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# On capacity and torsional rigidity

M. van den Berg and G. Buttazzo

## ABSTRACT

We investigate extremal properties of shape functionals which are products of Newtonian capacity  $\text{cap}(\bar{\Omega})$ , and powers of the torsional rigidity  $T(\Omega)$ , for an open set  $\Omega \subset \mathbb{R}^d$  with compact closure  $\bar{\Omega}$ , and prescribed Lebesgue measure. It is shown that if  $\Omega$  is convex, then  $\text{cap}(\bar{\Omega})T^q(\Omega)$  is (i) bounded from above if and only if  $q \geq 1$ , and (ii) bounded from below and away from 0 if and only if  $q \leq \frac{d-2}{2(d-1)}$ . Moreover a convex maximiser for the product exists if either  $q > 1$ , or  $d = 3$  and  $q = 1$ . A convex minimiser exists for  $q < \frac{d-2}{2(d-1)}$ . If  $q \leq 0$ , then the product is minimised among all bounded sets by a ball of measure 1.

## 1. Introduction and main results

Several classical inequalities of mathematical physics are of the following form. Let  $F$  and  $H$  be strictly positive set functions defined on a suitable collection  $\mathfrak{C}$  of open sets in  $\mathbb{R}^d$ , and which satisfy scaling relations

$$F(t\Omega) = t^{\beta_1}F(\Omega), \quad H(t\Omega) = t^{\beta_2}H(\Omega), \quad t > 0,$$

where  $t\Omega$  is homothety of  $\Omega$ , and  $\beta_1, \beta_2$  are constants. Then the shape functional

$$G(\Omega) = H(\Omega)F(\Omega)^{-\beta_2/\beta_1},$$

is invariant under homotheties, and in some cases this quantity is minimal (respectively, maximal) for some open set  $\Omega^* \in \mathfrak{C}$ ,

$$G(\Omega) \geq G(\Omega^*) \quad (\text{respectively, } G(\Omega) \leq G(\Omega^*)), \quad \Omega \in \mathfrak{C}.$$

The Faber–Krahn, Krahn–Szegő, and Kohler–Jobin inequalities are of this form. See, for example, the seminal text [11]. In a recent paper [3], a more general set of inequalities was investigated. These are of the following form: let  $q \in \mathbb{R}$ , and consider the shape functional

$$G(\Omega) = H(\Omega)F(\Omega)^q.$$

Then, unless  $q = -\beta_2/\beta_1$ , this product is not scaling invariant. However, denoting by  $|\Omega|$  the Lebesgue measure of  $\Omega$ , the quantity

$$\frac{H(\Omega)F(\Omega)^q}{|\Omega|^{(\beta_2+q\beta_1)/d}}$$

is scaling invariant. The case where  $H$  is the principal Dirichlet eigenvalue, and  $F$  is the torsional rigidity was analysed in [3]. In the present paper, we investigate, in the spirit of [11],

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the case where  $H$  is the Newtonian capacity, and  $F$  is the torsional rigidity. Since the Newtonian capacity is most easily defined for compact subsets of  $\mathbb{R}^d$ ,  $d \geq 3$ , we restrict ourselves to open sets  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 3$  which are precompact. In that case, the Newtonian capacity scales as a power  $\beta_2 = d - 2$  of the homothety.

Throughout this paper we let  $\Omega$  be a non-empty, open, bounded set in Euclidean space  $\mathbb{R}^d$ ,  $d \geq 3$ . For a set  $A \subset \mathbb{R}^d$ , we denote by  $\bar{A}$  its closure,  $\text{diam}(A) = \sup\{|x - y| : x \in A, y \in A\}$  its diameter, and  $r(A) = \sup\{r \geq 0 : (\exists x \in A), (B_r(x) \subset A)\}$  its inradius, where  $B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}$  is the ball of radius  $r$  centred at  $x$ . Before we state the main results we recall some basic facts about the torsion function, torsional rigidity, and Newtonian capacity.

The torsion function for an open set  $\Omega$  with finite measure is the solution of

$$-\Delta u = 1, \quad u \in H_0^1(\Omega),$$

and is denoted by  $u_\Omega$ . It is convenient to extend  $u_\Omega$  to all of  $\mathbb{R}^d$  by defining  $u_\Omega = 0$  on  $\mathbb{R}^d \setminus \Omega$ . It is well known that  $u_\Omega$  is non-negative, bounded ([2, 4 6, 12]), and monotone increasing with respect to  $\Omega$ , that is

$$\Omega_1 \subset \Omega_2 \Rightarrow u_{\Omega_1} \leq u_{\Omega_2}.$$

The torsional rigidity of  $\Omega$ , or torsion for short, is denoted by

$$T(\Omega) = \|u_\Omega\|_1,$$

where  $\|\cdot\|_p$ ,  $1 \leq p \leq \infty$  denotes the usual  $L^p$  norm. It follows that

$$\Omega_1 \subset \Omega_2 \Rightarrow T(\Omega_1) \leq T(\Omega_2), \tag{1}$$

and that the torsion satisfies the scaling property

$$T(t\Omega) = t^{d+2}T(\Omega), \quad t > 0. \tag{2}$$

Moreover  $T$  is additive on unions of disjoint families of open sets:

$$T(\cup_{i \in I} \Omega_i) = \sum_{i \in I} T(\Omega_i).$$

It is straightforward to verify that if  $E(a)$ , with  $a = (a_1, a_2, \dots, a_d) \in \mathbb{R}_+^d$ , is the ellipsoid

