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ROBIN'S INEQUALITY FOR 20-FREE INTEGERS

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Abstract

In 1984, Robin showed that the Riemann Hypothesis for ζ is equivalent to demonstrating $\sigma(n) < e^\gamma n \log\log n$ for all $n > 5040$. Robin's inequality has since been proven for various infinite families of power-free integers: 5-free integers, 7-free integers, and 11-free integers. We extend these results to cover 20-free integers.

In 1984, Robin gave an equivalent statement of the Riemann Hypothesis for ζ involving the divisors of integers.

Theorem 1 (Robin [11]). *The Riemann Hypothesis is true if and only if for all $n > 5040$,*

$$\sigma(n) < e^\gamma n \log\log n, \tag{RI}$$

where $\sigma(n)$ is the sum of divisors function and γ is the Euler–Mascheroni constant.

Since then, (RI) has become known as Robin's inequality. There are twenty-six known counterexamples to (RI), of which 5040 is the largest [5].

Robin's inequality has been proven for various infinite families of integers, in particular the t-free integers. Recall that n is called *t-free* if n is not divisible by the tth power of any prime number, and *t-full* otherwise. In 2007, Choie, Lichiardopol, Moree, and Solé [4] showed that (RI) holds for all 5-free integers greater than 5040. Then, in 2012, Planat and Solé [12] improved this result to (RI) for 7-free integers greater than 5040, which was followed by Broughan and Trudgian [3] with (RI) for 11-free integers greater than 5040 in 2015. By updating Broughan and Trudgian's work, we prove our main theorem.

Theorem 2. *Robin's inequality holds for 20-free integers greater than 5040.*

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Since there are no 20-full integers less than 5041, we may give a cleaner statement for Robin's theorem.

Corollary 1. *The Riemann Hypothesis is true if and only if* (RI) *holds for all* 20-*full integers.*

1. A Bound for t-free Integers

Solé and Planat [12] introduced the generalised Dedekind Ψ function

$$\Psi_t(n) := n\prod_{p|n}(1+p^{-1}+\cdots+p^{-(t-1)}) = n\prod_{p|n}\frac{1-p^{-t}}{1-p^{-1}}.$$

Since

$$\sigma(n) = n\prod_{p^a||n}(1+p^{-1}+\cdots+p^{-a}),$$

we see that $\sigma(n)\le\Psi_t(n)$, provided that n is t-free. Thus, we study the function

$$R_t(n) := \frac{\Psi_t(n)}{n\log\log n}.$$

By Proposition 2 of [12], it is sufficient to consider R_t only at the primorial numbers $p_n\# = \prod_{k=1}^n p_k$ where p_k is the kth prime. Compare this to the role of colossally abundant numbers in (RI) by Robin [11].

Using equation (2) of Broughan and Trudgian [3], we have for $n\ge 2$

$$R_t(p_n\#) = \frac{p_n\#\prod_{p\le p_n}\frac{1-p^{-t}}{1-p^{-1}}}{p_n\#\log\log p_n\#} = \frac{\prod_{p>p_n}(1-p^{-t})^{-1}}{\zeta(t)\log\vartheta(p_n)}\prod_{p\le p_n}(1-p^{-1})^{-1}$$

where $\vartheta(x)$ is the Chebyshev function $\sum_{p\le x}\log p$.

In Sections 2 and 3, we construct two non-increasing functions, $g_B(w;t)$ and $g_\infty(w;t)$ such that for some constants x_0, B we have for $x_0\le p_n\le B$

$$g_B(p_n;t)\ge R_t(p_n\#)\exp(-\gamma)$$

and for $p_n > B$

$$g_\infty(p_n;t)\ge R_t(p_n\#)\exp(-\gamma).$$

For a given $t\ge 2$, if we can show that all t-free numbers $5\,040 < n \le p_k\#$ satisfy (RI), that $g_B(p_k;t) < 1$ and that $g_\infty(B;t) < 1$, then we are done.

2. Deriving $g_B(p_n; t)$

We start with some lemmas.

Lemma 1. *Let ρ be a non-trivial zero of the Riemann zeta function with positive imaginary part not exceeding* $3 \cdot 10^{12}$. *Then* $\Re\rho = 1/2$.

