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WARING'S PROBLEM IN THIN SETS AND MIXED MOMENTS OF THE RIEMANN ZETA FUNCTION

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Abstract

This thesis is devoted to two topics in Analytic number theory, namely, that of Waring type problems in thin sets and mixed moments of the Riemann zeta function.

We begin by examining the expected asymptotic formula of the representation function in Waring's problem over sums of three cubes, both with and without multiplicities, thereby establishing its validity in the former case and deriving a lower bound in the latter.

A separate discussion is devoted to the investigation of the above setting for small exponents. We obtain upper bounds for the number of variables needed to represent every sufficiently large integer in the prescribed way for the exponents $2, 3$ and 4. We make use of the minor arc analysis in the case $k = 2$ and combine it with an intrincate major arc counterpart to deduce an almost all result for the analogous Lagrange's four-square theorem where the variables are restricted to the sums of three cubes.

We complete the circle method part of the thesis by examining the analogous problem in which the sums of three cubes are replaced by sums of t positive l-th powers, the desired objective in such a context being the accomplishment of some uniformity in the number s of variables needed. Those considerations are discussed and partially achieved when t lies in two particular ranges.

The second part of the thesis comprises the investigation of mixed third moments of the Riemann zeta function. We establish an asymptotic evaluation of the moment at hand in three different situations: one in which the corresponding coefficients are rational numbers in a suitable range, another one in which the coefficients are linearly independent over $\mathbb{Q}$, and the last one in which one of the coefficients equals minus the other one. In certain cases we are able to provide explicit account of lower order terms.

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To my family and friends on both sides of the Atlantic Ocean for their moral support and presence.

Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Contents

Chapter 1

Introduction

1.1 Diophantine problems and diagonal equations

Since its first appearance in 1918 in investigations of Hardy and Ramanujan [54] concerning the asymptotic evaluation of the partition function, which we define as the number of ways of writing a natural number n as

$$n = x_1 + \ldots + x_j, \qquad x_i \in \mathbb{N}, \quad j \in \mathbb{N}$$

where $x_1 \leq x_2 \leq \ldots \leq x_j$, the circle method has been one of the most powerful exploited techniques in the analysis of problems of additive nature in the theory of numbers. From its very early stages, it has been employed in various different areas of analytic number theory and has become a prominent tool in the analysis of diophantine equations, which are polynomial equations over the integers. The subsequent developments and refinements of it have led to noteworthy advances in the understanding of solutions of general systems of homogeneous equations, providing remarkably satisfactory results in the instance when the number of variables of the corresponding polynomials is quite large with respect to the degree of those polynomials.

In the analysis of diophantine equations with fewer variables one often encounters situations in which the number of solutions, if any, is expected, or at least conjectured to be small. Under such circumstances then algebraic geometric arguments are often employed to derive such conclusions. We take this opportunity to draw the reader's attention to the relatively recent celebrated resolution of Fermat's Last Theorem [160], which states that no triple $(x, y, z) \in \mathbb{N}^3$ satisfies the equation

$$x^n + y^n = z^n$$

for $n \geq 3$, or Faltings' theorem [41], which proves the finiteness of the rational points of curves of genus greater than 1 defined over a number field, to provide some of the most prominent results wherein the aforementioned approach has found success. In contrast, when the number of variables is reasonably large then one usually expects to have many solutions, and succesful applications of the circle method when attempting to shew the existence of any of those often achieve so by actually providing strong lower bounds in the number of solutions of the corresponding systems of equations.

For the sake of transparency and further convenience we introduce for fixed $k \geq 2$ and $a_i \in \mathbb{Z}$, not all with the same sign whenever k is even, the collection of equations

$$a_1 x_1^k + \ldots + a_s x_s^k = 0, \qquad x_i \in \mathbb{N},$$

which we will henceforth refer to as diagonal equations. Of particular interest among the diophantine problems are these equations because of their connection with problems in additive number theory and because the results involving those may be applied to obtain non-trivial consequences concerning the resolution of more general systems of homogeneous equations and the density of rational points on algebraic varieties. It is also worth noting that the development of the techniques to obtain sharper results in the analysis of diagonal equations entails refinements of estimates for some exponential sums which in turn deliver bounds for fractional parts of polynomials and have implications in the theory of equidistribution. Likewise, progress in the resolution of these equations often relies on improvements in estimates of mean values of those exponential sums which in turn might have applications for other subjects in analytic number theory.

It seems appropiate to remark that the extensive investigations by scholars concerning diagonal equations in many variables via the circle method have led to some of the most formidable results attained by the approach thereof, and the first three chapters of the present memoir will be primarily devoted to the analysis of problems involving those. For the purpose of illustrating the history of the aforementioned equations we shall give an exhaustive account of the main results within this circle of ideas in the next sections.

Nonetheless, there has also been a considerable amount of work devoted to the analysis of diagonal equations in which the number of variables is reasonably small as compared to the degree of those, and herein other analytic and algebraic geometric methods play a not unimportant role. When the equations in hand are symmetric, a consideration of a handful of examples lends credibility to the belief that there are fewer solutions other than the diagonal ones, and we should note herein that establishing such a result is called a paucity problem. Albeit the following list that will be presented shortly does not pretend to give account of all of the problems pertaining to the situation at hand, it is convenient to mention first excursions concerning, for $k \geq 3$, the asymptotic evaluation of

$$x_1^k + x_2^k = y_1^k + y_2^k,$$

where $x_1, x_2, y_1, y_2 \in \mathbb{N}$, and connected problems involving the estimation of the cardinality $\nu_k(x)$ of the set of integers up to x which have more than one representation as a sum of two k-th powers. We find it appropiate to refer the reader to the series of memoirs of Hooley [66], [67], [68], [69], [72] that shew the bound $\nu_k(x) = O_{\varepsilon,k}(x^{5/3k+\varepsilon})$ by employing sieve methods in conjunction with Deligne's estimates. It is also worth mentioning the papers of Greaves [47], [48], which establish a weaker bound by utilising Weil's estimates instead, and the work of Skinner and Wooley [125], which sharpens Hooley's estimate by making use of an elementary argument relying on a result of Bombieri and Pila [11] for counting integral points on curves. The current best bound for large k due to Browning [17] is of the shape $\nu_k(x) \ll x^{2/3k+\varepsilon}$ and is based on an argument of Heath-Brown [64] concerning the number of rational points on curves in the projective space. For the sake of completeness, though, we refer the reader to further refinements of Browning and Heath-Brown [19] and Salberger [122] for intermediate values of k.

We also find it convenient to mention the result of Browning and Heath-Brown [18] on the investigation of the number of solutions of the equation

$$x_1^k + x_2^k + x_3^k = x_4^k + x_5^k + x_6^k, \qquad 1 \leq x_i \leq P.$$

By making use of ideas concerning lattice points in conjunction with algebraic geometric arguments, the aforementioned authors show that the cardinality of the set of off-diagonal solutions is $o(P^3)$ whenever $k \geq 33$. As a consequence, they derive an asymptotic formula for the cardinality of numbers that are expressible as sums of three k-th powers, improving upon an earlier approach of Wooley [166] employing the circle method. It is worth mentioning the work of Salberger [121], wherein he derives the same conclusion for $k \geq 25$. Salberger and Wooley have further obtained the paucity of diagonal solutions of the equation

$$x_1^k + \ldots + x_s^k = x_{s+1}^k + \ldots + x_{2s}^k, \qquad 1 \leq x_i \leq P$$

in the instances when $k \geq (2s)^{4s}$.

It may also seem appropiate to illustrate our exposition by considering the connected problem concerning the system of equations

$$x_1^k + x_2^k + x_3^k = x_4^k + x_5^k + x_6^k,$$

$$x_1 + x_2 + x_3 = x_4 + x_5 + x_6,$$

where $1 \leq x_i \leq P$, and herein we shall mention the work of Greaves, who deduced that the cardinality of the set of non-diagonal solutions thereof is $O(P^{17/6+\varepsilon})$. Skinner and Wooley [126] sharpened this to $O(P^{8/3+1/(k-1)+\varepsilon})$ in a subsequent paper by generalising the methods employed in [125]. These bounds have now been superseded by the work of Salberger [120], which yields $O(P^{5/2+\varepsilon})$ for $k \geq 6$. As was mentioned earlier, the focus of the present memoir is on the instance where the circle method is applicable, and we shall not discuss the above situation again in our work.

1.2 Waring's problem

The first explicit mention of the simplest case of Waring's problem probably dates back to the 3rd century AD in the hellenistic mathematics treatise Arithmetica. Although Diophantus claims therein to know how to prove that every natural number can be written as a sum of four squares, it wasn't until 1770 that a rigorous proof was written down by Lagrange, even Euler and Fermat failing to find success on such a problem. We refer the interested reader to [55], [104], [106] for different proofs of the theorem.

On that same year, Waring conjectured a generalization of the above by asking whether for fixed $k \in \mathbb{N}$ there exists a number s depending on k for which every natural number is expressible as

$$n = x_1^k + \ldots + x_s^k \tag{1.2.1}$$

for non-negative integers x_i. On denoting by $g(k)$ to the minimum such s, if it exists, he further claimed without proof that $g(2) = 4$ (which was a consequence of Lagrange's theorem), $g(3) = 9$, $g(4) = 19$ and so on. By the end of the nineteenth century though, the existence of $g(k)$ was only known for a handful of exponents, and it wasn't until 1909 when Hilbert [65] proved it for all k by using a combinatorial argument relying on some intrincate polynomial identities.

Very shortly after that breakthrough, Wieferich [159] and Kempner [90] showed that $g(3) = 9$, the statement $g(4) = 19$ being established by Balasubramanian, Dress and Deshouillers [4], [5] in 1986, the identity $g(5) = 37$ by Chen [24] in 1964 and $g(6) = 73$ by Pillai [109] in 1940 . Before giving account of the progress being made concerning the evaluation of $g(k)$ for general k and for the purpose of merely illustrating such a discussion, we find it appropiate to observe that the number

$$n = 2^k \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 1$$

satisfies $n < 3^k$. Consequently, the most economical way of expressing n as a sum of k-th powers would only involve 1 and 2^k as summands and would yield the bound

$$g(k) \geq \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor + 2^k - 2.$$

Moreover, after the work of many mathematicians, it is now known that the above inequality is an equality whenever

$$2^k \big\{(3/2)^k\big\} + \lfloor (3/2)^k \rfloor \leq 2^k. \tag{1.2.2}$$

If the above conclusion does not hold then

$$g(k) = 2^k + \lfloor (3/2)^k \rfloor + \lfloor (4/3)^k \rfloor - \theta,$$

where θ is 2 or 3 depending on whether

$$\lfloor (4/3)^k \rfloor \lfloor (3/2)^k \rfloor + \lfloor (4/3)^k \rfloor + \lfloor (3/2)^k \rfloor$$

equals or exceeds 2^k respectively. It also seems appropiate to mention that condition (1.2.2) does not hold only for at most a finite number of cases (see Mahler [99]), and that the work of Kubina and Wunderlich [91] yields (1.2.2) whenever $k \leq 471,600,000$. Describing the long history of contributions involving such evaluations is hardly the point of this discussion, whence in the interest of curtailing our digression we refer the reader to [39], [40], [105], [151].

In view of the above discussion it transpires that the function $g(k)$ is determined by the behaviour of small k-th powers. The more modern variant of the problem avoids that limitation by seeking to minimise the number of variables to represent instead every sufficiently large number. We denote for further convenience by $G(k)$ to the smallest positive integer s for which every sufficiently large integer can be written as a sum of s positive k-th powers. It appears at first glance that a consideration of the representation of small values of n, as discussed above, already delivers robust lower bounds on $g(k)$. The problem then reduces to show via the circle method that the numbers larger than that already considered are expressible employing at most that many variables, whilst the only lower bounds for $G(k)$ that one may deduce esentially arise from local solubility constraints.

It seems apparent that the latter framework is much harder, and as will be detailed shortly, matching the upper bounds deduced via the circle method with the lower ones stemming from the local solubility analysis lies beyond the reach of current technology in most of the cases.

As a prelude to our forthcoming discussion we find it appropiate to briefly mention the results available on the literature concerning the evaluation of $G(k)$. It might be worth noting first that Lagrange's theorem for squares in conjunction with the observation that a sum of three squares cannot be congruent to 7 modulo 8 already delivered $G(2) = 4$. In 1939, Davenport [30] incorporated the exponent $k = 4$ to the above list by shewing that $G(4) = 16$

employing a sophisticated version of the diminishing ranges argument. In fact, his proof is accomplished by deriving a suitable minor arc estimate involving 14 variables, and so the latter author further deduces that every large positive integer not congruent to 15 or 16 modulo 16 is expressible as a sum of at most 14 biquadrates. The enthusiast reader shall be referred to the work of Vaughan [137], [139] or Kawada and Wooley [87] for further refinements in the above setting within this circle of ideas.

Nevertheless, the above evaluations are the only ones known, and the current best upper bounds for $G(k)$ obtained via the circle method are still quite far from the conjectured values. In order to describe the history of the progress and sharpening of the estimates for the aforementioned function rather precisely, we feel obliged to provide to the reader with a gentle introduction to the Hardy-Littlewood method, which we defer to the next section. Nonetheless, it is worth mentioning the bound $G(3) \leq 7$ obtained by Linnik [96] in 1943 avoiding the use of the circle method. We shall not give an overall account of recent progress in the understanding of sums of cubes herein, but nevertheless we content ourselves to mention that Vaughan [140] further deduced a lower bound on the number of representations of large positive integers as sums of seven cubes of the expected order via the circle method by combining a sixth moment estimate involving cubes obtained in [139] in conjunction with ideas from [136] and a delicate analysis. Wooley further simplified the proof to obtain such upper bound by refining the aforementioned estimate in [166].

1.3 Notation

We will give an overview of the most basic approach to Waring's problem utilising the Hardy-Littlewood method promptly, but for the sake of preciseness some basic notation which will be used throughout the entire memoir is required. For functions $f(t)$ and $g(t)$, the abbreviation $f(t) \ll g(t)$ will mean as is customary that there exists some constant $C > 0$ for which $|f(t)| \leq C|g(t)|$, and $f(t) \gg g(t)$ will denote the inequality $|f(t)| \geq C|g(t)|$ (this is often referred to as Vinogradov's notation in the literature). We will also write $A \asymp B$ when $A \ll B \ll A$. Whenever the symbols $\ll$, $\gg$ or $\asymp$ appear in the restrictions pertaining to a sum it will mean that the corresponding tuples are subjected to the underlying inequality for some particular choice of the implicit constant, the precise value not having any impact in the subsequent estimation. As usual in analytic number theory, for each $x \in \mathbb{R}$ then $e(x)$ will mean $\exp(2\pi i x)$, and for each natural number q then $e(x/q)$ will be written as $e_q(x)$. We will write $a \le \mathbf{V} \le b$ for $\mathbf{V} = (v_1, \ldots, v_n)$ when $a \le v_i \le b$ for $1 \le i \le n$. When ε appears in any bound, it will mean that the bound holds for every $\varepsilon > 0$, though the implicit constant then may depend on ε. We adopt the convention that unless specified, whenever we write δ in the computations we mean that there exists a positive constant δ for which the bound holds. We write $p^r || n$ to denote that $p^r | n$ but $p^{r+1} \nmid n$. The function $||x||$ denotes the distance to the nearest integer.

1.4 The application of the circle method to Waring's problem

We begin our discussion by fixing first $s, k \in \mathbb{N}$, taking a natural number n, which the reader should think of as being large, introducing the parameter $P = n^{1/k}$ and considering the representation function

$$R_{s,k}(n) = \text{card}\Big\{(x_1, \ldots, x_s) \in \mathbb{N}^s : \quad n = x_1^k + \ldots + x_s^k\Big\}.$$

The circle method employs Fourier Analysis to obtain quantitative information about the above function, and in order to illustrate this idea we find it convenient to introduce the Weyl sum

$$f(\alpha) = \sum_{1 \le x \le P} e(\alpha x^k). \tag{1.4.1}$$

Then by employing the basic orthogonality relation

$$\int_0^1 e(\alpha h) d\alpha = \begin{cases} 1 & \text{when } h = 0 \\ 0 & \text{when } h \in \mathbb{Z} \setminus \{0\} \end{cases} \tag{1.4.2}$$

we obtain the identity

$$R_{s,k}(n) = \int_0^1 f(\alpha)^s e(-\alpha n) d\alpha,$$

which expresses $R_{s,k}(n)$ in terms of the n-th Fourier coefficient of $f(\alpha)^s$, whence it transpires that controlling $R_{s,k}(n)$ amounts to understanding the behaviour of the exponential sum $f(\alpha)$. We will shortly provide the reader with an overview of the ideas to analyse the above integral, but first we find it appropiate to mention that assuming that integers of the same size should have similar number of representations then one would expect to obtain $R_{s,k}(n) \asymp n^{s/k-1}$.

It might be worth noting first that on taking a/q for $a, q \in \mathbb{Z}$ with $0 \leq a < q$ and $(a, q) = 1$ then by sorting the terms into arithmetic progressions modulo q one finds that

$$f(a/q) = \frac{P}{q} \sum_{r=1}^{q} e(ar^k/q) + O(q).$$

Consequently, assuming that the sum involved in the above expression is non-zero, it appears at first glance that whenever q is sufficiently small then the above equation is an asymptotic formula and implies that $|f(a/q)|$ is fairly big, and one expects to derive a similar conclusion for α close to rational numbers with small denominator. If, on the contrary, α does not satisfy the above property then one would expect the argument αx^k to be distributed rather randomly modulo 1 and hence to obtain plenty of cancelation when summing over x.

Motivated by such an observation, we will dissect the unit interval into major arcs, which will be sufficiently narrow intervals centered at rational numbers with small denominator, and minor arcs, which will be the complement of those. The above discussion then lends credibility to the expectation that the main contribution to the integral will be arising from the major arcs. History enthusiasts may find it interesting to know that such a terminology stems from the original treatment of Hardy and Littlewood involving the use of Cauchy's integral formula to express $R_{s,k}(n)$ as a contour integral over a circumference on the complex plane.

In order to put these ideas into effect, we take $\delta > 0$ to be a small parameter, consider $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $0 \leq a \leq q \leq P^\delta$ and with the property that $(a, q) = 1$ and introduce

$$\mathfrak{M}(a, q) = \Big\{\alpha \in [0, 1) : \ |\alpha - a/q| \leq \frac{P^\delta}{n}\Big\}.$$

We define the major arcs $\mathfrak{M}$ to be the union of the above intervals, and the minor arcs will then be $\mathfrak{m} = [0,1)\backslash\mathfrak{M}$. We also introduce, for further convenience, the auxiliary functions

$$S(q,a) = \sum_{r=1}^{q} e(\alpha r^k), \qquad v(\beta) = \int_0^P e(\beta\gamma^k)d\gamma.$$

We take $\alpha \in \mathfrak{M}(a,q)$ and write $\alpha = a/q + \beta$. Then, by sorting the terms into arithmetic progressions modulo q one obtains

$$f(\alpha) = \sum_{r=1}^{q} e(ar^k/q) \sum_{\substack{x\leq P \\ x\equiv r \pmod q}} e(\beta x^k).$$

It transpires that the derivative of the argument inside the inner sum on the above equation is fairly small, whence an application of Riemann-Stieltjes integration already suffices to obtain the asymptotic relation

$$f(\alpha) \sim q^{-1}S(q,a)v(\beta).$$

We find it appropiate to mention that a slightly more sophisticated approach involving a truncated version of Poisson summation formula in conjunction with estimates for complete exponential sums delivers an error term of the size $O(q^{1/2+\varepsilon})$ in the above formula.

Integrating over each individual arc and summing over pairs (a,q) satisfying the above properties we deduce that

$$\int_{\mathfrak{M}} f(\alpha)^s e(-\alpha n)d\alpha \sim \mathfrak{S}(n,P^\delta)J(n;P^\delta)$$

where

$$\mathfrak{S}(n;P^\delta) = \sum_{q\leq P^\delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \big(q^{-1}S(q,a)\big)^s e(-an/q)$$

and

$$J(n;P^\delta) = \int_{|\beta|\leq P^\delta n^{-1}} v(\beta)^s e(-\beta n)d\beta.$$

A careful analysis of the exponential sum $S(q,a)$ reveals that whenever $(a,q) = 1$ then

$$S(q,a) \ll q^{1-1/k},$$

which already suffices to provide, for s sufficiently large in terms of k, the convergence of

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}\big(q^{-1}S(q,a)\big)^{s}e(-an/q),$$

the bound $\mathfrak{S}(n)\ll 1$ and the asymptotic relation $\mathfrak{S}(n;P^{\delta})\sim\mathfrak{S}(n)$. The reader may find it useful to mention that $\mathfrak{S}(n)$ is often referred to as the *singular series*. It is a noteworthy feature that by using the multiplicative nature of the terms inside the series in conjunction with the bounds for $S(q,a)$ already mentioned and some basic orthogonality relations one may write it as the infinite product

$$\mathfrak{S}(n)=\prod_{p}\sigma(p),$$

where the local factors $\sigma(p)$ are defined as

$$\sigma(p)=\lim_{h\to\infty}p^{h(1-s)}\mathrm{card}\Big\{(x_1,\ldots,x_s)\in\big(\mathbb{Z}/p^h\mathbb{Z}\big)^{s}:x_1^k+\ldots+x_s^k\equiv n\ (\mathrm{mod}\ p^h)\Big\}.$$

Using some elementary combinatorial arguments one can derive a lower bound for the function inside the above limit not depending on h whenever $s\geq 4k$, and a refined estimate for $S(q,a)$ when one restricts q to prime powers combined with the previous observation delivers the lower bound $\mathfrak{S}(n)\gg 1$. We should note that the analysis of the congruence problem, often referred to as *local solubility* analysis, can be further refined to obtain sharper conclusions concerning the restriction on s. In particular, every bound for $G(k)$ that we mention in the discussion will embody these local solubility refinements in order to ensure that the estimate $\mathfrak{S}(n)\gg 1$ holds, but we shall not give account of those henceforth.

Likewise, a simple argument involving integration by parts yields the bound

$$v(\beta)\ll\frac{P}{(1+n|\beta|)^{1/k}}.$$

The above estimate already delivers the convergence of the *singular integral*, which we define by

$$J(n)=\int_{-\infty}^{\infty}v(\beta)^{s}e(-\beta n)d\beta,$$

subject to the condition $s \geq k+1$, and provides the asymptotic relation $J(n; P^\delta) \sim J(n)$. It follows from Fourier's Integral Theorem that

$$J(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{s/k-1},$$

whence by the preceding discussion one gets

$$\int_{\mathfrak{M}} f(\alpha)^s e(-\alpha n) d\alpha \sim \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{s/k-1} \mathfrak{S}(n).$$

We should remark that by employing all of the refinements mentioned throughout the above discussion, the above formula can be established whenever $s \geq \max(5, k+1)$. It transpires then that the major arcs are rather well understood and the restrictions in the number of variables stemming from the corresponding analysis thereof are essentially optimal. In contrast, the minor arc toolbox available is somewhat more limited, and any refinements on upper bounds for $G(k)$ essentially have their reliance on developments in the minor arc machinery. In what follows we shall give a brief account of the history of the progress in the minor arc analysis in connection with Waring's problem for large k.

Our journey begins in 1916 with the pointwise bound

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll P^{1-\delta 2^{1-k}+\varepsilon}$$

obtained by Weyl [157] employing his *Weyl differencing* technique. This non-trivial saving in conjunction with a near-optimal bound for the 4-th moment of $f(\alpha)$ stemming from an elementary divisor estimate argument and a refined major arc analysis already suffices to deduce the upper bound

$$G(k) \leq (k-2)2^{k-1} + 5,$$

which was accomplished by Hardy and Littlewood [51]. In 1938, Hua [73] further exploited the Weyl differencing ideas and incorporated them to the analysis of mean values over the unit interval, thus obtaining

$$\int_0^1 |f(\alpha)|^{2^j} d\alpha \ll P^{2^j - j + \varepsilon} \tag{1.4.3}$$

whenever $1 \leq j \leq k$. Combining the above equation with the pointwise bound earlier mentioned one obtains for $s \geq 2^k + 1$ the estimate

$$\int_{\mathfrak{m}} |f(\alpha)|^s d\alpha \ll \Big(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\Big)^{s-2^k} \int_0^1 |f(\alpha)|^{2^k} d\alpha \ll P^{s-k-2^{1-k}\delta+\varepsilon}.$$

The preceding discussion in conjunction with the above major arc discussion then yields

$$\int_0^1 f(\alpha)^s e(-\alpha n) d\alpha \sim \mathfrak{S}(n) \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{s/k-1}, \tag{1.4.4}$$

which in turn implies the bound $G(k) \leq 2^k + 1$ attributed to the latter author.

1.5 Large sieve inequality

The western modern approach of problems within this circle of ideas often relies on the application of the large sieve inequality, whence it has been thought pertinent to defer the above description on the progress in Waring's problem in order to have first a separate discussion about the proof and use of such a powerful utensil. Its first appearance on the stage dates back to the work of Linnik [95] on estimates for the cardinality of sets of natural numbers missing some congruence classes modulo a prime number for a given collection of primes. The modern shape of the large sieve is also applicable to the latter setting but is stated in a more general framework involving L^2-type estimates and has proved to be a very robust tool in many other contexts in analytic number theory. In order to describe the statement of the inequality at hand, we introduce first for $M, N \in \mathbb{Z}$ with $N \geq 1$ and a sequence of real numbers (c_n) the trigonometric polynomial

$$T(x) = \sum_{n=M+1}^{M+N} c_n e(nx).$$

Theorem 1.5.1. *(Large sieve inequality) Consider $\delta > 0$. Let $R \in \mathbb{N}$ and let x_r $(1 \leq r \leq R)$ be a set of real numbers with the property that*

$$||x_r - x_s|| \geq \delta, \qquad r \neq s.$$

Then one has

$$\sum_{r=1}^{R} |T(x_r)|^2 \leq C(N + \delta^{-1}) \sum_{n=M+1}^{M+N} |c_n|^2$$

for some constant $C > 0$.

For the purpose of illustrating our exposition with the circle of ideas underlying such inequalities we will shortly provide a sketch of the proof of the above theorem, but we should note first that if $R = 1$ and $c_n = e(-nx_1)$ then the equality

$$\sum_{r=1}^{R} |T(x_r)|^2 = N \sum_{n=M+1}^{M+N} |c_n|^2$$

holds. Similarly, a simple computation involving orthogonality reveals that

$$\int_0^1 \sum_{r=1}^{R} |T(x + r/R)|^2 dx = R \sum_{n=M+1}^{M+N} |c_n|^2,$$

whence an application of the mean value theorem yields the existence of some $x \in [0, 1]$ for which

$$\sum_{r=1}^{R} |T(x + r/R)|^2 \geq R \sum_{n=M+1}^{M+N} |c_n|^2.$$

The reader may note that the corresponding set of R points are R^{-1} spaced apart modulo 1. In view of the above discussion, it transpires that Theorem 1.5.1 as currently stated is sharp (up to a constant). It might also be convenient to observe that it is a noteworthy feature that whenever N is significantly smaller than δ^{-1} then the diagonal contribution esentially dominates over the non-diagonal one.

We find it desirable to mention that the first appearance of such a formulation in the literature dates back to Davenport and Halberstam [34] in 1966, who delivered the above inequality with $\frac{11}{5} \max(\delta^{-1}, N)$ replacing $C(N+\delta^{-1})$. This bound was ultimately improved independently by Montgomery and Vaughan [100] and Selberg [124], that replaced the aforementioned factor by $(N + \delta^{-1})$, which in view of the preceding discussion is essentially best possible (see Bombieri and Davenport [9] for some intermediate results). Because of its simplicity, we shall prove herein a version of the above due to Gallagher [45]. To this end we shall include first a technical lemma.