$$E(a) = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d \frac{x_i^2}{a_i^2} < 1 \right\},$$

then

$$u_{E(a)}(x) = \frac{1}{2} \left( \sum_{i=1}^d \frac{1}{a_i^2} \right)^{-1} \left( 1 - \sum_{i=1}^d \frac{x_i^2}{a_i^2} \right),$$

and

$$T(E(a)) = \frac{\omega_d}{d+2} \left( \prod_{i=1}^d a_i \right) \left( \sum_{i=1}^d \frac{1}{a_i^2} \right)^{-1}, \tag{3}$$

where

$$\omega_d = \frac{\pi^{d/2}}{\Gamma((d+2)/2)}$$

is the Lebesgue measure of a ball  $B_1$  with radius 1 in  $\mathbb{R}^d$ . We put

$$\tau_d = T(B_1) = \frac{\omega_d}{d(d+2)}.$$

The de Saint-Venant inequality (see, for instance, [11, Chapter V]) asserts that

$$T(\Omega) \leq T(\Omega^*), \quad (4)$$

where  $\Omega^*$  is any ball with  $|\Omega| = |\Omega^*|$ . It follows by scaling that

$$\frac{T(\Omega)}{|\Omega|^{(d+2)/d}} \leq \frac{\tau_d}{\omega_d^{(d+2)/d}} = \frac{1}{d(d+2)\omega_d^{2/d}}. \quad (5)$$

Below we recall some basic facts about the Newtonian capacity  $\text{cap}(K)$  of a compact set  $K \subset \mathbb{R}^d$ ,  $d \geq 3$ . There are several equivalent definitions of  $\text{cap}(K)$  of which we choose

$$\text{cap}(K) = \inf \left\{ \int_{\mathbb{R}^d} |D\varphi|^2 dx : \varphi_{K_\varepsilon} \geq 1, \varphi \in C_0^1(\mathbb{R}^d), \varepsilon > 0 \right\},$$

where  $\varphi_{K_\varepsilon}$  is the restriction of  $\varphi$  to  $K_\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, K) < \varepsilon\}$ . It follows that

$$K_1 \subset K_2 \Rightarrow \text{cap}(K_1) \leq \text{cap}(K_2), \quad (6)$$

and that the capacity satisfies the scaling property

$$\text{cap}(tK) = t^{d-2} \text{cap}(K), \quad t > 0. \quad (7)$$

Moreover if  $\{K_i, i \in I\}$  is a countable family of compact sets such that  $\cup_{i \in I} K_i$  is compact, then

$$\text{cap}(\cup_{i \in I} K_i) \leq \sum_{i \in I} \text{cap}(K_i).$$

It was reported in [8, pp. 260] that the Newtonian capacity of an ellipsoid was computed in volume 8, [5, pp. 103–104]. The formula is given in terms of an elliptic integral, and reads

$$\text{cap}(\overline{E(a)}) = \frac{4\pi^{d/2}}{\Gamma(d/2)} \mathfrak{e}(a)^{-1}, \quad (8)$$

where

$$\mathfrak{e}(a) = \int_0^\infty \left( \prod_{i=1}^d (a_i^2 + t) \right)^{-1/2} dt. \quad (9)$$

We put

$$\kappa_d = \text{cap}(\overline{B_1}) = \frac{4\pi^{d/2}}{\Gamma((d-2)/2)},$$

so that

$$\text{cap}(\overline{E(a)}) = \frac{2\kappa_d}{d-2} \mathfrak{e}(a)^{-1}. \quad (10)$$

The isoperimetric inequality for Newtonian capacity (see [11]) asserts that for all compact sets  $K \subset \mathbb{R}^d$ ,  $d \geq 3$ ,

$$\text{cap}(K) \geq \text{cap}(K^*),$$

where  $K^*$  is a closed ball with  $|K| = |K^*|$ . It follows by scaling that

$$\frac{\text{cap}(K)}{|K|^{(d-2)/d}} \geq \frac{\kappa_d}{\omega_d^{(d-2)/d}}. \quad (11)$$

The shape functional we consider in the present paper is

$$G_q(\Omega) = \frac{\text{cap}(\overline{\Omega})T(\Omega)^q}{|\Omega|^{1+q+2(q-1)/d}}, \quad (12)$$

defined for a bounded open set  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 3$ . By (2) and (7), we obtain that  $G_q$  is scaling invariant. With the definitions above, we have

$$G_q(B_1) = \frac{\kappa_d \tau_d^q}{\omega_d^{1+q+2(q-1)/d}}.$$

Since the ball  $\Omega^*$  with measure  $|\Omega^*| = |\Omega|$  maximises the torsional rigidity  $T(\Omega)$  (de Saint-Venant), and its closure minimises the Newtonian capacity  $\text{cap}(\overline{\Omega})$ , competition enters in the minimisation or maximisation problems for the functional in (12).

All of our main results are for  $d \geq 3$ , and are as follows.

