The sculpture *Manifold*: a band from a surface, a surface from a band

Bernd Krauskopf and Hinke M. Osinga
Department of Engineering Mathematics
University of Bristol, Queen's Building
Bristol BS8 1TR
United Kingdom
b.krauskopf@bristol.ac.uk
h.m.osinga@bristol.ac.uk

Benjamin Storch
Brynawel
Plas Llwyn Owen
Llanbrynmair
Powys SY19 7BG
United Kingdom
ben@benjaminstorch.co.uk

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Abstract

The steel sculpture *Manifold* consists of an 8 cm wide closed band of stainless steel that winds around in an intricate way, curving and coming very close to itself. It is based on a complicated mathematical surface, known as the Lorenz manifold, which has an important role in organising the chaotic dynamics of the well-known Lorenz equations. Namely, this surface consists of all points that, under the force field generated by the Lorenz equations, end up at the origin of the three-dimensional phase space. This is special because all other points go to the famous Lorenz butterfly attractor. The Lorenz manifold can be found and represented numerically by a set of smooth closed curves consisting of points that lie at the same geodesic distance (given by the length of the shortest path on the surface) from the origin. Any band between two such curves illustrates an aspect of the geometry of the surface.

As is explained in this paper, the sculpture *Manifold* represents a choice of band that is motivated by aesthetic, practical and mathematical considerations. The goal was to create an element of dynamicism while only hinting at the underlying surface.

1 Introduction

The existence of the sculpture *Manifold* is a direct result of the 2006 Bridges Conference held in London, where the authors met for the first time. It was a meeting of minds between two mathematicians with a complicated and aesthetically intruiging surface and an artist and metal craftsman who has been researching the creation of surfaces with negative curvature for several years.

The development of numerical methods for the computation of complicated surfaces such as the Lorenz manifold, introduced in more detail in Section 2, has been the research topic of Krauskopf and Osinga for quite a number of years. As it turns out, the result of their computational method translates directly into crochet instructions [9]. After mounting the crocheted work with some wires, a reasonably accurate three-dimensional image of the Lorenz manifold can be obtained. This representation of the surface in crochet attracted considerable media attention and was presented at the 2006 Bridges Conference in London. Starting from a discussion of the curvature properties of the Lorenz manifold, the plan was born to produce an artwork based on it in steel with the techniques perfected by Storch [12]. The idea was that this would allow for a better representation of the geometry of the surfaces, which is characterized by both positive and negative curvature [11]. The challenge in such an endeavour is to balance aesthetic considerations with the considerable practical problems of staying close to the mathematically prescribed form of the surface.

2 The Lorenz manifold

The Lorenz manifold is a two-dimensional surface of initial conditions of the well-known Lorenz equations [7], which are given as the differential equations

$$\begin{cases} \dot{x} &= \sigma(y-x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= xy - \beta z. \end{cases} \tag{1}$$

The right-hand side of the Lorenz equations (1) can be thought of as defining an instantaneous force of a given magnitude and direction that depends on the position in the three-dimensional (x,y,z)-space. Given a point particle at some initial position in (x,y,z)-space, the solution curve or trajectory through this point is the path that the particle travels under the influence of the force field (1). Trajectories can be computed by numerical integration. The Lorenz equations were introduced by Edward Lorenz in the 1960s as an example of a deterministic system that exhibits unpredictible chaotic dynamics [13]; see [3] for a popular account.

For the classic parameter values $\sigma = 10$, $\rho = 28$, and $\beta = 2\frac{2}{3}$ one finds the famous Lorenz butterfly attractor by integration from practically any initial position. The Lorenz manifold, on the other hand, consists of all initial positions in (x,y,z)-space for which the particle ends up at the origin $(x,y,z) = (0,0,0)$ instead. Mathematically, the origin is a saddle point with one unstable and two stable directions, and the Lorenz manifold is its stable manifold. As such, dynamical systems theory guarantees that the Lorenz manifold is a two-dimensional smooth surface. This surface is an important geometric object that helps one to understand chaos; see [1, 2, 8] for details.