Lemma 1.5.1. *(Sobolev) Suppose that $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{C}$ has a continuous derivative. Then one has*

$$\Big|f\Big(\frac{a+b}{2}\Big)\Big| \leq \frac{1}{b-a}\int_a^b |f(x)|dx + \frac{1}{2}\int_a^b |f'(x)|dx.$$

Proof. We observe first that integration by parts yields

$$\int_{(a+b)/2}^{b} f(x)dx = \frac{b-a}{2} f\Big(\frac{a+b}{2}\Big) - \int_{(a+b)/2}^{b} f'(x)(x-b)dx$$

and

$$\int_a^{(a+b)/2} f(x)dx = \frac{b-a}{2} f\Big(\frac{a+b}{2}\Big) - \int_a^{(a+b)/2} f'(x)(x-a)dx.$$

Therefore, summing the above equations, taking absolute values and applying the triangle inequality delivers the desired result. □

Proof of Theorem 1.5.1. For the purpose of getting a sharper constant in the inequality at hand it is desirable to introduce the parameter $K = M + 1 + \lfloor \frac{N}{2} \rfloor$ and write $T(x) = e(Kx)U(x)$, where

$$U(x) = \sum_{n=-\lfloor \frac{N}{2} \rfloor}^{N-\lfloor \frac{N}{2} \rfloor - 1} c_{n+K} e(nx).$$

It will therefore suffice to bound the mean square of $U(x)$ instead. Indeed, Lemma 1.5.1 applied to $U(x)^2$ yields

$$\begin{aligned} \sum_{r=1}^{R} |U(x_r)|^2 &\leq \delta^{-1} \sum_{r=1}^{R} \int_{x_r-\delta/2}^{x_r+\delta/2} |U(x)|^2 dx + \sum_{r=1}^{R} \int_{x_r-\delta/2}^{x_r+\delta/2} |U(x)U'(x)|dx \\ &\leq \delta^{-1} \int_0^1 |U(x)|^2 dx + \int_0^1 |U(x)U'(x)|dx, \qquad (1.5.1) \end{aligned}$$

where in the last line we used the spacing conditon modulo 1 of the sequence (x_r). It then transpires that orthogonality delivers

$$\int_0^1 |U(x)|^2 dx = \sum_{n=M+1}^{M+N} |c_n|^2$$

and

$$\int_0^1 |U'(x)|^2 dx \leq (\pi N)^2 \sum_{n=M+1}^{M+N} |c_n|^2.$$

Moreover, Cauchy's inequality in conjunction with the above lines yields

$$\int_0^1 |U(x)U'(x)| dx \leq \Big(\int_0^1 |U(x)|^2 dx\Big)^{1/2}\Big(\int_0^1 |U'(x)|^2 dx\Big)^{1/2} \leq N\pi \sum_{n=M+1}^{M+N} |c_n|^2,$$

whence by the preceding discussion one obtains

$$\sum_{r=1}^{R} |T(x_r)|^2 \leq (\pi N + \delta^{-1}) \sum_{n=M+1}^{M+N} |c_n|^2,$$

as desired.

The use of the large sieve inequality plays a prominent role in the analysis of pointwise estimates of exponential sums, and we shall give an account of how these ideas have been put into effect in the scene of the refinements on the estimates for $G(k)$ in the next section. It also seems appropiate to mention that the use of the large sieve inequality is of great importance in Chapter 4, whence we find it worth sketching how the application of such a utensil ultimately leads to pointwise bounds for Weyl sums since our treatment herein is inspired by the argument utilised in that context. To this end, a generalisation of the large sieve inequality for l dimensions, whose proof is slightly more intrincate, is required. It has been thought preferable to include instead Gallagher's proof of the one-dimensional version for the sake of prioritising concision over completeness since it already suffices to illustrate the underlying principles.

The aforementioned l-dimensional version of the large sieve (see [141] for a simplified proof) is due to Huxley [77] and Wilson [161] and was originally conceived to extend earlier sieve type bounds and other related results involving the distribution of primes in arithmetic progressions in the context of number fields.

Theorem 1.5.2. *Let $l \geq 2$ and take $\delta_j > 0$ for $1 \leq j \leq l$. Suppose that Γ is a non-empty set of points $\gamma \in \mathbb{R}^l$ with the property that for each $\gamma, \gamma' \in \Gamma$ with $\gamma \neq \gamma'$ the inequality*

$$||\gamma_j - \gamma_j'|| > \delta_j$$

holds for at least one $j \leq l$, where γ_j, γ_j' denote the j-th component of the *corresponding vector. Take natural numbers N_j for $1 \leq j \leq l$. Let $\mathcal{N}$ denote the set of integer tuples $\boldsymbol{n} = (n_1, \ldots, n_l)$ with $1 \leq n_j \leq N_j$ and consider the weighted exponential sum*

$$S(\boldsymbol{\alpha}) = \sum_{\boldsymbol{n} \in \mathcal{N}} c_{\boldsymbol{n}} e(\boldsymbol{n} \cdot \boldsymbol{\alpha})$$

for complex numbers $c_{\boldsymbol{n}}$. Then one has

$$\sum_{\gamma \in \Gamma} |S(\gamma)|^2 \ll \sum_{\boldsymbol{n} \in \mathcal{N}} |c_{\boldsymbol{n}}|^2 \prod_{j=1}^{l} (N_j + \delta_j^{-1}).$$

Equipped with such a result we are ready to sketch how to derive pointwise bounds for Weyl sums employing mean values of such sums. To this end we define, for any integer $x \in \mathbb{N}$ and fixed $k \in \mathbb{N}$ the vector $\nu(x) = (x, x^2, \ldots, x^k)$ and consider, for $\boldsymbol{\alpha} \in [0,1)^k$ the Weyl sum

$$f_k(\boldsymbol{\alpha}) = \sum_{x=1}^{P} e(\boldsymbol{\alpha} \cdot \nu(x)). \tag{1.5.2}$$

We also take, for convenience, the mean value

$$J_{s,k}(P) = \int_{[0,1)^k} |f_k(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha}. \tag{1.5.3}$$

We further introduce a set

$$\mathcal{M} \in [1, P], \qquad M = \mathrm{card}(\mathcal{M}),$$

and observe that for any $m \in \mathcal{M}$ then one has

$$f_k(\boldsymbol{\alpha}) = \sum_{x=1+m}^{P+m} e(\boldsymbol{\alpha} \cdot \nu(x-m)) = \int_0^1 g(m, \beta) \sum_{y=1+m}^{P+m} e(-\beta y) d\beta,$$

where

$$g(m, \beta) = \sum_{x=1}^{2P} e(\boldsymbol{\alpha} \cdot \nu(x-m) + \beta x),$$

and where in the last step we applied orthogonality. Therefore, averaging over the set $\mathcal{M}$ yields

$$f_k(\boldsymbol{\alpha}) \ll M^{-1} \sum_{m\in\mathcal{M}} \sup_{\beta\in[0,1)} |g(m,\beta)| \int_0^1 \min(P, ||\beta||^{-1}) d\beta$$
$$\ll M^{-1} \log(2P) \sup_{\beta\in[0,1)} \sum_{m\in\mathcal{M}} |g(m,\beta)|. \tag{1.5.4}$$

It seems pertinent to observe that

$$\boldsymbol{\alpha}\cdot\nu(x-m) = \alpha_k x^k + \nu_{k-1}(x)\cdot\gamma(m) + \sum_{j=1}^{k}(-m)^j\alpha_j,$$

where $\gamma(m)$ is a $(k-1)$-tuple whose entries are polynomials on m. We also define, for further convenience and $\boldsymbol{n} = (n_1, \dots, n_{k-1}) \in \mathbb{N}^{k-1}$ the complex numbers

$$a(\boldsymbol{n}) = \sum_{x_1,\dots,x_s} e\big(\alpha_k(x_1^k + \dots + x_s^k) + \beta(x_1 + \dots + x_s)\big),$$

where in the above sum the variables $x_i \in \mathbb{N}$ satisfy the system of equations

$$x_1^h + \dots + x_s^h = n_h, \qquad (1 \leq h \leq k-1).$$

It then transpires that an application of Holder's inequality in (1.5.4) delivers

$$|f_k(\boldsymbol{\alpha})|^{2s} \ll M^{-1}(\log(2P))^{2s} \sum_{m\in\mathcal{M}} \Big|\sum_{\boldsymbol{n}} a(\boldsymbol{n}) e\big(\gamma(m)\cdot\boldsymbol{n}\big)\Big|^2,$$

where in the above line $\boldsymbol{n} = (n_1, \dots, n_{k-1})$ runs over integer tuples satisfying $1 \leq n_h \leq sP^h$. It may be noticeable that the right side of the above inequality is eager to be estimated by the $(k-1)$-dimensional version of the large sieve inequality. For the purpose of avoiding a prolix discussion we refer the reader to Chapter 5 of Vaughan [141] for the verification of the spacing condition of the vectors $\gamma(m)$, and confine ourselves to state without proof that for a suitable choice of minor arcs $\mathfrak{m}$ and set $\mathcal{M}$ satisfying card$(\mathcal{M}) \asymp P$ one can verify that the latter condition holds for $\delta_h = P^{-h}$. Therefore, by the preceding discussion in conjunction with Theorem 1.5.2 one has that whenever $\boldsymbol{\alpha} \in \mathfrak{m}$

then

$$|f_k(\boldsymbol{\alpha})|^{2s} \ll (\log(2P))^{2s} P^{k(k-1)/2-1} \sum_{\boldsymbol{n}} |a(\boldsymbol{n})|^2$$
$$\ll (\log(2P))^{2s} P^{k(k-1)/2-1} J_{s,k-1}(P),$$

where we remind the reader that $J_{s,k-1}(P)$ was defined in (1.5.3). We find it appropiate to mention that whenever $s \geq k(k-1)/2$ then it is a consequence of the recent resolution of the main conjecture in Vinogradov's mean value theorem [15] that

$$J_{s,k-1}(P) \ll P^{2s-k(k-1)/2+\varepsilon},$$

whence taking $s = k(k-1)/2$ in the above equation and combining such an estimate with that of the preceding line yields, for $\boldsymbol{\alpha} \in \mathfrak{m}$, the bound

$$f_k(\boldsymbol{\alpha}) \ll P^{1-1/k(k-1)+\varepsilon},$$

as desired. The reader shall rest assured that the next subsection will be partially devoted to give account of the history of Vinogradov's mean value theorem.

1.6 Developments in the minor arc machinery

1.6.1 Asymptotic formula in Waring's problem

We should remark first that all of the approaches mentioned in the minor arc analysis in Section 1.4 further deliver an asymptotic formula for the representation function, and thus find it desirable to have a succinct discussion describing the refinements in the number of variables to secure the validity of such a formula. To this end we define for convenience by $\tilde{G}(k)$ to the smallest s for which (1.4.4) holds. Then, as was noted above, Hardy and Littlewood [51] obtained the bound $\tilde{G}(k) \leq (k-2)2^{k-1}+5$, and Hua improved it to $\tilde{G}(k) \leq 2^k+1$. The latter bound was further refined by Vaughan [136], [138] by showing that $\tilde{G}(k) \leq 2^k$ whenever $k \geq 3$. By utilising a stronger version of the estimate (1.4.3) due to Heath-Brown in conjunction with ideas from [136] then Boklan [7] accomplished the sharpening of the aforementioned bound and obtained $\tilde{G}(k) \leq 7 \cdot 2^{k-3}$ whenever $k \geq 6$. It is a noteworthy feature that hirtherto the foremost bounds for $3 \leq k \leq 5$ are those of Vaughan, while the work of Boklan yields the current best one when $k = 6$. For intermediate values of k the reader shall be referred to [8], [107] [177] in the interest of curtailing the exposition.

For larger k, Vinogradov's mean value theorem plays a prominent role, whence for the purpose of illustrating our discussion we shall introduce the corresponding formulation and elucidate its relevance to the problem at hand. To this end, we find it pertinent to draw the reader's attention to the definitions of $f_k(\boldsymbol{\alpha})$ and $J_{s,k}(P)$ in (1.5.2) and (1.5.3) respectively. A simple argument employing orthogonality and the triangle inequality in conjunction with the consideration of the diagonal solutions already delivers the lower bound $P^{2s-k(k+1)/2} + P^s \ll J_{s,k}(P)$, and leads to the question on whether the estimate

$$J_{s,k}(P) \ll P^{s+\varepsilon} + P^{2s-k(k+1)/2},$$

often referred to as the main conjecture in Vinogradov's mean value theorem, holds or not. The problem of estimating $J_{s,k}(P)$ has an extensive history which we shall shortly and briefly give account of herein and dates back to Vinogradov [148], but for the time being we find it worth noting first that on recalling (1.4.1) and using orthogonality then

$$\begin{aligned}\int_0^1 |f(\alpha)|^{2s} d\alpha &= \sum_{|n_i| \leq sP^i} \int_{[0,1)^k} |f(\boldsymbol{\alpha})|^{2s} e(-\alpha_{k-1}n_{k-1} - \ldots - \alpha_1 n_1) d\boldsymbol{\alpha} \\ &\ll P^{k(k-1)/2} J_{s,k}(P). \qquad (1.6.1)\end{aligned}$$

The reader may recall that progress in the investigation concerning the asymptotic formula at hand esentially hinges on refinements on the estimates for the mean value on the left side of the previous equation, whence in view of the above line of inequalities it transpires that sufficiently strong bounds for $J_{s,k}(P)$ deliver robust estimates for such a mean value which would ultimately lead to establish the validity of such a formula.

Estimates for $\tilde{G}(k)$ employing these circle of ideas were provided first by Vinogradov [148], who accomplished the bound $\tilde{G}(k) \leq 183k^9(\log k + 1)^2$. By improving the estimates for Vinogradov's mean value theorem, Hua [75] refined the latter to $\tilde{G}(k) \leq (4 + o(1))k^2 \log k$. Meanwhile, the work of Linnik [97], Karatsuba [85] and Stechkin [131] led, whenever $s \geq k$ to the satisfactory bound

$$J_{s,k}(P) \ll P^{2s-\frac{1}{2}k(k+1)+\eta_{s,k}}$$

where $\eta_{s,k} = \frac{1}{2}k^2(1-1/k)^{\lfloor s/k \rfloor}$. The reader may find it worth noting that with this notation then a combination of the above bound and (1.6.1) yields

$$\int_0^1 |f(\alpha)|^{2s} d\alpha \ll P^{2s-k+\eta_{s,k}}.$$

Wooley [163] further refined this estimate by showing that, roughly speaking, one may take $\eta_{s,k} \approx k^2 e^{-2s/k^2}$ whenever $s \leq k^2 \log k$ and $\eta_{s,k} \approx (\log k)^3 e^{-3s/2k^2}$ for $s > k^2 \log k$, and managed to accomplish $\tilde{G}(k) \leq (2+o(1))k^2 \log k$ by employing such an improvement in conjunction with standard techniques and the large sieve inequality. In a later memoir, Ford [42] combined such a bound with a more recondite framework of ideas involving an iterative argument reminiscent to that of Wooley [165] in order to establish $\tilde{G}(k) \leq (1+o(1))k^2 \log k$.

Progress concerning Vinogradov's mean value theorem and the asymptotic formula remained stubborn until the seminal work of Wooley [172] and the introduction of his new *efficient congruencing* technique, which established the main conjecture for the range $s \geq k(k+1)$ and the corresponding bound $\tilde{G}(k) \leq 2k^2 + 2k - 3$. The interested reader shall be referred to the paper of Ford and Wooley [43] or Wooley [173] for the purpose of having a better understanding of subsequent improvements within this circle of ideas. We also find it worth drawing the reader's attention to minor refinements in the bound for $\tilde{G}(k)$ accomplished in work of the latter author [171] by making use of the progress available at the time in Vinogradov's mean value theorem in conjunction with a novel mean value estimate over the minor arcs that shares some resemblance with an estimate which we shall deduce herein in Chapter 4.

This whole avenue of new ideas eventually emerged into the resolution of the main conjecture for the whole range of s by Wooley [175] when $k = 3$, and by Bourgain, Demeter and Guth [15] when $k \geq 4$ by making use of decoupling ideas, the case $k = 2$ being classical, and independently by Wooley [177] employing the more flexible efficient congruencing approach for $k \geq 3$. Therein the new estimates in Vinogradov's mean value theorem are employed in conjunction with the ideas from [171] and other manoeuvres in order to obtain the bound

$$\tilde{G}(k) \leq k^2 - k + \sqrt{8k} + O(1),$$

which remains the best one hirtherto. We find it appropiate to mention that some of the ideas involved in the proof of such a bound are used in Chapter 2 to deduce the validity of an analogous asymptotic formula in another context.

1.6.2 Minor arc developments and bounds on $G(k)$

As was previously foreshadowed, all of the minor arc approaches mentioned so far accomplished the validity of the asymptotic formula for the representation function at hand. In contrast, the most prominent improvements in the upper bounds for $G(k)$ depart from the aforementioned treatments in that the variables are restricted to particular subsets of the natural numbers for the purpose of saving more variables in the corresponding minor arc analysis. The first approach in this direction was accomplished by Vinogradov in a long series of papers, the novelty of the new ideas employed in almost all of his memoirs lying on the so-called *diminishing ranges* argument. This method had already been introduced by Hardy-Littlewood [52] and further exploited by Davenport (see [29], [32]). Describing such ideas extensively is hardly the point of this gentle introduction, but for the purpose of illustrating our discussion we shall give a sketch of it in its simplest form.

To this end we take $t \geq 2$, define

$$P_1 = \frac{1}{6}P, \quad P_{j+1} = \frac{1}{2}P_j^{1-1/k}, \quad j \leq t-1,$$

and consider the set of integers $\mathcal{U}$ of the shape

$$u = x_1^k + \ldots + x_t^k, \qquad P_j < x_j \leq 2P_j, \quad j \leq t.$$

We also introduce for convenience the exponential sum

$$S(\alpha) = \sum_{u \in \mathcal{U}} e(\alpha u).$$

The reader may observe that in view of the restrictions on the size of the variables, the only solutions to the equation

$$x_1^k + \ldots + x_t^k = y_1^k + \ldots + y_t^k, \qquad P_j < x_j, y_j \leq 2P_j. \tag{1.6.2}$$

are the diagonal ones (the ones satisfying $x_j = y_j$ for all j). It then transpires that

$$|\mathcal{U}| \asymp P^{k-k(1-1/k)^t} \tag{1.6.3}$$

and that

$$\int_0^1 |S(\alpha)|^2 \ll P^{k-k(1-1/k)^t}, \tag{1.6.4}$$

where the reader should observe that the above mean value counts the number of solutions of the equation (1.6.2), which in turn delivers a saving of $P^{k-k(1-1/k)^t}$ over the trivial bound. This estimate, in conjunction with pointwise bounds of exponential sums derived employing similar ideas and the large sieve inequality already yields bounds of the shape $G(k) \leq Ck \log k$.

We shall not give account of all of the extensive work accomplished by Vinogradov and others (see [23], [135], [144], [145], [146], [147], [149], [150]) concerning estimates of that flavour, but nonetheless find it desirable sketching some of the main ideas to obtain such bounds. For such purposes we take $l \geq 2$, consider $X = P^{1/2}$ and

$$X_1 = \frac{1}{6}X, \quad X_{j+1} = \frac{1}{2}X_j^{1-1/k}, \qquad j \leq l-1, \tag{1.6.5}$$

and denote for every $m \in \mathbb{N}$ by $Q_l(m)$ to the number of solutions of the equation

$$m = x_1^k + \ldots + x_l^k, \qquad X_j < x_j \leq 2X_j.$$

We further introduce, for convenience, the exponential sum

$$H(\alpha) = \sum_{X/2 \leq p < X} \sum_{m \leq X^k} Q_l(m) e(\alpha p^k m),$$

where p runs over the prime numbers. An application of Cauchy's inequality then delivers

$$|H(\alpha)|^2 \ll X \sum_{X/2 \leq p < X} \Big| \sum_{m \leq X^k} Q_l(m) e(\alpha p^k m) \Big|^2,$$

If $\alpha \in \mathfrak{m}$ then one may verify a suitable spacing condition modulo 1 required for the application of the large sieve inequality and combine it with the estimate (1.6.3) to obtain

$$|H(\alpha)|^2 \ll X^{k+1} \sum_{m \leq X^k} Q_l(m)^2 \ll X^{2k+1-k(1-1/k)^l},$$

which then yields

$$|H(\alpha)| \ll X^{k+1/2-k(1-1/k)^l/2}.$$

The reader may observe that this approach saves $X^{1/2-k(1-1/k)^l/2}$ over the trivial bound, which would entail saving a factor of X^c for some fixed $c > 0$ whenever $l = k \log k + C''k$ for fixed $C'' > 0$. Therefore, combining both this estimate and (1.6.4) for the choice $t = k \log k + C'k$ for a big enough constant C' yields

$$\int_{\mathfrak{m}} |S(\alpha)|^2 |H(\alpha)| d\alpha \ll |\mathcal{U}|^{5/2} P^{1/2-k-\delta}$$

for some $\delta > 0$, which, in conjunction with a careful major arc analysis delivers $G(k) \leq 3k(\log k + C)$ for some constant $C > 0$. We find it worth mentioning the bound

$$G(k) \leq 3k(\log k + 9)$$

achieved by Vinogradov [152] by means of the ideas described above. A careful inspection of the manoeuvres underlying the previous discussion reveals that any pointwise minor arc estimate obtained using diminishing ranges type bounds and employing the above circle of ideas could save a factor of say $P^{1/(4k \log k)}$ per variable over the trivial bound. However, the fact that the sizes of the variables are all different makes the analysis less flexible, leaving one in the recalcitrant position of having to employ $k \log k$ variables to only save a factor of $P^{1/4}$.

In a subsequent memoir, Vinogradov [153] further refined the above estimate to

$$G(k) \leq 2k\big(\log k + 2 \log\log k + \log\log\log k + 13/2\big) \tag{1.6.6}$$

for large k. Most of the minor arc saving therein is provided by employing the diminishing ranges argument for counting solutions to (1.6.2) in order to bound the corresponding mean value in a similar fashion as in the earlier work. However, as opposed to the previous approaches, instead of utilising a pointwise bound on the minor arcs relying on estimates for the number of solutions of (1.6.2), the latter author considers a more complicated exponential sum which he estimates by making use of the large sieve inequality in conjunction with an intrincate argument involving diminishing ranges ideas and the use of Vinogradov's mean-value theorem. The use of this pointwise minor arc estimate only entails introducing Ck more variables at the cost of employing $4k \log\log k$ extra variables in the mean value analysis that ultimately saves roughly speaking $k \log k$ variables over the previous approaches.

The estimate of Vinogradov remained unbeaten for a period of more than 25 years, and was in the end superceded by Karatsuba's work [86], which delivered

$$G(k) \leq 2k(\log k + \log\log k + 6k) \tag{1.6.7}$$

whenever $k \geq 4000$, the novelty of which had its reliance on a sharper pointwise minor arc bound. We find it desirable to remark that in the latter memoir there is no improvement in the minor arc saving stemming from the corresponding mean value considered, since the mean value employed therein is of the strenght of those appertaining to diminishing ranges ideas. As was outlined above, the previous approach of Vinogradov relied on a pointwise bound of an exponential sum involving Ck variables on diminishing ranges which was obtained making use of Vinogradov's mean-value theorem, thereby providing a saving of $P^{c/(\log k)^2 \log\log k}$. In contrast, Karatsuba considers instead symmetric diagonal equations involving k-th powers of square-free numbers which are of the same size.

In what follows we shall give an account of a sketched version of the argument employed by the latter author. For such purposes it has been thought pertinent to introduce for $l \in \mathbb{N}$ the parameters

$$P_j = P^{(1-1/k)^{j-1}/k}, \qquad 1 \leq j \leq l$$

and consider the set of square-free numbers defined by

$$\mathcal{V}_l = \Big\{ p_1 \cdots p_l : \quad P_j/4 \leq p_j \leq P_j/2,\ p_j \text{ prime} \Big\}.$$

Observe that the elements of $\mathcal{V}_l$ are of size $P^{1-(1-1/k)^l}(\log P)^{-l}$. The latter author then obtains an estimate for the number $J(P)$ of solutions of the equation

$$x_1^k + \ldots + x_l^k = y_1^k + \ldots + y_l^k, \qquad x_i, y_i \in \mathcal{V}_l \tag{1.6.8}$$

of the shape $J(P) \ll P^{2l-k+(k-l)(1-1/k)^l}$. While these equations provide savings which are slightly inferior to those stemming from the diminishing ranges ideas, they are still robust enough to deliver stronger pointwise estimates when applied in conjunction with the large sieve inequality. We find it worth noting that mean values involving variables of the same shape and size offer wider flexibility when applying the large sieve inequality to obtain those estimates. In order to illustrate these ideas we take X as in (1.6.5) and introduce the exponential sum

$$W(\alpha) = \sum_{X/2 \leq p < X} \sum_{x \in \mathcal{V}_l} e(\alpha p^k x^k), \tag{1.6.9}$$

where as customary p runs over the primes. We also find it worth defining $r_l(y)$ as the number of solutions of the equation

$$y = x_1^k + \ldots + x_l^k, \qquad x_i \in \mathcal{V}_l.$$

It is apparent that an application of Holder's inequality yields

$$\begin{aligned} |W(\alpha)|^{2l} &\ll X^{2l-1} \sum_{X/2 \leq p < X} \Big| \sum_{x \in \mathcal{V}_l} e(\alpha p^k x^k) \Big|^{2l} \\ &= X^{2l-1} \sum_{X/2 \leq p < X} \Big| \sum_{y \leq X^k} r_l(y) e(\alpha p^k y) \Big|^2, \end{aligned}$$

whence we have reached a position from which to bound the above term via the large sieve inequality, and thus get, ignoring the verification of the spacing modulo 1, the estimate

$$|W(\alpha)|^{2l} \ll X^{2l-1+k} \sum_{x \leq X^k} r_l(y)^2 \ll X^{4l-1+(k-l)(1-1/k)^l},$$

whenever $\alpha \in \mathfrak{m}$, where the sum involved in the above equation counts the number of solutions of (1.6.8) and where we applied the bound right after that equation. It then transpires that the choice $l = \lfloor k(\log k + 2 \log\log k) \rfloor$ leads to the bound

$$\sup_{\alpha\in\mathfrak{m}}|W(\alpha)| \ll X|\mathcal{V}_l|P^{-1/(4k\log k+c_k)},$$

wherein the constant c_k satisfies $c_k = o(4k\log k)$ and can be made explicit.

We feel obliged to mention that in the original treatment of Karatsuba there is no explicit mention to the large sieve inequality, but the manouvers deployed therein are very much reminiscent of its proof. However, his approach is somewhat less efficient and ends up delivering the weaker estimate

$$W(\alpha) \ll P^{1-1/24k\log k},$$

which nonetheless suffices to sharpen substantially Vinogradov's previous bound. The author's approach then achieves most of the minor arc saving by utilising a mean value estimate involving a p-adic analogue of the diminishing ranges argument of the same strength, thus employing $2k\log k + 2\log\log k + Ck$ variables in due course, and hence only necessitating $C'k$ extra copies of $W(\alpha)$ to provide the rest of the minor arc saving required.

The seminal paper of Vaughan [139], which completely changed the scenery in the subject and laid the foundations of the modern approach to problems within this circle of ideas made significant improvements in the bounds on $G(k)$ for small k and other related questions involving small exponents. However, the ideas underpinning these developments in the memoir barely managed to provide a small refinement of the bound (1.6.7). The main novelty of the paper is the use of variables of the same size that are smooth.