**THEOREM 1.** (i) *If  $q \in \mathbb{R}$ , then*

$$\sup\{G_q(\Omega) : \Omega \text{ open and bounded}\} = +\infty.$$

(ii) *If  $q \leq 0$ , then*

$$\min\{G_q(\Omega) : \Omega \text{ open and bounded}\} = G_q(B_1), \quad (13)$$

*with equality if and only if  $\Omega$  is (up to sets of capacity 0) a ball in  $\mathbb{R}^d$ .*

(iii) *If  $q > 0$ , then*

$$\inf\{G_q(\Omega) : \Omega \text{ open and bounded}\} = 0.$$

**THEOREM 2.** (i) *If  $q < 1$ , then*

$$\sup\{G_q(\Omega) : \Omega \text{ open, bounded and convex}\} = +\infty.$$

(ii) *If  $q \geq 1$ , then*

$$\begin{aligned} \sup\{G_q(\Omega) : \Omega \text{ open, bounded and convex}\} \\ \leq \frac{2^{(d+2)/2} d^{3q-2+d(q+1)}}{d-2} G_q(B_1). \end{aligned} \quad (14)$$

*If  $q > 1$ , then the variational problem in the left-hand side of (14) has a maximiser, say  $\Omega^+$ . For any such maximiser,*

$$\frac{\text{diam}(\Omega^+)}{r(\Omega^+)} \leq 2d \left( \frac{2^{(d+2)/2} d^{(1+q)d+3q-2}}{d-2} \right)^{d/(2(q-1))}. \quad (15)$$

(iii) *If  $q = 1$  and  $d = 3$ , then the variational problem in the left-hand side of (14) has a maximiser, say  $\Omega^+$ . For any such maximiser,*

$$\frac{\text{diam}(\Omega^+)}{r(\Omega^+)} \leq 2 \cdot 3^8 e^{3^7}. \quad (16)$$

**THEOREM 3.** (i) *If  $q > (d-2)/(2(d-1))$ , then*

$$\inf\{G_q(\Omega) : \Omega \text{ open, bounded and convex}\} = 0. \quad (17)$$

(ii) *If  $0 < q \leq (d-2)/(2(d-1))$ , then*

$$\inf\{G_q(\Omega) : \Omega \text{ open, bounded and convex}\} \geq \frac{1}{2d^{d+(d+2)q}} G_q(B_1). \quad (18)$$

*If  $0 < q < (d-2)/(2(d-1))$ , then the variational problem in the left-hand side of (18) has a convex minimiser, say  $\Omega^-$ . For any such minimiser,*

$$\frac{\text{diam}(\Omega^-)}{r(\Omega^-)} \leq 2d \left( 2d^{d+(d+2)q} \right)^{\frac{d(d-1)}{d-2-2q(d-1)}}. \quad (19)$$

We were unable to prove the existence or non-existence of a maximiser for the left-hand side of (14) for  $q = 1$  and  $d > 3$ . In these higher dimensional cases, there is a lack of compactness. For example, if

$$a_\varepsilon = \underbrace{(1, \dots, 1)}_{d-k}, \underbrace{(\varepsilon, \dots, \varepsilon)}_k,$$

and if  $k \geq 3$ , then  $\lim_{\varepsilon \rightarrow 0} G_1(E(a_\varepsilon))$  exists and is strictly positive. Similarly we were unable to prove the existence of a minimiser of the left-hand side of (18) at the critical point  $q = (d-2)/(2d-2)$ .

The proofs of Theorems 1–3 are deferred to Sections 2–4, respectively. A key ingredient in these proofs is John's ellipsoid theorem [9]. This theorem asserts that for any open, bounded convex set  $\Omega$  in  $\mathbb{R}^d$  there exists a translation and rotation of  $\Omega$ , again denoted by  $\Omega$ , and an open ellipsoid  $E(a)$  such that

$$E(a/d) \subset \Omega \subset E(a). \quad (20)$$

Moreover, among all ellipsoids in  $\Omega$ ,  $E(a/d)$  has maximal measure.

Finally in Section 5 we discuss the optimisation of a functional over all open bounded planar convex sets with fixed measure, and which involves the logarithmic capacity and torsional rigidity.

## 2. Proof of Theorem 1

*Proof.* To prove the assertion under (i), we let  $\Omega$  be the disjoint union of an open ball  $B'$  of measure  $1/2$  and an open ellipsoid  $E(b_\varepsilon)$ , with  $b_\varepsilon = (L_\varepsilon, \dots, L_\varepsilon, \varepsilon)$ , of measure  $1/2$ , where

$$L_\varepsilon = (2\omega_d \varepsilon)^{1/(1-d)}.$$

We have, by (8) and (9),

$$\begin{aligned} \text{cap}(\overline{E(b_\varepsilon)}) &= \frac{4\pi^{d/2}}{\Gamma(d/2)} \left( \int_0^\infty dt (L_\varepsilon^2 + t)^{(1-d)/2} (\varepsilon^2 + t)^{-1/2} \right)^{-1} \\ &\geq \frac{4\pi^{d/2}}{\Gamma(d/2)} \left( \int_0^\infty dt (L_\varepsilon^2 + t)^{(1-d)/2} t^{-1/2} \right)^{-1} \\ &= \frac{4\pi^{(d-1)/2} \Gamma((d-1)/2)}{\Gamma(d/2) \Gamma((d-2)/2)} L_\varepsilon^{d-2}, \end{aligned}$$

where we have used formulae [7, 8.380.3 and 8.384.1]. Hence

$$\begin{aligned} G_q(\Omega) &= \text{cap}(\overline{B' \cup E(b_\varepsilon)}) T(B' \cup E(b_\varepsilon))^q \\ &\geq \text{cap}(\overline{E(b_\varepsilon)}) T(B')^q \\ &\geq \frac{4\pi^{(d-1)/2} \Gamma((d-1)/2)}{\Gamma(d/2) \Gamma((d-2)/2)} T(B')^q (2\omega_d \varepsilon)^{(d-2)/(1-d)}, \end{aligned}$$

which tends to  $+\infty$  as  $\varepsilon \downarrow 0$ .