While the Lorenz manifold is well defined as a two-dimensional surface, there is not explicit or implicit equation that describes all the points on it (as is the case, for example, for a sphere). It is defined only 'indirectly' by the property that it consists of trajectories that end up at the special point $(x,y,z) = (0,0,0)$. As a result, the Lorenz manifold can only be solved via numerical approximation. There are several techniques for doing this; we refer to [6] for an overview where the Lorenz manifold is used throughout to illustrate the differences between the methods. Our approach is based on the idea that one can build up the surface from near the origin as a set of concentric smooth closed curves. The first such curve or ring is chosen as a small circle in the plane that is spanned by the two attracting directions; this choice amounts to a linear approximation of the surface that can be found directly via a straightforward eigenvector computation. A new ring is added to the collection at each step. It is found by determining the points that lie at a given small distance from and, under the force field, pass through the ring that was added in the previous step. In this way, the surface is grown with equal speed in all radial directions. Each computed ring corresponds to a so-called geodesic level set, that is, the arclength of the shortest path to the origin over the surface is (approximately) the same for all points on a ring. We refer to [4, 5] for the mathematical details of this method.

Figure 1 shows the Lorenz manifold computed with our method up to the geodesic level set at distance 140.75. The surface is rendered transparent except for a 20-units wide band of geodesic level sets that covers the last computed rings. Two different view points are shown, where each time the z-axis is the vertical axis in the picture. A first observation is that the Lorenz manifold is symmetric, that is, invariant under rotation over 180° about the z-axis. This is a direct consequence of the respective symmetry of the Lorenz equations (1). In particular, the z-axis itself is invariant under the dynamics and it is part of the Lorenz manifold.

3 From surface to sculpture

From an aesthetic point of view, the Lorenz manifold is a new complex form with a fascinating geometry. We find its geometry particularly appealing because the smooth complex form gives a sense of motion and dynamics to the static object. While it is not immediately clear that the surface is related to point particles

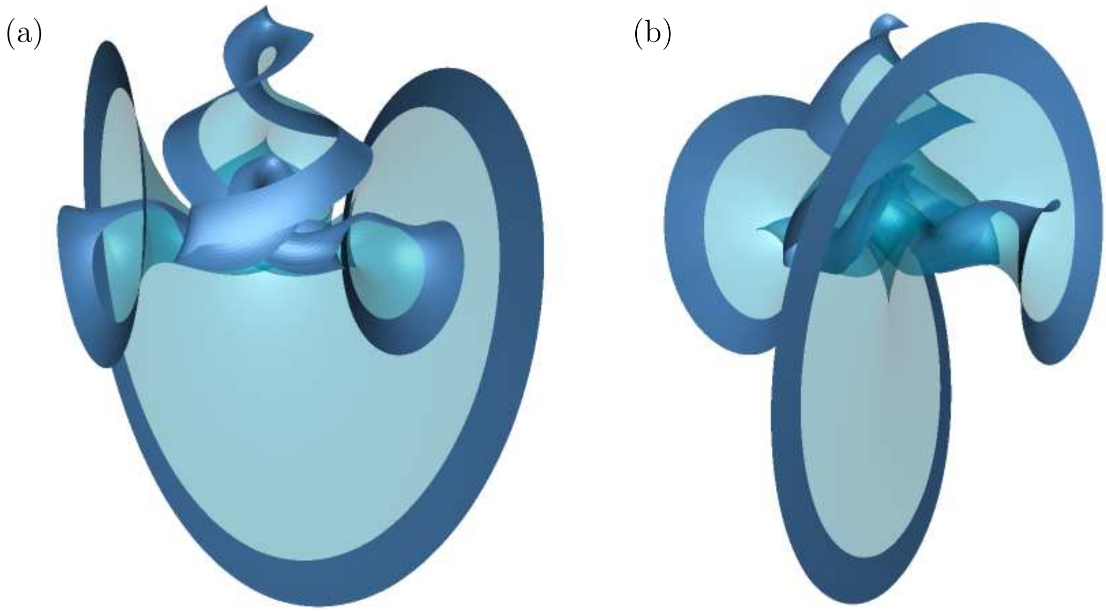

Figure 1: The Lorenz manifold computed as a collection of geodesic level sets. A darker outer band is shown on the transparently rendered surface. The two panels show two different view points obtained by rotation about the z-axis, which is the vertical axis in the images.

flowing to the origin, the shape does conjure up an image of gracious movement and, as such, somewhat captures the underlying mathematical meaning. The complexity of its geometry, however, presents a serious challenge when one wants to turn the Lorenz manifold into a sculpture.