In order to put ideas into effect, we find it convenient to define, for parameters $P, R > 0$ the set of smooth numbers

$$\mathcal{A}(P,R) = \{n \in [1,X]\cap\mathbb{N} : p \mid n \text{ and } p \text{ prime} \Rightarrow p \leq R\}.$$

The latter author considers the equation

$$x_1^k + \ldots + x_s^k = y_1^k + \ldots + y_s^k, \quad x_j, y_j \in \mathcal{A}(P,R) \tag{1.6.10}$$

whose number of solutions can be expressed by orthogonality as a mean value of smooth Weyl sums, relates it to the number of solutions of

$$x^k + m^k(x_1^k + \ldots + x_{s-1}^k) = y^k + m^k(y_1^k + \ldots + y_{s-1}^k), \tag{1.6.11}$$

wherein

$$x \leq P, \quad y \leq P, \quad x_j, y_j \in \mathcal{A}(P^{1-\theta}, R), \quad P^{\theta} < m \leq \min(P, P^{\theta} R)$$

for some $0 < \theta \leq 1/k$ via an application of Holder's inequality in conjunction with the use of the structure of smooth numbers and runs an iterative argument involving a differencing process. The mean value estimates cognate to the equations involving smooth variables obtained in the latter memoir are of the same strenght as those stemming from diminishing ranges ideas. Nonetheless, the use of smooth numbers in this context provides greater flexibility for both the estimations of the number of solutions of (1.6.10) and the investigations of pointwise upper bounds via the large sieve inequality combined with an application of Holder's inequality and the mean value estimates obtained thereof. The latter author then employs the same argument that was outlined in (1.6.9) utilising instead variables belonging to the set $\mathcal{A}(P, R)$, which ultimately leads to a pointwise bound that saves a factor of roughly $P^{1/4k \log k}$ and enables him to deduce the estimate

$$G(k) \leq 2k\left(\log k + \log\log k + 1 + \log 2 + O\Big(\frac{\log\log k}{\log k}\Big)\right). \tag{1.6.12}$$

Despite obtaining such modest improvement, a modification of the approach followed by Vaughan permitted Wooley [162], [164] to essentially reduce the above upper bound by a factor of 2, namely

$$G(k) < k\left(\log k + \log\log k + 2 + \log 2 + O\Big(\frac{\log\log k}{\log k}\Big)\right). \tag{1.6.13}$$

His treatment departs from the previous one in the differencing process when running the iterative argument, and delivers a bound for the number of solutions of (1.6.10) of the shape $P^{2s-k+ke^{1-2s/k}}$. The earlier strategy for obtaining pointwise minor arc estimates combined with the aforementioned bound thus essentially yields a power saving of $P^{1/2k \log k}$ and suffices to establish (1.6.13). We find it desirable to finish the discussion by noting that an analogous bound to (1.6.13) has also been obtained by the latter author [167] without the $\log 2$ summand by deriving a pointwise estimate for smooth Weyl sums which ultimately saves a factor of $P^{1/k \log k}$.

1.7 Waring's problem in thin sets

The main body of the present work shall comprise results in both additive number theory and the subject of moments of the Riemann zeta function, and succintly motivating and describing those pertaining to the use of the circle method is the purpose of the present section.

We begin our discussion by drawing the readers attention back to the problem of representing every sufficiently large integer n as

$$n = x_1^k + \ldots + x_s^k. \tag{1.7.1}$$

It transpires that a natural direction that one may pursue is that of restricting each of the underlying variables to thin subsets in the above equation and analyse the analogous questions which were addressed for the original problem. In order to progress in the discussion, some notation is required. We thus denote for convenience by $G_A(k)$ to the least positive integer s with the property that every sufficiently large natural number n posseses a representation of the shape (1.7.1) with $x_i \in A$. We recall the bound (1.6.13) to the end of noting that a consideration of the set of l-th powers then lends credibility to the heuristic that, for $A \subset \mathbb{N}$ nicely distributed over arithmetic progressions and satisfying the property $|A \cap [1, N]| \asymp N^\alpha$ for some $0 < \alpha < 1$ then one would ideally expect to achieve

$$G_A(k) \leq \frac{k}{\alpha}\big(\log(k/\alpha) + O(\log\log(k/\alpha)\big). \tag{1.7.2}$$

The reader may find it useful to note that the above bound would hold for the aforementioned set of l-th powers in view of (1.6.13). However, the methods available to anaylise the classical Waring's problem fail hugely to deliver such conclussions for even particular non-trivial examples, and so the above problem, as currently stated, is very hard. It is also worth noting that when the set A satisfies $|A \cap [1, N]| \gg N^{1-\varepsilon}$ then some of the techniques employed in the field may be applicable to deliver quite sharp conclusions which greatly simplify things. We deliberately avoid such a situation in the present memoir and concentrate on instances for which such a property is not known to hold.

A significant portion of the analysis in the additive number theoretic sections is devoted to the examination of the above problems for $A = \mathscr{C}$, where $\mathscr{C}$

is the set of natural numbers represented as the sums of three positive cubes, it being only conjectured that $\mathscr{C}$ have positive density, the best current lower bound on the cardinality of the set being $\mathcal{N}(X) = |\mathscr{C} \cap [1, X]| \gg X^{\beta}$, where

$$\beta = 0.91709477, \tag{1.7.3}$$

due to Wooley [170]. The first result that shall be presented herein and is discussed in Chapter 2 concerns the analysis of the validity of the asymptotic formula for the corresponding representation function. For such purposes we introduce for fixed k, s and $n \in \mathbb{N}$ the counting function $R_{k,s}(n)$, which we define as the number of solutions of

$$n = (x_{1,1}^3 + x_{1,2}^3 + x_{1,3}^3)^k + \ldots + (x_{s,1}^3 + x_{s,2}^3 + x_{s,3}^3)^k$$

for $x_{i,j} \in \mathbb{N}$.

Theorem 1.7.1. *Let $s \geq 9k^2 - k + 2$. Then, there exists a constant $\delta > 0$ such that*

$$R_{k,s}(n) = \Gamma\big(4/3\big)^{3s}\Gamma\big(1 + 1/k\big)^{s}\Gamma\big(s/k\big)^{-1}\mathfrak{S}(n)n^{s/k-1} + O(n^{s/k-1-\delta}),$$

where $\mathfrak{S}(n)$ here is the product of some local densities and satisfies $\mathfrak{S}(n) \gg 1$.

The counting of the multiplicity of each sum of three cubes enables one to express $R_{s,k}(n)$ as the Fourier coefficient of an exponential sum that we can treat using conventional methods. Experts may then recognise the difficulty associated to the problem when one instead removes the counting of the multiplicities of each sum of three cubes. Attaining an analogous evaluation with the current knowledge seems out of reach, and we content ourselves with deriving a non-trivial lower bound by obtaining an asymptotic formula for a suitable representation function involving smooth numbers. Without further delay, it seems appropiate to define $r(n)$ by the number of solutions of (1.7) but with each sum of three cubes involved being counted just once.

Theorem 1.7.2. *Let s be any positive integer with $s \geq 9k^2 - k + 3$. One has the lower bound*

$$r(n) \gg n^{\beta s/k-1},$$

where β was defined right after (1.7.3).

It is a noteworthy feature that the preceding lower bound may be the best possible estimate attainable with the current knowledge available.

We shift our focus to the problem of minimising the number of variables s needed to represent every sufficiently large number as a sum of k-th powers of sums of three cubes, it being discussed in Chapter 3, and denote for convenience by $G_3(k)$ to the minimum such s. It is often the case in additive problems involving small powers of positive integers that some arguments utilised to analyse those are not applicable in the setting of bigger exponents, such an observation motivating having a discussion about small values of k. A naive approach to bounding $G_3(k)$ would then be to replace each sum of three cubes by the specialisation $3x^3$, and this suggests a bound of the shape $G_3(k) \leq G(3k)$. Before giving account of the results attained in our work it is worth recalling the estimates $G(6) \leq 24$ due to Vaughan-Wooley [142], $G(9) \leq 47$ and $G(12) \leq 72$ due to Wooley [172].

Theorem 1.7.3. *One has $G_3(2) \leq 8$, $G_3(3) \leq 17$ and $G_3(4) \leq 57$.*

The above result confirms that we actually use the three integral cubes non-trivially in our argument. The key ingredient to the proof is a pointwise bound for some exponential sum over the minor arcs that we obtain by following the treatment of Vaughan [139]. We find it desirable to mention that the mean value estimate utilised in the problem for $k = 4$ makes use of a bound for a mean value of smooth Weyl sums of exponent 12. Nonetheless, it seems pertinent to point out that the classical diminishing ranges argument is more efficient than the approach taken herein and ultimately saves a handful of variables.

In view of the above result for squares, experts may wonder whether the minor arc analysis pertaining to that instance could be combined with a careful major arc treatment to derive an almost all result for the analogous Lagrange's four-square theorem when one restricts the variables to lie on $\mathscr{C}$. Confirming this belief is, inter alia, the purpose of the following theorem discussed in Chapter 3.

Theorem 1.7.4. *Almost all natural numbers n have a representation of the shape*

$$n = (x_1^3+x_2^3+x_3^3)^2+(x_4^3+x_5^3+x_6^3)^2+(x_7^3+x_8^3+x_9^3)^2+(x_{10}^3+x_{11}^3+x_{12}^3)^2, \qquad x_i \in \mathbb{N}.$$

It seems worth anticipating that the bound on the cardinality $E(N)$ of exceptional set that is derived in this memoir is of the shape

$$E(N) \ll N(\log N)^{-4/31+\varepsilon}.$$

We find it desirable to note that one could utilise ideas from the paper [21] to get $E(N) \ll N^{1-\delta}$ for some (possibly microscopic) $\delta > 0$, the achievement of such a refinement hardly being the point of this memoir in view of the poor description available of the constant δ.

The last section comprising the circle method part of this thesis examines the analogous problem when one replaces the set of integers represented by sums of three cubes in the above setting by that of numbers represented as a sum of t positive l-th powers. The purpose of our investigation in this new setting is to establish uniform bounds with respect to l in the number of variables that enables one to assure the validity of a lower bound of the right order of magnitude for the corresponding representation function rather than to sharpen the k dependence of such bounds. In order to embark in such an endeavour, we find convenient to define, for $k, l, t \in \mathbb{N}$ the set

$$\mathcal{T}_t = \left\{ x_1^l + \ldots + x_t^l : \ \mathbf{x} \in \mathbb{N}^t \right\}$$

and anticipate that we shall restrict our attention to the analysis of the solubility of (1.2.1) for the choice $A = \mathcal{T}_t$ with t lying on different regimes.

For such purposes, it seems worth fixing k and begin by considering the case $t = C(k)l$, where $C(k)$ is a fixed integer-valued function. It is worth mentioning that the sharpest estimate available for the cardinality of the above set is of the shape $|\mathcal{T}_t \cap [1, N]| \gg N^{c_k}$ for some constant $0 < c_k < 1$, such a conclusion being easily derived by a routinary diminishing ranges argument. Following the preceding discussion and in view of the heuristics that suggested (1.7.2) it becomes natural to consider the problem of finding an integer $s(k)$ only depending on k with the property that every sufficiently large integer n can be expressed as

$$n = (x_{1,1}^l + \ldots + x_{1,C(k)l}^l)^k + \ldots + (x_{s(k),1}^l + \ldots + x_{s(k),C(k)l}^l)^k, \qquad x_{i,j} \in \mathbb{N}.$$

The reader may observe that for fixed k then the right side of the above equation consists of sums of $C'l$ positive integral l-th powers gathered in groups and raised to the power k for some constant $C' = C'(k) > 0$ depending on k. It then transpires that accomplishing such an endeavour seems even harder than deriving the bound $G(l) \leq C'l$, which would in turn be a big breakthrough in view of (1.6.13). Despite not being able to achieve such a result, we derive the following weaker version comprising the introduction of k-th powers of natural integers for the purpose of handling a wider range of the major arcs.

Theorem 1.7.5. *Let $k, l \geq 2$ and let $C(k)$ be an integer valued function with the property that $C(k) \geq \log k + 4$. Let $t = C(k)l$. Then there is some parameter satisfying $s_0(k) = k^2 + O(k)$ such that for $s \geq s_0(k)$ and every sufficiently large n one has*

$$n = (x_{1,1}^l + \ldots + x_{1,t}^l)^k + \ldots + (x_{s,1}^l + \ldots + x_{s,t}^l)^k + \sum_{i=1}^{4} y_i^k, \tag{1.7.4}$$

wherein $y_i, x_{i,j} \in \mathbb{N}$.

It is worth noting that the bound (1.6.13) reveals that for sufficiently large k the summands involving the diagonal forms in the above equation are remarkably exploited in order to attain such a representation. The novel idea of the proof relies on a pointwise bound of a suitable exponential sum over the minor arcs that is uniform on l which we obtain via an application of the large sieve inequality. It is a noteworthy feature that the approach taken herein further establishes a lower bound in the number of representations of the expected size, the diminishing ranges approach failing to achieve such an endeavour.

Before describing another result concerning a different regime for t it is convenient to define $\xi_0(k,l) = \lceil l/2\big(\log l + \log\big(k(k+1)\big) + 2\big)\rceil$. We find it desirable to note that with the current knowledge available one may only be able to achieve $|\mathcal{T}_{\xi_0(k,l)} \cap [1,N]| \gg N^{1-c/k^2 l}$ for some $c > 0$.

Theorem 1.7.6. *Let $k, l \geq 2$ and take $\xi \geq \xi_0(k,l)$ and $s \geq s_0(k)$, where $s_0(k)$ is a parameter satisfying $s_0(k) = k^2 + O(k)$. Then every sufficiently large n can be represented as*

$$n = (x_{1,1}^l + \ldots + x_{1,\xi}^l)^k + \ldots + (x_{s,1}^l + \ldots + x_{s,\xi}^l)^k,$$

where $x_{i,j} \in \mathbb{N}$.

The reader should observe that the above result is uniform in l and does not require any additional summands of different nature at the cost of taking diagonal forms with more variables.

We find it desirable to complete this section by anticipating that the mean value estimates utilised in the course of the minor arc analysis pertaining to each of the problems either makes use of available mean value estimates corresponding to the instance when $A = \mathbb{N}$ in the above setting or utilises mean values associated to the natural polynomial structure cognate to the corresponding sets. It therefore transpires that in view of the poor bounds in the cardinality of the corresponding sets analysed, the bounds in the number of variables deduced in this memoir for large k are a long way off from the desired ones.

1.8 Moments of the Riemann zeta function

The use of analytic methods in the investigation of the properties of the Riemann zeta function to derive results concerning the distribution of prime numbers and related problems has a long history and in modern form esentially dates back to the unconditional proof of the prime number theorem by Hadamard and de la Vallée Poussin in 1896. Shortly after, a formidable amount of work following various different avenues has led to suprisingly good results in analytic number theory, the corresponding conjectural counterparts pertaining to those results being in most occasions far out of reach. Providing an extensive historical overview of the progress within the field is hardly the point of the present introduction. We shall nonetheless give a brief account of how moments of the Riemann zeta function and problems within this circle of ideas play a prominent role in the area and may be applied in the investigation of prime numbers to the end of motivating their analysis. For such purposes it seems worth defining the aforementioned Riemann zeta function by means of the formula

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\qquad \mathrm{Re}(s)>1,$$

and note that it can be analytically extended to a meromorphic function on the plane. We further introduce, for convenience, the function

$$\psi(x) = \sum_{n \leq x} \Lambda(n),$$

wherein $\Lambda(n)$ denotes the von Mangoldt function, defined by means of the relation

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{else.} \end{cases}$$

We also find it useful to write $\psi_0(x) = \psi(x)$ if x is not a perfect power and $\psi(x) - \frac{1}{2}\Lambda(x)$ when it is. It is a noteworthy feature that the classical proof of the prime number theorem utilising complex analytic methods has its genesis in the formula

$$\psi_0(x) = x - \sum_{\substack{\rho = \beta + i\gamma \\ |\gamma| < T}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}) + R(x, T),$$

wherein the above sum runs over the zeros of $\zeta(s)$ and the function $R(x, T)$ is essentially of small size. The reader may also find it useful to recall, if necessary, that the prime number theorem is equivalent to the assertion $\psi_0(x) \sim x$. In view of the above consideration, it transpires that improvements pertaining to the error term in the preceding asymptotic relation may have their reliance on zero-free regions of the Riemann-zeta function. A careful examination of the preceding formula may also reveal that results cognate to the existence of primes in short intervals may potentially stem from the sharpening of zero-density estimates, such being an avenue explored by a not unformidable number of scholars. The interested reader shall be referred to Huxley [78] or Baker, Harman and Pintz [3].

The aforementioned bounds for the number of zeros in rectangles of large height may be derived employing various techniques, the estimates of moments of zeta or certain Dirichlet polynomials having the potential of achieving so (see Ingham [81], Jutila [83]), and among those, that of utilising pointwise estimates of the shape

$$\zeta(1/2 + it) \ll t^\delta \tag{1.8.1}$$

for $0 < \delta < 1$ plays a prominent role, as it becomes apparent after a perusal of the papers of, inter alia, Huxley [78] or Jutila [83].

The problem of estimating (1.8.1) has a long history and dates back to the work of Weyl, it being the first instance in the literature for which the convexity bound $t^{1/4}$ was broken, thereby obtaining $\delta = 1/6$ instead. Considerations of space preclude us from providing an account of the extensive sequel of improvements in this direction, but for the purpose of illustrating the discussion we shall content ourselves by mentioning that the foremost bound due to Bourgain [13] achieves (1.8.1) with $\delta = 13/84 + \varepsilon$ by combining decoupling estimates with the previous work of Bombieri-Iwaniec [10] and Huxley [79]. We take this as an opportunity to emphasize that it is expected that

$$\zeta(1/2 + it) \ll_{\varepsilon} t^{\varepsilon},$$

such a conjecture often being referred to as the Lindelöf Hypothesis.

The above estimate, as previously foreshadowed, would have many implications in the analytic theory of numbers but seems completely out of reach with the current technology available. Nevertheless, the problem of estimating the above on average, in a suitable sense, turns out to be more tractable. For such purposes, it seems worth defining for real $k > 0$ the integral

$$M_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt.$$

We shall give a succint overview of what is known about the above object, but nonetheless find it worth mentioning to the end of further motivating the analysis of moments that the Lindelöf hypothesis is equivalent to the bound

$$M_k(T) \ll T^{1+\varepsilon} \qquad \text{for all } k \in \mathbb{N}. \tag{1.8.2}$$

The first succesful asymptotic evaluation dates back to the early work of Hardy-Littlewood (1918) concerning the second moment, to which the reader shall be referred to [134, Theorem 7.3], by making a clever use of the approximate functional equation for $\zeta(s)$ in a suitable shape. After the efforts of many scholars invested in such an endeavour, it currently takes the form

$$M_1(T) = T \log(T/2\pi) + (2\gamma - 1)T + E_1(T)$$

with $E_1(T)$ satisfying bounds of the shape $E_1(T) \ll T^{\Delta}$ for some fixed $\Delta > 0$, the sharpest of which follows after work of Bourgain and Watt [16] and may be taken to be $\Delta = 1515/4816+\varepsilon$. We find it desirable to conclude the discussion concerning the second moment by presenting the estimate

$$\zeta(1/2+it)^2 \ll (\log t)^4 + (\log t) \max E_1\{t \pm (\log t)^2\}$$

due to Heath-Brown [61]. It then transpires that upper bounds for $E_1(T)$ deliver pointwise estimates for $|\zeta(1/2+it)|$, such an approach having the recalcitrant limitation that $E_1(T) = \Omega(T^{1/4})$, it being an immediate consequence of the formula

$$\int_0^T E_1(t)^2 dt = \frac{2}{3}(2\pi)^{-1}\frac{\zeta(3/2)^4}{\zeta(3)}T^{3/2} + O(T^{5/4}\log^2 T)$$

provided by the latter author [58]. Eight years after the aforementioned milestone of Hardy-Littlewood, Ingham [80] utilised the approximate functional equation for $\zeta(s)^2$ that at that time had recently been introduced by Hardy-Littlewood [53] to obtain

$$M_2(T) = \frac{1}{2\pi^2}T\log^4 T + O(T\log^3 T), \tag{1.8.3}$$

the deficiency in such an approach having its reliance, inter alia, on the concomitant aspect that the corresponding error term pertaining to the aforementioned approximation is too large. These complications were circumvented in the work of Heath-Brown [59] by utilising instead an approximate functional equation for $|\zeta(1/2+it)|^4$, thereby considerably reducing the error term arising after the application of such an approximation, and ultimately led to the evaluation

$$M_2(T) = TP_4\big(\log T\big) + E_2(T), \tag{1.8.4}$$

wherein $P_4(x)$ is a quartic polynomial and $E_2(T) \ll T^{7/8+\varepsilon}$. It seems desirable to add that the off-diagonal term contributes to the lower order terms in the above formula, the analysis of which makes a crucial use of estimates of Kloosterman sums due to Weil [155]. We find it useful to add for the purpose of illustrating the historical discussion that Atkinson [1] had previously obtained an analogous formula to that of (1.8.4) for a smooth version of the fourth moment, a recalcitrant aspect of such a result being that it does not provide any effective information about the error term in (1.8.3).

The above bound on $E_2(T)$ was eventually refined in the work of Zavorotnyi [178] and Ivic-Motohashi [82] by making use of the spectral theory of automorphic forms, thus ultimately leading to the estimate

$$E_2(T) \ll T^{2/3+\varepsilon},$$

which essentially remains best possible.

However, the extensive investigations done pertaining to the asymptotic behaviour of higher moments, despite the large efforts invested in such an endeavour, have been so far conjectural. This avenue was first pursued in the work of Conrey-Gosh [26] in their paper concerning the asymptotic evaluation of $M_3(T)$ which ultimately led to the conjectural result

$$\int_0^T |\zeta(1/2+it)|^6 dt \sim \frac{42}{9!} \prod_p \Big((1-1/p)^4 \big(1 + \frac{4}{p} + \frac{1}{p^2}\big)\Big),$$

the main idea latent in the argument having its reliance on a conjectural formula for the second moment of the Riemann zeta function twisted by the square of a Dirichlet polynomial of length T, an analogous formula holding if the length of the corresponding polynomial be $T^{1/2-\varepsilon}$. To the end of providing a historical background it shall be noted that hirtherto it was believed that

$$M_k(T) \sim c_k T (\log T)^{k^2} \tag{1.8.5}$$

would hold for $k > 0$, but the precise value of the constants c_k had not been provided for any k.

The previous paper was followed by a memoir of Conrey-Gonek [27] which encompassed a method to conjecturally evaluate both the sixth and the eighth moment by pursing a not dissimilar approach in conjunction with new ideas concerning mean values of long Dirichlet polynomials and the use of the δ-method of Duke, Friedlander and Iwaniec to compute divisor correlation sums.

The conjectural investigations of the asymptotic formula for the moments were independently culminated by Keating-Snaith [89] with the incorporation of Random Matrix theory to the picture, which ultimately led to establishing (1.8.5) with precise values for the constants c_k. It should nonetheless be noted for the purpose of merely illustrating the present introduction that further conjectural examinations pertaining to higher order moments have been pursued by Conrey-Farmer-Keating-Rubinstein-Snaith [25] by bringing into use the so-called recipe, thereby delivering analogous conclusions of the shape

$$M_k(T) = TP_{k^2}\big(\log T\big) + O(T^{1/2+\varepsilon})$$

for $k \geq 3$, wherein $P_{k^2}(x)$ is a degree-k^2 polynomial.

We shall shift our focus to the analysis of lower and upper bounds for moments, such a consideration providing more flexibility to the extent that under further assumptions, sharp estimates may be achieved for real $k > 0$ rather than just for integers. Lower bounds of the shape

$$M_k(T) \gg T(\log T)^{k^2}$$

were already established for integral $2k$ by Ramachandra [117], such a result being extended to positive rational numbers in the work of Heath-Brown [60] and accomplished for real $k > 0$ under the assumption of RH by Ramachandra [116]. Such a condition was ultimately removed in a paper of Radziwill and Soundararajan [115] when $k \geq 1$.

As was foreshadowed earlier, unconditional upper bounds for arbitrarily large k of the expected shape are a long way off, the weaker inequality (1.8.2) being equivalent to the Lindelof Hypothesis. It should not therefore come as a surprise that the majority of the results obtained in this direction are conditional. The first investigations following this trend are due to Ramachandra [117], [118] and Heath-Brown [62], thus delivering the bounds

$$M_k(T) \ll T(\log T)^{k^2}$$

for $0 \leq k \leq 2$ under the assumption of RH, such a range being further extended to 2.18 in a later paper of Radziwill [114]. However, it wasn't till the breakthrough of Soundararajan [128] that an analogous result was established for all $k > 0$ at the cost of deriving the weaker estimate

$$M_k(T) \ll T(\log T)^{k^2+\varepsilon},$$

the ε in the exponent ultimately being removed in a subsequent paper of Harper

[56]. We find it desirable to conclude this discussion by remarking that the best unconditional result improving earlier work of many others accomplishes the estimate $M_k(T) \ll T(\log T)^{k^2}$ for the range $0 \leq k \leq 2$ and is due to Heap-Radziwill-Soundararajan [57].

1.9 Mixed moments of the Riemann zeta function

We shall complete the introduction with a description of the problems concerning moments of the Riemann zeta function in this memoir. We find it desirable to anticipate, as opposed to what the reader may have guessed in view of the brevity of the former section, that the body of this thesis shall comprise a longer discussion devoted to the analysis of such problems than the circle method counterpart, the main reasons lying on both the prolixity of some of the routinary computations associated to such investigations and the considerations of space and time in the former section devoted to the historical introduction of those problems.

Hirtherto, little attention has been paid to the problem of examining mixed moments of the type

$$I_{\boldsymbol{a}}(T) = \int_0^T \zeta(1/2 + ia_1 t) \cdots \zeta(1/2 + ia_k t) dt \tag{1.9.1}$$

for $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k$ and $k \in \mathbb{N}$, such a collection of integrals being an interesting source of examples from which to look for similar, or perhaps dissimilar behaviour exhibited in the analysis of moments of L-functions. We find it worth announcing that the central object of study in this memoir will be the above mixed moment for $k = 3$, space and time limitations forcing us to defer analogous considerations for $k = 4$. For the sake of clarity we find it worth defining, for positive real numbers $a, b, c \in \mathbb{R}_+$ the integral

$$I_{a,b,c}(T) = \int_0^T \zeta(1/2 + iat)\zeta(1/2 - ibt)\zeta(1/2 - ict) dt.$$

In Chapter 5 we use the approximate functional equation for each of the zeta factors to obtain several results which we now describe.

Theorem 1.9.1. *Let $T > 0$ and $a, b, c \in \mathbb{N}$ with the property that $(a, b, c) = 1$. Then, whenever $a < c \leq b$ one has the asymptotic evaluation*

$$I_{a,b,c}(T) = \sigma_{a,b,c}T + E_{a,b,c}(T),$$

where $\sigma_{a,b,c} > 1$ is a computable constant and

$$E_{a,b,c}(T) \ll T^{1-1/2a+1/2c} + T^{3/4+a/4c}.$$

If a weak version of the abc conjecture is assumed, the above error term may be refined to obtain

$$\begin{aligned} E_{a,b,c}(T) \ll &T^{1/2+a/(a+c)+\varepsilon} \\ &+ (\log T)^2\big(T^{3/4} + T^{3/4-a/4c+(a,c)/2c} + T^{3/4-a/4b+(a,b)/2b}\big). \end{aligned}$$

We next shift our focus to the case when $a, b, c \in \mathbb{R}$. Experts in the community may recognise the little amount of correlation of the zeta factors in (1.9.1) that one expects whenever these coefficients are linearly independent. Quantifying this belief is the end of the following theorem.

Theorem 1.9.2. *Let $T > 0$ and $a, b, c \in \mathbb{R}$ be algebraic numbers linearly independent over $\mathbb{Q}$. Then one has*

$$I_{a,b,c}(T) \sim T.$$

The last part of the study in Chapter 5 is devoted to the examination of the case $a = c$, a new framework of ideas being required in order to succesfully accomplish the asymptotic evaluation.