To prove the assertion under (ii), we recall (4), and infer that  $T^q(\Omega) \geq T^q(\Omega^*)$  for  $q \leq 0$ . This implies (13) by (5) and (11).

To prove (iii), we let  $Q \subset \mathbb{R}^d$  be a cube with  $|Q| = 1$ . Let  $N \in \mathbb{N}$  be arbitrary. The cube  $Q$  contains  $N^d$  open disjoint cubes each of measure  $N^{-d}$ . Each open cube contains an open

ball with radius  $1/(2N)$ . Let  $Q_N$  be the union of these  $N^d$  open balls. Since  $Q_N \subset Q$  we have  $\text{cap}(\overline{Q_N}) \leq \text{cap}(\overline{Q})$ . On the other hand, additivity and scaling properties of the torsion give

$$T(Q_N) = N^d(2N)^{-(d+2)}T(B_1) = 2^{-d-2}N^{-2}T(B_1).$$

Furthermore,

$$|Q_N| = \frac{\omega_d}{2^d}.$$

Hence

$$\begin{aligned} \inf \{G_q(\Omega) : \Omega \text{ open and bounded}\} &\leq \frac{\text{cap}(\overline{Q})T^q(Q_N)}{|Q_N|^{1+q+2(q-1)/d}} \\ &= \frac{2^{d-2}}{\omega_d^{1+q+2(q-1)/d}} \text{cap}(\overline{Q})T^q(B_1)N^{-2q}. \end{aligned}$$

This implies (17) since  $q > 0$ , and  $N \in \mathbb{N}$  was arbitrary.  $\square$

### 3. Proof of Theorem 2

*Proof.* To prove (i) we consider the open ellipsoid  $E(a_\varepsilon)$  with  $a_\varepsilon = (1, \varepsilon, \dots, \varepsilon)$ . We have

$$\begin{aligned} |E(a_\varepsilon)| &= \omega_d \varepsilon^{d-1}, \\ T(E(a_\varepsilon)) &= \frac{\omega_d}{d+2} \frac{\varepsilon^{d+1}}{d-1+\varepsilon^2}, \\ \text{cap}(\overline{E(a_\varepsilon)}) &= \begin{cases} 4\pi\varepsilon(\log(\varepsilon^{-1}))^{-1}(1+o(1)), & d=3, \varepsilon \downarrow 0, \\ \frac{2\pi^{d/2}(d-3)}{\Gamma(d/2)}\varepsilon^{d-3}(1+o(1)), & d>3, \varepsilon \downarrow 0, \end{cases} \end{aligned}$$

where we have used the formulae on [8, p. 260]. Hence

$$G_q(E(a_\varepsilon)) = \begin{cases} C_3\varepsilon^{2(q-1)/3}(\log \varepsilon^{-1})^{-1}(1+o(1)), & d=3, \varepsilon \downarrow 0, \\ C_d\varepsilon^{2(q-1)/d}(1+o(1)), & d>3, \varepsilon \downarrow 0, \end{cases}$$

where  $C_d$  is a positive constant depending only on  $d$ . Since  $q < 1$ , we obtain the desired result by letting  $\varepsilon \downarrow 0$ .

To prove (ii), we first observe that the formulae for  $|E(a)|$ ,  $\text{cap}(\overline{E(a)})$ , and  $T(E(a))$  are symmetric in  $a_1, \dots, a_d$ . Without loss of generality we may therefore assume here, and throughout this paper, that  $a_1 \geq a_2 \geq \dots \geq a_d$ . By inclusion, and (20) we have

$$d^{-d}\omega_d \prod_{i=1}^d a_i = |E(a/d)| \leq |\Omega| \leq |E(a)| = \omega_d \prod_{i=1}^d a_i, \quad (21)$$

and

$$T(\Omega) \geq T(E(a/d)) = \frac{\omega_d}{d^{d+2}(d+2)} \left( \prod_{i=1}^d a_i \right) \left( \sum_{i=1}^d \frac{1}{a_i^2} \right)^{-1} \geq \frac{\tau_d}{d^{d+2}} \left( \prod_{i=1}^d a_i \right) a_d^2. \quad (22)$$

We have by (8),

$$\epsilon(a) \geq \int_0^{a_d^2} dt \left( \prod_{i=1}^d (a_i^2 + t) \right)^{-1/2}$$

$$\begin{aligned}
&\geq a_d^2 \left( \prod_{i=1}^d (a_i^2 + a_d^2) \right)^{-1/2} \\
&\geq 2^{-d/2} a_d^2 \left( \prod_{i=1}^d a_i^2 \right)^{-1/2} \\
&= 2^{-d/2} \left( \prod_{i=1}^d a_i \right)^{-1} a_d^2.
\end{aligned} \tag{23}$$

By (9), (10) and (23), taking into account that  $\Gamma(z+1) = z\Gamma(z)$ ,  $z > 0$ ,

$$\text{cap}(\bar{\Omega}) \leq \text{cap}(\overline{E(a)}) \leq \frac{2^{(d+2)/2} \kappa_d}{d-2} \left( \prod_{i=1}^d a_i \right) a_d^{-2}. \tag{24}$$

By (1) and (3),

$$T(\Omega) \leq T(E(a)) \leq d\tau_d \left( \prod_{i=1}^d a_i \right) a_d^2. \tag{25}$$

