A Note on Stochastic Complementarity for the Applied Researcher

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Abstract

In their discussion of stochastic complementarity, Manzini et al. (2018) show that the intuitively appealing correlation criterion does not—in general—satisfy the axiom of monotonicity: products classified as complements can turn into substitutes following an increase in their joint purchases. In this note, we show that, however, by restricting attention to mixed logit models along the lines of Gentzkow (2007)'s, one can prevent such violations.

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1 Introduction

Manzini et al. (2018) propose various criteria to classify two products or services as complements. The proposed criteria satisfy three requirements:

(a) they do not rely on price/income or choice set variation,

(b) they only rely on observed data on bundle-level consumption/purchase probabilities, and

(c) they place no restriction on the data generating process.

Manzini et al. (2018) specifically attempt to go beyond the classical definition that two products are complements whenever their (compensated) cross-price elasticity is negative (e.g., Samuelson (1974)) and characterize complementarity also in cases of "items" for which there are no prices or choice set variation, such as a park and a museum or different business practices within a firm.[1] As a striking finding, Manzini et al. (2018) show that there cannot be any criterion for complementarity satisfying (a), (b), and (c) that also fulfils all of the desirable axioms they propose. In particular, the intuitively appealing correlation criterion (i.e., two products are complements if their purchases are positively correlated, see Brynjolfsson and Milgrom (2013)), in general violates the axiom of monotonicity:[2] products classified as complements can turn into substitutes following an increase in their joint purchases.

The purpose of this note is to show that by placing restrictions on the data generating process (point (c) above), as typically done by applied researchers, the correlation criterion can satisfy all the axioms suggested by Manzini et al. (2018)—including monotonicity. In particular, we demonstrate that by restricting attention to mixed logit models along the lines of Gentzkow (2007)'s, one can prevent violations of the monotonicity axiom.[3]

Our analysis clarifies that, in this class of models, violations of monotonicity by the correlation criterion relate to a central identification challenge pointed out by Gentzkow (2007): product-specific unobserved preference heterogeneity induces correlation in the purchases of different products, and this complicates the identification of complementarity and substitutability from observed purchase probabilities.[4] By restricting the extent of unobserved preference heterogeneity, it is however possible to unambiguously interpret correlation in the purchases of different products as a sign of complementarity or substitutability.

[1] For simplicity, in what follows we only talk about "products" and their "purchases." However, in the spirit of Manzini et al. (2018), our discussion applies to more abstract "items" such as a park and a museum.

[2] Even though the correlation criterion in general violates monotonicity, it satisfies the other two desirable axioms proposed by Manzini et al. (2018): duality and responsiveness. See Manzini et al. (2018) for details.

[3] Mixed logit models similar to the one we study in this note are used by, for example: Gentzkow (2007), Liu et al. (2010), Grzybowski and Verboven (2016), and Iaria and Wang (2019).

[4] More recently, the problem of identification in empirical models of demand for bundles is also studied by: Fox and Lazzati (2017), Allen and Rehbeck (2019), Iaria and Wang (2019), and Wang (2019).

2 Notation and Definitions

Suppose to observe data on the bundle-level purchase probabilities of two products, x and y. We define the sampling space of bundle-level purchase probabilities as

$$\mathbf{T} \equiv \left\{ (P_{xy}, P_x, P_y, P_0) \middle| P_k \in (0,1), \forall k; \sum P_k = 1 \right\},$$

where P_{xy} is the probability of the joint purchase of products x and y, P_x (respectively P_y) is the purchase probability of x but not y (respectively y but not x), and P_0 is the probability of the outside option—the choice of not purchasing any of the two products. $\mathbf{T}$ is the space of possible values (P_{xy}, P_x, P_y, P_0) can take. For simplicity, we rule out the "zero purchase probability" problem (Gandhi et al. (2019)) and assume that $P_k \in (0,1)$ for $k \in \{xy, x, y, 0\}$ and that $P_{xy} + P_x + P_y + P_0 = 1$.

We now define the correlation criterion and the monotonicity axiom, and briefly illustrate why—without further restrictions on the data generating process—the correlation criterion will violate monotonicity.