Theorem 1.9.3. *Let $a < b$ be natural numbers satisfying $(a, b) = 1$. Then one has the asymptotic formula*

$$I_{a,b,a}(T) \sim \zeta\big((a + b)/2\big)T \log T.$$

In Chapter 6, a different approach is taken to analyse (1.9.1), the theorem stemming from such an examination being of the same type as that of Theorem 1.9.1 with a refined error term in certain cases. We avoid giving account of the precise statement herein due to space limitation purposes.

The thesis is completed with a theorem in a dissimilar yet not entirely intrincate context, some of the techniques earlier used not being required therein in favour of a more explicit control of the corresponding lower order terms. We deduce a formula of a shape with few precedents in the literature save some third moments of quadratic Dirichlet L-function evaluation. We define for such purposes the integral

$$I(T)=\int_0^T \zeta(1/2+2it)\zeta(1/2-it)^2 dt \tag{1.9.2}$$

and anticipate the main theorem concerning its evaluation.

Theorem 1.9.4. *For $T>0$ one has the asymptotic formula*

$$I(T)=c_1T+c_2T^{7/8}+O\big(T^{3/4}(\log T)^3\big),$$

where the constants $c_1,c_2\in\mathbb{C}$ are defined by means of the equations

$$c_1=\frac{\zeta(3/2)^3}{2\zeta(3)}(3-i),$$

and

$$c_2=\frac{32(2\pi)^{1/8}}{7}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}i.$$

Chapter 2

Asymptotic formula and lower bounds for the representation functions

2.1 Introduction[1]

It is widely believed, but still unknown, that the set of integers $\mathscr{C}$ represented as a sum of three positive integral cubes has positive density. Hardy and Littlewood [50] first announced what is known as the Hypothesis-K, which asserts that for each $\varepsilon > 0$, the number of representations $r_k(n)$ of n as a sum of k positive integral k-th powers is $O(n^{\varepsilon})$. Although this conjecture is known to be false when $k = 3$ (see Mahler [98]), the weaker claim that

$$\sum_{n \leq X} r_k(n)^2 \ll X^{1+\varepsilon}, \tag{2.1.1}$$

known as Hypothesis K^* (see [71]), would allow one to show, through a standard Cauchy-Schwarz argument, that $\mathcal{N}(X) = |\mathscr{C} \cap [1, X]| \gg X^{1-\varepsilon}$. In fact, under some unproved assumptions on the zeros of some Hasse-Weil L-functions, Hooley ([70], [71]) and Heath-Brown [63] showed using different procedures that (2.1.1) holds for $k = 3$. Nevertheless, some unconditional progress has been made on strengthening lower bounds for $\mathcal{N}(X)$. By using methods of diminishing ranges, Davenport [31] obtained the bound $\mathcal{N}(X) \gg X^{47/54-\varepsilon}$. Later

[1]This chapter is based on material by the author [113] that is published in Mathematika.

on, Vaughan improved it to $\mathcal{N}(X) \gg X^{11/12-\varepsilon}$ by introducing smooth numbers in his "new iterative method" [139], and Wooley, extending the method to obtain non-trivial bounds for fractional moments of smooth Weyl sums, improved the estimate in a series of papers ([166], [168], [174]), the best current one being $\mathcal{N}(X) \gg X^{\beta}$, where $\beta = 0.91709477$.

A vast number of results can be found in the literature on problems involving equations over special subsets of the integers. The Green-Tao Theorem [49], which proves the existence of arbitrarily long arithmetic progressions over the primes is an example of such problems when the special set is the set of prime numbers. Other instances where the set $\mathscr{C}$ is involved include some correlation estimates for sums of three cubes by Brüdern and Wooley [22], and lower bounds of the shape $N_3(\mathscr{C}, X) \gg X^{5/2-\varepsilon}$, by Balog and Brüdern [6]. The parameter $N_3(\mathscr{C}, X)$ here denotes the number of triples with entries in $\mathscr{C} \cap [1, X]$ whose entries averages lie on $\mathscr{C}$ as well.

In this paper we investigate the asymptotic formula for Waring's problem when the set of k-th powers of integers is replaced by the set of k-th powers of elements of $\mathscr{C}$, but before stating the main result that we obtain here it is convenient to introduce some notation. Let $k \geq 2$ and $n \in \mathbb{N}$. Take $P = n^{1/3k}$. For every vector $\mathbf{v} \in \mathbb{R}^n$ and parameters $a, b \in \mathbb{R}$ we will write $a \leq \mathbf{v} \leq b$ to denote that $a \leq v_i \leq b$ for $1 \leq i \leq n$. We take the function $T(\mathbf{x}) = x_1^3 + x_2^3 + x_3^3$, and consider the weights

$$r_3(x) = \mathrm{card}\Big\{\mathbf{x} \in \mathbb{N}^3 : \ x = T(\mathbf{x}), \ \ \mathbf{x} \leq P\Big\}$$

and the set

$$\mathcal{X}_n = \Big\{(x_1, \ldots, x_s) \in \mathscr{C}^s, \qquad n = \sum_{i=1}^{s} x_i^k\Big\}.$$

Define the functions

$$R(n) = \sum_{\mathbf{x} \in \mathcal{X}_n} r_3(x_1) \cdots r_3(x_s), \qquad\qquad r(n) = \sum_{\mathbf{x} \in \mathcal{X}_n} 1, \tag{2.1.2}$$

which count the number of representations of n as a sum of k-th powers of integers represented as sums of three positive cubes, counted with and without multiplicities respectively. Take the singular series associated to the problem, defined as

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}\Big(q^{-3}\sum_{1\leq\mathbf{r}\leq q}e\big(aT(\mathbf{r})^k/q\big)\Big)^s e\big(-an/q\big). \tag{2.1.3}$$

The main result of this paper establishes an asymptotic formula for $R(n)$. For such purpose, it is convenient to introduce the parameter $H(k)=9k^2-k+2$.

Theorem 2.1.1. *Let $s\geq H(k)$. Then, there exists a constant $\delta>0$ such that*

$$R(n)=\Gamma\big(4/3\big)^{3s}\Gamma\big(1+1/k\big)^s\Gamma\big(s/k\big)^{-1}\mathfrak{S}(n)n^{s/k-1}+O(n^{s/k-1-\delta}),$$

where the singular series satisfies $\mathfrak{S}(n)\gg 1$.

Our proof of Theorem 2.1.1 is based on the application of the Hardy-Littlewood method. In order to discuss the constraint of the previous result on the number of variables, we define first $\tilde{G}(k)$ as the minimum number such that for $s\geq\tilde{G}(k)$, the anticipated asymptotic formula in the classical Waring's problem holds. We remind the reader that as a consequence of Vinogradov's mean value theorem, Bourgain [14] showed that $\tilde{G}(k)\leq k^2-k+O(\sqrt{k})$. The lack of understanding of the cardinality of the set $\mathscr{C}$ mentioned at the beginning of the paper both weakens the minor arc bounds and prevents us from having a better understanding of its distribution over arithmetic progressions, which often comes into play on the major arc analysis. The methods used in this memoir then are based on arguments in which in most of the sums of three cubes employed in the representation, all but one of the cubes is fixed in the associated analysis. Consequently, the constraint for the number of variables that we obtain here is asymptotic to the bound for $\tilde{G}(3k)$ mentioned above.

The problem becomes more challenging when we remove the counting of the multiplicities, and even if getting an asymptotic formula seems out of reach, Theorem 2.1.1 can be used to obtain a non-trivial lower bound. However, the whole strategy relies on an estimate for the L^2-norm of the sequence $r_3(x)$ of the shape

$$\sum_{x\leq X}r_3(x)^2\ll X^{7/6+\varepsilon} \tag{2.1.4}$$

that follows after an application of Hua's Lemma [141, Lemma 2.5]. Instead of taking that approach, we restrict the triples to lie on

$$\mathcal{C}(P) = \big\{\mathbf{x} \in [1, P]^3 :\ x_1, x_2 \in \mathcal{A}(P, P^\eta)\big\},$$

where $\eta > 0$ is a small enough fixed parameter and

$$\mathcal{A}(X, R) = \{n \in [1, X] \cap \mathbb{N} : p \mid n \text{ and } p \text{ prime} \Rightarrow p \leq R\},$$

and make use of the stronger estimate

$$\sum_{x \leq X} s_3(x)^2 \ll X^{1+\nu} \tag{2.1.5}$$

due to Wooley [174, Theorem 1.2], where $s_3(x) = \mathrm{card}\big\{\mathbf{x} \in \mathcal{C}(P) :\ x = T(\mathbf{x})\big\}$ and $\nu = 0.08290523$. It transpires that one should then have some control of the order of magnitude of the analogous function of $R(n)$ when we impose that restriction on the triples. For such matters, we define for each $n \in \mathbb{N}$ the aforementioned counting function

$$R_\eta(n) = \sum_{\mathbf{X} \in \mathcal{X}_n} s_3(x_1) \cdots s_3(x_s). \tag{2.1.6}$$

We also introduce Dickman's function, defined for real x by

$$\rho(x) = 0 \text{ when } x < 0,$$

$$\rho(x) = 1 \text{ when } 0 \leq x \leq 1,$$

$$\rho \text{ continuous for } x > 0,$$

$$\rho \text{ differentiable for } x > 1$$

$$x\rho'(x) = -\rho(x-1) \text{ when } x > 1.$$

Theorem 2.1.2. *Let s be any positive integer with $s \geq H(k)$. Then, there exists $\delta > 0$ such that*

$$\begin{aligned} R_\eta(n) =& \Gamma\big(4/3\big)^{3s} \Gamma\big(1 + 1/k\big)^s \Gamma\big(s/k\big)^{-1} \rho\big(1/\eta\big)^{2s} \mathfrak{S}(n) n^{s/k-1} \\ &+ O(n^{s/k-1}(\log n)^{-\delta}), \end{aligned}$$

where the singular series satisfies $\mathfrak{S}(n) \gg 1$.

An application of this theorem then, together with equation (2.1.5) and some other arguments yield the following result, which improves substantially the bound that one could obtain if no restriction on the triples was made.

Theorem 2.1.3. *Let s be any positive integer with $s \geq H(k)+1$. One has the lower bound*

$$r(n) \gg n^{(1-\nu)s/k-1},$$

where ν was defined right after (2.1.5).

It is worth noting that the preceding lower bound may be the best possible estimate attainable with the current knowledge available. The final question that will be addressed here is the constraint on the number of variables that guarantees the existence of solutions. For such purpose, we define $G_3(k)$ as the minimum integer such that for all $s \geq G_3(k)$ then $r(n) \geq 1$ holds for sufficiently large integers. We apply a previous result of Wooley [162] to obtain the following bound.

Theorem 2.1.4. *Let $k \in \mathbb{N}$. Then,*

$$G_3(k) \leq 3k\big(\log k + \log\log k + O(1)\big).$$

Our proofs for the main theorems of the paper are based on the application of the Hardy-Littlewood method. In Section 2.2, we apply a mean value estimate related to that of Vinogradov to bound the minor arc contribution. Section 2.3 deals with estimates of complete exponential sums and other related sums. In Sections 2.4 we discuss the local solubility of the problem and some properties of the singular series and include a brief proof of Theorem 2.1.4. Using the Riemann-Stieltjes integral we give an approximation of $f(\alpha)$ over the major arcs in Section 2.5. In Section 2.6 we study the singular integral, we obtain an asymptotic formula for the major arcs and we include a proof of Theorem 2.1.1. Section 2.7 is devoted to the study of the asymptotic formula when we introduce smooth numbers, and Theorem 2.1.3 is then proven in Section 2.8 via an application of Theorem 2.1.2. We have also included a small appendix in which we improve the constraint on the number of variables needed in Theorem 2.1.1 for small exponents by using restriction estimates.

Notation. Unless specified, any lower case letter $\mathbf{x}$ written in bold will denote a triple of integers (x_1, x_2, x_3). For any scalar λ and any vector $\mathbf{x}$ we write $\lambda\mathbf{x}$ for the vector $(\lambda x_1, \lambda x_2, \lambda x_3)$. When $R, V \in \mathbb{Z}^d$ then $R \equiv V \pmod q$ will mean that $R_i \equiv V_i \pmod q$ for all $1 \leq i \leq d$.

2.2 Minor arc estimate.

We obtain estimates for certain moments of an exponential sum on the minor arcs which we now define. Fix $s, k \geq 2$ and consider

$$f(\alpha) = \sum_{\mathbf{x} \leq P} f_{\mathbf{x}}(\alpha), \qquad \text{where} \quad f_{\mathbf{x}}(\alpha) = \sum_{1 \leq x \leq P} e\big(\alpha T(\mathbf{x}, x)^k\big)$$

and $\mathbf{x} \in \mathbb{N}^2$. Recalling (2.1.2), note that by orthogonality it follows that

$$R(n) = \int_0^1 f(\alpha)^s e(-\alpha n) d\alpha.$$

The purpose of this section is to bound the minor arc contribution of this integral. In order to make further progress we make use of a Hardy-Littlewood dissection in our analysis. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $0 \leq a \leq q \leq P^{\xi}$ and $(a, q) = 1$ with $\xi < \frac{s}{s+2}$, consider

$$\mathfrak{M}(a, q) = \Big\{\alpha \in [0, 1) : \Big|\alpha - a/q\Big| \leq \frac{P^{\xi}}{qn}\Big\}. \tag{2.2.1}$$

Then the major arcs $\mathfrak{M}$ will be the union of these arcs and $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$ will be the minor arcs.

Proposition 2.2.1. *When s is any positive integer with $s \geq H(k)$ one has*

$$\int_{\mathfrak{m}} |f(\alpha)|^s \mathrm{d}\alpha \ll P^{3s-3k-\delta}.$$

Moreover, if $s \geq 3k(3k+1)$ then it follows that

$$\int_{\mathfrak{m}} |f(\alpha)|^s \mathrm{d}\alpha \ll P^{3s-3k-\xi+\varepsilon}.$$

Proof. We bound the previous integrals in terms of a mean value of that of Vinogradov and apply estimates derived from Wooley [177, Theorems 14.4, 14.5]. For such purpose, it is convenient to take the set $\mathfrak{B} = \mathfrak{m} \times [0, 1)^{k-1}$ and consider the exponential sums

$$G_{\mathbf{x}}(\boldsymbol{\alpha}) = \sum_{x\leq P} e\Big(\alpha_k T(x,\mathbf{x})^k + \sum_{j=1}^{k-1}\alpha_j x^{3j}\Big) \quad \text{and} \quad F(\boldsymbol{\alpha}) = \sum_{x\leq P} e\Big(\sum_{j=1}^{k}\alpha_j x^{3j}\Big). \tag{2.2.2}$$

We write $H(k) = 2t$ for some positive integer t. Using Hölder's inequality and orthogonality we find that

$$\begin{aligned}\int_{\mathfrak{m}} |f(\alpha)|^{2t} d\alpha &\ll P^{4t-2}\int_{\mathfrak{m}}\sum_{\mathbf{x}\leq P}|f_{\mathbf{x}}(\alpha)|^{2t}d\alpha \\ = P^{4t-2}\sum_{\mathbf{x}\leq P}\sum_{n_j}\int_{\mathfrak{B}}|G_{\mathbf{x}}(\boldsymbol{\alpha})|^{2t} e\Big(-\sum_{j=1}^{k-1}\alpha_j n_j\Big)d\boldsymbol{\alpha} &\ll P^{4t+3k(k-1)/2}\int_{\mathfrak{B}}|F(\boldsymbol{\alpha})|^{2t}d\boldsymbol{\alpha},\end{aligned} \tag{2.2.3}$$

where $(n_j)_j$ runs over the tuples with $1 \leq |n_j| \leq tP^{3j}$. Observe that by Weyl's inequality [141, Lemma 2.4] one has that

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{B}}|F(\boldsymbol{\alpha})| \ll P^{1-\delta},$$

whence the aforementioned pointwise bound and Theorem 14.5 of [177] with the choice $r = 3k-2$ deliver the estimate

$$\int_{\mathfrak{B}}|F(\boldsymbol{\alpha})|^{2t}d\boldsymbol{\alpha} \ll P^{2t-3k(k+1)/2-\delta}. \tag{2.2.4}$$

The above equation and (2.2.3) then yield the first part of the proposition. For the second part we use a small modification of Wooley [177, Theorem 14.4]. On that paper, the author, in a more general setting, takes the choice $\xi = 1$ and obtains a saving of X over the expected main term. It transpires that the same exact method can be applied to save X^{ξ} for $\xi < 1$. Thus, we have that for $s \geq 3k(3k+1)$ then

$$\int_{\mathfrak{B}}|F(\boldsymbol{\alpha})|^{s}d\boldsymbol{\alpha} \ll P^{s-3k(k+1)/2-\xi+\varepsilon}.$$

Replacing $2t$ by s in (2.2.3) and using the previous equation we get the desired result. □

2.3 Complete exponential sums

In this section we study the complete exponential sum associated to the problem and deduce some bounds involving this sum. For such purpose, it is convenient to define for $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ the expressions

$$S(q,a) = \sum_{1\leq \mathbf{r}\leq q} e_q\big(aT(\mathbf{r})^k\big) \qquad \text{and} \qquad S_k(q,a) = \sum_{r=1}^{q} e_q(ar^k).$$

Note that by orthogonality then one can rewrite $S(q, a)$ as

$$S(q,a) = q^{-1}\sum_{u=1}^{q} S_3(q,u)^3 S_k(q,a,-u), \quad \text{where } S_k(q,a,b) = \sum_{r=1}^{q} e_q(ar^k + br). \tag{2.3.1}$$

In what follows we provide bounds for $S(q, a)$ using estimates for $S_3(q, a)$ and $S_k(q, a, b)$. Observe that by the quasi-multiplicative structure of it then it suffices to investigate the instances when $q = p^l$ is a prime power.

Lemma 2.3.1. *Let $l \geq 2$, let p be a prime number and $a \in \mathbb{Z}$ with $(a, p) = 1$. Then,*

$$S(p^l, a) \ll \min(p^{3l-1}, lp^{3l-l/k+\varepsilon}).$$

Proof. Note that Vaughan [141, Theorem 7.1] yields the bound

$$S(p^l, a, -u) \ll p^{l(1-1/k)+\varepsilon}.$$

Therefore, an application of this estimate and Theorem 4.2 of [141] to equation (2.3.1) gives

$$S(p^l, a) \ll p^{2l-l/k+\varepsilon}\sum_{u=1}^{p^l}(u, p^l) \ll lp^{3l-l/k+\varepsilon}.$$

Observe that we can also deduce the bound $S(p^l, a, -u) \ll p^{l-1}$ from the proof[2] of Vaughan [141, Theorem 7.1], so the application of this estimate instead and the same procedure delivers $S(p^l, a) \ll p^{3l-1}$. □

When p is prime we can provide a more precise description of $S(p, a)$ by involving the sum $S_k(p, a)$ in its expression. Despite not using this refinement

[2] See in particular the argument following Vaughan [141, (7.16)]

in this chapter, we have included such analysis for further use in the upcoming one.

Lemma 2.3.2. *Let p be a prime number and $a \in \mathbb{Z}$ with $(a,p)=1$. Then,*

$$S(p,a) = p^2 S_k(p,a) + O(p^2).$$

In particular, one has the bound $S(p,a) \ll p^{5/2}$.

Proof. By equation (2.3.1) it follows that $S(p,a) = p^2 S_k(p,a) + E$, where

$$E = p^{-1} \sum_{1\leq u\leq p-1} S_3(p,u)^3 S_k(p,a,-u).$$

Using Vaughan [141, Lemma 4.3] to bound $S_3(p,u)$ and the work of Weil[3] [155] to bound $S_k(p,a,-u)$ we obtain the estimate $E \ll p^2$. Consequently, another application of the aforementioned lemma of Vaughan [141] to $S_k(p,a)$ delivers $S(p,a) \ll p^{5/2}$. □

The reader may notice that this result is then best possible since whenever $(k,p-1)>1$ then there is a positive proportion of positive integers $a \leq p$ for which $S_k(p,a) \gg p^{1/2}$, whence the above result delivers an asymptotic formula in those situations. It seems unclear whether the error term in the formula could be improved. Such improvement though would not have any impact in our work. For future purposes, it is convenient to define, for each $q \in \mathbb{N}$, the exponential sums

$$S_n(q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \big(q^{-3}S(q,a)\big)^s e_q(-na), \qquad S_s^*(q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \big|q^{-3}S(q,a)\big|^s \tag{2.3.2}$$

and to analyse their behaviour when summing over q.

Lemma 2.3.3. *Let $s \geq \max(4,k+1)$. One has*

$$\sum_{q\leq Q} S_s^*(q) \ll Q^{\varepsilon} \qquad \textit{and} \qquad \sum_{q\leq Q} |S_n(q)| \ll Q^{\varepsilon}, \tag{2.3.3}$$

[3] See Schmidt [123, Corollary 2F] for an elementary proof of this bound.

and for $s \geq \max(5, k+2)$ *it follows that*

$$\sum_{q \leq Q} q^{1/k} |S_n(q)| \ll Q^{\varepsilon} \qquad \textit{and} \qquad \sum_{q > Q} |S_n(q)| \ll Q^{\varepsilon - 1/k}. \tag{2.3.4}$$

Proof. To show (2.3.3) it suffices to prove the bound for $S_s^*(q)$ since one trivially has the inequality $|S_n(q)| \leq S_s^*(q)$. Applying Lemmata 2.3.1 and 2.3.2 we deduce that each of $S_n(p)$ and $S_s^*(p)$ is $O(p^{1-s/2})$, and each of $S_n(p^l)$ and $S_s^*(p^l)$ is $O\big(\min(p^{l-s}, l^s p^{l-ls/k+\varepsilon})\big)$ when $l \geq 2$. Consequently, whenever one has $s \geq \max(4, k+1)$ then using the fact that $S_s^*(q)$ is multiplicative we find that

$$\sum_{q \leq Q} S_s^*(q) \ll \prod_{p \leq Q} \big(1 + \sum_{l=1}^{\infty} S_s^*(p^l)\big) \ll \prod_{p \leq Q} (1 + C/p) \ll Q^{\varepsilon},$$

where $C > 0$ is some suitable constant. The first assertion of (2.3.4) follows by the same argument, and the second follows observing that then

$$\sum_{Q \leq q \leq 2Q} |S_n(q)| \ll Q^{\varepsilon - 1/k},$$

whence summing over dyadic intervals we obtained the desired result. □

2.4 Singular series

We give sufficient conditions in terms of the number of variables to ensure the local solubility of the problem and combine such work with the bounds obtained in the previous section to introduce and analyse the singular series associated to the problem. We also include a brief proof of Theorem 2.1.4. For such purposes, a little preparation is required. Let p a prime number and take $\tau \geq 0$ such that $p^{\tau} || 3k$. Let $\gamma = 2\tau + 1$, consider the set

$$\mathcal{M}_n(p^h) = \Big\{ \mathbf{Y} \in [1, p^h]^{3s} : \sum_{i=1}^{s} T(\mathbf{y}_i)^k \equiv n \pmod{p^h} \Big\}, \tag{2.4.1}$$

where $\mathbf{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_s)$ with $\mathbf{y}_i \in \mathbb{N}^3$, and the subset

$$\mathcal{M}_n^*(p^h) = \Big\{ \mathbf{Y} \in \mathcal{M}_n(p^h) : p \nmid y_{1,1},\ p \nmid T(\mathbf{y}_1) \Big\},$$

where $\mathbf{y}_1 = (y_{1,1}, y_{1,2}, y_{1,3})$. Define as well the quantities $M_n(p^h) = |\mathcal{M}_n(p^h)|$ and $M_n^*(p^h) = |\mathcal{M}_n^*(p^h)|$. Here the reader may want to observe that the divisibility restrictions on the above definition are imposed for a latter application of Hensel's Lemma. Before showing that under some constraint in the number of variables then $M_n^*(p^\gamma) > 0$, we first provide an accurate description of the set

$$\mathcal{M}_{3,3}(p^h) = \Big\{T(\mathbf{x}) : \ \mathbf{x} \in \Big(\mathbb{Z}/p^h\mathbb{Z}\Big)^3, \ (x_1, p) = 1\Big\}$$

that will be used throughout the whole argument.

Lemma 2.4.1. *Let $h \in \mathbb{N}$. Then, whenever $p \neq 3$ one finds that*

$$\mathcal{M}_{3,3}(p^h) = \mathbb{Z}/p^h\mathbb{Z}. \tag{2.4.2}$$

For the case $p = 3$ one has $\mathcal{M}_{3,3}(3) = \mathbb{Z}/3\mathbb{Z}$ and when $h \geq 2$ then

$$\mathcal{M}_{3,3}(3^h) = \Big\{x \in \mathbb{Z}/3^h\mathbb{Z} : \quad x \not\equiv 4 \pmod 9, \ x \not\equiv 5 \pmod 9\Big\}.$$

Proof. When $p \neq 3$, we can assume that $h = 1$, since an application of Hensel's Lemma would then yield the case $h \geq 2$. For a better description of the argument, it is convenient to define the counting functions

$$N_n(p) = \text{card}\Big\{\mathbf{x} \in \big(\mathbb{Z}/p\mathbb{Z}\big)^3 : \ T(\mathbf{x}) \equiv n \pmod p\Big\},$$

$$N_{n,4}(p) = \text{card}\Big\{\mathbf{y} \in \big(\mathbb{Z}/p\mathbb{Z}\big)^4 : \ y_1^3 + y_2^3 + y_3^3 - ny_4^3 \equiv 0 \pmod p\Big\}.$$

Observe that by making a distinction for the tuples counted in $N_{n,4}(p)$ regarding the divisibility of y_4 by p one has that $(p-1)N_n(p) = N_{n,4}(p) - N_0(p)$. Note that when $(n, p) = 1$ then the work of Weil [156] on equations over finite fields leads to

$$|N_{n,4}(p) - p^3| \leq 6(p-1)p \qquad \text{and} \qquad |N_0(p) - p^2| \leq 2(p-1)\sqrt{p}.$$

Consequently, one finds that $N_n(p) = p^2 + E_p$ with $|E_p| \leq 6p + 2\sqrt{p}$, and hence $N_n(p) \geq 1$ for $p \geq 7$. Observe as well that $N_n(2) \geq 1$ and $N_n(5) \geq 1$ follow trivially. This implies that there is at least one solution to the equation

$$x^3 + y^3 + z^3 \equiv n \pmod p, \qquad (x, p) = 1, \tag{2.4.3}$$

and when $n = 0$ then $(1, -1, 0)$ is also a solution for (2.4.3), whence the preceding discussion yields (2.4.2).

When $p = 3$ then the case $h = 1$ is trivial. Note that since cubes can only be $\pm 1 \pmod 9$, the only residues which cannot be written as sums of three cubes are 4 and 5. For $h = 3$, a slightly tedious computation reveals that the only residues not represented as sums of three cubes are the ones congruent to 4 or 5 (mod 9). Therefore, a routine application of Hensel's Lemma delivers the proof for $h \geq 4$.

$\square$

The previous lemma asserts that the local solubility of the problem studied here only differs from the local solubility of the original Waring's problem at the prime 3. This conclusion is gathered in the following statement.

Lemma 2.4.2. *Suppose that* $s \geq \frac{p}{p-1}\big(k, p^{\tau}(p-1)\big)$ *when* $p \neq 2, 3$ *or* $p = 2$ *and* $\tau = 0$, *that* $s \geq \frac{9}{4}\big(k, \phi(3^{\gamma})\big)$ *when* $p = 3$, *that* $s \geq 2^{\tau+2}$ *when* $p = 2$ *and* $\tau > 0$ *with* $k > 2$, *and that* $s \geq 5$ *when* $p = k = 2$. *Then one has* $M_n^*(p^{\gamma}) > 0$.