By (21), (24), (25),  $a_1 \geq a_2 \geq \dots \geq a_d$ , and  $q > 1$ , we obtain,

$$\begin{aligned}
G_q(\Omega) &\leq \frac{\text{cap}(\overline{E(a)}) T(E(a))^q}{|E(a/d)|^{1+q+2(q-1)/d}} \\
&\leq \frac{2^{(d+2)/2} \kappa_d}{d-2} (d\tau_d)^q (d^{-d} \omega_d)^{-(1+q+2(q-1)/d)} \left( \prod_{i=1}^d a_i \right)^{2(1-q)/d} a_d^{2q-2} \\
&= \frac{2^{(d+2)/2} d^{3q-2+d(q+1)}}{d-2} G_q(B_1) \left( \prod_{i=1}^d a_i \right)^{2(1-q)/d} a_d^{2q-2} \\
&\leq \frac{2^{(d+2)/2} d^{3q-2+d(q+1)}}{d-2} G_q(B_1).
\end{aligned} \tag{26}$$

This proves (14).

To prove the existence of a maximiser, we observe that if the left-hand side of (14) equals  $G_q(B_1)$ , then  $B_1$  is a maximiser which satisfies (15). If the left-hand side of (14) is strictly greater than  $G_q(B_1)$ , we let  $\Omega$  be bounded, open, and convex, and such that

$$G_q(\Omega) > G_q(B_1). \tag{27}$$

By the third inequality in (26), (27),  $q > 1$ , and  $a_1 \geq a_2 \geq \dots \geq a_d$ , we find that

$$a_1 \leq \beta_d^{d/(2(q-1))} a_d, \tag{28}$$

where  $\beta_d$  is the coefficient of  $G_q(B_1)$  in the right-hand side of (26). Since

$$\text{diam}(\Omega) \leq \text{diam}(E(a)) \leq 2a_1, \tag{29}$$

and

$$r(\Omega) \geq r(E(a/d)) = \frac{a_d}{d}, \tag{30}$$



we obtain by (28)–(30),

$$\frac{\text{diam}(\Omega)}{r(\Omega)} \leq 2d \left( \frac{2^{(d+2)/2} d^{3q-2+d(q+1)}}{d-2} \right)^{d/(2(q-1))}. \quad (31)$$

Let  $(\Omega_n)$  be a maximising sequence for the left-hand side of (14). Since this supremum is scaling invariant, we fix  $r(\Omega_n) = 1$ . By (31),  $\text{diam}(\Omega_n) \leq L$  for some  $L < \infty$ , and for all  $n$ . By taking translations of  $(\Omega_n)$ , these translates are contained in a closed ball  $B_L$  of radius  $L$ . Since the Hausdorff metric is compact on the space of convex, compact sets in  $B_L$ , there exists a subsequence of  $(\overline{\Omega_n})$ , again denoted by  $(\overline{\Omega_n})$  which converges in the Hausdorff (and in the complementary Hausdorff) metric to an element say  $\tilde{\Omega}^+$ . Set  $\Omega^+ = \text{int}(\tilde{\Omega}^+)$ . Note that  $\Omega^+$  is an open, bounded, convex set which is non-empty since  $\Omega^+$  has inradius 1. Furthermore measure, torsion, capacity, and diameter are all continuous with respect to this metric. Hence

$$G_q(\Omega^+) = \lim_{n \rightarrow \infty} G_q(\Omega_n),$$

and  $\Omega^+$  is a maximiser which satisfies (15).

To prove (iii), we let  $q = 1$  and  $d = 3$ . Let  $\Omega$  be an element of a maximising sequence. We may assume that

$$G_1(B_1) \leq G_1(\Omega) \leq \frac{\text{cap}(\overline{E(a)}) T(E(a))}{|E(a/3)|^2}. \quad (32)$$

We obtain an upper bound on  $\text{cap}(\overline{E(a)})$  by obtaining a lower bound on  $\epsilon(a)$ . By (9), we have

$$\begin{aligned} \epsilon(a_1, a_2, a_3) &\geq \int_0^\infty (a_1^2 + t)^{-1/2} (a_2^2 + t)^{-1} dt \\ &= \frac{2}{(a_1^2 - a_2^2)^{1/2}} \log \left( \frac{a_1}{a_2} + \left( \frac{a_1^2}{a_2^2} - 1 \right)^{1/2} \right) \\ &\geq \frac{2}{a_1} \log(a_1 a_2^{-1}). \end{aligned} \quad (33)$$

By (8) for  $d = 3$ , and (33),

$$\text{cap}(\overline{E(a)}) \leq \kappa_3 a_1 (\log(a_1 a_2^{-1}))^{-1}.$$

Since  $\Omega \subset B_{a_1}$ , we also have

$$\text{cap}(\overline{\Omega}) \leq \text{cap}(\overline{B_{a_1}}) = \kappa_3 a_1.$$

Hence

$$\text{cap}(\overline{\Omega}) \leq \kappa_3 a_1 \min \left\{ 1, (\log(a_1 a_2^{-1}))^{-1} \right\}.$$

In addition,

$$T(E(a)) \leq 3\tau_3 a_1 a_2 a_3^3, \quad |E(a/3)| = 3^{-3} \omega_3 a_1 a_2 a_3.$$

Summarising, from (32) we obtain

$$G_1(\Omega) \leq 3^7 G_1(B_1) \cdot \frac{a_3}{a_2} \cdot \min \{ 1, (\log(a_1 a_2^{-1}))^{-1} \}. \quad (34)$$

