A NOTE ON MULTIPLICATIVE AUTOMATIC SEQUENCES, II

OLEKSIY KLURMAN AND PÄR KURLBERG

ABSTRACT. We prove that any q-automatic multiplicative function $f : \mathbb{N} \to \mathbb{C}$ either essentially coincides with a Dirichlet character, or vanishes on all sufficiently large primes. This confirms a strong form of a conjecture of J. Bell, N. Bruin, and M. Coons.

1. INTRODUCTION

Automatic sequences play important role in computer science and number theory. For a detailed account of the theory and applications we refer the reader to the classical monograph [2]. One of the applications of such sequences in number theory stems from a celebrated theorem of Christol [5] (also cf. [6]), which asserts that in order to show the transcendence of the power series $\sum_{n\geq 1} f(n)z^n$ over $\mathbb{F}_q[z]$ it is enough to establish that the function $f : \mathbb{N} \to \overline{\mathbb{F}}_q$ is *not* q-automatic. In this note, rather than working within the general set up, we confine ourselves to functions with range in $\mathbb{C}$. There are several equivalent definitions of automatic (or more precisely, q-automatic) sequences. It will be convenient for us to use the following one.

Definition 1.1. The sequence $f : \mathbb{N} \to \mathbb{C}$ is called q-automatic if the q-kernel of it, defined as a set of subsequences

$$K_q(f) = \left\{ \{f(q^i n + r)\}_{n\geq 0} |\; i \geq 1, 0 \leq r \leq q^i - 1 \right\},$$

is finite.

We remark that any $q-$automatic sequence takes only finitely many values, since it is a function on the states of finite automata. A function $f : \mathbb{N} \to \mathbb{C}$ is called multiplicative if $f(mn) = f(m)f(n)$ for all pairs $(m, n) = 1$. The question of which multiplicative functions are q-automatic attracted considerable attention of several authors including [17], [15], [7], [16], [3], [1], [14], [12]. In particular, the following conjecture was made in [3].

Conjecture 1.2 (Bell-Bruin-Coons)**.** For any multiplicative q-automatic function $f : \mathbb{N} \to \mathbb{C}$ there exists an eventually periodic function $g : \mathbb{N} \to \mathbb{C}$, such that $f(p) = g(p)$ for all primes p.

DEPARTMENT OF MATHEMATICS, KTH ROYAL INSTITUTE OF TECHNOLOGY, STOCKHOLM

E-mail addresses: lklurman@gmail.com, kurlberg@math.kth.se.

Date: Wednesday 11th December, 2019.

P.K. was partially supported by the Swedish Research Council (2016-03701).

Some progress towards this conjecture has been made when f is assumed to be completely multiplicative. In particular, Schlage-Puchta [16] showed that a completely multiplicative q-automatic sequence which does not vanish is almost periodic. Hu [10] improved on that result by showing that the same conclusion holds under a slightly weaker hypothesis. Allouche and Goldmakher [1] considered a related question classifying what they called "mock" Dirichlet characters. Finally Li [14] and the authors [12] very recently proved Conjecture 1.2 when f is additionally assumed to be completely multiplicative. Moreover, the methods of [12] settled Conjecture 1.2 fully under the assumption of the Generalized Riemann Hypothesis. The key tool there was a variant of Heath-Brown's results [9] on Artin's primitive root conjecture. In this article, we develop an alternative approach, and unconditionally prove a strong form of Conjecture 1.2.

Theorem 1.3. *Let $q \geq 2$ and let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative q-automatic sequence. Then, there exists a Dirichlet character χ and an integer $Q \geq 1$, such that either $f(n) = \chi(n)$, for all $(n, Q) = 1$ or $f(p) = 0$ for all sufficiently large p.*

We remark that J. Konieczny [13], very recently and independently established a variant of Theorem 1.3 using different methods relying on the structure theory of automatic sequences.

Acknowledgments. The authors would like to thank J.-P. Allouche from whom they learnt about the question considered in this note and for his interest and encouragement. We are also very grateful to the referee for pointing out the possibility that our original argument can be simplified, as well as for valuable comments and suggestions.

2. Proof of the main result

Let $\mathbb{P}_0$ the set of primes for which $f(p) = 0$. Since f is q-automatic, it is well-known that the image of $f : \mathbb{N} \to \mathbb{C}$ is finite and therefore if we define the sets

$$\mathbb{P}_{>1} = \{p \text{ prime} : |f(p^e)| > 1 \text{ for some } e \geq 1\}$$

and

$$\mathbb{P}_{<1} = \{p \text{ prime} : |f(p^e)| < 1 \text{ for some } e \geq 1\},$$

then, from multiplicativity of f we easily deduce $|\mathbb{P}_{>1}|, |\mathbb{P}_{<1}| < \infty$. We begin by assuming additionally that $|\mathbb{P}_0| < \infty$.