Proof. If $p \neq 3$ then Lemma 2.4.1 implies that the local solubility for each of these primes is equivalent to that of the original Waring's problem, hence Vaughan [141, Lemma 2.15] yields $M_n^*(p^{\gamma}) > 0$. Here the reader might want to observe that the definition for γ taken here is different from the one in Vaughan [141, (2.25)], so one may have to apply Lemma 2.13 of [141] as well. For the case $p = 3$, Lemma 2.4.1 delivers

$$\Big|\mathcal{M}_{3,3}(3^{\gamma}) \cap U(\mathbb{Z}/3^{\gamma}\mathbb{Z})\Big| = 4 \cdot 3^{\gamma-2},$$

where $U(\mathbb{Z}/3^{\gamma}\mathbb{Z})$ denotes the group of units of $\mathbb{Z}/3^{\gamma}\mathbb{Z}$. Therefore, using Vaughan [141, Lemma 2.14] we get that $M_n^*(3^{\gamma}) > 0$ whenever $s \geq \frac{9}{4}\big(k, \phi(3^{\gamma})\big)$. $\square$

Observe that by the combination of Lemma 2.4.1 and Vaughan [141, Lemma 2.14] then for $u \geq 9k/4$ we find that the form $T(\mathbf{x}_1)^k + \cdots + T(\mathbf{x}_u)^k$ covers all the residue classes modulo 3^k. Take now s such that every sufficiently large number can be written as a sum of s integral $3k$-th powers. Given a large integer n, we can find integral triples $\mathbf{x}_1, \ldots, \mathbf{x}_u$ for which

$$n \equiv T(\mathbf{x}_1)^k + \cdots + T(\mathbf{x}_u)^k \pmod{3^k} \quad \text{and } 1 \leq \mathbf{x}_i \leq 3^k.$$

Fixing any one such choice of the $\mathbf{x}_i$, we can also find integers $x_1, \ldots, x_s$ satisfying

$$x_1^{3k} + \cdots + x_s^{3k} = 3^{-k}\Big(n - \big(T(\mathbf{x}_1)^k + \cdots + T(\mathbf{x}_u)^k\big)\Big).$$

Here the reader may find convenient to observe that the term on the right side of the equality is still large. Therefore, we obtain the representation

$$n = T(\mathbf{x}_1)^k + \cdots + T(\mathbf{x}_u)^{3k} + 3^k\big(x_1^{3k} + \cdots + x_s^{3k}\big).$$

Noting that the sums of three cubes on the right side have been replaced by the specialization $3x^3$, one gets $G_3(k) \leq u+s$, and hence by Wooley [162, Corollary 1.2.1] we have that $G_3(k) \leq 3k\big(\log k+\log\log k+O(1)\big)$, which yields Theorem 2.1.4. As experts will realise, one could apply the ideas of Wooley [167] to obtain a refinement of the shape

$$G_3(k) \leq 3k\left(\log k + \log\log 3k + \log 3 + 2 + O\Big(\frac{\log\log k}{\log k}\Big)\right)$$

by using instead other specializations. For the sake of brevity though we omit making such analysis here.

Finally we combine work of this section with estimates from the previous one to analyse the singular series. Observe that by (2.3.2) we can rewrite the singular series, defined in (2.1.3), as

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} S_n(q).$$

To express the above series as a product of local densities it is convenient to define the infinite sum

$$\sigma(p) = \sum_{l=0}^{\infty} S_n(p^l)$$

for each prime p.

Proposition 2.4.1. *Let $s \geq \max(5, k+2)$. Then, one has*

$$\mathfrak{S}(n) = \prod_p \sigma(p), \tag{2.4.4}$$

the series $\mathfrak{S}(n)$ converges absolutely and $\mathfrak{S}(n) \ll 1$. Moreover, if s satisfy the *conditions of Lemma 2.4.2 one gets* $\mathfrak{S}(n) \gg 1$.

Proof. Using the estimates mentioned at the beginning of the proof of Lemma 2.3.3 we find

$$\sum_{l=1}^{\infty} |S_n(p^l)| \ll p^{1-s/2} + p^{k-s} + \sum_{l \geq k+1} l^s p^{l-ls/k+\varepsilon} \ll p^{-3/2}. \tag{2.4.5}$$

Therefore, (2.4.4) holds by multiplicativity, $\mathfrak{S}(n)$ converges absolutely and $\mathfrak{S}(n) \ll 1$. In order to prove the lower bound, we recall first (2.3.2) and (2.4.1) to deduce that orthogonality then yields

$$\sum_{l=0}^{h} S_n(p^l) = M_n(p^h) p^{h(1-3s)}.$$

If s satisfies the conditions of Lemma 2.4.2, an application of Hensel's Lemma gives the lower bound $M_n(p^h) \geq p^{(3s-1)(h-\gamma)}$, which combined with the above equation implies that $\sigma(p) \geq p^{-(3s-1)\gamma}$. Consequently, the previous estimate and (2.4.5) deliver the lower bound $\mathfrak{S}(n) \gg 1$. □

2.5 Approximation on the major arcs.

In this section we use a simple argument involving the Riemman-Stieltjes integral and integration by parts to approximate $f(\alpha)$ by an auxiliary function on the major arcs. We also provide bounds for this function. For such purposes, we introduce first some notation. Let $\alpha \in [0, 1)$ and $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a, q) = 1$. Denote $\beta = \alpha - a/q$ and define the aforementioned auxiliary function

$$V(\alpha, q, a) = q^{-3} S(q, a) v(\beta), \qquad \text{where } \ v(\beta) = \int_{[0,P]^3} e\big(\beta T(\mathbf{x})^k\big) d\mathbf{x}. \tag{2.5.1}$$

Lemma 2.5.1. *Let* $q < P$. *Then one has that*

$$f(\alpha) = V(\alpha, q, a) + O\big(P^2 q(1 + n|\beta|)\big).$$

Proof. Before embarking on our task, it is convenient to define the sums

$$K_{\mathbf{r}}(\beta) = \sum_{\substack{\mathbf{x}\leq P \\ \mathbf{x}\equiv\mathbf{r} \pmod q}} e\big(F_\beta(\mathbf{x})\big), \qquad B_r(x) = \sum_{\substack{0<z\leq x \\ z\equiv r \pmod q}} 1,$$

where $F_\beta(\mathbf{x}) = \beta T(\mathbf{x})^k$. Observe that by sorting the summation into arithmetic progressions modulo q we find that

$$f(\alpha) = \sum_{\mathbf{r}\leq q} e_q\big(aT(\mathbf{r})^k\big) K_{\mathbf{r}}(\beta). \tag{2.5.2}$$

For each $\mathbf{r} \in \mathbb{N}^3$ write $\mathbf{r} = (\mathbf{r}_1, r_3)$. Then by Abel's summation formula we obtain

$$K_{\mathbf{r}}(\beta) = B_{r_3}(P)\sum_{\mathbf{x}_1} e\big(F_\beta(\mathbf{x}_1, P)\big) - \int_0^P \sum_{\mathbf{x}_1} \frac{\partial}{\partial z} e\big(F_\beta(\mathbf{x}_1, z)\big) B_{r_3}(z)dz,$$

where $\mathbf{x}_1$ runs over pairs $\mathbf{x}_1 \in [1, P]^2$ with $\mathbf{x}_1 \equiv \mathbf{r}_1 \pmod q$. Consequently, using the equation $B_{r_3}(x) = x/q + O(1)$ and integration by parts one gets

$$K_{\mathbf{r}}(\beta) = q^{-1}\int_0^P \sum_{\mathbf{x}_1} e\big(F_\beta(\mathbf{x}_1, z)\big)dz + O\big(q^{-2}P^2(1 + n|\beta|)\big).$$

We repeat the exact same procedure for the first two variables to obtain

$$K_{\mathbf{r}}(\beta) = q^{-3}v(\beta) + O\big(q^{-2}P^2(1 + n|\beta|)\big),$$

whence the combination of the above expression with (2.5.2) yields the result claimed above. $\square$

In order to make further progress, we provide an upper bound for $v(\beta)$ in the following lemma. This lemma will be used throughout the major arc analysis.

Lemma 2.5.2. *Let $\beta \in \mathbb{R}$. One has that*

$$v(\beta) \ll \frac{P^3}{(1 + n|\beta|)^{1/k}}.$$

Proof. Let $\mathbf{y} \in [0,P]^2$ and set $C_{\mathbf{y}} = y_1^3 + y_2^3$. Define the auxiliary function

$$B_{\mathbf{y}}(y) = (3k)^{-1} y^{1/k-1} (y^{1/k} - C_{\mathbf{y}})^{-2/3}.$$

Note that by a change of variables one can rewrite $v(\beta)$ as

$$v(\beta) = \int_{\mathbf{y}\in[0,P]^2} \int_{N_{\mathbf{y}}}^{M_{\mathbf{y}}} B_{\mathbf{y}}(y) e(\beta y) dy d\mathbf{y}, \tag{2.5.3}$$

where $N_{\mathbf{y}} = C_{\mathbf{y}}^k$ and $M_{\mathbf{y}} = (P^3 + C_{\mathbf{y}})^k$. Observe first that when $|\beta| \leq n^{-1}$ then one trivially gets

$$v(\beta) \ll \int_{\mathbf{y}\in[0,P]^2} \int_{N_{\mathbf{y}}}^{M_{\mathbf{y}}} B_{\mathbf{y}}(y) dy d\mathbf{y} \ll P^3.$$

For the case $|\beta| > n^{-1}$ we split the integral into

$$v(\beta) \ll I_1 + I_2 + I_3,$$

where

$$I_i = \int_{\mathbf{x}\in\mathcal{T}_i} e\big(\beta T(\mathbf{x})^k\big) d\mathbf{x} \quad \text{for each } i \in \{1,2,3\}$$

and the sets of integration taken are

$$\mathcal{T}_1 = \Big\{\mathbf{x} \in [0,P]^3 : \quad \mathbf{x} \leq |\beta|^{-1/3k}\Big\}, \quad \mathcal{T}_2 = \Big\{\mathbf{x} \in [0,P]^3 : \quad x_2, x_3 > |\beta|^{-1/3k}\Big\},$$

$$\mathcal{T}_3 = \Big\{\mathbf{x} \in [0,P]^3 : \quad x_3 > |\beta|^{-1/3k}, \qquad x_1, x_2 \leq |\beta|^{-1/3k}\Big\}.$$

Note that for the first integral one has $I_1 \leq |\mathcal{T}_1| \ll |\beta|^{-1/k}$. For the other two it is convenient to define the parameter $T_\beta = (|\beta|^{-1/k} + C_{\mathbf{y}})^k$ and consider the set $\mathcal{M}_2 = [0,P] \times [|\beta|^{-1/3k}, P]$. Then, by applying integration by parts we find that

$$I_2 = \int_{\mathbf{y}\in\mathcal{M}_2} \int_{T_\beta}^{M_{\mathbf{y}}} B_{\mathbf{y}}(y) e(\beta y) d\mathbf{y} dy \ll |\beta|^{-1} \int_{\mathbf{y}\in\mathcal{M}_2} B_{\mathbf{y}}(T_\beta) d\mathbf{y},$$

where we used the fact that the function $B_{\mathbf{y}}(y)$ is decreasing. Observe that whenever $\mathbf{y} \in \mathcal{M}_2$ then one has $|\beta|^{-1/k} \leq C_{\mathbf{y}}$, which delivers the estimate

$$I_2 \ll |\beta|^{-1+2/3k} \int_{\mathbf{y}\in\mathcal{M}_2} C_{\mathbf{y}}^{1-k} d\mathbf{y} \ll |\beta|^{-1+2/3k} \int_{|\beta|^{-1/3k}\leq x} x^{4-3k} dx \ll |\beta|^{-1/k}.$$

Likewise, we introduce the set $\mathcal{M}_3 = [0, |\beta|^{-1/3k}]^2$ to handle I_3. Using the same argument we get that

$$I_3 \ll |\beta|^{-1} \int_{\mathbf{y}\in\mathcal{M}_3} B_{\mathbf{y}}(T_\beta) d\mathbf{y},$$

and applying the fact that $C_{\mathbf{y}} \ll |\beta|^{-1/k}$ whenever $\mathbf{y} \in \mathcal{M}_3$ to bound $B_{\mathbf{y}}(T_\beta)$ then we obtain

$$I_3 \ll |\beta|^{-1} \int_{\mathbf{y}\in\mathcal{M}_3} |\beta|^{1-1/3k} d\mathbf{y} \ll |\beta|^{-1/k}.$$

The combination of the bounds for I_1, I_2 and I_3 yields the result of the lemma.

□

2.6 The asymptotic formula for $R(n)$.

We compute the size of the singular integral and use the work and the bounds obtained in the previous sections to obtain an asymptotic formula for $R(n)$. Whenever $s \geq k+1$ define the aforementioned singular integral by

$$J(n) = \int_{-\infty}^{\infty} v(\beta)^s e(-\beta n) d\beta.$$

Consider the set

$$\mathcal{S} = \Big\{(\mathbf{y}_1, y_1, \ldots, \mathbf{y}_s, y_s) \in \mathbb{R}^{3s} : \quad \mathbf{y}_i \in [0, P]^2, \quad N_{\mathbf{y}_i} \leq y_i \leq M_{\mathbf{y}_i}\Big\}.$$

Then, recalling (2.5.3) and using a change of variables it follows that

$$\begin{aligned} J(n) &= \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \int_{\mathbf{Y}\in\mathcal{S}} \prod_{i=1}^{s} B_{\mathbf{y}_i}(y_i) e\Big(\beta\Big(\sum_{i=1}^{s} y_i - n\Big)\Big) d\mathbf{Y} d\beta \\ &= \lim_{\lambda\to\infty} \int_0^{3^k sn} \phi(v) \frac{\sin\big(2\pi\lambda(v-n)\big)}{\pi(v-n)} dv, \end{aligned}$$

where we have taken

$$\phi(v) = \int_{\mathbf{Y}\in\mathcal{S}'} B_{\mathbf{y}_s}(\gamma_v) \prod_{i=1}^{s-1} B_{\mathbf{y}_i}(y_i)\ d\mathbf{Y}, \qquad \text{with } \gamma_v = v - \sum_{i=1}^{s-1} y_i$$

and $\mathcal{S}' \subset \mathbb{N}^{3s-1}$ is the set determined by the underlying inequalities. Observe that $\phi(v)$ is a function of bounded variation, whence by Fourier's Integral Theorem it follows that $J(n) = \phi(n)$. To obtain a precise formula for $J(n)$ it is convenient to introduce the subset $\mathcal{Y} \subset [0, n]^{s-1}$ defined by the constraint $0 \leq \gamma_n \leq n$. Then, by several subsequent changes of variables and the formula of the Euler's Beta function one has that

$$J(n) = 3^{-3s}k^{-s}\Gamma\big(1/3\big)^{3s}\int_{\mathcal{Y}} \gamma_n^{1/k-1}\Big(\prod_{i=1}^{s-1} y_i^{1/k-1}\Big)\ dy_1 \cdots dy_{s-1}$$
$$= \Gamma\big(4/3\big)^{3s}\Gamma\big(1+1/k\big)^{s}\Gamma\big(s/k\big)^{-1}n^{s/k-1}. \qquad (2.6.1)$$

We are now equipped to compute an asymptotic formula for the major arc contribution, which we define by

$$R_{\mathfrak{M}}(n) = \int_{\mathfrak{M}} f(\alpha)^s e(-\alpha n) d\alpha.$$

Proposition 2.6.1. *Let* $s \geq \max(5, k+2)$. *Then,*

$$R_{\mathfrak{M}}(n) = \Gamma\big(4/3\big)^{3s}\Gamma\big(1+1/k\big)^{s}\Gamma\big(s/k\big)^{-1}\mathfrak{S}(n)n^{s/k-1} + O(n^{s/k-1-\delta}).$$

Proof. For the sake of simplicity we consider the auxiliary function $f^*(\alpha)$ for $\alpha \in [0, 1)$ by putting

$$f^*(\alpha) = V(\alpha, q, a)$$

when $\alpha \in \mathfrak{M}(a, q) \subset \mathfrak{M}$ and $f^*(\alpha) = 0$ for $\alpha \in \mathfrak{m}$. We remind the reader that $V(\alpha, q, a)$ was defined in (2.5.1). Recalling Lemma 2.5.1 then whenever $\alpha \in \mathfrak{M}(a, q)$ one has that

$$f(\alpha)^s - f^*(\alpha)^s \ll P^{2s}q^s(1 + n|\beta|)^s + P^2q(1 + n|\beta|)|f^*(\alpha)|^{s-1}.$$

Integrating over the major arcs, which were defined in (2.2.1), we find that

$$\int_{\mathfrak{M}} |f(\alpha)^s - f^*(\alpha)^s| d\alpha \ll P^{2s+\xi(s+2)}n^{-1} + P^{3s-3k-1+\xi}\sum_{q \leq P^{\xi}} S^*_{s-1}(q), \qquad (2.6.2)$$

and hence Lemma 2.3.3 and the assumption on ξ stated before (2.2.1) implies that the above integral is $O(P^{3s-3k-\delta})$. Observe that by the same lemma and

Lemma 2.5.2 respectively we have

$$\sum_{P^{\xi}<q} |S_n(q)| = O(P^{-\delta}) \qquad \text{and} \qquad \int_{|\beta|>\frac{P^{\xi}}{qn}} |v(\beta)|^s d\beta = O\big(P^{3s-3k} q^{\delta} P^{-\delta\xi}\big)$$

for $q \leq P^{\xi}$ and some $\delta > 0$. Combining these observations with the aforementioned lemmata and equations (2.3.4) and (2.6.2) we obtain

$$R_{\mathfrak{M}}(n) = \mathfrak{S}(n)J(n) + O(n^{s/k-1-\delta}),$$

and hence (2.6.1) delivers the result. □

Theorem 2.1.1 then follows applying Propositions 2.2.1, 2.4.1 and 2.6.1.

2.7 Asymptotic formula over the smooth numbers.

In this section we investigate the asymptotic formula for the representation function when two of the variables of each triple lie on the smooth numbers. The strategy for bounding the integral over the minor arcs of $g(\alpha)$ combines arguments of Section 2.2 with major arc techniques. We define the major arcs $\mathfrak{N}$ to be the union of

$$\mathfrak{N}(a,q) = \Big\{\alpha \in [0,1) : \big|\alpha - a/q\big| \leq q^{-1}(\log P)^{\kappa} P^{-3k}\Big\} \tag{2.7.1}$$

with $0 \leq a \leq q \leq (\log P)^{\kappa}$ and $(a,q) = 1$, where $\kappa = 1/5$. We take the minor arcs $\mathfrak{n} = [0,1) \setminus \mathfrak{N}$. Similarly, when $1 \leq X \leq L$ with $L = P^{1/3k}$, define $\mathfrak{W}(X)$ as the union of the arcs

$$\mathfrak{W}(\mathbf{a},q) = \Big\{\boldsymbol{\alpha} \in [0,1)^k : |\alpha_j - a_j/q| \leq q^{-1}XP^{-3j} \quad (1 \leq j \leq k)\Big\}$$

with $0 \leq \mathbf{a} \leq q \leq X$ and $(q,a_1,\dots,a_k) = 1$. For the sake of simplicity we write

$$\mathfrak{W} = \mathfrak{W}(L), \qquad \mathfrak{P} = \mathfrak{W}\big((\log P)^{\kappa}\big),$$

and we take the minor arcs $\mathfrak{w} = [0,1)^k \setminus \mathfrak{W}$ and $\mathfrak{p} = [0,1)^k \setminus \mathfrak{P}$. First we prove a lemma which permits us to have a saving over the trivial bound for some

Weyl sum on $\mathfrak{p}$. For such purposes we consider the exponential sum associated to the Vinogradov's system

$$f_k(\boldsymbol{\alpha}; X) = \sum_{1 \leq x \leq X} e\big(\alpha_1 x + \ldots + \alpha_k x^k\big).$$

Lemma 2.7.1. *Let X be any real positive number sufficiently big in terms of k, let μ be a real number such that $\mu^{-1} > 4k(k-1)$ and let γ denote a real number with $X^{-\mu} \leq \gamma \leq 1$. Then whenever $|f_k(\boldsymbol{\alpha}; X)| \geq \gamma X$, there exist an integer $q \in \mathbb{N}$ and a tuple $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ with $(q, a_1, \ldots, a_k) = 1$ and $1 \leq q \ll \gamma^{-k-\varepsilon}$ and such that*

$$|q\alpha_j - a_j| \ll \gamma^{-k-\varepsilon} X^{-j} \qquad (1 \leq j \leq k).$$

Proof. Suppose that $|f_k(\boldsymbol{\alpha}; X)| \geq \gamma X$. Then, applying Wooley [172, Theorem 1.6] we obtain that there exist $q \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{N}^k$ with $(q, a_1, \ldots, a_k) = 1$ such that $1 \leq q \leq X^{1/k}$ and

$$|q\alpha_j - a_j| \leq X^{1/k-j} \qquad (1 \leq j \leq k). \tag{2.7.2}$$

In order to make further progress it is convenient to define the auxiliary function

$$T(\boldsymbol{\alpha}; q, \mathbf{a}) = q + |q\alpha_1 - a_1| X + \cdots + |q\alpha_k - a_k| X^k.$$

By Theorems 7.1, 7.2 and 7.3 of Vaughan [141] one has that

$$|f_k(\boldsymbol{\alpha}; X)| \ll q^{\varepsilon} X T(\boldsymbol{\alpha}; q, \mathbf{a})^{-1/k} + T(\boldsymbol{\alpha}; q, \mathbf{a}).$$

Observe that equation (2.7.2) yields $T(\boldsymbol{\alpha}; q, \mathbf{a}) \ll X^{1/k}$, which implies that $q^{\varepsilon} X T(\boldsymbol{\alpha}; q, \mathbf{a})^{-1/k}$ is the term dominating in the previous estimate. Therefore, by the preceding discussion and the fact that $q \leq T(\boldsymbol{\alpha}; q, \mathbf{a})$ we obtain

$$\gamma X \leq |f_k(\boldsymbol{\alpha}; X)| \ll X T(\boldsymbol{\alpha}; q, \mathbf{a})^{-1/k+\varepsilon},$$

which gives $T(\boldsymbol{\alpha}; q, \mathbf{a}) \ll \gamma^{-k-\varepsilon}$ and delivers the lemma. □

We are now equipped to prove the minor arc estimate. Recalling (2.1.6) and the definition of $\mathcal{C}(P)$ made after (2.1.4), observe that by orthogonality one has that

$$R_\eta(n) = \int_0^1 g(\alpha)^s e(-\alpha n) d\alpha, \qquad \text{where } g(\alpha) = \sum_{\mathbf{x}\in\mathcal{C}(P)} e\big(\alpha T(\mathbf{x})^k\big).$$

In what follows we show that the minor arc contribution is smaller than the expected main term by combining the previous lemma and other minor arc estimates with some major arc ideas.

Proposition 2.7.1. *Whenever s is any positive integer with $s \geq H(k)$ one has*

$$\int_{\mathfrak{n}} |g(\alpha)|^s \mathrm{d}\alpha \ll P^{3s-3k}(\log P)^{-\delta}.$$

Proof. We write $H(k) = 2t$ for some positive integer t. Recalling (2.2.2) and using the same argument as in (2.2.3) it follows that

$$\int_{\mathfrak{n}} |g(\alpha)|^{2t} d\alpha \ll P^{4t+3k(k-1)/2} \int_{\mathfrak{n}} \int_{[0,1)^{k-1}} |F(\boldsymbol{\alpha})|^{2t} d\boldsymbol{\alpha}. \tag{2.7.3}$$

Observe that we can estimate the above integral by

$$\int_{\mathfrak{n}} \int_{[0,1)^{k-1}} |F(\boldsymbol{\alpha})|^{2t} d\boldsymbol{\alpha} \ll \int_{\mathfrak{w}} |F(\boldsymbol{\alpha})|^{2t} d\boldsymbol{\alpha} + \int_{\mathfrak{W}\setminus\mathfrak{P}} |F(\boldsymbol{\alpha})|^{2t} d\boldsymbol{\alpha}. \tag{2.7.4}$$

Note as well that combining Vaughan [141, Theorem 5.2] with Bourgain's result on Vinogradov's mean value theorem [14, Theorem 1.1] one has that

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{w}} |F(\boldsymbol{\alpha})| \ll P^{1-\delta}.$$

We then bound the first term on the right side of (2.7.4) via an application of the above pointwise bound and Wooley [177, Theorem 14.5] in the same way as in (2.2.4) to obtain

$$\int_{\mathfrak{w}} |F(\boldsymbol{\alpha})|^{2t} d\boldsymbol{\alpha} \ll P^{2t-3k(k+1)/2-\delta}.$$

In order to estimate the second one we provide a major arc analysis. For such purpose we consider the auxiliary function

$$V(\boldsymbol{\alpha}; q, \mathbf{a}) = q^{-1} S(q, \mathbf{a}) I(\boldsymbol{\alpha} - \mathbf{a}/q),$$

where

$$S(q,\mathbf{a})=\sum_{r=1}^{q}e_q\big(a_1r^3+\ldots+a_kr^{3k}\big),\qquad I(\boldsymbol{\beta})=\int_0^P e(\beta_1\gamma^3+\ldots+\beta_k\gamma^{3k})d\gamma.$$

For the sake of conciseness, we define for $\boldsymbol{\alpha}\in[0,1)^k$ the function

$$V(\boldsymbol{\alpha})=V(\boldsymbol{\alpha};q,\mathbf{a})$$

when $\boldsymbol{\alpha}\in\mathfrak{W}(\mathbf{a},q)\subset\mathfrak{W}$ and $V(\boldsymbol{\alpha})=0$ for $\boldsymbol{\alpha}\in\mathfrak{w}$. Observe that by Vaughan [141, Theorem 7.2] then whenever $\boldsymbol{\alpha}\in\mathfrak{W}$ it follows that

$$F(\boldsymbol{\alpha})-V(\boldsymbol{\alpha})\ll L,$$

whence the triangle inequality then yields

$$|F(\boldsymbol{\alpha})|^{2t-2}-|V(\boldsymbol{\alpha})|^{2t-2}\ll LP^{2t-3}.$$

Therefore, combining the fact that $\mathrm{mes}(\mathfrak{W})\ll L^{k+1}P^{-3k(k+1)/2}$ with the above estimate we get

$$\int_{\mathfrak{W}}|F(\boldsymbol{\alpha})|^{2t-2}d\boldsymbol{\alpha}-\int_{\mathfrak{W}}|V(\boldsymbol{\alpha})|^{2t-2}d\boldsymbol{\alpha}\ll P^{2t-2-3k(k+1)/2-\delta}.$$

On the other hand, note that Vaughan [141, Theorems 7.1, 7.3] gives

$$V(\boldsymbol{\alpha})\ll q^{\varepsilon}P\Big(q+|q\alpha_1-a_1|P^3+\cdots+|q\alpha_k-a_k|P^{3k}\Big)^{-1/3k},$$

and consequently, it follows that

$$\int_{\mathfrak{W}}|F(\boldsymbol{\alpha})|^{2t-2}d\boldsymbol{\alpha}\ll P^{2t-2-3k(k+1)/2}.$$

We finally apply Lemma 2.7.1 to $F(\boldsymbol{\alpha})$ to obtain that when $\boldsymbol{\alpha}\in\mathfrak{p}$ then one has $F(\boldsymbol{\alpha})<P(\log P)^{-\delta}$ for some $\delta>0$. Therefore, combining this bound with the above major arc estimate we obtain

$$\int_{\mathfrak{W}\setminus\mathfrak{P}}|F(\boldsymbol{\alpha})|^{2t}d\boldsymbol{\alpha}\ll P^{2t-3k(k+1)/2}(\log P)^{-\delta}.$$

The preceding discussion and equations (2.7.3) and (2.7.4) imply the proposition. $\square$

Next we introduce some properties of the smooth numbers concerning their density and distribution over arithmetic progressions which will be used throughout the argument for the approximation of $g(\alpha)$ over the major arcs. For such purposes, it is convenient to define

$$A_r(m) = \sum_{\substack{x\in\mathcal{A}(m,P^\eta)\\ x\equiv r \pmod q}} 1.$$

Lemma 2.7.2. *Let $q, m \in \mathbb{N}$ with $q \leq P^\eta$ and $P^\eta < m \leq P$. Then for each $0 \leq r \leq q-1$ one has*

$$A_r(m) = q^{-1}m\rho\Big(\frac{\log m}{\eta\log P}\Big) + O\Big(\frac{m}{\log m}\Big),$$

where the function $\rho(x)$ was defined before Theorem 2.1.2.