If the supremum in the left-hand side of (14) equals  $G_1(B_1)$ , then  $B_1$  is a maximiser which satisfies (16). If not then we may assume that  $G_1(\Omega) > G_1(B_1)$ . This, together with (34),

yields  $a_3 \geq 3^{-7}a_2$ ,  $a_1 \leq a_2e^{3^7}$ . These inequalities imply that  $a_1/a_3 \leq 3^7e^{3^7}$ . Hence (29) and (30) yield

$$\frac{\text{diam}(\Omega)}{r(\Omega)} \leq 2 \cdot 3^8 e^{3^7}.$$

The remaining part of the proof follows similar lines as those in the proof of (ii).  $\square$

#### 4. Proof of Theorem 3

*Proof.* To prove (i), we consider as  $\Omega$  the ellipsoid  $E(a)$  with

$$a = (a_1, \dots, a_1, a_d), \quad a_1 \geq a_d, \quad a_1^{d-1}a_d = 1,$$

where  $a_d \in (0, 1)$  is arbitrary. Since  $E(a) \subset B_{a_1}$ , we have by (6), (7) and (10),

$$\text{cap}(\overline{E(a)}) \leq \kappa_d a_1^{d-2}. \quad (35)$$

By (3) and (35),

$$G_q(E(a)) \leq \frac{\kappa_d a_1^{d-2}}{\omega_d^{1+q+2(q-1)/d}} \left( \frac{\omega_d}{d+2} \frac{a_1^2 a_d^2}{(d-1)a_d^2 + a_1^2} \right)^q \leq \frac{\kappa_d a_1^{d-2} a_d^{2q}}{\omega_d^{1+2(q-1)/d} (d+2)^q}.$$

Since  $a_1 = a_d^{-1/(d-1)}$ , we have

$$G_q(E(a)) \leq \frac{\kappa_d}{\omega_d^{1+2(q-1)/d} (d+2)^q} a_d^{-(d-2)/(d-1)+2q},$$

and since  $a_d \in (0, 1)$  was arbitrary, we obtain (17).

To prove (ii), we let  $a_1 \geq a_2 \geq \dots \geq a_d$ . We have

$$\text{cap}(\overline{\Omega}) \geq \text{cap}(\overline{E(a/d)}) = d^{2-d} \text{cap}(\overline{E(a)}) = \frac{2\kappa_d}{d^{d-2}(d-2)} (\epsilon(a))^{-1}. \quad (36)$$

In order to obtain an upper bound on  $\epsilon(a)$ , we have by using the inequality  $(x^2 + t)^{1/2} \geq 2^{-1/2}(x + t^{1/2})$ , and the change of variables  $t = \theta^2$ ,

$$\begin{aligned} \epsilon(a) &\leq \left( \prod_{i \leq d-3} a_i^{-1} \right) \int_0^\infty dt \left( (a_{d-2}^2 + t)(a_{d-1}^2 + t)(a_d^2 + t) \right)^{-1/2} \\ &\leq \left( \prod_{i \leq d-3} a_i^{-1} \right) \int_0^\infty dt \left( (a_{d-2}^2 + t)(a_{d-1}^2 + t)t \right)^{-1/2} \\ &\leq 2 \left( \prod_{i \leq d-3} a_i^{-1} \right) \int_0^\infty dt \left( (a_{d-2} + t^{1/2})(a_{d-1} + t^{1/2})t^{1/2} \right)^{-1} \\ &= 4 \left( \prod_{i \leq d-3} a_i^{-1} \right) \int_0^\infty d\theta \left( (a_{d-2} + \theta)(a_{d-1} + \theta) \right)^{-1} \\ &= 4 \left( \prod_{i \leq d-2} a_i^{-1} \right) \left( 1 - \frac{a_{d-1}}{a_{d-2}} \right)^{-1} \log(a_{d-2}/a_{d-1}), \end{aligned} \quad (37)$$

where the product over the empty set in the right-hand side of (37) is defined to be equal to 1, and where the case  $a_{d-2} = a_{d-1}$  follows by taking the appropriate limit in the right-hand side of (37). It is elementary to verify that

$$(1-x)^{-1} \log(x^{-1}) \leq \log(e/x), \quad 0 < x < 1,$$

and

$$\lim_{x \uparrow 1} (1-x)^{-1} \log(x^{-1}) = 1.$$

This gives by (37),

$$\epsilon(a) \leq 4 \left( \prod_{i \leq d-2} a_i^{-1} \right) \log(ea_{d-2}/a_{d-1}). \quad (38)$$

Hence by (36) and (38),

$$\text{cap}(\bar{\Omega}) \geq \frac{\kappa_d}{2d^{d-2}(d-2)} \left( \prod_{i \leq d-2} a_i \right) (\log(ea_{d-2}/a_{d-1}))^{-1}. \quad (39)$$

By (21), (22), and (39),

$$G_q(\Omega) \geq \frac{1}{2d^{d-2+(d+2)q}(d-2)} G_q(B_1) \frac{a_d^{2q-1}}{a_{d-1}} \left( \prod_{i=1}^d a_i \right)^{2(1-q)/d} (\log(ea_{d-2}/a_{d-1}))^{-1}. \quad (40)$$

The  $a$ -dependence in the right-hand side of (40) is scaling invariant. It is convenient to choose  $\prod_{i=1}^d a_i = 1$ . We then have

$$\frac{a_{d-2}}{a_{d-1}} = \left( \prod_{i \leq d-3} a_i^{-1} \right) a_{d-1}^{-2} a_d^{-1} \leq a_{d-1}^{-(d-1)} a_d^{-1}.$$

