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# Gipps' Model of Highway Traffic

R. Eddie Wilson

Department of Engineering Mathematics, University of Bristol, Bristol BS8 1TR,  
United Kingdom

**Abstract.** We consider the car-following model introduced by P.G. Gipps (1981). This model is of practical importance as it powers the UK Transport Research Laboratory highway simulation package SISTM. Firstly, a brief derivation of the model is given in simplified circumstances. Second, we show how uniform flow solutions and a speed-headway function may be derived. Finally, we consider the linear stability of uniform flow solutions. Conditions for stability and onset of instability are derived.

## 1 Introduction

This paper is concerned with the mathematical model of highway traffic introduced by Gipps (1981) [2], which forms the kernel of the UK Transport Research Laboratory's simulation package SISTM [7].

Recently there has been a great deal of interest in the nonlinear dynamics of highway traffic models, originating with the papers of Kerner and Konhäuser (1993) [4] and Bando *et al* (1995) [1]. Linear stability of so-called *uniform flow* solutions, and the different types of nonlinear wave which the models support, have been studied extensively.

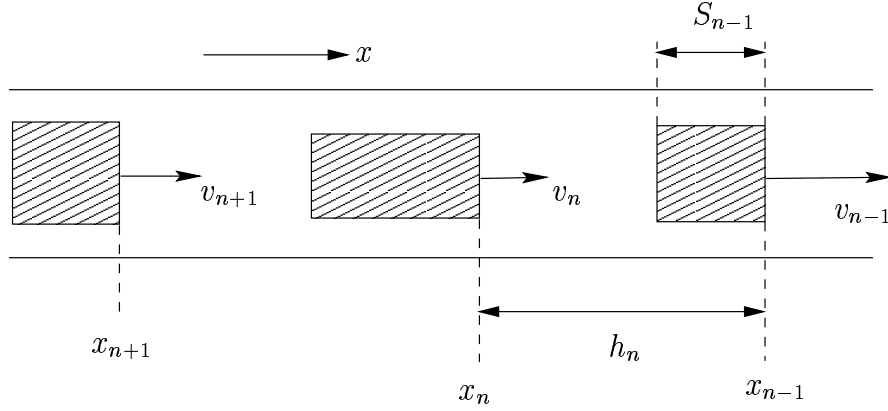
The Bando, Kerner-Konhäuser and other models in the mathematics and physics literature have a rather simple form compared with those implemented in traffic engineers' simulation software. The message of this paper is that a mathematical analysis of some of the more complicated engineering models is possible.

Section 2 gives an overview of the derivation of Gipps' model; section 3 derives uniform flow solutions and an *effective speed-headway function*; section 4 gives a brief analysis of the linear stability of uniform flow.

## 2 Brief Derivation of Gipps' Model

For simplicity, we consider a single lane highway only; multi-lane scenarios with lane changing effects are not considered. Further, we will later assume that vehicles and drivers have identical characteristics.

Fig. 1 shows the basic situation. Vehicles move in an increasing  $x$  direction, with displacements  $x_1, x_2, \dots$ , and with positive velocities  $v_1, v_2, \dots$ , with vehicle indices increasing in the upstream (negative  $x$ ) direction. The *headway*



**Fig. 1.** Definitions of displacement  $x$ , velocity  $v$ , headway  $h$ , and length of vehicle  $S$ . Vehicles travel to the right in an increasing  $x$  direction

of vehicle  $n$  is defined by  $h_n := x_{n-1} - x_n$ , and  $S_{n-1}$  gives the length of each vehicle, so that  $h_n < S_{n-1}$  corresponds to vehicles colliding.

The fundamental premise of Gipps' model is that drivers choose their speeds according to the following behavioural rule.