Proof. It follows from Montgomery and Vaughan [102, Theorem 7.2] and the argument of the proof of Vaughan [139, Lemma 5.4]. $\square$

We are now equipped to provide the approximation for $g(\alpha)$. In fact, we prove here a generalized version for future use in one of our forthcoming article. For such purpose, we take constants $0 \leq C_1 < C_2$ and $C_3 > 0$. Let $Q > 0$, let $m \in \mathbb{N}$ and define the exponential sum

$$g_{Q,m}(\alpha) = \sum_{\mathbf{x}\in\mathcal{B}} e\big(\alpha T(m\mathbf{x})^k\big),$$

where

$$\mathcal{B} = \Big\{\mathbf{x} \in \mathbb{N}^3 :\quad C_1Q < x_1 \leq C_2Q,\quad x_2, x_3 \in \mathcal{A}(C_3Q, Q^\eta)\Big\}.$$

Despite making the choice $\kappa = 1/5$ in the definition (2.7.1), the following lemma contains a result which makes no use of that choice and remains valid for the range $0 < \kappa < 1$.

Lemma 2.7.3. *Let $\alpha \in \mathfrak{N}(a,q)$, where $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q) = 1$ and $q \leq (\log)^\kappa$. Let $m \in \mathbb{N}$ with $(m,q) = 1$. Take $Q > 0$ with the property that*

$mQ \asymp P$ and consider $\beta = \alpha - a/q$. Then,

$$g_{Q,m}(\alpha) = V_{Q,m}(\alpha, q, a) + O\big(E(Q)\big),$$

where we take $E(Q) = Q^3(\log Q)^{\kappa-1}\log\log Q$ and

$$V_{Q,m}(\alpha, q, a) = q^{-3}S(q,a)\rho\big(1/\eta\big)^2 \int_{\mathbf{x}\in\mathcal{S}_Q} e\big(F_m(\mathbf{x})\big)d\mathbf{x}$$

with $F_m(\mathbf{x}) = \beta T(m\mathbf{x})^k$ and

$$\mathcal{S}_Q = \Big\{\mathbf{x} \in \mathbb{R}^3: \quad C_1Q \leq x_1 \leq C_2Q, \quad 0 \leq x_2, x_3 \leq C_3Q\Big\}.$$

Proof. For ease of notation, we omit the subscripts for the rest of the proof. We combine the ideas of the proof of Lemma 2.5.1 with the analysis of the distribution of smooth numbers discussed above. For such purposes, it is convenient to define first

$$K_{\mathbf{r}}(\beta; m) = \sum_{\substack{\mathbf{x}\in\mathcal{B} \\ \mathbf{x}\equiv\mathbf{r} \pmod{q}}} e\big(F_m(\mathbf{x})\big).$$

Observe that by sorting the summation into arithmetic progressions modulo q one has

$$g_{Q,m}(\alpha) = \sum_{\mathbf{r}\leq q} e_q\big(aT(m\mathbf{r})^k\big)K_{\mathbf{r}}(\beta; m). \tag{2.7.5}$$

For each $\mathbf{r} \in \mathbb{N}^3$, write $\mathbf{r} = (\mathbf{r}_1, r_3)$ with $\mathbf{r}_1 = (r_1, r_2)$. Then, we find that

$$K_{\mathbf{r}}(\beta; m) = \sum_{\mathbf{x}_1} \int_0^{C_3Q} e\big(F_m(\mathbf{x}_1, x)\big)dA_{r_3}(x),$$

where the integral on the right side is the Riemann-Stieltjes integral and $\mathbf{x}_1$ runs over the set

$$\mathcal{C}_{\mathbf{r}_1} = \Big\{\mathbf{x}_1 \in \mathbb{N}^2, \quad C_1Q < x_1 \leq C_2Q, \quad x_2 \in \mathcal{A}(C_3Q, Q^\eta), \quad \mathbf{x}_1 \equiv \mathbf{r}_1 \pmod{q}\Big\}.$$

The reader may find useful to observe that the contribution coming from the set $[0, Q/\log Q]$ to the above integral is $O\big(q^{-3}Q^3(\log Q)^{-1}\big)$, whence integration by parts yields

$$K_{\mathbf{r}}(\beta;m) = A_{r_3}(C_3Q)\sum_{\mathbf{x}_1} e\big(F_m(\mathbf{x}_1, C_3Q)\big) - \int_{Q/\log Q}^{C_3Q} \sum_{\mathbf{x}_1} \frac{\partial}{\partial z} e\big(F_m(\mathbf{x}_1, z)\big) A_{r_3}(z)dz + O\big(q^{-3}Q^3(\log Q)^{-1}\big).$$

We observe for convenience that

$$(mQ)^{3k}|\beta| \ll n|\beta| \ll q^{-1}(\log Q)^{\kappa}.$$

Therefore, the integral of the error term that arises when we approximate $A_{r_3}(z)$ in the above equation is bounded above by

$$\int_{Q/\log Q}^{C_3Q} \sum_{\mathbf{x}_1} \Big|\frac{\partial F_m}{\partial z}(\mathbf{x}_1, z)\Big| \frac{z}{\log z} dz \ll q^{-3}Q^3(\log Q)^{\kappa-1}.$$

Before embarking in the process of giving a better description of the above equation, we recall that an application of the mean value theorem gives that for any $w \in [Q/\log Q, C_3Q]$ then

$$\Big|\rho\Big(\frac{1}{\eta}\Big) - \rho\Big(\frac{\log w}{\eta \log Q}\Big)\Big| \ll \frac{\log\log Q}{\log Q}.$$

Consequently, by Lemma 2.7.2 and the preceding discussion we obtain

$$K_{\mathbf{r}}(\beta;m) = q^{-1}\rho\big(1/\eta\big)C_3Q\sum_{\mathbf{x}_1} e\big(F_m(\mathbf{x}_1, C_3Q)\big) - q^{-1}\rho\big(1/\eta\big)\int_{Q/\log Q}^{C_3Q} z\sum_{\mathbf{x}_1} \frac{\partial}{\partial z} e\big(F_m(\mathbf{x}_1, z)\big)dz + O\big(q^{-3}E(Q)\big),$$

and hence integration by parts yields

$$K_{\mathbf{r}}(\beta;m) = q^{-1}\rho\big(1/\eta\big)\int_0^{C_3Q} \sum_{\mathbf{x}_1} e\big(F_m(\mathbf{x}_1, z)\big)dz + O\big(q^{-3}E(Q)\big),$$

where we implicitly used that the term arising after evaluating at the end-points and the contribution of the interval $[0, Q/\log Q]$ to the above integral is $O(q^{-3}Q^3(\log Q)^{-1})$. Likewise, applying a similar procedure for the second variable one gets

$$K_{\mathbf{r}}(\beta;m)=q^{-2}\rho\big(1/\eta\big)^2\int_{\mathbf{y}\in[0,C_3Q]^2}\sum_x e\big(F_m(x,\mathbf{y})\big)d\mathbf{y}+O\big(q^{-3}E(Q)\big),\quad (2.7.6)$$

where x runs over the range $C_1Q<x\leq C_2Q$ and $x\equiv r_1 \pmod q$. For the first variable, we follow the same procedure as in Lemma 2.5.1 to obtain

$$\sum_{x\equiv r_1 \pmod q} e\big(F_m(x,\mathbf{y})\big)=q^{-1}\int_{C_1Q}^{C_2Q} e\big(F_m(x,\mathbf{y})\big)dx+O\big(1+q^{-1}(\log Q)^{\kappa}\big).$$

Consequently, combining the above equation with (2.7.6) we get

$$K_{\mathbf{r}}(\beta;m)=q^{-3}\rho\big(1/\eta\big)^2\int_{\mathbf{x}\in\mathcal{S}_Q} e\big(F_m(\mathbf{x})\big)d\mathbf{x}+O\big(q^{-3}E(Q)\big).$$

Observe that since $(m,q)=1$ then a change of variables yields

$$S(q,am^{3k})=S(q,a).$$

The lemma then follows by the preceding discussion and (2.7.5). □

Corollary 2.7.1. *Let $\alpha\in\mathfrak{N}(a,q)$ where $a\in\mathbb{Z}$, $q\in\mathbb{N}$ with $(a,q)=1$ and $q\leq(\log P)^{\kappa}$. Consider $\beta=\alpha-a/q$. One has that*

$$g(\alpha)=W(\alpha,q,a)+O(P^3(\log P)^{\kappa-1}\log\log P),$$

where on recalling (2.5.1) we take

$$W(\alpha,q,a)=q^{-3}S(q,a)\rho(1/\eta)^2v(\beta).$$

Proof. Note that $g(\alpha)=g_{P,1}(\alpha)$ with the choices $C_1=0$, $C_2=1$ and $C_3=1$. The result is then a consequence of the previous lemma. □

In the rest of the section we deduce an asymptotic formula for the contribution over the major arcs. For such purposes, consider the integral

$$R_{\mathfrak{N}}(n)=\int_{\mathfrak{N}} g(\alpha)^s e(-\alpha n)d\alpha.$$

Define the auxiliary function $g^*(\alpha)$ for $\alpha \in [0, 1)$ by putting

$$g^*(\alpha) = W(\alpha, q, a)$$

when $\alpha \in \mathfrak{N}(a, q) \subset \mathfrak{N}$ and $g^*(\alpha) = 0$ for $\alpha \in \mathfrak{n}$.

Proposition 2.7.2. *Let $s \geq \max(5, k+2)$. Then, there exists a constant $\delta > 0$ such that*

$$\begin{aligned} R_{\mathfrak{N}}(n) =& \Gamma\big(4/3\big)^{3s}\Gamma\big(1 + 1/k\big)^{s}\Gamma\big(s/k\big)^{-1}\rho\big(1/\eta\big)^{2s}\mathfrak{S}(n)n^{s/k-1} \\ &+ O(n^{s/k-1}(\log n)^{-\delta}). \end{aligned}$$

Proof. Observe that by the definition (2.7.1) one has that for $\alpha \in \mathfrak{N}(a, q)$ and $\beta = \alpha - a/q$ the bounds

$$(1 + n|\beta|)^{-(s-1)/k} \geq (\log P)^{-(s-1)\kappa/k} \geq (\log P)^{(s-1)(\kappa-1)}.$$

Consequently, Lemmata 2.5.2 and Corollary 2.7.1 yield

$$g(\alpha)^s - g^*(\alpha)^s \ll P^{3s}(\log P)^{\kappa-1+\varepsilon}(1 + n|\beta|)^{-(s-1)/k},$$

whence integrating over the major arcs we obtain that

$$\int_{\mathfrak{N}} \big|g(\alpha)^s - g^*(\alpha)^s\big| d\alpha \ll P^{3s-3k}(\log P)^{-\delta}.$$

Observe that by Lemmata 2.3.3 and 2.5.2 respectively we have

$$\sum_{q>(\log P)^{\kappa}} |S_n(q)| \ll (\log P)^{-\delta} \quad \text{and} \quad \int_{|\beta|>\frac{(\log P)^{\kappa}}{qn}} |v(\beta)|^s d\beta \ll P^{3s-3k}q^{\delta}(\log P)^{-\delta\kappa},$$

where in the second integral $q \leq (\log P)^{\kappa}$. Consequently, the combination of the aforementioned lemmas, equations (2.3.4) and (2.6.1) and the above bounds deliver the theorem. $\square$

Theorem 2.1.2 then follows by the application of Proposition 2.4.1, the estimate for the minor arcs in Proposition 2.7.1 and the above proposition.

2.8 Lower bound for $r(n)$.

In this section we prove Theorem 2.1.3 via an application of Theorem 2.1.2. The main idea is to show that the contribution to $R_\eta(n)$ of tuples whose coordinates have many representations as sums of three positive integral cubes is fairly small. We present first some notation and a simple lemma which will be used in the proof. Recalling the definition for $s_3(n)$ described before (2.1.5) and the parameter ν presented right after that equation, let $\theta = \nu/k$ and consider the set

$$S_K(n) = \Big\{ m \in \mathbb{N} : \ 1 \leq m \leq n^{1/k} : \ s_3(m) > K n^{\theta} \Big\},$$

where $K > 0$.

Lemma 2.8.1. *Let $n \in \mathbb{N}$ and $K > 0$. Then*

$$\sum_{m \in S_K(n)} s_3(m) \ll K^{-1} n^{1/k}.$$

Proof. It follows by noting that

$$\sum_{m \in S_K(n)} s_3(m) \ll K^{-1} n^{-\theta} \sum_{m \in S_K(n)} s_3(m)^2 \ll K^{-1} n^{1/k},$$

where in the last step we used (2.1.5). □

Define $R_1(n)$ as the contribution to $R_\eta(n)$ of tuples $\mathbf{X} \in \mathscr{C}^s$ for which $x_i \in S_K(n)$ for some i. Likewise, let $R_0(n)$ be the contribution to $R_\eta(n)$ of tuples $\mathbf{X} \in \mathscr{C}^s$ with $x_i \notin S_K(n)$ for every $1 \leq i \leq s$. Observe that with this notation then one has

$$R_\eta(n) = R_0(n) + R_1(n). \tag{2.8.1}$$

Note that by orthogonality, Theorem 2.1.2 and Lemma 2.8.1 one finds that

$$\begin{aligned} R_1(n) &\ll \sum_{m \in S_K(n)} s_3(m) \int_0^1 g(\alpha)^{s-1} e\big(-\alpha(n - m^k)\big)\, d\alpha \\ &\ll n^{(s-1)/k-1} \sum_{m \in S_K(n)} s_3(m) \ll K^{-1} n^{s/k-1}. \end{aligned}$$

whenever $s - 1 \geq H(k)$. Therefore, taking K to be big enough in terms of k

and s, we have by Theorem 2.1.2 and (2.8.1) that $R_\eta(n) \asymp R_0(n)$. Since each representation of n as a sum of k-th powers of elements of $\mathscr{C}$ is counted at most $s_3(x_1)\dots s_3(x_s) \leq K^s n^{s\theta}$ times by $R_0(n)$, we find that

$$r(n) \gg n^{s/k-1-s\theta} = n^{(1-\nu)s/k-1},$$

which delivers Theorem 2.1.3.

2.9 Asymptotic formula for small powers

We improve the constraint on the number of variables in Theorem 2.1.1 for the cases $2 \leq k \leq 7$ by interpolating between some restriction estimates and mean value bounds for Weyl sums over the minor arcs computed in Proposition 2.2.1. In the following lemma we present the aforementioned restriction estimate bounds, but first define $r(k) = 2^k$ for $2 \leq k \leq 3$ and $r(k) = k(k+1)$ when $4 \leq k \leq 7$. For the sake of conciseness, we omit writing the dependence on k for the rest of the section.

Lemma 2.9.1. *One has that*

$$\int_0^1 |f(\alpha)|^r \mathrm{d}\alpha \ll P^{13r/4-3k+\varepsilon}.$$

Proof. Note that recalling the definition of $r_3(n)$ before (2.1.2) we can rewrite the exponential sum $f(\alpha)$ as

$$f(\alpha) = \sum_{x \leq 3P^3} r_3(x)e(\alpha x^k).$$

We then apply mean value estimates of Bourgain [12, (1.6)] when $k = 2$, Hughes and Wooley [76, Theorem 4.1] for the case $k = 3$ and the work of Wooley [177, Corollary 1.4] when $4 \leq k \leq 7$ to obtain

$$\int_0^1 |f(\alpha)|^r d\alpha \ll P^{3r/2-3k+\varepsilon}\Big(\sum_{1 \leq m \leq 3P^3} r_3(m)^2\Big)^{r/2}.$$

Observe that the cited result for the case $4 \leq k \leq 7$ is the weighted version of Vinogradov's mean value theorem. As experts will realise, we can apply such result to obtain the estimate that we use herein via a similar argument than

k	2	3	4	5	6	7
s	24	63	134	216	316	435
t	23.4331	62.9722	133.4783	215.3978	315.9897	434.9924

Table 2.1

the one used in (2.2.3). The lemma then follows by combining the previous bound with (2.1.4). $\square$

Before describing the rest of the proof it is convenient to introduce some parameters. Take $h = \lfloor (k+1)/2 \rfloor$. Consider $p = 1 + r/4\xi_0$ and the exponents $q = p/(p-1)$ and $t = r/p + 3k(3k+1)/q$, where ξ_0 is defined as the positive root of the quadratic equation which is obtained by imposing the condition

$$\xi_0 = 1 - 1/(t - 2h + 1).$$

Let $s = \lceil t \rceil$. Both the values of s and t are gathered in Table 2.1. The following statement improves the number of variables obtained in Proposition 2.2.1 by interpolating the estimates that we get in the second part of Proposition 2.2.1 with Lemma 2.9.1.

Proposition 2.9.1. *One has that*

$$\int_{\mathfrak{m}} |f(\alpha)|^s \mathrm{d}\alpha \ll P^{3s-3k-\delta},$$

where we take the minor arcs $\mathfrak{m}$ to be as described right after (2.2.1) with ξ on the range $\xi_0 < \xi < 1 - 1/(s - 2h + 1)$.

Proof. By Hölder's inequality, Proposition 2.2.1 and Lemma 2.9.1 we obtain that

$$\begin{aligned}\int_{\mathfrak{m}} |f(\alpha)|^t \mathrm{d}\alpha &\ll \Big(\int_0^1 |f(\alpha)|^r d\alpha\Big)^{1/p}\Big(\int_{\mathfrak{m}} |f(\alpha)|^{3k(3k+1)} d\alpha\Big)^{1/q} \\ &\ll \big(P^{13r/4-3k+\varepsilon}\big)^{1/p}\big(P^{27k^2+6k-\xi+\varepsilon}\big)^{1/q} \ll P^{3t-3k-\delta},\end{aligned}$$

from where the lemma follows by observing that $t < s$. $\square$

The rest of the appendix is devoted to make a refinement of the argument used in Proposition 2.6.1 to enlarge the major arcs by taking ξ on the range described above and win one variable for the cases $k = 3, 6$ and 7. Let $q < P$. Denote by $N(q, P)$ the number of solutions of the congruence

$$T(\mathbf{x}_1)^k + \ldots + T(\mathbf{x}_h)^k \equiv T(\mathbf{y}_1)^k + \ldots + T(\mathbf{y}_h)^k \pmod{q},$$

where $0 \le \mathbf{x}_i, \mathbf{y}_i \le P$. By expressing q as the product of prime powers, using the structure of the ring of integers of these prime powers and noting that the number of primes dividing q is bounded by q^ε we obtain $N(q, P) \ll q^{\varepsilon - 1} P^{2h}$, and hence orthogonality yields

$$\sum_{a=1}^{q} |f(\beta + a/q)|^{2h} \ll qN(q, P) \ll q^\varepsilon P^{2h}. \tag{2.9.1}$$

Now consider the difference function $D(\alpha) = f(\alpha) - f^*(\alpha)$. By the triangle inequality one has

$$|f(\alpha)^s - f^*(\alpha)^s| \ll F_1(\alpha) + F_2(\alpha),$$

where

$$F_1(\alpha) = |f(\alpha)|^{2h} |D(\alpha)| \left(|f^*(\alpha)|^{s-2h-1} + |D(\alpha)|^{s-2h-1}\right)$$

and $F_2(\alpha) = |D(\alpha)||f^*(\alpha)|^{s-1}$. The integral over the major arcs for $F_2(\alpha)$ is bounded in the same way as in equation (2.6.2), and by combining Lemmata 2.5.1 and 2.5.2 with equation (2.9.1) we get

$$\int_{\mathfrak{M}} F_1(\alpha) d\alpha \ll P^{3s-3k+\xi-1+\varepsilon} \sum_{q \le P^\xi} S^*_{s-2h-1}(q) + P^{3s-3k+(s-2h+1)\xi - s + 2h} \sum_{q \le P^\xi} q^{\varepsilon - 1}.$$

Using the fact that $\xi < 1 - 1/(s - 2h + 1)$ and Lemma 2.3.3 we obtain that the previous integral is $O(P^{3s-3k-\delta})$. Therefore, by the preceding discussion, the argument following (2.6.2) and Propositions 2.4.1 and 2.9.1 then the conclusion of Theorem 2.1.1 holds for the values of s in Table 2.1.

Chapter 3

Waring's problem in sums of three cubes for small exponents

3.1 Upper bounds on the number of variables for small exponents[1]

As the above heading anticipates, the purpose of the present subsection is to provide an upper bound for the minimum s with the property that every sufficiently large integer can be represented as the sum of s positive k-th powers of integers, each of which is represented as the sum of three positive cubes for the cases $2 \leq k \leq 4$.

3.1.1 Introduction

Additive problems involving small powers of positive integers have led to a vast development of new ideas and techniques in the application of the Hardy-Littlewood method which often cannot be extended to the setting of general k-th powers. Finding the least number s such that for every sufficiently large integer n then

$$n = x_1^k + \ldots + x_s^k, \tag{3.1.1}$$

where $x_i \in \mathbb{N}$, might be among the most studied examples. We denote such number s by $G(k)$. Let $\mathscr{C}$ be the set of integers represented as a sum of

[1] This section is based on a paper [110] by the author that has been accepted in the Journal of the Australian Mathematical Society.

three positive integral cubes. In this memoir we shall be concerned with the function $G_3(k)$, which we define as the minimum s such that (3.1.1) is soluble with $x_i \in \mathscr{C}$, for the cases $2 \leq k \leq 4$.

Providing the precise value of $G(k)$ is still an open question for most k, the cases $k = 2, 4$ being precisely the only ones solved. Lagrange showed in 1770 that $G(2) = 4$ and Davenport [30] proved in 1939 the identity $G(4) = 16$, and though it is believed that $G(3) = 4$, the best current upper bound is $G(3) \leq 7$ due to Linnik [96].

Not very much is known about $\mathscr{C}$. In fact, it isn't even known whether it has positive density or not, the best current lower bound on the cardinality of the set being

$$\mathcal{N}(X) = |\mathscr{C} \cap [1, X]| \gg X^{\beta},$$

where $\beta = 0.91709477$, due to Wooley [174]. We note that under some unproved assumptions on the zeros of some Hasse-Weil L-functions, Hooley ([70], [71]) and Heath-Brown [63] showed using different procedures that

$$\sum_{n \leq X} r_3(n)^2 \ll X^{1+\varepsilon},$$

where $r_3(n)$ is the number of representations of n as sums of three positive integral cubes, which implies by applying a standard Cauchy-Schwarz argument that $\mathcal{N}(X) \gg X^{1-\varepsilon}$. This lack of understanding of the cardinality of the set also prevents us from understanding its distribution over arithmetic progressions, which often comes into play on the major arc analysis. Therefore, even if the exponents $k = 2, 4$ are well understood for the original problem, it turns out to be much harder when we restrict the variables to lie on $\mathscr{C}$. In this paper we establish the following bounds for $G_3(k)$.

Theorem 3.1.1. *One has* $G_3(2) \leq 8$, $G_3(3) \leq 17$ *and* $G_3(4) \leq 57$.

We are far from knowing whether these estimates are good or bad, since the only lower bounds that we have for the above quantities are $4 \leq G(3) \leq G_3(3)$ and $16 = G(4) \leq G_3(4)$. For the case $k = 2$ though we can actually do better. We take, for convenience, an integer $j \geq 0$, and observe that the only solutions to

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 33 \cdot 2^{12j} \tag{3.1.2}$$

with $x_i \in \mathbb{N} \cup \{0\}$ are either

$$x_1 = 5 \cdot 2^{6j},\ x_2 = 2^{1+6j},\ x_3 = 2^{1+6j},\ x_4 = 0,$$

or

$$x_1 = 4 \cdot 2^{6j},\ x_2 = 3 \cdot 2^{6j},\ x_3 = 2^{1+6j},\ x_4 = 2^{1+6j},$$

or

$$x_1 = 4 \cdot 2^{6j},\ x_2 = 4 \cdot 2^{6j},\ x_3 = 2^{6j},\ x_4 = 0$$

or the solutions corresponding to permutations of the above. This may be seen by taking the equation (3.1.2) modulo 8, realising that one must have $2 \mid x_i$ for every i, iterating the process and noting that the only solutions to the equation

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 33$$

satisfy either

$$\{y_1, y_2, y_3, y_4\} = \{5, 2, 2, 0\} \text{ or } \{y_1, y_2, y_3, y_4\} = \{4, 3, 2, 2\}$$

or

$$\{y_1, y_2, y_3, y_4\} = \{4, 4, 1, 0\}.$$

However, one has $4 \cdot 2^{6j} \equiv 4 \pmod 9$ and $5 \cdot 2^{6j} \equiv 5 \pmod 9$, and no number congruent to 4 or 5 $\pmod 9$ can be written as the sum of three cubes. Therefore, there are infinitely many numbers not represented as sums of at most four squares of sums of three cubes. The preceding remark implies then the bound $5 \leq G_3(2)$.

Our proof of Theorem 3.1.1 is based on the application of the Hardy-Littlewood method. In the setting of this paper, the constraint which prevents us from taking fewer variables is the treatment of the minor arcs discussed in Subsection 3.1.2. In order to analyse them we utilise an argument of Vaughan [139, Lemma 3.4] to bound certain families of exponential sums over the minor arcs together with non-optimal estimates of sums of the shape

$$\sum_{x \leq X} a_x^2, \quad \text{where } a_x = \text{card}\big\{\mathbf{x} \in \mathbb{N}^3 :\ x = x_1^3 + x_2^3 + x_3^3,\ \ x_2, x_3 \in \mathcal{A}(P, P^\eta)\big\}$$

with $\eta > 0$ being a small enough parameter and

$$\mathcal{A}(Y,R) = \{n \in [1,Y] \cap \mathbb{N} : p \mid n \text{ and } p \text{ prime} \Rightarrow p \leq R\}.$$

Here, the reader may find it useful to observe that it is a consequence of Montgomery and Vaughan [102, Theorem 7.2] that

$$\operatorname{card}\big(\mathcal{A}(P,P^{\eta})\big) = c_{\eta}P + O\big(P/\log P\big)$$

for some constant $c_{\eta} > 0$ that only depends on η. In order to briefly discuss the outcome that follows after applying the argument of Vaughan we introduce the exponential sum

$$W(\alpha) = \sum_{M/2\leq p\leq M}\ \sum_{H/2\leq h\leq H} b_h e(\alpha p^{3k}h^k), \tag{3.1.3}$$

where $M, H > 0$ and b_h are weights which the reader should think of being a_h and p runs over prime numbers. It is worth mentioning that in order to make the argument work, the parameters M and H must be subjected to the constraint $\max(M^{5-1/k}, M^{2^{k-1}}) \leq H$. The saving over the natural bound HM for $W(\alpha)$ obtained with the method is roughly speaking of size $M^{1/2}H^{-1/24}$, which makes the estimate obtained worse than trivial for $k \geq 5$.