Proof. See Theorem 1 of [7]. □

Lemma 2. *Let* $B = 2.169 \cdot 10^{25}$. *Then we have*

$$|\vartheta(x) - x| \leq \frac{1}{8\pi}\sqrt{x}\log^2 x \quad \textit{for } 599 \leq x \leq B.$$

Proof. Given that one knows the Riemann Hypothesis to height T, [1] tells us that we may use Schoenfeld's bounds from [10] but restricted to B such that

$$4.92\sqrt{\frac{B}{\log B}} \leq T.$$

Using $T = 3 \cdot 10^{12}$ from Lemma 1 we find $B = 2.169 \cdot 10^{25}$ is admissible. □

Lemma 3. *Let* $\log x \geq 55$. *Then*

$$|\vartheta(x) - x| \leq 1.388 \cdot 10^{-10}x + 1.4262\sqrt{x}$$

or

$$|\vartheta(x) - x| \leq 1.405 \cdot 10^{-10}x.$$

Proof. From Table 1 of [6] we have for $x > \exp(55)$

$$|\psi(x) - x| \leq 1.388 \cdot 10^{-10}x$$

so that by Theorem 13 of [9] we get, again for $x > \exp(55)$, that

$$|\vartheta(x) - x| \leq 1.388 \cdot 10^{-10}x + 1.4262\sqrt{x}.$$

The second bound follows trivially. □

Lemma 4. *Take B as above and define*

$$C_1 = \int_B^\infty \frac{(\vartheta(t) - t)(1 + \log t)}{t^2 \log^2 t}\, dt.$$

Then $C_1 \leq 2.645 \cdot 10^{-9}$.

Proof. We split the integral at $X_0 = \exp(2000)$, apply Lemma 3 and consider

$$1.405 \cdot 10^{-10} \int_B^{X_0} \frac{1+\log t}{t \log^2 t} \mathrm{d}t \leq 1.430 \cdot 10^{-10} \int_B^{X_0} \frac{\mathrm{d}t}{t \log t} \leq 5.055 \cdot 10^{-10}.$$

For the tail of the integral, we use

$$|\vartheta(x) - x| \leq 30.3 x \log^{1.52} x \exp(-0.8\sqrt{\log x})$$

from Corollary 1 of [8], valid for $x \geq X_0$. We can then majorise the tail with

$$30.3 \int_{X_0}^{\infty} \frac{\log t \exp(-0.8\sqrt{\log t})}{t} \mathrm{d}t$$

which is less than $2.139 \cdot 10^{-9}$.

□

Lemma 5. *Take B, C_1 as above and let $599 \leq x \leq B$. For $t > 1$, define*

$$w(t) = \frac{(\log t + 3)\sqrt{B} - (\log B + 3)\sqrt{t}}{4\pi\sqrt{tB}}.$$

Then

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \geq \frac{\exp(-\gamma)}{\log x} \exp\left(\frac{1.02}{(x-1)\log x} + \frac{\log x}{8\pi\sqrt{x}} + C_1 + w(x)\right).$$

Proof. Let M be the Meissel-Mertens constant

$$M = \gamma + \sum_p (\log(1 - 1/p) + 1/p).$$

Then by 4.20 of [9] we have

$$\left| \sum_{p \leq x} \frac{1}{p} - \log\log x - M \right| \leq \frac{|\vartheta(x) - x|}{x \log x} + \int_x^{\infty} \frac{|\vartheta(t) - t|(1 + \log t)}{t^2 \log^2 t} \mathrm{d}t.$$

Since $599 \leq x \leq B$ we can use Lemma 2 to bound the first term with

$$\frac{\log x}{8\pi\sqrt{x}}.$$

We can split the integral at B and over the range $[B, \infty)$ use the bound from Lemma 4. This leaves the range $[x, B]$ where we can use Lemma 2 and a straightforward integration yields a contribution of

$$\frac{(\log x + 3)\sqrt{B} - (\log B + 3)\sqrt{x}}{4\pi\sqrt{xB}} = w(x).$$

We then simply follow the method used to prove Theorem 5.9 of [6] with our bounds in place of

$$\frac{\eta_k}{k\log^k x}+\frac{(k+2)\eta_k}{(k+1)\log^{k+1} x}.$$

□

We also need Lemma 2 of [12].