*What speed should I travel at now, given the behaviour of the vehicle in front one reaction time ago. If the vehicle in front comes to a stop at what I think is its hardest rate, and one reaction time later I commence braking at my hardest rate, I must come to a stop safely.*

Coming to a stop safely means not encroaching in the safety distance, so that  $h_n - S_{n-1} > 0$ . The final headway may be computed as the distance travelled by vehicle  $n - 1$  in coming to a stop, minus the distance travelled by vehicle  $n$  in coming to a stop, plus the initial headway  $h_n(t)$  when vehicle  $n$  commences braking. One may thus obtain

$$\left[ \frac{v_{n-1}(t)^2}{2\hat{B}_{n-1}} \right] - \left[ \frac{\tau_n}{2} [v_n(t) + v_n(t + \tau_n)] + \frac{v_n(t + \tau_n)^2}{2B_n} + \theta_n v_n(t + \tau_n) \right] + h_n(t) \geq S_{n-1}. \quad (1)$$

Here we suppose that  $\tau_n > 0$  is the (constant) reaction time of driver  $n$ ,  $B_n > 0$  is the (constant) braking rate of vehicle  $n$ , and  $\hat{B}_{n-1} > 0$  is the (constant) braking rate of vehicle  $n - 1$ , as estimated by vehicle  $n$ . To derive (1), apply constant acceleration formula and a trapezoidal approximation of the distance travelled by vehicle  $n$  in the time interval  $(t, t + \tau_n)$ . In addition, following Gipps [2], we have included a heuristic *safety margin* term involving a (constant) notional delay  $\theta_n > 0$ . As we showed in [5], Gipps' model may not have solutions for all time  $t > 0$  if the safety margin term is absent.

From now on we suppose that all vehicles and drivers have identical characteristics, and drop subscripts from parameters.

Equation (1) may be re-arranged as a quadratic inequality

$$\left[\frac{1}{2B}\right] v_n(t+\tau)^2 + \left[\frac{\tau}{2} + \theta\right] v_n(t+\tau) - \left[(h_n(t) - S) + \frac{\tau}{2}v_n(t) + \frac{v_{n-1}(t)^2}{2\hat{B}}\right] \leq 0, \quad (2)$$

for  $v_n(t+\tau)$ . If we suppose for simplicity that driver  $n$  wishes to drive at the greatest possible safe speed (physical characteristics of the vehicle may prevent this), then equality is attained, and the positive (i.e. physical) solution of the quadratic equation yields

$$v_n(t+\tau) = -B\left(\frac{\tau}{2} + \theta\right) + \sqrt{\left[B^2\left(\frac{\tau}{2} + \theta\right)^2 + B\left\{2\{h_n(t) - S\} - \tau v_n(t) + \frac{v_{n-1}(t)^2}{\hat{B}}\right\}\right]}, \quad (3)$$

or

$$v_n(t+\tau) = F(h_n(t), v_n(t), v_{n-1}(t)), \quad (4)$$

where it may be shown that the partial derivatives satisfy  $D_1F > 0$ ,  $D_2F < 0$ ,  $D_3F > 0$ . So the speed of vehicle  $n$  at time  $t+\tau$  is given by an increasing function of the headway and speed of the vehicle in front at time  $t$ , and a decreasing function of its own speed at time  $t$ .

We obtain a closed system for the evolution of all vehicle headways and velocities by supplementing (4) with

$$\frac{d}{dt}h_n(t) = v_{n-1}(t) - v_n(t), \quad (5)$$

and by supplying appropriate initial and boundary data.

### 3 Uniform Flow Solutions

We now consider steady state solutions of (4,5), where all vehicles translate along at the same speed with the same time independent headway: these are called *uniform flow* solutions.

We have  $v_n(t) = v^* \forall n, t$ , and  $h_n(t) = h^* \forall n, t$ , so using (4), we require

$$v^* = F(h^*, v^*, v^*), \quad (6)$$

which implies a functional relationship  $v^* = V(h^*)$  between the speed and headway of uniform flow solutions. In other models (e.g. Bando, Kerner-Konhäuser), this *speed-headway* or *optimal velocity* function is a model parameter, however in our case it is determined indirectly by drivers' other

prescribed behavioural laws. This idea of an *effective* speed-headway function, i.e. one that is implied by the model rather than specified explicitly, was discussed in detail for a variety of other traffic equations by Holland (1998) [3].

In the case of Gipps' standard model (3), one may solve a quadratic to isolate  $v^*$  in (6), to give  $V$  explicitly. See [5] for details.