Proposition 2.1. *Let $f : \mathbb{N} \to \mathbb{C}$ be a q-automatic multiplicative function and suppose that $|\mathbb{P}_0| < \infty$. Then there exists a Dirichlet character χ, such that $f(p^k) = \chi(p^k)$ for all $p \notin \mathbb{P}_{>1} \cup \mathbb{P}_{<1} \cup \mathbb{P}_0$ and any $k \geq 1$.*

Proof. Since f is q-automatic, we have that the kernel $K_q(f)$ is finite. By the pigeonhole principle, there exist positive integers $i_1 \neq i_2$ such that

$f(q^{i_1}n+1) = f(q^{i_2}n+1)$ for all $n \geq 1$. If $n = m\prod_{p\in\mathbb{P}_{>1}\cup\mathbb{P}_{<1}\cup\mathbb{P}_0} p$ then

$$\frac{f(q^{i_1}m\prod_{p\in\mathbb{P}_{>1}\cup\mathbb{P}_{<1}\cup\mathbb{P}_0} p+1)}{f(q^{i_2}m\prod_{p\in\mathbb{P}_{>1}\cup\mathbb{P}_{<1}\cup\mathbb{P}_0} p+1)} = 1 \neq 0, \tag{2.1}$$

for all $m \geq 1$. The conclusion now immediately follows from the following result by Elliott and Kish (stated as Theorem 1 in the arXiv preprint.)

Theorem 2.2 ([8], Theorem 2)**.** *Let integers $a > 0, b, B,$ satisfy $\Delta = aB - Ab \neq 0$. Set $\delta = 6(a,A)(aA)^2\Delta^3$. If a multiplicative complex-valued function g satisfies*

$$g\left(\frac{an+b}{An+B}\right) = c \neq 0$$

on all but finitely positive integers, n, then there is a Dirichlet character $\pmod{\delta}$ with which g coincides on all primes that do not divide δ.

We remark that the result is stated only for completely multiplicative functions, but the proof is in fact also valid for multiplicative functions. Alternatively, one could also apply correlation formulas developed in [11] to get the result. □

In what follows we will assume that $|\mathbb{P}_0| = \infty$ and show that in this case $f(p) = 0$ for all sufficiently large primes p. We begin with a convenient reduction: defining g as

$$g(n) = \begin{cases} 1, & \text{if } f(n) \neq 0, \\ 0, & \text{if } f(n) = 0, \end{cases}$$

we note that sequence $(g(n))_n$ is also q-automatic, and has the same zero set as the sequence $(f(n))_n$. Thus, replacing f by g, it is therefore enough to prove the claim for binary valued functions $f : \mathbb{N} \to \{0,1\}$.

To facilitate our discussion, we introduce the set

$$\widehat{\mathbb{P}_1} := \{p \text{ prime} : f(p^e) = 1 \text{ for some } e \geq 1\}.$$

Since the case of finite $\widehat{\mathbb{P}_1}$ is trivial, we from now on assume that $\widehat{\mathbb{P}_1}$ is infinite.

We note that if f is q-automatic, then there exists $k_0 = k_0(f)$ with the following property: if for some $i \geq 1$ and $0 \leq r \leq q^i - 1$ the equality $f(q^i n + r) = 0$ holds for all integer $n \in [1, k_0]$, then $f(q^i n + r) = 0$ for all $n \geq 1$. A useful consequence is the following:

Corollary 2.3. *Given any arithmetic progression $\{q^l m + r\}_{m\geq 1}$, with $r < q^l$, so that $f(q^l m + r) = 1$ for infinitely many m, there exists $m_0 \leq k_0$ such that $f(q^l m_0 + r) = 1$.*

We next show that non-vanishing of f in certain progressions is either trivial or infinite.

Lemma 2.4. *Assume that $|\widehat{\mathbb{P}_1}| = \infty$. Given $r, l \in \mathbb{N}$ such that $f(r) = 1$ and $q^l > r$, there exists infinitely many m such that $f(q^l m + r) = 1$.*

Proof. Let $p_i \in \widehat{\mathbb{P}_1}$ denote distinct primes such that $p_i > r$ and select exponents e_i so that $f(p_i^{e_i}) = 1$, and denote $P_i = p_i^{e_i}$. We may then select a sequence of finite disjoint sets I_j such that $\prod_{i\in I_j} P_i \equiv 1 \mod q^l$. Letting $n_j = r \cdot \prod_{i\in I_j} P_i$ we find, using multiplicativity and that $(r, P_i) = 1$, that $f(n_j) = 1$, and that $n_j \equiv r \mod q^l$. Consequently we can write $n_j = q^l m_j + r$, and the proof is concluded. □

Let $q_1, \ldots, q_{k_0+1} \in \mathbb{P}_0$ denote distinct primes with $q_i > q$ for all $1 \leq i \leq k_0 + 1$. Define $Q := \prod_{i=1}^{k_0+1} q_i$. By the Chinese remainder theorem, there exists an interval $I \subset \mathbb{Z}/Q^2\mathbb{Z}$ with $|I| = k_0 + 1$ such that all integers $n \in I$ have the property that at least one $q_i || n$. In particular, $f(n) = 0$ for all $n \in I$. Further, letting $L = \phi(Q^2)$, we note that $q^L \equiv 1 \mod Q^2$.

