_1| = \dots = |E_\ell| = 2, \min(E_1) < \dots < \min(E_\ell)} \prod_{j=1}^{\ell} u_{\min(E_j)}. \end{aligned}$$

Let us now count the number of pairings using the algorithm described below: First, suppose $\min(E_1) = r_1, \dots, \min(E_\ell) = r_\ell$ are fixed and $r_1 < r_2 < \dots < r_\ell$. Next, once r_ℓ is fixed, we choose $\max(E_\ell)$ which is strictly larger than r_ℓ , and this shall result in $(2\ell - r_\ell)_+$ possibilities (where $(\cdot)_+$ denotes the positive part of the expression). Then, after E_ℓ and $r_{\ell-1}$ are fixed, we choose $\max(E_{\ell-1})$ which is strictly larger than $r_{\ell-1}$ and not in E_ℓ , so $(2\ell - 2 - r_{\ell-1})_+$ possibilities are formed in this step. The subsequent steps are performed in the same manner. Hence,

$$\begin{aligned} b_{2\ell}(u_1, \dots, u_{2\ell}; 1, \dots, 1) &= \mathbb{E}[B_{u_1} B_{u_2} \dots B_{u_{2\ell}}] \\ &= \sum_{1 \leq r_1 < r_2 < \dots < r_\ell \leq 2\ell} \prod_{j=1}^{\ell} [(2j - r_j)_+ u_{r_j}] \\ &= \sum_{1 \leq r_1 < r_2 < \dots < r_\ell, r_j < 2j} \prod_{j=1}^{\ell} [(2j - r_j) u_{r_j}]. \end{aligned}$$

For example, we have

$$b_4(u_1, u_2, u_3, u_4; 1, 1, 1, 1) = \mathbb{E}[B_{u_1} B_{u_2} B_{u_3} B_{u_4}] = 2u_1 u_2 + u_1 u_3.$$

2.2. Let us now compute $\mathbb{E}[B_{u_1}^2 B_{u_2}^2 \dots B_{u_\ell}^2]$, which is a particular case of the previous computation with $(u_1, u_2, \dots, u_{2\ell})$ replaced by $(u_1, u_1, u_2, u_2, \dots, u_\ell, u_\ell)$. For $0 \leq e_j \leq 2$, $1 \leq j \leq \ell$, let us compute the coefficient of $u_1^{e_1} u_2^{e_2} \dots u_\ell^{e_\ell}$. Note that we need $e_1 + \dots + e_\ell = \ell$.

We have to count the number of pairings (E_1, \dots, E_ℓ) of $\{1, \dots, 2\ell\}$ such that $\min(E_j)$ (i.e. the smallest of the two elements of E_j) is equal to 1 or 2 for e_1 values of $j \in \{1, \dots, \ell\}$ (i.e. no value of $j \in \{1, \dots, \ell\}$ if $e_1 = 0$, one value of j if $e_1 = 1$, two values of j if $e_1 = 2$), $\min(E_j)$ is equal to 3 or 4 for e_2 values of j , $\min(E_j)$ is equal to 5 or 6 for e_3 values of j , and so on.

We will do this counting as follows: we first look at the number of possible choices we have for the pairs having $2\ell - 1$ or 2ℓ as their smallest element, then for each of these choices, we look at the number of possibilities we have for the pairs having $2\ell - 3$ or $2\ell - 2$ as their smallest element, then we look at the pairs starting at $2\ell - 5$ and $2\ell - 4$, and so on until the pairs starting at 1 or 2.

If $e_\ell = 0$, we have no choice to make for the pairs starting at $2\ell - 1$ or 2ℓ (since in this case there is no such pair to choose), so this gives 1 possibility. If $e_\ell = 1$, we need to get the set $\{2\ell - 1, 2\ell\}$ in the pairing, so this gives again 1 possibility. If $e_\ell = 2$,

we have to choose two pairs starting at $2\ell - 1$ or 2ℓ : this choice cannot be made, and then there is 0 possibility.