Lemma 6 (Solé and Planat [12]). *For $n \geq 2$,*

$$\prod_{p>p_n}\frac{1}{1-p^{-t}}\leq \exp(2/p_n).$$

Putting all this together, we have the following.

Lemma 7. *Let $w(t)$ be as per Lemma 5. Now define*

$$g_B(p_n;t)=\frac{\exp\left(\frac{2}{p_n}+\frac{1.02}{(p_n-1)\log p_n}+\frac{\log p_n}{8\pi\sqrt{p_n}}+C_1+w(p_n)\right)\log p_n}{\zeta(t)\log\left(p_n-\frac{\sqrt{p_n}\log^2 p_n}{8\pi}\right)}.$$

Then for $t\geq 2$ and $599\leq p_n\leq B=2.169\cdot 10^{25}$ we have $g_B(p_n;t)$ non-increasing in n and $R_t(p_n\#)\leq \exp(\gamma)g_B(p_n;t)$.

3. Deriving $g_\infty(p_n;t)$

We will need a further bound.

Theorem 3. *For $x\geq 767\,135\,587$,*

$$\prod_{p\leq x}\frac{p}{p-1}\leq e^{\gamma}\log x\exp\left(\frac{1.02}{(x-1)\log x}+\frac{1}{6\log^3 x}+\frac{5}{8\log^4 x}\right).$$

Proof. This is the last display on page 245 of [6] with $k=3$ so that $\eta_k=0.5$. □

We can now deduce

Theorem 4. *Define*

$$g_\infty(p_n;t)=\frac{\exp\left(\frac{2}{p_n}+\frac{1.02}{(p_n-1)\log p_n}+\frac{1}{6\log^3 p_n}+\frac{5}{8\log^4 p_n}\right)\log p_n}{\zeta(t)\log\left(p_n-1.338\cdot 10^{-10}p_n-1.4262\sqrt{p_n}\right)}.$$

Then for $t\geq 2$ and $\log p_n\geq 55$ we have

$$R_t(p_n\#)\leq e^{\gamma}g_\infty(p_n;t)$$

and $g_\infty(p_n;t)$ is non-increasing in n.

4. Computations

The proof rests on Briggs' work [2] on the colossally abundant numbers, which implies (RI) for $5040 < n \leq 10^{(10^{10})}$. We extend this result with the following theorem.

Theorem 5. *Robin's inequality holds for all* $5040 < n \leq 10^{(10^{13.114\,85})}$.

Proof. We implemented Brigg's algorithm from [2] but using extended precision (100 bits) and interval arithmetic to carefully manage rounding errors. The final n checked was

$$29\,996\,208\,012\,611\# \cdot 7\,662\,961\# \cdot 44\,293\# \cdot 3\,271\# \cdot 666\# \cdot 233\# \cdot 109\# \cdot 61\# \\ \cdot 37\# \cdot 23\# \cdot 19\# \cdot (13\#)^2 \cdot (7\#)^4 \cdot (5\#)^3 \cdot (3\#)^{10} \cdot 2^{19}.$$

□

Corollary 2. *Robin's inequality holds for all* $13\# \leq n \leq 29\,996\,208\,012\,611\#$.

We are now in a position to prove Theorem 2. We find that

$$g_B(29\,996\,208\,012\,611; 20) < 1$$

and

$$g_\infty(B; 20) < 1$$

and the result follows.

5. Comments

In terms of going further with this method, we observe that both

$$g_B(29\,996\,208\,012\,611; 21) > 1$$

and

$$g_\infty(B; 21) > 1$$

so one would need improvements in both. We only pause to note that one of the inputs to Dusart's unconditional bounds that feed into g_∞ is again the height to which the Riemann Hypothesis is known[3], so the improvements from Lemma 1 could be incorporated.

Finally, we observe that if $R_t(p_n\#)$ could be shown to be decreasing in n, then our lives would have been much easier.

[3]Dusart uses $T \geq 2\,445\,999\,556\,030$.

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