We can now arrive at a contradiction by constructing a sequence of integer r_i's so that on one hand $f(r_i) = 1$ for all i, while at the same time, some $r_i \mod Q^2$ must fall in I and consequently $f(r_i) = 0$.

To construct such a sequence we argue as follows. Given r_1 such that $f(r_1) = 1$, select s_1 so that $l_1 := s_1 L$ has the property that $q^{l_1} > r$. By Lemma 2.4, there exist infinitely many m such that $f(q^{l_1}m + r_1) = 1$, and thus, by Corollary 2.3, there exists $m_1 \leq k_0$ such that

$$r_2 := q^{l_1} m_1 + r_1$$

has the property that $f(r_2) = 1$. We also note that $r_2 \equiv m_1 + r_1 \mod Q^2$.

Proceeding inductively, given r_i with properties as above, we can define

$$r_{i+1} := q^{l_i} m_i + r_i$$

so that $f(r_{i+1}) = 1$, $m_i \leq k_0$, and $r_{i+1} \equiv r_i + m_i \mod Q^2$ (in particular, take $l_i = Ls_i$ with s_i large enough so that $q^{l_i} > r_i$.)

Now, as all $m_i \leq k_0$, we find that $\{r_i \mod Q^2\}$ must intersect I, consequently $f(r_i) = 0$ for some i and we have arrived at a contradiction.

Remark 2.5. The proof can be modified to give a slightly stronger conclusion, namely if $|\mathbb{P}_0| = \infty$ then $|\widehat{\mathbb{P}_1}| < \infty$. From here one can easily get more refined information about the set of values of $\{f(p^n)\}_{n\geq 1}$. We leave the details to the interested reader.

REFERENCES

[1] J.-P. Allouche and L. Goldmakher. Mock characters and the Kronecker symbol. *J. Number Theory*, 192:356–372, 2018.

[2] J.-P. Allouche and J. Shallit. *Automatic sequences*. Cambridge University Press, Cambridge, 2003. Theory, applications, generalizations.

[3] J. P. Bell, N. Bruin, and M. Coons. Transcendence of generating functions whose coefficients are multiplicative. *Trans. Amer. Math. Soc.*, 364(2):933–959, 2012.

[4] J. P. Bell, M. Coons, and K. G. Hare. The minimal growth of a k-regular sequence. *Bull. Aust. Math. Soc.*, 90(2):195–203, 2014.

[5] G. Christol. Ensembles presque periodiques k-reconnaissables. *Theoret. Comput. Sci.*, 9(1):141–145, 1979.

[6] G. Christol, T. Kamae, M. Mendès France, and G. Rauzy. Suites algébriques, automates et substitutions. *Bull. Soc. Math. France*, 108(4):401–419, 1980.
[7] M. Coons. (Non)automaticity of number theoretic functions. *J. Théor. Nombres Bordeaux*, 22(2):339–352, 2010.
[8] P. D. T. A. Elliott and J. Kish. Harmonic analysis on the positive rationals. Determination of the group generated by the ratios $(an + b)/(An + B)$. *Mathematika*, 63(3):919–943, 2017.
[9] D. R. Heath-Brown. Artin's conjecture for primitive roots. *Quart. J. Math. Oxford Ser. (2)*, 37(145):27–38, 1986.
[10] Y. Hu. Subword complexity and non-automaticity of certain completely multiplicative functions. *Adv. in Appl. Math.*, 84:73–81, 2017.
[11] O. Klurman. Correlations of multiplicative functions and applications. *Compos. Math.*, 153(8):1622–1657, 2017.
[12] O. Klurman and P. Kurlberg. A note on multiplicative automatic sequences. *C. R. Math. Acad. Sci. Paris*, 357(10):752–755, 2019.
[13] J. Konieczny. On multiplicative automatic sequences. *arXiv:1905.11981*, 2019.
[14] S. Li. On completely multiplicative automatic sequences. *arXiv:1903.04385*, 2019.
[15] J.-C. Schlage-Puchta. A criterion for non-automaticity of sequences. *J. Integer Seq.*, 6(3):Article 03.3.8, 5, 2003.
[16] J.-C. Schlage-Puchta. Completely multiplicative automatic functions. *Integers*, 11:A31, 8, 2011.
[17] S. Yazdani. Multiplicative functions and k-automatic sequences. *J. Théor. Nombres Bordeaux*, 13(2):651–658, 2001.