For each given choice of the pairs starting at $2\ell - 1$ or 2ℓ , let us look at the number of choices we have for the pairs starting at $2\ell - 3$ or $2\ell - 2$. If $e_{\ell-1} = 0$, we get 1 possibility, since we have no choice to make. If $e_{\ell-1} = 1$, we need a set in the pairing whose smallest element is $2\ell - 3$ or $2\ell - 2$. If it is $2\ell - 2$, there are two choices for the largest element of the pair ($2\ell - 1$ and 2ℓ), but $2e_\ell$ of them are taken by the last set (i.e. with the largest minimal element) of the pairing, so it remains $2 - 2e_\ell$ choices. If the smallest element is $2\ell - 3$, we have one more choice for the largest element of the pair ($2\ell - 2$), which gives $3 - 2e_\ell$ choices. Putting all the choices together gives $5 - 4e_\ell$ choices for the corresponding pair if $e_{\ell-1} = 1$. If $e_{\ell-1} = 2$, we have to find a pair with minimum $2\ell - 2$, and then a pair with minimum $2\ell - 3$, which gives $2 - 2e_\ell$ choices for the first pair, and then $(1 - 2e_\ell)_+$ choices for the second pair (three choices, $2 + 2e_\ell$ being taken by the pairs chosen before). For $e_{\ell-1} = 2$, we then get $(2 - 2e_\ell)(1 - 2e_\ell)_+$ possibilities.

For $1 \leq j \leq \ell - 2$, and for each choice of the pairs whose minimal element is strictly larger than $2j$, let us count the number of possibilities we have for the pairs starting at $2j - 1$ or $2j$. We get 1 possibility if $e_j = 0$. If $e_j = 1$, we need a pair starting with $2j - 1$ or $2j$. For a pair starting with $2j$, there are $2\ell - 2j$ choices, $2(e_{j+1} + \dots + e_\ell)$ being already taken by pairs previously constructed. For a pair starting with $2j - 1$, we have one more choice. The total gives the following number of possibilities:

$$\begin{aligned} & (2\ell - 2j - 2(e_{j+1} + \dots + e_\ell))_+ + (2\ell + 1 - 2j - 2(e_{j+1} + \dots + e_\ell))_+ \\ & = (4(\ell - j) + 1 - 4(e_{j+1} + \dots + e_\ell))_+, \end{aligned}$$

the last equality coming from the fact that for $a \in \mathbb{Z}$, $(2a)_+ + (2a + 1)_+ = (4a + 1)_+$. For $e_j = 2$, we need a pair starting with $2j$ and then a pair starting with $2j - 1$. For the first pair, we have $(2\ell - 2j - 2(e_{j+1} + \dots + e_\ell))_+$ choices and then, for the second pair, $(2\ell - 2j - 1 - 2(e_{j+1} + \dots + e_\ell))_+$. This gives the following number of possibilities:

$$(2\ell - 2j - 2(e_{j+1} + \dots + e_\ell))_+(2\ell - 1 - 2j - 2(e_{j+1} + \dots + e_\ell))_+.$$

We deduce the following formula:

$$\begin{aligned} \beta_\ell(u_1, \dots, u_\ell) &= b_\ell(u_1, \dots, u_\ell, 2, \dots, 2) = \mathbb{E}[B_{u_1}^2 \dots B_{u_\ell}^2] \\ &= \sum_{e_1, \dots, e_\ell \in \{0, 1, 2\}, e_1 + \dots + e_\ell = \ell} u_1^{e_1} u_2^{e_2} \dots u_\ell^{e_\ell} \prod_{j, e_j = 1} (4(\ell - j) + 1 - 4(e_{j+1} + \dots + e_\ell))_+ \\ &\times \prod_{j, e_j = 2} [(2\ell - 2j - 2(e_{j+1} + \dots + e_\ell))_+(2\ell - 1 - 2j - 2(e_{j+1} + \dots + e_\ell))_+] \end{aligned}$$

This formula can be slightly simplified. Indeed, since $e_1 + \dots + e_\ell = \ell$, we have

$$\ell - j - (e_{j+1} + \dots + e_\ell) = e_1 + \dots + e_j - j.$$

Hence,

$$\begin{aligned} \beta_\ell(u_1, \dots, u_\ell) &= b_\ell(u_1, \dots, u_\ell, 2, \dots, 2) = \mathbb{E}[B_{u_1}^2 \dots B_{u_\ell}^2] \\ &= \sum_{e_1, \dots, e_\ell \in \{0, 1, 2\}, e_1 + \dots + e_\ell = \ell} u_1^{e_1} u_2^{e_2} \dots u_\ell^{e_\ell} \prod_{j, e_j = 1} (4(e_1 + \dots + e_j - j) + 1)_+ \\ &\times \prod_{j, e_j = 2} [2(e_1 + \dots + e_j - j)_+(2(e_1 + \dots + e_j - j) - 1)_+]. \end{aligned}$$