This gives with (40),

$$G_q(\Omega) \geq \frac{1}{2d^{d-2+(d+2)q}(d-2)} G_q(B_1) \frac{a_d^{2q-1}}{a_{d-1}} (\log(e/(a_{d-1}^{d-1} a_d)))^{-1}. \quad (41)$$

Since

$$x^{1/(d-1)} \log(e/x) \leq d-1, \quad 0 < x < 1,$$

we have, with  $x = a_{d-1}^{d-1} a_d$ ,

$$(\log(e/(a_{d-1}^{d-1} a_d)))^{-1} \geq a_{d-1} a_d^{1/(d-1)} (d-1)^{-1}.$$

This, together with (41), gives

$$\begin{aligned} G_q(\Omega) &\geq \frac{1}{2d^{d-2+(d+2)q}(d-2)(d-1)} G_q(B_1) a_d^{2q - \frac{d-2}{d-1}} \\ &\geq \frac{1}{2d^{d+(d+2)q}} G_q(B_1) a_d^{2q - \frac{d-2}{d-1}}. \end{aligned} \quad (42)$$

This proves (18) since  $a_d \in (0, 1]$ , and  $q \leq (d-2)/(2(d-1))$ .

To prove the existence of a minimiser, we observe that if the left-hand side of (18) equals  $G_q(B_1)$ , then  $B_1$  is a minimiser which satisfies (19). If the left-hand side of (18) is strictly less than  $G_q(B_1)$ , we let  $\Omega$  be bounded and convex such that

$$G_q(\Omega) < G_q(B_1). \quad (43)$$

By (42) and (43), we infer

$$a_d \geq \left( \frac{1}{2d^{d+(d+2)q}} \right)^{(d-1)/(d-2-2q(d-1))}. \quad (44)$$

Since  $\prod_{i=1}^d a_i = 1$ , and  $a_1 \geq a_2 \geq \dots \geq a_d$ , we have  $a_1 \leq a_d^{1-d}$ . By (29), (30) and (44), we obtain

$$\frac{\text{diam}(\Omega)}{r(\Omega)} \leq 2da_d^{-d} \leq 2d \left( 2d^{d+(d+2)q} \right)^{\frac{d(d-1)}{d-2-2q(d-1)}}. \quad (45)$$

The proof of the existence of a minimiser is similar to the proof of the existence of a maximiser in Theorem 1(ii), and has been omitted. If  $\Omega^-$  is a minimiser, then, by continuity of diameter and inradius,  $\Omega^-$  satisfies (45). This proves (19).  $\square$

### 5. The logarithmic capacity

We briefly recall some basic properties of the logarithmic capacity of a compact set  $K$  in  $\mathbb{R}^2$ . Let  $\mu$  be a probability measure supported on  $K$ , and let

$$I(\mu) = \iint_{K \times K} \log \left( \frac{1}{|x-y|} \right) \mu(dx) \mu(dy).$$

Furthermore let

$$V(K) = \inf \{ I(\mu) : \mu \text{ a probability measure on } K \}.$$

The logarithmic capacity of  $K$  is denoted by  $\text{cap}(K)$ , and is the non-negative real number

$$\text{cap}(K) = e^{-V(K)}.$$

It shares some of the properties of the Newtonian capacity. In particular, if  $K_1$  and  $K_2$  are compact sets in  $\mathbb{R}^2$  with  $K_1 \subset K_2$ , then  $\text{cap}(K_1) \leq \text{cap}(K_2)$ . Moreover,  $\text{cap}(K)$  is invariant under translations and rotations of  $K$ , and

$$\text{cap}(K) \geq \text{cap}(K^*), \quad (46)$$

where  $K^*$  is the disc with  $|K| = |K^*|$ . See [1] for some refinements. Finally for a homothety,

$$\text{cap}(tK) = t \text{cap}(K), \quad t > 0. \quad (47)$$

The classic treatise [10] gives various planar domains for which the logarithmic capacity can be computed analytically. In particular, for the ellipse with semi axes  $a_1$  and  $a_2$ ,

$$\text{cap}(\overline{E(a_1, a_2)}) = \frac{1}{2}(a_1 + a_2). \quad (48)$$

For an open, bounded, convex planar set  $\Omega$ , we define the functional

$$H_q(\Omega) = \frac{\text{cap}(\overline{\Omega}) T^q(\Omega)}{|\Omega|^{(1+4q)/2}}.$$

In particular, we have

$$H_q(B_1) = \frac{\tau_2^q}{\omega_2^{(1+4q)/2}} = \frac{1}{8^q \pi^{q+1/2}}.$$

We immediately see that by (2), and (47) that  $H_q(t\Omega) = H_q(\Omega)$ ,  $t > 0$ . Our main result is the following.

THEOREM 4. (i) If  $q \geq 1/2$ , then

$$\sup\{H_q(\Omega) : \Omega \text{ open, bounded, planar, and convex}\} \leq 2^{1+5q}H_q(B_1). \quad (49)$$

(ii) If  $q > 1/2$ , then the left-hand side of (49) has an open, bounded, planar, and convex maximiser. For any such maximiser, say  $\Omega^+$ ,

$$\frac{\text{diam}(\Omega^+)}{r(\Omega^+)} \leq \frac{2^{14q}}{2q-1}. \quad (50)$$

(iii) If  $q < 1/2$ , then

$$\sup\{H_q(\Omega) : \Omega \text{ open, bounded, planar, and convex}\} = +\infty. \quad (51)$$

(iv) If  $q \leq 1/2$ , then

$$\inf\{H_q(\Omega) : \Omega \text{ open, bounded, planar, and convex}\} \geq 2^{-2(1+2q)}H_q(B_1). \quad (52)$$