Correlation Criterion. According to this criterion, two products are classified as complements if their purchases are positively correlated, see Brynjolfsson and Milgrom (2013). Manzini et al. (2018) illustrate that the correlation criterion partitions the sampling space $\mathbf{T}$ into three subsets $\mathbf{C}$, $\mathbf{I}$, and $\mathbf{S}$: the collections of possible values of (P_{xy}, P_x, P_y, P_0) that exhibit complementarity, independence, and substitutability. $(P_{xy}, P_x, P_y, P_0) \in \mathbf{C}$ if and only if $\frac{P_{xy}}{P_{xy}+P_y} > P_{xy} + P_x$. Note that this condition is equivalent to $P_{xy}P_0 > P_x P_y$. Similarly, $(P_{xy}, P_x, P_y, P_0) \in \mathbf{I}$ if and only if $P_{xy}P_0 = P_x P_y$ and $(P_{xy}, P_x, P_y, P_0) \in \mathbf{S}$ if and only if $P_{xy}P_0 < P_x P_y$.

Monotonicity Axiom. This axiom embodies the coherence of the principle of revealed preferences. Monotonicity requires that if $(P_{xy}, P_x, P_y, P_0) \in \mathbf{C}$, $(P'_{xy}, P'_x, P'_y, P'_0) \in \mathbf{T}$ with $P'_{xy} \geq P_{xy}$, $P'_x \leq P_x$, and $P'_y \leq P_y$, then $(P'_{xy}, P'_x, P'_y, P'_0) \in \mathbf{C}$. Symmetrically, if $(P_{xy}, P_x, P_y, P_0) \in \mathbf{S}$, $(P'_{xy}, P'_x, P'_y, P'_0) \in \mathbf{T}$ with $P'_{xy} \leq P_{xy}$, $P'_x \geq P_x$, and $P'_y \geq P_y$, then $(P'_{xy}, P'_x, P'_y, P'_0) \in \mathbf{S}$. The monotonicity axiom requires that, if two products are complements for some (P_{xy}, P_x, P_y, P_0), then they cannot be substitutes or independent for any $(P'_{xy}, P'_x, P'_y, P'_0)$ with larger joint purchases and smaller or equal purchases of each single product, and vice versa (i.e., substitutes cannot turn into complements following a decrease in their joint purchases).

Correlation Criterion and Violation of Monotonicity. In general, the correlation criterion does not satisfy the monotonicity axiom. We illustrate this using a counter-example due to Manzini et al. (2018). Suppose that $(P_{xy}, P_x, P_y, P_0) \in \mathbf{C}$ and consider another possible value from $\mathbf{T}$ such that $P'_{xy} = P_{xy} + \epsilon$, $P'_x = P_x$, $P'_y = P_y$, $P'_0 = P_0 - \epsilon$ for a small enough ϵ. It then follows that $P'_{xy}P'_0 = (P_{xy} + \epsilon)(P_0 - \epsilon)$ and for $\epsilon \to P_0$, $P'_{xy}P'_0 \to 0$. This implies that, for $\epsilon \to P_0$, $P'_{xy}P'_0 < P'_xP'_y$ and $(P'_{xy}, P'_x, P'_y, P'_0) \in \mathbf{S}$, violating monotonicity.

As discussed by Manzini et al. (2018), the correlation criterion will *in general* satisfy the other two desirable axioms of duality and responsiveness. The restrictions we propose to the data generating process do not affect this, while preventing violations of the monotonicity axiom.

3 Gentzkow (2007)'s Model and Monotonicity

Manzini et al. (2018) resolve the inconsistency between the correlation criterion and the monotonicity axiom by proposing a weaker version of the axiom while maintaining an unrestricted data generating process (DGP). Differently, we add structure to the DGP and resolve the inconsistency within the more specific context of a mixed logit model of demand for bundles along the lines of Gentzkow (2007), Liu et al. (2010), Grzybowski and Verboven (2016), and Iaria and Wang (2019). The purchase probabilities of the mixed logit model we investigate can be expressed as:

$$
\begin{aligned}
\text{den}(\delta, \Gamma, \mu_i) = &\quad 1 + \exp(\delta_x + \mu_{ix}) + \exp(\delta_y + \mu_{iy}) \\
&+ \exp(\delta_x + \mu_{ix} + \delta_y + \mu_{iy} + \Gamma) \\
P_0 = P_0(\theta, F_\theta) = &\quad \int \frac{1}{\text{den}(\delta, \Gamma, \mu_i)} dF_\theta(\mu_i) \\
P_z = P_z(\theta, F_\theta) = &\quad \int \frac{\exp(\delta_z + \mu_{iz})}{\text{den}(\delta, \Gamma, \mu_i)} dF_\theta(\mu_i); \text{ for } z = x, y \\
P_{xy} = P_{xy}(\theta, F_\theta) = &\quad \int \frac{\exp(\delta_x + \mu_{ix} + \delta_y + \mu_{iy} + \Gamma)}{\text{den}(\delta, \Gamma, \mu_i)} dF_\theta(\mu_i),
\end{aligned} \tag{1}
$$

where $\delta = (\delta_x, \delta_y)$ and $\delta_x + \delta_y + \Gamma$ are the average utilities associated to purchasing, respectively: only product x, only product y, and the bundle xy. Following Gentzkow (2007), we interpret Γ as the extra portion of average utility associated to the joint purchase of x and y. For example, if $\Gamma > 0$, individuals obtain a higher average utility from the joint rather than the separate purchase of x and y.