The steel sculpture uses the Lorenz manifold as inspiration. Rather than creating its entire image, we explored the idea of selecting only part of the surface as a representation of the entire geometric object. The selected band on the surface shown in Figure 1 has the elemental aspect of a surface, while at the same time it gives the impression of a one-dimensional curve with its own dynamics. The mathematical meaning of the band and its association with geodesic level sets provides added insight into the geometry of the Lorenz manifold when the band is visualised together with the surface. However, without the underlying surface the band becomes an interesting object in its own right. It has such a complex geometry that it is not immediately obvious what the underlying surface actually looks like.

Figure 2 shows the actual steel sculpture (left column) together with a computer rendering of the respective band on the Lorenz manifold (right column). We found it quite amazing how well the steel band captures the shape of the entire two-dimensional surface, at least for someone who knows what the Lorenz manifold looks like. While the small band communicates the dynamics of a one-dimensional curve, it also reveals important elements of the two-dimensional geometry. Overall, the sculpture *Manifold* captures the complexity of the Lorenz manifold in a subtle way by how it curves in the three-dimensional space. Note that the band is closed, but not knotted or twisted: it is orientable, that is, topologically equivalent to a simple annulus. In other words, the steel band has two sides, which are distinguished by their surface finish: one side is highly polished and the other is wire brushed.

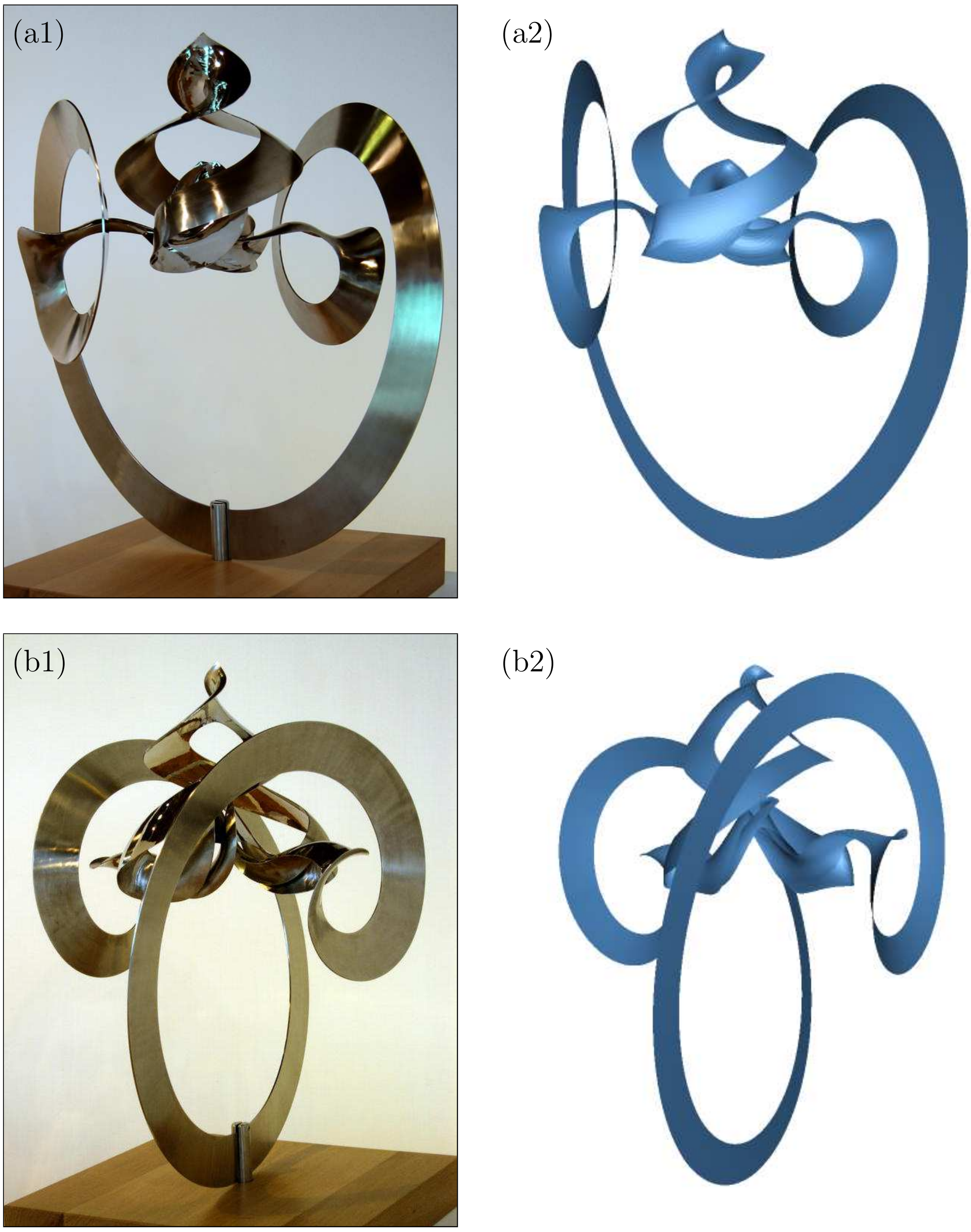

Figure 2: The sculpture *Manifold* (left column) and its computer-generated equivalent (right column); the two view points are the same as in Figure 1.