Note that for safety reasons, we expect (at least in steady situations) that the spacing of vehicles will grow larger as the vehicles travel faster. For example, the United Kingdom Highway Code [6] states that drivers should maintain a two second gap, which corresponds to an increasing (and linear) speed-headway function  $V$ .

In general, differentiating  $V(h) = F(h, V(h), V(h))$  with respect to  $h$  gives

$$V'(h) = \frac{D_1 F}{1 - D_2 F - D_3 F}, \quad (7)$$

where each partial derivative is evaluated at  $(h, V(h), V(h))$ . So for  $V$  to be increasing, we require

$$D_2 F + D_3 F < 1, \quad (8)$$

since  $D_1 F > 0$ . Thus equation (8) must hold if our traffic model is to have sensible uniform flow solutions.

## 4 Overview of Stability Analysis

We now consider briefly the stability or otherwise of uniform flow solutions of (4,5). For simplicity, we place  $N$  vehicles on a circular road so that vehicle 1 follows vehicle  $N$ , and seek solutions in the form

$$h_n(t) = h^* + \tilde{h}_n(t), \quad v_n(t) = V(h^*) + \tilde{v}_n(t), \quad (9)$$

where  $\tilde{h}(t)$  and  $\tilde{v}(t)$  are assumed small. Linearisation gives the differential delay type system

$$\begin{aligned} \tilde{v}_n(t + \tau) &= (D_1 F)\tilde{h}_n(t) + (D_2 F)\tilde{v}_n(t) + (D_3 F)\tilde{v}_{n-1}(t), \\ \tilde{h}'_n(t) &= \tilde{v}_{n-1}(t) - \tilde{v}_n(t), \end{aligned} \quad (10)$$

where partial derivatives are evaluated at  $(h^*, V(h^*), V(h^*))$ . In [5], we performed an analysis of the system of maps obtained by replacing  $\tilde{h}'(t)$  with the trapezium rule method of step length  $\tau$ .

Here we seek solutions of the continuous time system (10) in the form  $\tilde{v}_n = \Re[c_v \exp(\lambda t + i\xi n)]$  and  $\tilde{h}_n = \Re[c_h \exp(\lambda t + i\xi n)]$ , where  $\xi = 2\pi/N$ , and  $c_h, c_v$  are complex constants. This yields a pair of algebraic equations from which  $c_h$  and  $c_v$  may be eliminated, to give

$$\lambda [e^{\lambda\tau} - (D_2 F) - (D_3 F)e^{-i\xi}] = (D_1 F) (e^{-i\xi} - 1), \quad (11)$$

relating  $\lambda$  to the spatial wave number parameter  $\xi$ .

Let us analyse the case  $\xi = 0$  corresponding to spatially independent perturbations. We obtain  $\lambda = 0$ , which gives the neutral stability within the family of uniform flow solutions, and

$$\lambda = \frac{1}{\tau} \ln (D_2 F + D_3 F) . \quad (12)$$

A sufficient condition for the stability of spatially independent modes is thus

$$|D_2 F + D_3 F| < 1 , \quad (13)$$

c.f. equation (8).

Let us now analyse the opposite extreme where  $\xi = \pi$ , which corresponds to short wavelength perturbations which repeat every other vehicle. Substitution in (11) gives the transcendental equation

$$\lambda [e^{\lambda\tau} - (D_2 F) + (D_3 F)] = -2(D_1 F) . \quad (14)$$

By considering the signs of the partial derivatives of  $F$ , one observes that real solutions are all negative (i.e., stable). For brevity, we perform here a partial analysis analysis of the onset of instability of the  $\xi = \pi$  mode, where  $\lambda = i\omega$  ( $\omega$  real, nonzero). Equating real and imaginary parts yields

$$\cos \omega\tau = D_2 F - D_3 F \quad (< 0) , \quad \text{and} \quad \omega \sin \omega\tau = 2(D_1 F) , \quad (15)$$

respectively. Now  $\omega$  may be eliminated from these two equations to give a single (albeit complicated) relationship between the partial derivatives which must hold at the onset of instability of the  $\xi = \pi$  mode. For Gipps' standard model (3), this may be analysed graphically, and unstable parameter regimes identified.

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