In order to have nonzero terms, we need $e_1 + \dots + e_j \geq j$. Under this assumption, we can remove positive parts, and get

$$\begin{aligned} \beta_\ell(u_1, \dots, u_\ell) &= b_\ell(u_1, \dots, u_\ell, 2, \dots, 2) = \mathbb{E}[B_{u_1}^2 \dots B_{u_\ell}^2] \\ &= \sum_{e_1, \dots, e_\ell \in \{0, 1, 2\}, e_1 + \dots + e_\ell = \ell, e_1 + \dots + e_j \geq j} u_1^{e_1} u_2^{e_2} \dots u_\ell^{e_\ell} \prod_{j, e_j=1} (4(e_1 + \dots + e_j - j) + 1) \\ &\times \prod_{j, e_j=2} [2(e_1 + \dots + e_j - j)(2(e_1 + \dots + e_j - j) - 1)]. \end{aligned}$$

Example 1. For $\ell = 3$, taking (e_1, e_2, e_3) : $(2, 1, 0)$, $(2, 0, 1)$, $(1, 2, 0)$, $(1, 1, 1)$, we have the corresponding coefficients as follows:

$$\begin{aligned} (4(3 - 2) + 1)[2(2 - 1)(2(2 - 1) - 1)] &= 10, \\ (4(3 - 3) + 1)[2(2 - 1)(2(2 - 1) - 1)] &= 2, \\ (4(1 - 1) + 1)[2(3 - 2)(2(3 - 2) - 1)] &= 2, \\ (4(1 - 1) + 1)(4(2 - 2) + 1)(4(3 - 3) + 1) &= 1. \end{aligned}$$

Hence

$$\begin{aligned} \beta_\ell(u_1, u_2, u_3) &= b_\ell(u_1, u_2, u_3, 2, 2, 2) = \mathbb{E}[B_{u_1}^2 B_{u_2}^2 B_{u_3}^2] \\ &= 10u_1^2 u_2 + 2u_1^2 u_3 + 2u_1 u_2^2 + u_1 u_2 u_3. \end{aligned}$$

2.3. Let us now compute $m_\ell(u_1, \dots, u_\ell; 1, \dots, 1) = \mathbb{E}[M_{u_1} \dots M_{u_\ell}]$, where $M_u = B_u^2 - u$. We can check, by first conditioning on $(B_u)_{u \geq 0}$:

$$m_\ell(u_1, \dots, u_\ell; 1, \dots, 1) = \mathbb{E}[M_{u_1} \dots M_{u_\ell}] = \mathbb{E}[G_{u_1} G'_{u_1} \dots G_{u_\ell} G'_{u_\ell}],$$

where $G_u = B_u + N_u$, $G'_u = B_u - N_u$, the variable N_u being centered Gaussian of variance u , all the variables $(N_u)_{u \geq 0}$ being independent and independent of the Brownian motion B . The covariance structure of G and G' is given as follows: for $u \neq v$,

$$\mathbb{E}[G_u G_v] = \mathbb{E}[G_u G'_v] = \mathbb{E}[G'_u G'_v] = \mathbb{E}[B_u B_v] = \min(u, v),$$

and

$$\mathbb{E}[G_u G'_u] = 0.$$

We can then compute $\mathbb{E}[G_{u_1} G'_{u_1} \dots G_{u_\ell} G'_{u_\ell}]$ in the same way as $\mathbb{E}[B_{u_1}^2 \dots B_{u_\ell}^2]$ via the pairings of $\{1, \dots, 2\ell\}$. The resulting terms are almost the same as the previous setting, except that each pairing of the form $\{2j - 1, 2j\}$ kills the corresponding term since $\mathbb{E}[G_u G'_u] = 0$. Hence, we proceed with the same counting of possible pairings as described in the previous section, except that we shall forbid the pairs of the form $\{2j - 1, 2j\}$. The counting stays unchanged for $e_j = 0$. It is also the same for $e_j = 2$, since $2j - 1$ and $2j$ are in two different pairs. There is one less pairing choice for $e_j = 1$, hence, we have to subtract one from all the factors corresponding to $e_j = 1$, which gives

$$\begin{aligned} m_\ell(u_1, \dots, u_\ell; 1, \dots, 1) &= \mathbb{E}[M_{u_1} \dots M_{u_\ell}] \\ &= \sum_{e_1, \dots, e_\ell \in \{0, 1, 2\}, e_1 + \dots + e_\ell = \ell, e_1 + \dots + e_j \geq j} u_1^{e_1} u_2^{e_2} \dots u_\ell^{e_\ell} \prod_{j, e_j=1} (4(e_1 + \dots + e_j - j)) \\ &\times \prod_{j, e_j=2} [2(e_1 + \dots + e_j - j)(2(e_1 + \dots + e_j - j) - 1)]. \end{aligned}$$

Note that all terms such that there exists j with $e_1 + \dots + e_j = j$ and $e_j \geq 1$ are equal to zero. This reduces the number of terms we need to consider.

