GEODESIC PARAMETRIZATION OF GLOBAL INVARIANT MANIFOLDS OR WHAT DOES THE EQUADIFF 2003 POSTER SHOW?

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We demonstrate the use of an algorithm to compute a two-dimensional global invariant manifold as a sequence of approximate geodesic level sets. The resulting information of the parametrization by geodesic distance can be used to visualize and even crochet the manifold. This is illustrated with the example of the stable manifold of the origin in the Lorenz system, which is also shown on the Equadiff 2003 poster.

Applications often give rise to mathematical models in the form of a low-dimensional system of ordinary differential equations. Well-known examples are periodically forced systems, Chua's circuit and the Lorenz system[3], which we use as the guiding example below. In order to understand the global dynamics one needs to find invariant objects, such as equilibria, periodic orbits and, if they are of saddle-type, also their stable and unstable manifolds. These manifolds are global objects that need to be computed numerically.

For definitenes, we consider here only the case of a saddle point of a vector field in $\mathbb{R}^3$. To be even more specific, we use throughout the example of the stable manifold of the origin in the Lorenz system[3]

$$\begin{cases} \dot{x} = \sigma(y - x)\,, \\ \dot{y} = \varrho x - y - xz\,, \\ \dot{z} = xy - \beta z\,. \end{cases} \tag{1}$$

We use the standard values of the parameters $\sigma = 10$, $\varrho = 28$, and $\beta = 8/3$. The stable manifold of the origin globally organizes how trajectories approach and subsequently follow the well-known Lorenz attractor. This global manifold has emerged as a kind of benchmark example for computing global (un)stable manifolds.

In this paper we explain how the manifold can found by computing geodesic level sets for increasing geodesic distance — effectively finding a set of growing smooth topological circles in $\mathbb{R}^3$. The idea of viewing the manifold as a one-parameter family of topological circles is also used in Refs. [1,2]. A version of our algorithm to compute two-dimensional (un)stable manifolds of maps is described in Ref. [4]. We refer to Ref. [7] for a more complete overview over recent methods for computing global manifolds.

Let us now consider the stable manifold $W^s(0)$ in the Lorenz system (1); for examples of other two-dimensional (un)stable manifolds see Refs. [5,7]. Let $d_g(x,y)$ denote the geodesic distance between points $x, y \in W^s(0)$, that is, the arclength of the shortest path in $W^s(0)$ connecting x and y. The geodesic parametrization of $W^s(0)$ is the family

$$\{S_\eta\}_{\eta>0} \quad \text{where } S_\eta := \{x \in W^s(0) \mid d_g(x,0) = \eta\}\,. \tag{2}$$

An important property is that $\{S_\eta\}_{\eta>0}$ does not depend on the dynamics on $W^s(0)$, but only on its geometry. The manifold $W^s(0)$ is smooth and tangent to the stable eigenspace $E^s(0)$ at 0. Hence, there exists $0 < \eta_{\max} \leq \infty$ such that $\mathbf{S}_\eta$ is a single smooth topological circle without self-intersections for all $0 < \eta < \eta_{\max}$ [9]. It appears that for the manifold $W^s(0)$ in the Lorenz system we have that $\eta_{\max} = \infty$.

Our algorithm approximates a finite set of growing geodesic level sets S_{η_i}, represented by a circular list M_i of mesh points. Inbetween mesh points of M_i we use linear interpolation to obtain the piece-wise linear, continuous representation C_i of S_{η_i}. The manifold $W^s(0)$ itself is enlarged at step i by adding a band, which is given as a suitable triangulation generated by the points in M_{i-1} and M_i. The resulting mesh is nice (not distorted and with a controlled interpolation error), subject to certain accuracy parameters. In fact, the mesh that we compute is so regular that it can be used as a crochet pattern; see already Fig. 1.

To start a computation we choose the first geodesic circle S_{η_0} as a small circle in $E^s(0)$ around the origin. Hence, the mesh representation M_0 is a finite set of equidistant points on a circle in $E^s(x_0)$ at some prescribed distance δ from 0.

Suppose now that we have computed M_0 up to M_i and need to find the points of M_{i+1} forming the next approximated geodesic level set at geodesic distance $\eta_{i+1} = \eta_i + \Delta_i$. Here Δ_i is a prescribed increment that is adapted according to the local curvature along geodesics. The mesh M_{i+1} is obtained pointwise. Let $r \in M_i$ and define $\mathbf{F}_r$ as the plane through r

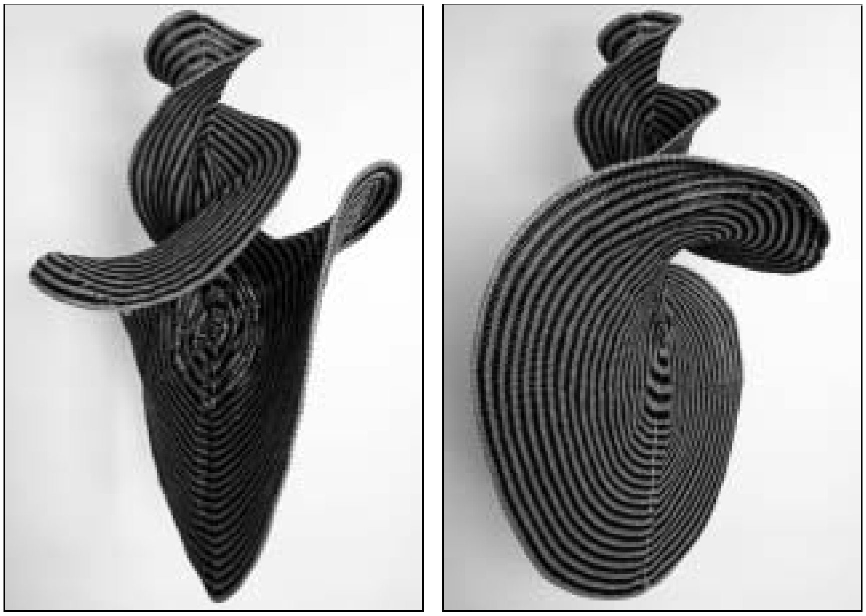

Figure 1. Two views of the stable manifold of the Lorenz system, crocheted by H.O. up to geodesic distance 106.75.