A naive approach to bounding $G_3(k)$ would then be to replace each sum of three cubes by the specialisation $3x^3$, and this suggests a bound of the shape $G_3(k) \leq G(3k)$. With this idea in mind, the bounds $G(6) \leq 24$ due to Vaughan and Wooley [142], $G(9) \leq 47$ and $G(12) \leq 72$ due to Wooley [176] reveal that the methods used in this memoir improve what would have been the trivial approach and confirms that we are actually using the three integral cubes non-trivially in our argument. For the cases $k = 2, 3$ we combine the pointwise bound obtained for $W(\alpha)$ over the minor arcs with some restriction estimates involving the coefficients a_m. When $k = 4$ we instead use a bound for a mean value of smooth Weyl sums of exponent 12. The estimate for $W(\alpha)$ obtained here is then robust enough to enable us to gain 15 variables from the trivial approach over the minor arcs and allows us to prune back to a narrower set of major arcs.

The purpose of the present section is to derive upper bounds for the minimum number of variables that guarantee the existence of solutions to equation

(3.1.1) for smaller values of k. As experts may expect, the minor arc arguments in the analysis of the previous chapter, as opposed to those employed in the present one, rely on estimates stemming from Vinogradov's Mean Value Theorem [177]. Moreover, the major arc discussion follows a standard approach, and the author incorporates the three cubes in the analysis of the singular series. In constrast, the major arc manoeuvres herein entail fixing the two smooth cubes in order to provide robust approximations of the corresponding exponential sums on a wider set of major arcs, and the pruning operations deployed in the discussion involve both these approximations and minor arc type estimates. Moreover, the absence of two of the cubes in the corresponding singular series makes the local solubility analysis more tedious and somewhat different than that of the aforementioned chapter.

In the next section, we shall employ the minor arc bound for the case $k = 2$ obtained in the present section to derive an almost all result for the analogue of Lagrange's four square theorem when the variables are restricted to the set of sums of three positive cubes. Moreover, we shall make use of the approximations of the exponential sums obtained herein in the pruning process and apply several major arc type lemmata deduced in this work. Nevertheless, we shall also incorporate the three cubes in the analysis of the singular series involving four squares, which in turn entails having a rather delicate discussion of a different nature regarding the behaviour of such a series. This strategy requires the use of thinner major arcs and pruning operations that have little resemblance to the manouvres deployed in the present section.

The paper is organized as follows. In Subsection 3.1.2 we use Vaughan methods to estimate $W(\alpha)$ and provide bounds for the contribution of the minor arcs which are good enough for our purposes when $k = 2, 3$. We approximate the generating functions of the problem on a narrower set of major arcs in Subsection 3.1.3. In Subsections 3.1.4, 3.1.5 and 3.1.6 we only consider the exponents $k = 2, 3$, whereas in Subsection 3.1.7 we prove the theorem for $k = 4$. Subsections 3.1.4 and 3.1.5 are devoted to the study of the singular series and the singular integral respectively. We then prune back to the narrower set of arcs to show a lower bound for the major arc contribution in Subsection 3.1.6.

3.1.2 Minor arc estimates

As mentioned in the introduction, we provide an estimate for the exponential sum $W(\alpha)$ by using methods of Vaughan. We make use of a Hardy-Littlewood dissection and combine both the bound for $W(\alpha)$ and a restriction estimate of a certain mean value to bound the minor arc contribution for the cases $k = 2, 3$. We also remark that the estimate for $W(\alpha)$ is also used in Subsections 3.1.6 and 3.1.7 to prune the major arcs back to a narrower set of arcs. Before going into the proof of the main lemma, it is convenient to write

$$S_k(q, a) = \sum_{r=1}^{q} e_q(ar^k). \tag{3.1.4}$$

We also introduce the multiplicative function $\tau_k(q)$ by defining $\tau_2(q) = q^{-1/2}$ and

$$\tau_k(p^{uk+v}) = p^{-u-1} \quad \text{when } u \geq 0 \text{ and } 2 \leq v \leq k$$

and

$$\tau_k(p^{uk+1}) = kp^{-u-1/2} \quad \text{when } u \geq 0$$

for $k \geq 3$. Observe that with this definition then one has the bound

$$\tau_k(q) \ll q^{-1/k}, \tag{3.1.5}$$

and the proof of Theorem 4.2 of [141] yields

$$q^{-1} S_k(q, a) \ll \tau_k(q). \tag{3.1.6}$$

Lemma 3.1.1. *Let $2 \leq k \leq 4$. Take parameters $H, M > 0$ with the property that $\max(M^{5-1/k}, M^{2^{k-1}}) \leq H$. Let $\alpha \in [0, 1)$. Suppose that $\alpha = a/q + \beta$, where $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and such that $q \leq Y$, and $|\beta| \leq q^{-1}Y^{-1}$, where Y is a parameter in the range $M^k \leq Y \leq H^k M^{2k}$. Then the exponential sum $W(\alpha)$, defined in (3.1.3), satisfies*

$$W(\alpha) \ll H^{\varepsilon} \left(HM + \frac{\tau_k(q) HM^2}{1 + M^{3k} H^k |\alpha - a/q|} \right)^{1/2} \left(\sum_{H/2 \leq h \leq H} |b_h|^2 \right)^{1/2}. \tag{3.1.7}$$

Proof. For the sake of simplicity we will not write the limits of summation for p and h throughout the rest of the subsection. We apply Cauchy-Schwarz to obtain

$$W(\alpha) \ll \Big(\sum_h |b_h|^2\Big)^{1/2} \Big(\sum_h \sum_{p_1,p_2} e\big(\alpha(p_1^{3k} - p_2^{3k})h^k\big)\Big)^{1/2}$$
$$\ll \Big(\sum_h |b_h|^2\Big)^{1/2} \Big(HM + E(\alpha)\Big)^{1/2}, \tag{3.1.8}$$

where the term HM comes from the diagonal contribution and

$$E(\alpha) = \sum_h \sum_{p_2<p_1} e\big(\alpha(p_1^{3k} - p_2^{3k})h^k\big).$$

In order to estimate $E(\alpha)$ we will follow closely the argument employed in Vaughan [139, Lemma 3.4]. For a given pair of primes (p_1, p_2) we choose $b, r \in \mathbb{N}$ with $(b, r) = 1$ and such that $r \leq 2kH^{k-1}$ and

$$|\alpha(p_1^{3k} - p_2^{3k}) - b/r| \leq (2k)^{-1} r^{-1} H^{1-k}.$$

Then if $r > H$, an application of Weyl's inequality [141, Lemma 2.4] yields the bound

$$\sum_h e\big(\alpha(p_1^{3k} - p_2^{3k})h^k\big) \ll H^{1-2^{1-k}+\varepsilon} \ll H^{1+\varepsilon} M^{-1},$$

where we used the restriction on M at the beginning of the lemma. If on the other hand $r \leq H$ we combine Lemmata 6.1 and 6.2 of [141] with (3.1.6) to obtain

$$\sum_h e\big(\alpha(p_1^{3k} - p_2^{3k})h^k\big) \ll \frac{\tau_k(r)H}{1 + H^k\big|\alpha(p_1^{3k} - p_2^{3k}) - b/r\big|} + r^{1/2+\varepsilon}.$$

Consequently, one has that

$$E(\alpha) \ll E_0 + H^{1+\varepsilon} M + \sum_{(p_1,p_2)} H^{1/2+\varepsilon} \ll E_0 + H^{1+\varepsilon} M,$$

where

$$E_0 = \sum_{(p_1,p_2)\in\mathcal{A}} \frac{\tau_k(r)H}{1 + H^k\big|\alpha(p_1^{3k} - p_2^{3k}) - b/r\big|}$$

and $\mathcal{A}$ is the set of pairs (p_1, p_2) with $p_2 < p_1$ for which $r < (6k)^{-1} M^k$ and such that

$$\big|\alpha(p_1^{3k} - p_2^{3k}) - b/r\big| < 2^{-1} r^{-1/k} M H^{-k}.$$

Note that the contribution of the pairs for which one of the previous two restrictions doesn't hold is $O(HM)$. For each pair (p_1, p_2), define $n = p_2$, $l = p_1 - p_2$ and $D = \big((n+l)^{3k} - n^{3k}\big)/l$. Then one finds that

$$E_0 \ll \sum_{(n,l)} \frac{\tau_k(r)H}{1 + H^k\big|\alpha l D - b/r\big|}, \tag{3.1.9}$$

where (n,l) runs over pairs with $1 \le l \le M$ and $M/2 \le n \le M$ such that $(n+l, n) = 1$ and satisfying the existing bounds on r and $\big|\alpha l D - b/r\big|$.

We next choose for convenience $c, s \in \mathbb{N}$ satisfying $(c,s) = 1$ and with the property that $s \le H^k M^{-k}$ and

$$|\alpha l - c/s| \le s^{-1} M^k H^{-k}.$$

By the constraint imposed on M and H at the beginning of the lemma we obtain

$$\Big|\frac{c}{s} - \frac{b}{rD}\Big| sDr < DrM^kH^{-k} + \frac{1}{2}sr^{1-1/k}MH^{-k} < \frac{3k}{6k}M^{5k-1}H^{-k} + \frac{1}{2}sM^kH^{-k} \le 1.$$

Therefore, one has $crD = bs$, and hence the coprimality condition on r and b yields $r|s$. Let $s_0 = s/r$. We then have that $s_0 \mid D$, whence

$$E_0 \ll \sum_{s_0|s} \tau_k\Big(\frac{s}{s_0}\Big) \sum_{(n,l)} \frac{H}{1 + H^kD\big|\alpha l - c/s\big|},$$

where the sum on (n,l) runs over the same range described after (3.1.9) with the conditions $(n+l, n) = 1$ and $\big((n+l)^{3k} - n^{3k}\big)/l \equiv 0 \pmod{s_0}$. Once we fix l then using the above constraints one has that the number of such n is bounded above by $O\big((M/s_0 + 1)s_0^{\varepsilon}\big)$. Consequently, we obtain that

$$E(\alpha) \ll H^{1+\varepsilon}M + H^{\varepsilon}ME_1,$$

where

$$E_1 = \sum_{l \in \mathcal{L}} \frac{\tau_k(s)H}{1 + H^kM^{3k-1}\big|\alpha l - c/s\big|},$$

and $\mathcal{L}$ is the set of integers $l \le M$ for which $s < M^k/2$ and

$$|\alpha l - c/s| < M^{2-3k}H^{-k}.$$

Now we choose d, t with $(d,t) = 1$ satisfying $t \le M^{k+1}$ and

$$\bigl|\alpha - d/t\bigr| \le t^{-1}M^{-k-1}.$$

We then find that

$$\Bigl|\frac{c}{ls} - \frac{d}{t}\Bigr| slt < stM^{2-3k}H^{-k} + slM^{-k-1} < \frac{1}{2}M^{3-k}H^{-k} + \frac{1}{2} \le 1.$$

Therefore, one has $ct = dsl$ and hence $s|t$. Let $t_0 = t/s$. Then it follows that $t_0 \mid l$, and on defining $l_0 = l/t_0$ we obtain

$$E_1 \ll \sum_{t_0|t} \tau_k\Bigl(\frac{t}{t_0}\Bigr) \sum_{l_0 \le M/t_0} \frac{H}{1 + H^k M^{3k-1} l_0 t_0 \bigl|\alpha - d/t\bigr|} \ll \frac{\tau_k(t) H M^{1+\varepsilon}}{1 + H^k M^{3k}\bigl|\alpha - d/t\bigr|}.$$

If either $t \ge M^k/2$ or

$$|\alpha - d/t| \ge 2^{-1}t^{-1/k}H^{-k}M^{1-3k}$$

then we get $E_1 \ll HM^{\varepsilon}$ and we would be done. For the remaining cases one finds that

$$\Bigl|\frac{a}{q} - \frac{d}{t}\Bigr| qt < \frac{1}{2}qH^{-k}M^{1-3k}t^{1-1/k} + tY^{-1} < \frac{1}{2}YH^{-k}M^{-2k} + \frac{1}{2}M^kY^{-1} \le 1,$$

which implies that $a = d$ and $q = t$, and yields the bound

$$E(\alpha) \ll H^{1+\varepsilon}M + \frac{\tau_k(q)H^{1+\varepsilon}M^2}{1 + H^kM^{3k}\bigl|\alpha - a/q\bigr|}.$$

The combination of this estimate and (3.1.8) proves the lemma. $\square$

Before describing the application of this lemma in the minor arc treatment it is convenient to introduce some notation. Let n be a natural number and take $P = n^{1/(3k)}$. Define the parameters

$$\gamma(k) = \frac{3}{3 + \max(5 - 1/k, 2^{k-1})}, \qquad M = P^{\gamma(k)}, \qquad H = \max(M^{5-1/k}, M^{2^{k-1}}). \tag{3.1.10}$$

We observe for further purposes that

$$P^3 = M^3H. \tag{3.1.11}$$

Note that these choices for M and H maximize the saving obtained for $W(\alpha)$ over the trivial bound in the previous lemma. Take

$$H_1 = \Big(\frac{1}{2}\Big)^{1/3} H^{1/3}, \qquad H_2 = \Big(\frac{2}{3}\Big)^{1/3} H^{1/3}, \qquad H_3 = \Big(\frac{1}{6}\Big)^{1/3} H^{1/3}.$$

For every triple $\mathbf{x} \in \mathbb{R}^3$, consider the function $T(\mathbf{x}) = x_1^3 + x_2^3 + x_3^3$. Define the sets

$$\mathcal{H} = \Big\{(y, \mathbf{y}) \in \mathbb{N}^3 : \quad \frac{P}{2} \leq y \leq P, \qquad \mathbf{y} \in \mathcal{A}(P, P^\eta)^2\Big\},$$
$$\mathcal{W} = \Big\{(y, \mathbf{y}) \in \mathbb{N}^3 : \quad H_1 \leq y \leq H_2, \quad \mathbf{y} \in \mathcal{A}(H_3, P^\eta)^2\Big\},$$

and the corresponding weights

$$a_x = |\{\mathbf{x} \in \mathcal{H} : \ x = T(\mathbf{x})\}|, \qquad b_h = |\{\mathbf{x} \in \mathcal{W} : \ h = T(\mathbf{x})\}|,$$

where b_h is the choice that we make for the weights of $W(\alpha)$ in (3.1.3). We use a_x to define the weighted exponential sum

$$h(\alpha) = \sum_{x \leq 3P^3} a_x e(\alpha x^k).$$

Before describing how $h(\alpha)$ and $W(\alpha)$ play a role in the argument we first show upper bounds on the L^2-norms of the weights which will be used to estimate the minor arc contribution. Let $X > 0$, consider

$$f(\alpha; X) = \sum_{x \leq X} e(\alpha x^3), \qquad f(\alpha; X; X^\eta) = \sum_{x \in \mathcal{A}(X, X^\eta)} e(\alpha x^3)$$

and define the mean value

$$U(X) = \int_0^1 |f(\alpha; X)|^2 |f(\alpha; X; X^\eta)|^4 d\alpha.$$

It is a consequence of Wooley [174, Theorem 1.2] that $U(X) \ll X^{3+1/4-\tau}$, where $\tau = 0.00128432$. Consequently, on considering the underlying diophantine equations due to orthogonality, it follows that

$$\sum_{x \leq 3P^3} a_x^2 \leq U(P) \ll P^{3+1/4-\tau}, \qquad \sum_{H/2 \leq h \leq H} b_h^2 \leq U(H^{1/3}) \ll H^{13/12-\tau/3}. \tag{3.1.12}$$

The reader may note that we didn't write the entire decimal expression of τ, so the bound for $U(X)$ holds for a slightly bigger τ. Therefore, whenever we encounter bounds with the mean value $U(X)$ involved, we can omit the parameter ε in the exponents.

Take

$$s(k) = 2^k \qquad \text{when } k = 2, 3$$
$$t(2) = 4 \qquad \text{and} \qquad t(3) = 9. \tag{3.1.13}$$

For ease of notation we will just write s and t instead of $s(k)$ and $t(k)$ throughout the paper. Let $R(n)$ be the number of solutions of the equation

$$n = \sum_{i=1}^{t} T(p_i \mathbf{x}_i)^k + \sum_{i=t+1}^{s+t} T(\mathbf{x}_i)^k,$$

where $\mathbf{x}_i \in \mathcal{W}$ for $1 \leq i \leq t$ with $M/2 \leq p_i \leq M$ prime and $\mathbf{x}_i \in \mathcal{H}$ for $t+1 \leq i \leq s+t$. Note that by orthogonality then

$$R(n) = \int_0^1 h(\alpha)^s W(\alpha)^t e(-\alpha n) d\alpha.$$

Our goal throughout Subsections 3.1.2 to 3.1.6 is to obtain a lower bound for $R(n)$ for all sufficiently large n. For such purpose, we make use of a Hardy-Littlewood dissection in our analysis. When $1 \leq X \leq M^k$, we define the major arcs $\mathfrak{M}(X)$ to be the union of

$$\mathfrak{M}(a, q) = \Big\{\alpha \in [0, 1) : \Big|\alpha - a/q\Big| \leq \frac{X}{qn}\Big\} \tag{3.1.14}$$

with $0 \leq a \leq q \leq X$ and $(a, q) = 1$. For the sake of simplicity we write

$$\mathfrak{M} = \mathfrak{M}(M^k), \qquad \mathfrak{N} = \mathfrak{M}\big((6k)^{-1} H^{1/3}\big).$$

We define the minor arcs as $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$ and $\mathfrak{n} = [0, 1) \setminus \mathfrak{N}$. This dissection remains valid for the case $k = 4$ and will be used in Subsection 3.1.7. We then take $\alpha \in \mathfrak{m}$ and observe that by Dirichlet's approximation there exist non-negative integers a, q with $(a, q) = 1$ and $1 \leq q \leq nM^{-k}$ such that

$$|\alpha - a/q| \leq \frac{M^k}{qn}.$$

Consequently, one has $q > M^k$, and hence (3.1.12) and Lemma 3.1.1 yield the bound

$$W(\alpha) \ll H^{1/2+\varepsilon} M^{1/2} \Big(\sum_{h \leq H} b_h^2 \Big)^{1/2} \ll H^{1+1/24-\tau/6} M^{1/2}. \tag{3.1.15}$$

As was previously observed right after (3.1.3), the reader may find it useful to note that in view of (3.1.10) then the above estimate is worse than the trivial one HM whenever $k \geq 5$. This explains the reason why we have restricted our analysis in this memoir to the cases $2 \leq k \leq 4$. In the following proposition we combine this pointwise bound with some restriction estimates to bound the minor arc contribution.

Proposition 3.1.1. *When $k = 2, 3$ then one has that*

$$\int_{\mathfrak{m}} |h(\alpha)|^s |W(\alpha)|^t d\alpha \ll (HM)^t P^{3s-3k-\delta}. \tag{3.1.16}$$

Proof. Combining Bourgain [12, (1.6)] when $k = 2$ and the work of Hughes and Wooley [76, Theorem 4.1] when $k = 3$ with equation (3.1.12) we find that

$$\int_0^1 |h(\alpha)|^s d\alpha \ll P^{3s/2-3k+\varepsilon} \Big(\sum_{x \leq 3P^3} a_x^2 \Big)^{s/2} \ll P^{3s-3k+s/8-\delta}.$$

Therefore, an application of the pointwise bound on the minor arcs obtained in (3.1.15) yields the estimate

$$\int_{\mathfrak{m}} |h(\alpha)|^s |W(\alpha)|^t d\alpha \ll H^{t+t/24} M^{t/2} P^{3s-3k+s/8-\delta}.$$

We define for convenience the parameter $\xi(k)$ as $\xi(2) = 0$ and $\xi(3) = 7/92$, and deduce that the proposition then follows after noting by (3.1.10) that $H^{t/24} M^{t/2} P^{s/8} = M^t P^{-\xi(k)}$. For the purpose of this paper, knowing the existence of $\delta > 0$ for which (3.1.16) holds suffices. The reader may observe though that the precise saving over the expected main term that we obtain here is $H^{t\tau/6} P^{\xi(k)+s\tau/2-\varepsilon}$. □

3.1.3 Approximation of exponential sums over the major arcs

We adapt the argument of Vaughan [141, Theorem 4.1] to estimate the difference between the exponential sums $h(\alpha), W(\alpha)$ and their approximations over the major arcs. Let $\mathbf{y} \in [0, P]^2$ and set

$$C_{\mathbf{y}} = y_1^3 + y_2^3. \tag{3.1.17}$$

Let $\beta \in \mathbb{R}$ and let p be a prime number. Consider the integrals

$$v_{\mathbf{y}}(\beta) = \int_{P/2}^{P} e\Big(\beta\big(x^3 + C_{\mathbf{y}}\big)^k\Big)dx \quad \text{and} \quad v_{\mathbf{y},p}(\beta) = \int_{H_1}^{H_2} e\Big(\beta p^{3k}\big(x^3 + C_{\mathbf{y}}\big)^k\Big)dx. \tag{3.1.18}$$

Note that by a change of variables one finds that

$$v_{\mathbf{y}}(\beta) = \int_{M_{\mathbf{y}}}^{N_{\mathbf{y}}} B_{\mathbf{y}}(\gamma)e(\beta\gamma)d\gamma, \qquad v_{\mathbf{y},p}(\beta) = \int_{M_{\mathbf{y},p}}^{N_{\mathbf{y},p}} B_{\mathbf{y},p}(\gamma)e(\beta\gamma)d\gamma, \tag{3.1.19}$$

where the limits of integration taken are $M_{\mathbf{y}} = \big(P^3/8 + C_{\mathbf{y}}\big)^k$, $N_{\mathbf{y}} = \big(P^3 + C_{\mathbf{y}}\big)^k$, $M_{\mathbf{y},p} = \big(Hp^3/2 + C_{p\mathbf{y}}\big)^k$ and $N_{\mathbf{y},p} = \big(2Hp^3/3 + C_{p\mathbf{y}}\big)^k$, and the functions inside the integral are defined as

$$B_{\mathbf{y}}(\gamma) = \frac{1}{3k}\gamma^{1/k-1}(\gamma^{1/k} - C_{\mathbf{y}})^{-2/3}, \qquad B_{\mathbf{y},p}(\gamma) = \frac{1}{3kp}\gamma^{1/k-1}(\gamma^{1/k} - C_{p\mathbf{y}})^{-2/3}. \tag{3.1.20}$$

We introduce the auxiliary multiplicative function $w_k(q)$ defined for prime powers by taking

$$w_k(p^{3ku+v}) = \begin{cases} p^{-u-v/3k} & \text{when } u \geq 1 \text{ and } 1 \leq v \leq 3k, \\ p^{-1} & \text{when } u = 0 \text{ and } 2 \leq v \leq 3k, \\ p^{-1/2} & \text{when } u = 0 \text{ and } v = 1. \end{cases} \tag{3.1.21}$$

In order to discuss the approximation of $f(\alpha)$ on the major arcs, it is convenient to consider for $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ the sums

$$S_{\mathbf{y}}(q, a) = \sum_{r=1}^{q} e_q\Big(a\big(r^3 + C_{\mathbf{y}}\big)^k\Big) \qquad \text{and} \qquad V(\alpha, q, a) = q^{-1}\sum_{\mathbf{y}} S_{\mathbf{y}}(q, a)v_{\mathbf{y}}(\beta), \tag{3.1.22}$$

where $\mathbf{y}$ runs over the set $\mathcal{A}(P, P^\eta)^2$ of pairs of smooth numbers.

Lemma 3.1.2. *Suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$. Let $\alpha \in [0, 1)$ and $\beta = \alpha - a/q$. Then we have the estimate*

$$h(\alpha) - V(\alpha, q, a) \ll P^2 q^{1+\varepsilon} w_k(q)(1 + n|\beta|)^{1/2}.$$

Moreover, if $|\beta| \leq (2 \cdot 3^k kq)^{-1} P n^{-1}$ one finds that

$$h(\alpha) - V(\alpha, q, a) \ll P^2 q^{1+\varepsilon} w_k(q). \tag{3.1.23}$$

Proof. Let $b \in \mathbb{Z}$ and $\mathbf{y} \in \mathcal{A}(P, P^\eta)^2$. We define

$$S_{\mathbf{y}}(q, a, b) = \sum_{r=1}^{q} e_q\big(a\big(r^3 + C_{\mathbf{y}}\big)^k + br\big) \qquad \text{and} \qquad I_{\mathbf{y}}(b) = \int_{P/2}^{P} e\big(F(\gamma; b)\big) d\gamma, \tag{3.1.24}$$

where the function in the argument inside the integral is taken to be

$$F(\gamma; b) = \beta\big(\gamma^3 + C_{\mathbf{y}}\big)^k - b\gamma/q.$$

Both the complete exponential sum and the integral play a role in the analysis of the main and the error term. Observe that $h(\alpha)$ can be written as

$$h(\alpha) = \sum_{\mathbf{y} \in \mathcal{A}(P, P^\eta)^2} h_{\mathbf{y}}(\alpha), \qquad \text{with} \qquad h_{\mathbf{y}}(\alpha) = \sum_{P/2 \leq x \leq P} e\big(\alpha(x^3 + C_{\mathbf{y}})^k\big).$$

Then by sorting the summation into arithmetic progressions modulo q and applying orthogonality, it follows that

$$h_{\mathbf{y}}(\alpha) = q^{-1} \sum_{-q/2 < b \leq q/2} S_{\mathbf{y}}(q, a, b) \sum_{P/2 \leq x \leq P} e\big(F(x; b)\big),$$

whence using Vaughan [141, Lemma 4.2] we obtain

$$\begin{aligned} h_{\mathbf{y}}(\alpha) - q^{-1} S_{\mathbf{y}}(q, a) v_{\mathbf{y}}(\beta) =& q^{-1} \sum_{\substack{-B < b \leq B \\ b \neq 0}} S_{\mathbf{y}}(q, a, b) I_{\mathbf{y}}(b) \\ &+ O\Big(q^{-1} \log(H + 2) \sum_{-q/2 < b \leq q/2} \big|S_{\mathbf{y}}(q, a, b)\big|\Big), \end{aligned} \tag{3.1.25}$$

where $B = (H + 1/2)q$ and $H = \lceil 3^k k P^{-1} n|\beta| + 1/2 \rceil$. Note that by the quasi-multiplicative property, in order to bound $S_{\mathbf{y}}(q, a, b)$ it suffices to consider the case when q is a prime power. For such purposes, we take $q = p^{3ku+v}$. We observe first that by Vaughan [141, Theorem 7.1] one has that

$$S_{\mathbf{y}}(q, a, b) \ll q^{1-1/3k+\varepsilon}.$$

Moreover, when $v \geq 2$ and $u = 0$ we can deduce from the proof of the same theorem[2] that $S_{\mathbf{y}}(p^v, a, b) \ll p^{v-1}$. For the case $q = p$, the work of Weil[3] [155] yields the estimate $S_{\mathbf{y}}(p, a, b) \ll p^{1/2}$. Therefore, combining these bounds with the definition (3.1.21) one finds that

$$S_{\mathbf{y}}(q, a, b) \ll q^{1+\varepsilon} w_k(q). \tag{3.1.26}$$

Consequently, by (3.1.25) we have

$$h_{\mathbf{y}}(\alpha) - q^{-1} S_{\mathbf{y}}(q, a) v_{\mathbf{y}}(\beta) \ll q^{\varepsilon} w_k(q) \sum_{\substack{-B < b \leq B \\ b \neq 0}} |I_{\mathbf{y}}(b)| + q^{1+\varepsilon} w_k(q) \log(H + 2). \tag{3.1.27}$$

To treat the sum on the right-hand side we use the methods of the proof of Vaughan [141, Theorem 4.1]. In his analysis he classifies the range of integration of $I(b)$ according to the size of $|G'(\gamma)|$, where

$$G(\gamma) = \beta\gamma^k - b\gamma/q \qquad \text{and} \ \ I(b) = \int_0^X e\big(G(\gamma)\big) d\gamma.$$

We follow Vaughan's analysis closely, dividing the range of integration of $I_{\mathbf{y}}(b)$ according to the size of $|F'(\gamma; b)|$, to obtain

$$\sum_{\substack{-B < b \leq B \\ b \neq 0}} |I_{\mathbf{y}}(b)| \ll q^{1+\varepsilon} (1 + n|\beta|)^{1/2}.$$

Since $\log(H + 2) \ll (1 + n|\beta|)^{1/2}$ then

$$h_{\mathbf{y}}(\alpha) - q^{-1} S_{\mathbf{y}}(q, a) v_{\mathbf{y}}(\beta) \ll q^{1+\varepsilon} w_k(q) (1 + n|\beta|)^{1/2},$$

[2] See in particular the argument following [141, (7.16)]

[3] See Schmidt [123, Corollary 2F] for an elementary proof of this bound.

which implies the first statement of the lemma by summing over $\mathbf{y} \in \mathcal{A}(P, P^\eta)^2$. Note that when $|\beta| \le (2 \cdot 3^k kq)^{-1} P n^{-1}$ and $b \ne 0$ one has $|F'(x; b)| \ge |b|/(2q)$ and $H = 1$. Observing that $F'(x; b)$ is monotonic then partial integration yields

$$\sum_{\substack{-B<b\le B\\ b\neq 0}} |I_{\mathbf{y}}(b)| \ll \sum_{\substack{-B<b\le B\\ b\neq 0}} \frac{q}{|b|} \ll q^{1+\varepsilon}.$$

Combining this estimate with (3.1.27) and summing over $\mathbf{y} \in \mathcal{A}(P, P^\eta)^2$ we get (3.1.23). □

By applying similar methods we can obtain the same type of approximation for the exponential sum $W(\alpha)$. For $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and recalling (3.1.18) and (3.1.22) we introduce the auxiliary function

$$W(\alpha, q, a) = q^{-1} \sum_{\mathbf{y},p} S_{p\mathbf{y}}(q, a) v_{\mathbf{y},p}(\beta), \tag{3.1.28}$$

where $\mathbf{y} \in \mathcal{A}(H_3, P^\eta)^2$ and $M/2 \le p \le M$.