4 Creation of the steel sculpture

The sculpture *Manifold* was created by using the data of the Lorenz manifold as computed with the algorithm in [4, 5]. We selected a band derived from computed rings that correspond to a steel band of 8 cm in an overall sculpture of 70 cm in diameter.

The main challenge of creating the band from steel is that it is made by hammering flat segments into the required curved shape. The data of the respective rings was loaded into CAD software and processed to render templates for the sculpture. To this end, we utilized the symmetry of the Lorenz manifold, and thus, of the band. There are precisely two intersection lines with the z-axis in the computer-rendered object, which are located vertically at the top and bottom in Figure 2. A cut through those two intersection lines gives two halves that are each other's image under the symmetry. Each half was then divided into three sections: one large relatively flat section that roughly forms the bottom half of the band, and two smaller sections that form the upper half with the highest curvature. The selection of the three (pairs of) sections was further based on the necessity that they needed to be welded together along relatively straight sections in accessible locations. The three individual sections were created in the CAD software so that the steel pieces could be compared directly with the computer model. Furthermore, they were then manipulated in the CAD software to 'flatten them out' into pieces that could be laser-cut from flat sheet steel.

The two pairs of flattened pieces for the top half were cut from 1.2 mm stainless steel. The two flattened pieces corresponding to the symmetric bottom sections were cut from 3 mm stainless steel, which ensures that the bottom half is strong enough to support the more winding and, hence, heavier top half of the sculpture. The flat pieces were hammered into their desired shape by hand, which took place under constant comparison with three-dimensional computer-rendered images of the CAD model. The top part of the band continuously changes in direction and curvature. Therefore, the right supporting tools (stakes) needed to be applied to stretch and compress the sheet locally to the desired degree. While the flat sections already showed the most appropriate curvature in the plane, the relation between the inner and outer edge length is further altered in the transformation into the three-dimensional shape, where stretching of the inner or outer edge can be emphasised through the forming process. Furthermore, pieces were worked into their shape in pairs to ensure the symmetry of the overall sculpture. As can be seen in Figure 2, near the centre of the sculpture the steel band comes very close to itself. This made it necessary to polish and brush the respective sections before they were welded together.

The welding itself is also quite a challenge because the tension in the steel sections is difficult to balance during the welding process. The key is to keep stresses to a minimum and balance them in such a way that the overall required shape is attained. To achieve this, the four sections forming the top and central sections of the band are welded together first; here care must be taken to ensure that the resulting symmetric piece comes close to itself in the central region, but does not actually touch. Similarly, the two pieces that form the bottom half (up to about the centre of the outer spiral in Figure 2) are first welded together along the symmetry axis. Finally, the entire sculpture is welded together along the two remaining seams. The finished sculpture is mounted onto a wooden base via a 30 mm pin that holds the band at the bottom along the intersection line with the z-axis. In this way, the steel band can be rotated about the vertical axis of symmetry, so that it can be enjoyed from any angle.

The quality and precision of the sculpture is so good that it provides an almost-perfect rendering in steel of the computer-generated band. To illustrate the level of accuracy, we tried to match the view points chosen for the computer images with those of the photographs of the sculpture in Figure 2. There are some differences due to slightly different angles of projection, but overall the agreement is very convincing. However, nothing beats the experience of the real sculpture — and being able to look at it in detail and from all angles!

5 Conclusions

The steel sculpture *Manifold* was born out of the wish to create a flowing and dynamic form on the basis of a special mathematical surface. It could only ever have taken shape by combining mathematical ideas from dynamical systems theory with the specialised hammering technique that allows for the introduction of negative curvature into sheets of metal. In short, the sculpture is an example of a successful interaction between arts and mathematics. While its intriguing shape can be enjoyed without any knowlegde of the underlying mathematics, we hope that it may draw some viewers into the world of geometry.

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