Mixed logit model (1) allows for unobserved heterogeneity across individuals—denoted by the subscript i—in the product-specific intercepts $\mu_i = (\mu_{ix}, \mu_{iy})$, where we assume that

$$\mu_{iz} = \sigma_z \times v_i + \eta_{iz}, \; z \in \{x, y\}$$

is a function of the mutually independent and standardized[5] random variables $v_i \in \mathbb{R}$ and $\eta_{iz} \in \mathbb{R}$. Note that this specification, while simple, allows for flexible forms of correlation between μ_{ix} and μ_{iy}. Note that $\text{Corr}(\mu_{ix}, \mu_{iy}) \geq 0$ if and only if $\sigma_x \times \sigma_y \geq 0$.[6] Denote by F_θ the joint distribution of $\mu_i = (\mu_{ix}, \mu_{iy})$ across individuals conditional on $\theta = (\delta, \Gamma, \sigma) \in \mathbb{R}^5$. Denoting by $\mathscr{F} \ni F_\theta$ the set of all possible distributions, we can define

$$\mathbf{T} = \{(P_{xy}(\theta, F_\theta), P_x(\theta, F_\theta), P_y(\theta, F_\theta), P_0(\theta, F_\theta)) \mid \theta \in \mathbb{R}^5, F_\theta \in \mathscr{F}\}$$

and

$$\mathbf{C} = \{(P_{xy}(\theta, F_\theta), P_x(\theta, F_\theta), P_y(\theta, F_\theta), P_0(\theta, F_\theta)) \mid \theta \in \mathbb{R}^5, F_\theta \in \mathscr{F}, P_{xy}P_0 > P_xP_y\}.$$

I and **S** can be defined analogously to **C**. Next, we show that with a sign restriction on Γ and $\sigma_x \times \sigma_y$, i.e., when both are positive (negative) for all $\theta \in \mathbb{R}^5$, mixed logit model (1) does not violate the monotonicity axiom.

Proposition 1. *Suppose that the DGP is mixed logit model* (1). *It then follows that:*

- *If* $\Gamma > 0$ *and Corr*$(\mu_{ix}, \mu_{iy}) > 0$ *(i.e.,* $\sigma_x \times \sigma_y > 0$*), then* $\mathbf{T} = \mathbf{C}$.
- *If* $\Gamma < 0$ *and Corr*$(\mu_{ix}, \mu_{iy}) < 0$ *(i.e.,* $\sigma_x \times \sigma_y < 0$*), then* $\mathbf{T} = \mathbf{S}$.

Proof. See Online Supplement. □

Proposition 1 relates violations of monotonicity by the correlation criterion to a central identification challenge pointed out by Gentzkow (2007): μ_{ix} and μ_{iy} induce correlation in the purchases of different products, and this complicates the identification of complementarity and substitutability from observed purchase probabilities.[7] Our result shows that violations of monotonicity can be prevented when the correlation between μ_{ix} and μ_{iy} goes "in the same direction" as the sign of Γ. Even though this condition is not necessary (only sufficient), the examples in Figure 1 illustrate that when $\text{Corr}(\mu_{ix}, \mu_{iy})$ and the sign of Γ go "in opposite directions", violations of monotonicity are likely to arise.

In both panels of Figure 1, we depict whether each value of $\delta = (\delta_x, \delta_y)$ corresponds to purchase probabilities that belong to **C** (in white), to **S** (in light gray), or that violate the monotonicity axiom (in dark gray) in relation to a reference point. We assume that v_i, η_{ix}, η_{iy} are i.i.d. binary variables that equal -1 with probability 0.5 and 1 with probability 0.5. In panel (a), $\Gamma = 0.62, \sigma_x = 1$, and $\sigma_y = -1$ (violating the first condition of Proposition 1),