that is 'most perpendicular' to C_i at r. Then $W_s(0)$ intersects $\mathbf{F}_r$ in a well-defined one-dimensional curve locally near r. This curve is parametrized by integration time τ as the family of orbits starting in $\mathbf{F}_r$ and ending in C_i. This family of two-point boundary value problems can be followed by continuation, starting from the trivial solution, namely the solution for $\tau = 0$ consisting only of the point r. When the solution of the boundary value problem is reached for which the initial condition b_r in $\mathbf{F}_r$ is distance Δ_i from r then the point b_r is accepted as a new point of M_{i+1}.

Mesh points are added or removed during a computation in a way that guarantees a global bound on the interpolation error. The computation stops when either $\eta_{\max}$ or a prescribed total geodesic distance is reached. Our method is presently implemented for manifolds of dimension two in an ambient space of any dimension; see Refs. [5,7] for details.

Computing a set of approximate geodesic level results in a very regular mesh on $W^s(0)$ that only depends on the geometry of the manifold, and not on the dynamics on it. Furthermore, information on geodesic distance can be used to visualize the manifold; see also Ref. [6]. In fact, the mesh that we compute can be directly interpreted as a crochet pattern. The increment Δ_i determines the type of crochet stitch to be used for the next round. Where new mesh points were added or removed crochet stitches are made two-in-one or worked together, respectively. The total number of points in M_i correspond to the total number of crochet stitches in round i. Each round of crochet stitches of the appropriate length then generates a new band of the manifold.

Figure 1 shows a crochet model of the manifold $W^s(0)$ up to geodesic distance 106.75. Successive bands were made using alternating colours of yarn to highlight the different bands, of which there are 45 in total. The last round consists of 1065 crochet stitches. The crochet manifold is a very floppy object and quite out of shape, illustrating dramatically that it is in fact a topological disk. To embed this disk properly into $\mathbb{R}^3$, that is, to get the crochet model of $W^s(0)$ into shape, we used a loop of 3 mm garden wire around the entire outer perimeter, as well as along the invariant z-axis and the strong stable manifold. With the help of a computer generated image of $W^s(0)$, the crochet manifold was then bent into shape.

To further illustrate the geometric embedding of $W^s(0)$ into $\mathbb{R}^3$ on the computer it is possible to assign colour according to geodesic distance. Furthermore, one can only show every other band to obtain a see-through effect; see also Ref. [6]. However, as the manifolds starts to scroll into the Lorenz attractor, it becomes increasingly difficult to see how this happens exactly. As a remedy we consider only the part of $W^s(0)$ that lies inside a sphere of some fixed radius around the point $(0, 0, 27)$ on the z-axis right between the two symmetric saddle points.

Figure 2 shows the Lorenz manifold computed up to geodesic distance 151.75 inside the sphere of radius 60. The necessary clipping of the data was done with the visualization package GeomView[8], which was used for all of our computer images. Fig. 2 (a) shows all bands and Fig. 2 (b) — the main image on the Equadiff 2003 poster — shows only every second band. The color scheme indicating geodesic distance from the origin is visible in the inset images on the Equadiff 2003 poster and on the cover of this book.

In conclusion, computing the geodesic parametrisation of a global manifold results in a representation by a mesh of high quality. Furthermore, the information of the geodesic distance and the band structure of the mesh can be used effectively to visualize complicated manifolds, such as the Lorenz manifold discussed here. This provides an advanced tool for the study of global dynamics of low-dimensional vector fields — and many interesting and challenging crochet projects.

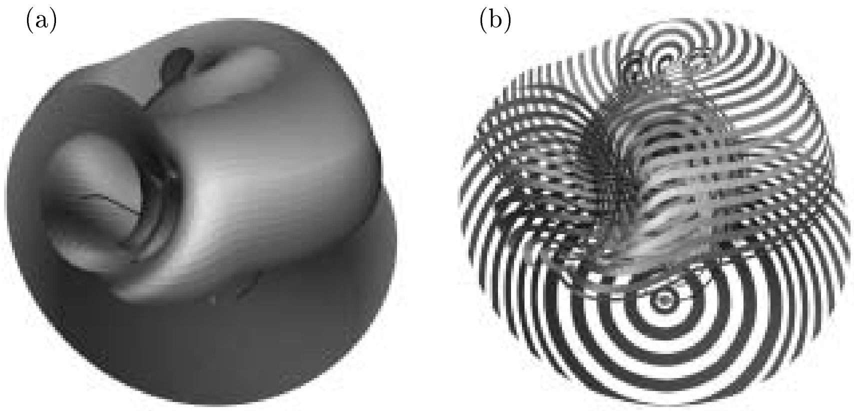

Figure 2. The part of the stable manifold of the Lorenz system (computed up to geodesic distance 151.75) that lies inside a sphere around $(0, 0, 27)$ of radius 60. In panel (a) all geodesic bands are shown, while in panel (b) only every other band is shown. Reprinted from *Computers and Graphics* 26, B. Krauskopf and H.M. Osinga, Visualizing the structure of chaos in the Lorenz system, pp. 815–823, © (2002), with permission from Elsevier.

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