(v) If  $q < 1/2$ , then the left-hand side of (52) has an open, bounded, planar, and convex minimiser. For any such minimiser, say  $\Omega^-$ ,

$$\frac{\text{diam}(\Omega^-)}{r(\Omega^-)} \leq 2^{2(3+2q)/(1-2q)}. \quad (53)$$

(vi) If  $q > 1/2$ , then

$$\inf\{H_q(\Omega) : \Omega \text{ open, bounded, planar, and convex}\} = 0.$$

*Proof.* (i) If  $E(a)$  is the John's ellipsoid for  $\Omega$ , then  $E(a/2) \subset \Omega \subset E(a)$  with  $a_1 \geq a_2$ . Furthermore,

$$\text{cap}(\overline{E(a)}) \leq a_1, \quad T(E(a)) \leq 2\tau_2 a_1 a_2^3, \quad |\Omega| \geq |E(a/2)| = \omega_2 a_1 a_2 / 4,$$

so that

$$H_q(\Omega) \leq 2^{1+5q} \frac{\tau_2^q}{\omega_2^{(1+4q)/2}} \left(\frac{a_2}{a_1}\right)^{q-1/2}. \quad (54)$$

This implies (49) since  $q \geq 1/2$ .

(ii) To prove (50), we have that either the supremum in the left-hand side of (49) is attained for a ball, in which case the maximiser exists and satisfies (50), or we may assume that  $H_q(\Omega) > H_q(B_1)$ . This implies, by (29) and (30),

$$\frac{\text{diam}(\Omega)}{r(\Omega)} \leq \frac{2^{14q}}{2q-1}.$$

The remaining part of the proof is similar to the corresponding parts in the proof of Theorem 2.

(iii) By (21), (22), and (48),

$$H_q(\Omega) \geq 2^{-2(1+2q)} \frac{\tau_2^q}{\omega_2^{(1+4q)/2}} \left(\frac{a_2}{a_1}\right)^{q-1/2}. \quad (55)$$

This implies (51) by letting  $a_2/a_1 \rightarrow 0$  in (55).

(iv) This follows from (55) and  $a_1 \geq a_2$ .

(v) Either the infimum in the left-hand side of (52) is attained for a ball, in which case the minimiser exists and satisfies (53), or we may assume that  $H_q(\Omega) < H_q(B_1)$ . By (55), (29), and (30)

$$\frac{\text{diam}(\Omega)}{r(\Omega)} \leq 2^{\frac{2(3+2q)}{1-2q}}.$$

The remaining part of the proof is similar to the corresponding parts in the proof of Theorem 2.

(vi) This follows by letting  $a_2/a_1 \rightarrow 0$  in (54).  $\square$

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### References

1. V. ANDRIEVSKII, W. HANSEN and N. NADIRASHVILI, ‘Isoperimetric inequalities for capacities in the plane’, *Math. Ann.* 292 (1992) 191–195.
2. M. VAN DEN BERG, ‘Estimates for the torsion function and Sobolev constants’, *Potential Anal.* 36 (2012) 607–616.
3. M. VAN DEN BERG, G. BUTTAZZO and A. PRATELLI, ‘On relations between principal eigenvalue and torsional rigidity’, *Commun. Contemp. Math.*, to appear.
4. M. VAN DEN BERG and T. CARROLL, ‘Hardy inequality and  $L^p$  estimates for the torsion function’, *Bull. Lond. Math. Soc.* 41 (2009) 980–986.
5. G. CHRYSTAL, ‘Electricity’, *Encyclopedia Britannica*, 9th edn (1879).
6. T. GIORGI and R. G. SMITS, ‘Principal eigenvalue estimates via the supremum of torsion’, *Indiana Univ. Math. J.* 59 (2010) 987–1011.
7. I. S. GRADSHTEYN and I. M. RYZHIK, *Table of integrals, series, and products* (Elsevier/Academic Press, Amsterdam, 2015).
8. K. ITÔ and H. P. MCKEAN, *Diffusion processes and their sample paths*, Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften Band 125 (Springer, Berlin–New York, 1974).
9. F. JOHN, *Extremum problems with inequalities as subsidiary conditions*. Studies and Essays Presented to R. Courant on his 60<sup>th</sup> Birthday, January 8, 1948 (Interscience Publishers, Inc., New York, NY, 1948).
10. N. S. LANDKOF, *Foundations of modern potential theory*, Die Grundlehren der mathematischen Wissenschaften 180 (Springer, New York–Heidelberg, 1972).
11. G. PÓLYA and G. SZEGÖ, *Isoperimetric inequalities in mathematical physics*, Annals of Mathematics Studies 27 (Princeton University Press, Princeton, NJ, 1951).
12. H. VOGT, ‘ $L_\infty$ -estimates for the torsion function and  $L_\infty$ -growth of semigroups satisfying Gaussian bounds’, *Potential Anal.* 51 (2019) 37–47.

M. van den Berg  
 School of Mathematics  
 University of Bristol  
 Fry Building, Woodland Road  
 Bristol BS8 1UG  
 United Kingdom

[mamvdb@bristol.ac.uk](mailto:mamvdb@bristol.ac.uk)

G. Buttazzo  
 Dipartimento di Matematica  
 Università di Pisa  
 Largo B. Pontecorvo 5  
 Pisa 56127  
 Italy

[giuseppe.buttazzo@dm.unipi.it](mailto:giuseppe.buttazzo@dm.unipi.it)