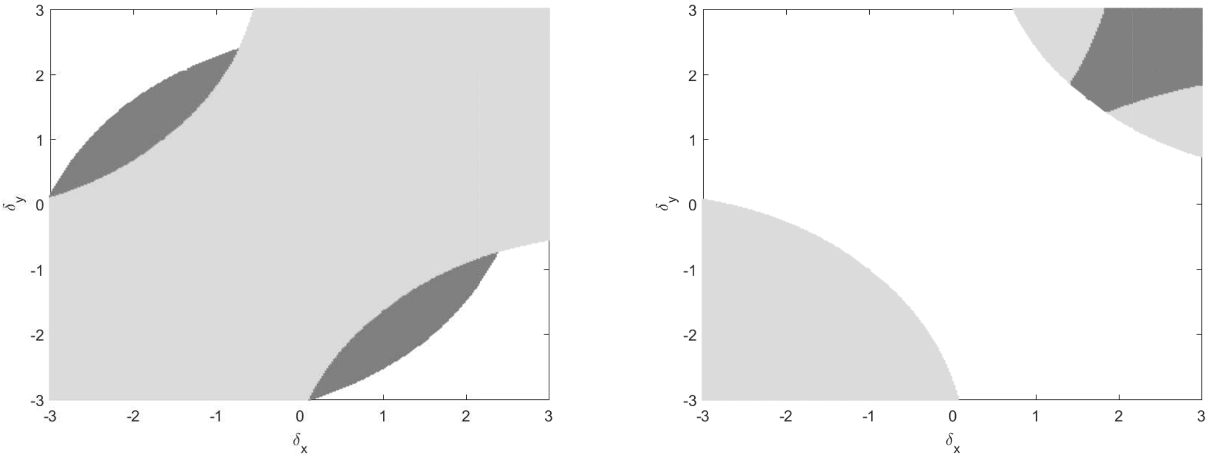
[5]By standardized, we intend with zero mean and unit standard deviation.

[6]$\text{COV}(\mu_{ix}, \mu_{iy}) = \mathbb{E}[(\sigma_x \times v_i + \eta_{ix})(\sigma_y \times v_i + \eta_{iy})] - \mathbb{E}[\sigma_x \times v_i + \eta_{ix}]\mathbb{E}[\sigma_y \times v_i + \eta_{iy}] = \sigma_x\sigma_y$. Then, $\text{Corr}(\mu_{ix}, \mu_{iy}) \geq 0$ if and only if $\sigma_x\sigma_y \geq 0$.

[7]See footnote 4 for recent references on the problem of identification in this class of models.

Figure 1: Two Numerical Examples of Violations of Monotonicity

(a) $\Gamma = 0.62, \sigma_x = 1, \sigma_y = -1$ (b) $\Gamma = -0.62, \sigma_x = 1, \sigma_y = 1$



while the reference point with respect to which we check monotonicity is $\delta = (3, 3)$ (in $\mathbf{S}$): the probabilities corresponding to the dark gray areas qualify for monotonicity but belong to $\mathbf{C}$.[8] In panel (b), instead, $\Gamma = -0.62, \sigma_x = 1$, and $\sigma_y = 1$ (violating the second condition of Proposition 1), while the reference point is $\delta = (0, 0)$ (in $\mathbf{C}$): the probabilities corresponding to the dark gray area qualify for monotonicity but belong to $\mathbf{S}$.[9]

[8]The probabilities corresponding to $(\delta_x, \delta_y) = (3, 3)$ are $(P_{xy}, P_x, P_y, P_0) = (0.8899, 0.0537, 0.0537, 0.0026) \in \mathbf{S}$. Then, take for example $(\delta'_x, \delta'_y) = (-1.5, 1.5)$ and the corresponding $(P'_{xy}, P'_x, P'_y, P'_0) = (0.2519, 0.0723, 0.5245, 0.1513)$. Note that $P'_x > P_x$, $P'_y > P_y$, and $P'_{xy} < P_{xy}$. However, $(P'_{xy}, P'_x, P'_y, P'_0) \in \mathbf{C}$, violating monotonicity.

[9]$(\delta_x, \delta_y) = (0, 0)$ corresponds to $(P_{xy}, P_x, P_y, P_0) = (0.2029, 0.2471, 0.2471, 0.3030) \in \mathbf{C}$. Then, consider for example $(\delta'_x, \delta'_y) = (2, 2)$, which corresponds to $(P'_{xy}, P'_x, P'_y, P'_0) = (0.5667, 0.18640.1864, 0.0605)$ with $P'_x < P_x$, $P'_y < P_y$, and $P'_{xy} > P_{xy}$. However, $(P'_{xy}, P'_x, P'_y, P'_0) \in \mathbf{S}$, violating monotonicity.

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