Example 2. For $\ell = 3$, we need $e_1 \geq 1$, and in fact $e_1 = 2$ since $e_1 = 1$ gives a term equal to zero. The sequence $(2, 0, 1)$ gives also a zero term since $e_1 + e_2 + e_3 = 3$ and $e_3 \geq 1$. The only non-zero term is given by $(2, 1, 0)$, and its coefficient is

$$4(3 - 2)[2(2 - 1)(2(2 - 1) - 1)] = 8.$$

Hence,

$$m_\ell(u_1, u_2, u_3; 1, 1, 1) = \mathbb{E}[M_{u_1} M_{u_2} M_{u_3}] = 8u_1^2 u_2.$$

For $\ell = 4$, we need $e_1 = 2$, and $e_4 = 0$. This gives the possibilities $(2, 2, 0, 0)$, $(2, 1, 1, 0)$, $(2, 0, 2, 0)$. The coefficients are

$$[2(2 - 1)(2(2 - 1) - 1)][2(4 - 2)(2(4 - 2) - 1)] = 24,$$

$$[2(2 - 1)(2(2 - 1) - 1)][4(3 - 2)][4(4 - 3)] = 32,$$

$$[2(2 - 1)(2(2 - 1) - 1)][2(4 - 3)(2(4 - 3) - 1)] = 4.$$

Hence

$$m_4(u_1, u_2, u_3, u_4; 1, 1, 1, 1) = \mathbb{E}[M_{u_1} M_{u_2} M_{u_3} M_{u_4}] = 24u_1^2 u_2^2 + 32u_1^2 u_2 u_3 + 4u_1^2 u_3^2.$$

2.4. The previous formula can be simplified by considering it as summing the trajectories of a random walk. If we denote

$$s_j := e_1 + e_2 + \dots + e_j - j,$$

the condition given in the sum becomes:

$$|s_{j+1} - s_j| \leq 1, s_j \geq 0, s_0 = s_\ell = 0,$$

i.e. s is a trajectory of a random walk with integer increments equal to $-1, 0$ or 1 , starting and ending at zero and remaining nonnegative. If we denote by $\mathcal{S}_{\ell,0}^+$ this set of trajectories, we deduce

$$\begin{aligned} m_\ell(u_1, \dots, u_\ell; 1, \dots, 1) &= \mathbb{E}[M_{u_1} \dots M_{u_\ell}] \\ &= \sum_{s \in \mathcal{S}_{\ell,0}^+} u_1^{s_1 - s_0 + 1} u_2^{s_2 - s_1 + 1} \dots u_\ell^{s_\ell - s_{\ell-1} + 1} \prod_{j=0}^{\ell-1} w_{s_j, s_{j+1}}, \end{aligned}$$

where

$$w_{k-1, k} = 2k(2k - 1), w_{k, k} = 4k, w_{k, k-1} = 1.$$

Note that the condition that the trajectories remain nonnegative can be removed, since the trajectories taking negative values necessarily result in a factor $w_{-1,0} = 0$.

We can then write

$$\begin{aligned} m_\ell(u_1, \dots, u_\ell; 1, \dots, 1) &= \mathbb{E}[M_{u_1} \dots M_{u_\ell}] \\ &= \sum_{s \in \mathcal{S}_{\ell,0}} u_1^{s_1 - s_0 + 1} u_2^{s_2 - s_1 + 1} \dots u_\ell^{s_\ell - s_{\ell-1} + 1} \prod_{j=0}^{\ell-1} w_{s_j, s_{j+1}}, \end{aligned}$$

where $\mathcal{S}_{\ell,0}$ is the set of all integer trajectories starting and ending at zero, with length ℓ and increments between -1 and 1 .

ACKNOWLEDGEMENT

We thank the anonymous reviewer for his/her careful reading and pointing out other combinatorial techniques to us, such as the application of hafnians studied in [1, 4]

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