Lemma 3.1.3. *Suppose that $(a, q) = 1$ and $(p, q) = 1$ for all primes with $M/2 \le p \le M$. Let $\alpha \in [0, 1)$ and $\beta = \alpha - a/q$. Then we have the estimate*

$$W(\alpha) - W(\alpha, q, a) \ll M H^{2/3} q^{1+\varepsilon} w_k(q)(1 + n|\beta|)^{1/2} (\log P)^{-1}.$$

Moreover, if $|\beta| \le (6kq)^{-1} H^{1/3} n^{-1}$ one finds that

$$W(\alpha) - W(\alpha, q, a) \ll M H^{2/3} q^{1+\varepsilon} w_k(q) (\log P)^{-1}.$$

Proof. In the same way as before, we can express the exponential sum $W(\alpha)$ as

$$W(\alpha) = \sum_{\mathbf{y},p} W_{\mathbf{y},p}(\alpha), \qquad \text{where} \quad W_{\mathbf{y},p}(\alpha) = \sum_{H_1 \le x \le H_2} e\big(\alpha p^{3k}(x^3 + C_{\mathbf{y}})^k\big),$$

and the parameter $C_{\mathbf{y}}$ was defined in (3.1.17). Sorting the summation into arithmetic progressions modulo q and applying orthogonality one has that

$$W_{\mathbf{y},p}(\alpha) = q^{-1} \sum_{-q/2<b\le q/2} S_{\mathbf{y}}(q, ap^{3k}, b) \sum_{H_1 \le x \le H_2} e\Big(\beta p^{3k}(x^3 + C_{\mathbf{y}})^k - \frac{bx}{q}\Big).$$

Recalling that $(q,p)=1$ then a change of variables yields

$$S_{\mathbf{y}}(q,ap^{3k})=S_{p\mathbf{y}}(q,a).$$

Therefore, the application of the argument of Vaughan [141, Theorem 4.1] in the same way as we did above leads to

$$W_{\mathbf{y},p}(\alpha)-q^{-1}S_{p\mathbf{y}}(q,a)v_{\mathbf{y},p}(\beta)\ll q^{1+\varepsilon}w_k(q)(1+n|\beta|)^{1/2},$$

and if $|\beta|\leq (6kq)^{-1}H^{1/3}n^{-1}$ then

$$W_{\mathbf{y},p}(\alpha)-q^{-1}S_{p\mathbf{y}}(q,a)v_{\mathbf{y},p}(\beta)\ll q^{1+\varepsilon}w_k(q),$$

which delivers the desired result by summing over the range of $(\mathbf{y},p)$ described in (3.1.28). □

3.1.4 Treatment of the singular series

Unless specified, in this subsection and the two upcoming ones we assume that $k=2,3$. We introduce some exponential sums and present upper bounds which we obtain making use of the arguments in Vaughan [141, Theorem 7.1]. We also discuss the congruence problem and introduce some divisibility constraints on $C_{\mathbf{y}_i}$ and $C_{p_i\mathbf{y}_i}$ to ensure local solubility. For further purposes, we remind the reader of the definition (3.1.13). For the rest of the paper, unless specified,

$$\mathbf{Y}=(\mathbf{y}_1,\ldots,\mathbf{y}_{s+t})\in\mathbb{N}^{2s+2t}\quad\text{and}\quad\mathbf{p}=(p_1,\ldots,p_t)$$

will denote tuples with $\mathbf{y}_i\in\mathcal{A}(P,P^{\eta})^2$ for $t+1\leq i\leq s+t$ and $\mathbf{y}_i\in\mathcal{A}(H_3,P^{\eta})^2$ for $1\leq i\leq t$, where p_i are primes satisfying $M/2\leq p_i\leq M$. Take $q\in\mathbb{N}$ and define

$$S_{\mathbf{Y},\mathbf{p}}(q)=q^{-s-t}\sum_{\substack{a=1\\(a,q)=1}}^{q}e(-an/q)\prod_{i=1}^{t}S_{p_i\mathbf{y}_i}(q,a)\prod_{i=t+1}^{s+t}S_{\mathbf{y}_i}(q,a).$$

The following technical lemma provides a straightforward upper bound for the previous exponential sum and will be used throughout the major arc treatment.

Lemma 3.1.4. *Assume that* $2\leq k\leq 4$. *Let* $m\geq 2$. *Take* $\alpha\leq\frac{m-1}{3k}$ *when*

$m \geq 3$ and $\alpha = 0$ for $m = 2$. Let $Q \geq 1$. Then, recalling (3.1.21) one has

$$\sum_{q \leq Q} q^\alpha w_k(q)^m \ll Q^\varepsilon.$$

Moreover, for the case $k = 4$ we also have

$$\sum_{q \leq Q} q\tau_4(q)^4 w_4(q) \ll Q^\varepsilon, \tag{3.1.29}$$

where $\tau_4(q)$ was defined just before Lemma 3.1.1.

Proof. By the multiplicative property of $w_k(q)$ it follows that

$$\sum_{q \leq Q} q^\alpha w_k(q)^m \ll \prod_{p \leq Q} \left(1 + \sum_{h=1}^{\infty} p^{h\alpha} w_k(p^h)^m\right) \ll \prod_{p \leq Q} \left(1 + Cp^{-1}\right) \ll Q^\varepsilon.$$

For the second estimate we use the bound $\tau_4(p^h)^4 \ll p^{-h}$ when $h \geq 2$ to obtain

$$\sum_{h=1}^{\infty} p^h \tau_4(p^h)^4 w_4(p^h) \ll p^{-3/2} + \sum_{h \geq 2} w_4(p^h) \ll p^{-1}.$$

Equation (3.1.29) follows then combining the above bound with multiplicativity. □

Lemma 3.1.5. *Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$. The functions $S_{\mathbf{y}}(q, a)$ and $S_{\mathbf{Y},\mathbf{p}}(q)$ defined above satisfy*

$$S_{\mathbf{y}}(q, a) \ll q^{1+\varepsilon} w_k(q), \qquad S_{\mathbf{Y},\mathbf{p}}(q) \ll q^{1+\varepsilon} w_k(q)^{s+t}. \tag{3.1.30}$$

As a consequence, for every $Q \geq 1$ and every $\alpha \leq \frac{s+t-1}{3k} - 1$ it follows that

$$\sum_{q \leq Q} q^\alpha |S_{\mathbf{Y},\mathbf{p}}(q)| \ll Q^\varepsilon \quad \text{and} \quad \sum_{q > Q} |S_{\mathbf{Y},\mathbf{p}}(q)| \ll Q^{\varepsilon-\alpha}. \tag{3.1.31}$$

Proof. On recalling (3.1.24) note that $S_{\mathbf{y}}(q, a) = S_{\mathbf{y}}(q, a, 0)$. Therefore, (3.1.26) yields $S_{\mathbf{y}}(q, a) \ll q^{1+\varepsilon} w_k(q)$, and hence (3.1.30) holds. This estimate and Lemma 3.1.4 imply the first inequality in (3.1.31). Finally, observe that as a consequence we have

$$\sum_{Q \leq q \leq 2Q} |S_{\mathbf{Y},\mathbf{p}}(q)| \ll Q^{\varepsilon-\alpha},$$

from where the second inequality of (3.1.31) follows by summing over dyadic intervals. $\square$

We apply the bounds obtained in the previous lemma to a collection of singular series and other related series. For such purpose, it is convenient to define, for tuples $(\mathbf{Y},\mathbf{p})$ and each prime p the sums

$$\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)=\sum_{q=1}^{\infty}S_{\mathbf{Y},\mathbf{p}}(q),\qquad\qquad \sigma(p)=\sum_{l=0}^{\infty}S_{\mathbf{Y},\mathbf{p}}(p^{l}).$$

Lemma 3.1.6. *The singular series $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)$ converges absolutely, the identity*

$$\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)=\prod_{p}\sigma(p) \tag{3.1.32}$$

holds and $0\leq\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)\ll 1$. Furthermore, on recalling (3.1.13) and (3.1.17) one has $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)\gg 1$ provided that:

1. *When $k=2$ one has $C_{p_i\mathbf{y}_i}\equiv 28 \pmod{108}$ for $1\leq i\leq t$ and $C_{\mathbf{y}_i}\equiv 28 \pmod{108}$ for $t+1\leq i\leq s+t$;*
2. *When $k=3$ one has $C_{p_i\mathbf{y}_i}\equiv 0 \pmod{162}$ for $1\leq i\leq t$ and $C_{\mathbf{y}_i}\equiv 0 \pmod{162}$ for $t+1\leq i\leq s+t$.*

As mentioned before, the constraints on $C_{\mathbf{y}_i}$ and $C_{p_i\mathbf{y}_i}$ ensure the local solubility of the problem. Note that the set of tuples with these divisibility conditions has positive density over the set of tuples without the restrictions since it follows from the proof of Lemma 5.4 of [139] that smooth numbers are well distributed on arithmetic progressions. Therefore, we are still able to get the expected lower bound for the major arc contribution. Observe though that the choices for the constraints are not unique, but for the purpose of this exposition it will suffice to study just one of the possible restrictions.

Proof. Note that the application of Lemma 3.1.5 yields the estimate

$$\sigma(p)-1\ll p^{-2}. \tag{3.1.33}$$

This bound and the multiplicative property of $S_{\mathbf{Y},\mathbf{p}}(q)$ imply (3.1.32), the convergence of the series $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)$ and its upper bound. To give a more arithmetic description of $\sigma(p)$ it is convenient to introduce

$$\mathcal{M}_n(p^h) = \Big\{\mathbf{X} \in [1, p^h]^{s+t} : \ n \equiv \sum_{i=1}^{t}(x_i^3+C_{p_i\mathbf{y}_i})^k + \sum_{i=t+1}^{s+t}(x_i^3+C_{\mathbf{y}_i})^k \pmod{p^h}\Big\}$$

and $M_n(p^h) = |\mathcal{M}_n(p^h)|$. Observe that by a standard argument making use of orthogonality we obtain the relation

$$\sum_{l=0}^{h} S_{\mathbf{Y},\mathbf{p}}(p^l) = p^{(1-s-t)h} M_n(p^h).$$

In view of (3.1.33) it transpires then that in order to prove the lower bound for $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)$ it will suffice to show that $p^{(1-s-t)h}M_n(p^h) \geq C_p$ for some positive constant C_p depending on p. For each p prime, take $\tau \geq 0$ for which $p^\tau \| 3k$. Define $\gamma = \gamma(p) = 2\tau + 1$ and

$$\mathcal{M}_n^*(p^\gamma) = \Big\{\mathbf{X} \in \mathcal{M}_n(p^\gamma) : \quad p \nmid x_1, \quad p \nmid (x_1^3 + C_{p_1\mathbf{y}_1})\Big\}.$$

We take $h \geq \gamma$ for convenience. Our priority for the rest of the lemma will be to show that $|\mathcal{M}_n^*(p^\gamma)| > 0$, since then an application of Hensel's Lemma will yield the bound $M_n(p^h) \geq p^{(s+t-1)(h-\gamma)}$.

For further discussion, it is convenient to consider for a fixed number $C \in \mathbb{N}$ the sets

$$\mathcal{T}_C(p^\gamma) = \Big\{x^3 + C \pmod{p^\gamma}\Big\}$$

and

$$\mathcal{T}_C^*(p^\gamma) = \Big\{x^3 + C \pmod{p^\gamma} : \ p \nmid x, \quad p \nmid (x^3 + C)\Big\}.$$

Let $p \equiv 1 \pmod 3$. Under this condition one has $p \geq 7$, so $\gamma = 1$ with $|\mathcal{T}_C(p)| = (p+2)/3$ and $|\mathcal{T}_C^*(p)| \geq 1$. If we denote the set of k-th powers of the above set by

$$\mathcal{T}_C^k(p^\gamma) = \Big\{y^k \pmod{p^\gamma} : \ y \in \mathcal{T}_C(p^\gamma)\Big\},$$

then one finds that

$$|\mathcal{T}_C^k(p)| \geq \lceil (p+2)/3k \rceil.$$

One can check that $|\mathcal{T}_C^k(7)| \geq 2$ for every $C \in \mathbb{N}$, and whenever $p > 7$ we find that

$$(s+t-1)\Big(\Big\lceil \frac{p+2}{3k} \Big\rceil - 1\Big) \geq p,$$

and hence Cauchy-Davenport [141, Lemma 2.14] delivers $|\mathcal{M}_n^*(p)|> 0$. When $p \equiv 2 \pmod 3$ and $p > 2$ then $\gamma = 1$ and we further get $|\mathcal{T}_C(p)| = p$ and $|\mathcal{T}_C^*(p)| \geq 1$, whence another application of Cauchy-Davenport [141, Lemma 2.14] yields $|\mathcal{M}_n^*(p)| > 0$. For the case $p = 2$ and $k = 2$ the divisibility contraints reduce the problem to the resolution of

$$y_1^6 + \cdots + y_8^6 \equiv n \pmod 8$$

with $y_i \in \mathbb{N}$ and $2 \nmid y_1$, which is straightforward. The case $k = 3$ is also trivial since then one would have $\gamma(2) = 1$. Likewise, if $p = 3$ one finds that whenever $C \equiv 1 \pmod{27}$ then

$$\mathcal{T}_C^2(27) = \{0, 1, 4, 13, 22\}$$

and $|\mathcal{T}_C^*(27)| = 3$, so $|\mathcal{M}_n^*(27)| > 0$ when $k = 2$ follows combining the constraints for $C_{p_i\mathbf{y}_i}$ and $C_{\mathbf{y}_i}$ described above and Vaughan [141, Lemma 2.14]. Finally, when $k = 3$ we make use of the conditions $C_{\mathbf{y}_i} \equiv 0 \pmod{81}$ and $C_{p_i\mathbf{y}_i} \equiv 0 \pmod{81}$ to reduce the problem to finding a solution for

$$y_1^9 + \ldots + y_{17}^9 \equiv n \pmod{243}$$

with $y_i \in \mathbb{N}$ and $3 \nmid y_1$. The solubility of this congruence is a consequence of Vaughan [141, Lemma 2.15]. □

3.1.5 Singular integral

In this subsection we analyse the size of the singular integral following the classical approach making use of Fourier's Integral Theorem. For each pair of tuples $(\mathbf{Y}, \mathbf{p})$ consider

$$J_{\mathbf{Y},\mathbf{p}}(n) = \int_{-\infty}^{\infty} V_{\mathbf{Y},\mathbf{p}}(\beta) e(-n\beta) d\beta, \quad \text{where } V_{\mathbf{Y},\mathbf{p}}(\beta) = \prod_{i=1}^{t} v_{\mathbf{y}_i,p_i}(\beta) \prod_{i=t+1}^{s+t} v_{\mathbf{y}_i}(\beta),$$

and $v_{\mathbf{y_i},p_i}(\beta)$ and $v_{\mathbf{y}_i}(\beta)$ were defined in (3.1.18).

Lemma 3.1.7. *One has that* $0 \leq J_{\mathbf{Y},\mathbf{p}}(n) \ll P^s H^{t/3} n^{-1}$. *Moreover, whenever* $(\mathbf{Y}, \mathbf{p})$ *satisfies* $M/2 \leq p_i \leq 51M/100$ *for* $1 \leq i \leq t$ *and* $\mathbf{y}_i \leq P/2$ *for* $t + 1 \leq i \leq s + t$ *then*

$$J_{\mathbf{Y},\mathbf{p}}(n) \gg P^s H^{t/3} n^{-1}. \tag{3.1.34}$$

In the following discussion we rewrite $J_{\mathbf{Y},\mathbf{p}}(n)$ as an integral whose size is easier to estimate. The conditions on the tuples described before ensure that we get a suitable range of integration for such integral. Note that the set of tuples on that range has positive density over the set of tuples without the restrictions, and hence we are still able to get the expected lower bound for the major arc contribution.

Proof. By using the expression of both $v_{\mathbf{y}}(\beta)$ and $v_{\mathbf{y},p}(\beta)$ in (3.1.19) we find that

$$J_{\mathbf{Y},\mathbf{p}}(n) = \lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\int_{\mathbf{x}\in\mathcal{S}} B_{\mathbf{Y},\mathbf{p}}(\mathbf{x})e\Big(\beta\Big(\sum_{i=1}^{s+t} x_i - n\Big)\Big)d\mathbf{x}d\beta,$$

where the function $B_{\mathbf{Y},\mathbf{p}}(\mathbf{x})$ is taken to be

$$B_{\mathbf{Y},\mathbf{p}}(\mathbf{x}) = \prod_{i=1}^{t} B_{\mathbf{y}_i,p_i}(x_i)\prod_{i=t+1}^{s+t} B_{\mathbf{y}_i}(x_i)$$

and we integrate over the set

$$\mathcal{S} = \prod[M_{\mathbf{y}_i,\mathbf{p}_i}, N_{\mathbf{y}_i,\mathbf{p}_i}] \times \prod[M_{\mathbf{y}_i}, N_{\mathbf{y}_i}].$$

Then by integrating on β and making the change of variables $v = \sum_{i=1}^{s+t} x_i$ we obtain

$$J_{\mathbf{Y},\mathbf{p}}(n) = \lim_{\lambda\to\infty}\int_{S_1}^{S_2}\phi(v)\frac{\sin\big(2\pi\lambda(v-n)\big)}{\pi(v-n)}dv,$$

where $\phi(v)$ is defined as

$$\phi(v) = \int_{\mathbf{x}\in\mathcal{S}'(v)} B_{\mathbf{y}_{s+t}}\Big(v - \sum_{i=1}^{s+t-1} x_i\Big)\prod_{i=1}^{t} B_{\mathbf{y}_i,p_i}(x_i)\prod_{i=t+1}^{s+t-1} B_{\mathbf{y}_i}(x_i)d\mathbf{x},$$

the subset $\mathcal{S}'(v) \subset \mathbb{R}^{s+t-1}$ denotes the tuples satisfying

$$x_i \in [M_{\mathbf{y}_i,p_i}, N_{\mathbf{y}_i,p_i}] \quad \text{for } 1 \le i \le t, \quad x_i \in [M_{\mathbf{y}_i}, N_{\mathbf{y}_i}] \quad \text{for } t+1 \le i \le s+t-1,$$

and

$$M_{\mathbf{y}_{s+t}} \le v - \sum_{i=1}^{s+t-1} x_i \le N_{\mathbf{y}_{s+t}} \tag{3.1.35}$$

and the above limits of integration are

$$S_1 = \sum_{i=1}^{t} M_{\mathbf{y}_i,\mathbf{p}_i} + \sum_{i=t+1}^{s+t} M_{\mathbf{y}_i}, \qquad S_2 = \sum_{i=1}^{t} N_{\mathbf{y}_i,\mathbf{p}_i} + \sum_{i=t+1}^{s+t} N_{\mathbf{y}_i}.$$

Since $\phi(v)$ is a function of bounded variation, it follows from Fourier's Integral Theorem [4] that $J_{\mathbf{Y},\mathbf{p}}(n) = \phi(n)$, which implies positivity. Note that combining the identity (3.1.11), the limits of integration defined after (3.1.19) and equation (3.1.20), we find that whenever $\mathbf{x} \in \mathcal{S}'(n)$ then it follows that $B_{\mathbf{y}_i,p_i}(x_i) \asymp H^{1/3}n^{-1}$ for $1 \leq i \leq t$ and $B_{\mathbf{y}_i}(x_i) \asymp Pn^{-1}$ for $t+1 \leq i \leq s+t-1$, and one further has

$$B_{\mathbf{y}_{s+t}}(n - \sum_{i=1}^{s+t-1} x_i) \asymp Pn^{-1}.$$

Therefore, combining the previous ideas we obtain the upper bound for $J_{\mathbf{Y},\mathbf{p}}(n)$ stated at the beginning of the lemma. Moreover, if $(\mathbf{Y},\mathbf{p})$ lies in the range described right after that bound, then there exist intervals $I_i \subset [M_{\mathbf{y}_i,p_i}, N_{\mathbf{y}_i,p_i}]$ for $1 \leq i \leq t$ and $I_i \subset [M_{\mathbf{y}_i}, N_{\mathbf{y}_i}]$ for $t+1 \leq i \leq s+t-1$ satisfying $|I_i| \asymp n$ and with the property that whenever $x_i \in I_i$ then (3.1.35) holds for $v = n$. Consequently, the preceding discussion yields (3.1.34). $\square$

For the sake of brevity we define the auxiliary functions $h^*(\alpha)$ and $W^*(\alpha)$ by putting

$$h^*(\alpha) = V(\alpha, q, a) \qquad \text{and} \qquad W^*(\alpha) = W(\alpha, q, a)$$

when $\alpha \in \mathfrak{M}(a,q) \subset \mathfrak{M}$ and $h^*(\alpha) = W^*(\alpha) = 0$ for $\alpha \in \mathfrak{m}$. Here the reader may want to recall (3.1.22) and (3.1.28). For the rest of the subsection we present some bounds for these functions.

Lemma 3.1.8. *Let $\beta \in \mathbb{R}$. For every prime p and $\mathbf{y} \in \mathbb{N}^2$ one has*

$$v_{\mathbf{y}}(\beta) \ll \frac{P}{1+n|\beta|} \qquad \text{and} \qquad v_{\mathbf{y},p}(\beta) \ll \frac{H^{1/3}}{1+n|\beta|}.$$

Moreover, whenever $\alpha \in \mathfrak{M}(a,q) \subset \mathfrak{M}$ one finds that

$$h^*(\alpha) \ll \frac{q^{\varepsilon} w_k(q) P^3}{1+n|\alpha - a/q|} \qquad \text{and} \qquad W^*(\alpha) \ll \frac{q^{\varepsilon} w_k(q) MH}{(1+n|\alpha - a/q|)(\log P)}.$$

[4] See the argument in Davenport [33, p. 21–22] or in [158, 7.43].

Proof. When $|\beta| \le n^{-1}$, the bound for $v_{\mathbf{y}}(\beta)$ follows observing that by (3.1.19) and the limits of integration taken after (3.1.19) then

$$v_{\mathbf{y}}(\beta) \ll \int_{M_{\mathbf{y}}}^{N_{\mathbf{y}}} y^{1/k-1}\big(y^{1/k} - C_{\mathbf{y}}\big)^{-2/3} dy \ll P.$$

The reader may find it useful to recall (3.1.17) in the above line. For the case $|\beta| > n^{-1}$, using the fact that $B_{\mathbf{y}}(y)$ is decreasing and integrating by parts we have that

$$v_{\mathbf{y}}(\beta) \ll |\beta|^{-1} B_{\mathbf{y}}(M_{\mathbf{y}}) \ll |\beta|^{-1} n^{1/3k-1},$$

which proves the statement. The case $v_{\mathbf{y},p}(\beta)$ is done in a similar way and follows after applying (3.1.11). Combining these estimates and Lemma 3.1.5 we get the bounds for $h^*(\alpha)$ and $W^*(\alpha)$. $\square$

3.1.6 Major arc contribution

In this subsection we show that the contribution of the set of narrow arcs $\mathfrak{N}$ is asymptotic to the expected main term. We prove then that the contribution of the remaining arcs is smaller by combining major and minor arc techniques and making use of Lemma 3.1.1.

Proposition 3.1.2. *There exists $\delta > 0$ such that*

$$\int_{\mathfrak{M}} h(\alpha)^s W(\alpha)^t e(-\alpha n) d\alpha = \sum_{\mathbf{Y},\mathbf{p}} \mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) J_{\mathbf{Y},\mathbf{p}}(n) + O(H^t M^t P^{3s-3k-\delta}),$$

where $(\mathbf{Y},\mathbf{p})$ lies in the range of summation described at the beginning of §4.

Proof. We note first that the triangle inequality yields

$$h(\alpha)^s - h^*(\alpha)^s \ll |h(\alpha) - h^*(\alpha)|\big(|h^*(\alpha)|^{s-1} + |h(\alpha) - h^*(\alpha)|^{s-1}\big).$$

Observe that by (3.1.10) and the definition (3.1.14) then whenever $\alpha \in \mathfrak{N}(a,q)$ one has that

$$(1 + n|\beta|)^{-1} \ge qH^{-1/3} \ge qP^{-1}$$

and

$$|\beta| \le (6kq)^{-1} H^{1/3} n^{-1} \le (2\cdot 3^k kq)^{-1} P n^{-1}$$

for n sufficiently large. Consequently, Lemma 3.1.2 applied to $|h(\alpha) - h^*(\alpha)|$ and Lemma 3.1.8 applied to $|h^*(\alpha)|$ in the above equation deliver

$$h(\alpha)^s - h^*(\alpha)^s \ll q^{1+\varepsilon} w_k(q)^s P^{3s-1}(1+n|\beta|)^{-s+1}, \tag{3.1.36}$$

and by the same reason then whenever $\alpha \in \mathfrak{N}(a,q)$ with $(p,q) = 1$ for all primes $M/2 \leq p \leq M$, Lemma 3.1.3 gives

$$W(\alpha)^t - W^*(\alpha)^t \ll M^t H^{t-1/3} q^{1+\varepsilon} w_k(q)^t (1+n|\beta|)^{-t+1}. \tag{3.1.37}$$

We also need a bound on the following quantity to exploit some orthogonality relation when averaging over q. Denote by $N(q,P)$ the number of solutions of the congruence

$$T(p_1\mathbf{x}_1)^k + T(p_2\mathbf{x}_2)^k \equiv T(p_3\mathbf{x}_3)^k + T(p_4\mathbf{x}_4)^k \pmod{q},$$

where $\mathbf{x}_i \in [1, H^{1/3}]^3$ and $M/2 \leq p_i \leq M$ with $q \in \mathbb{N}$. By expressing q as the product of prime powers, using the structure of the ring of integers modulo these prime powers and noting that the number of primes dividing q is $O\big((\log q)/\log\log q\big)$ we obtain

$$N(q,P) \ll q^{\varepsilon}(MH)^4(\log P)^{-4}\big(q^{-1} + P^{-1}\big), \tag{3.1.38}$$

where we also used the identity (3.1.11), and hence by orthogonality it follows that

$$\sum_{a=1}^{q} |W(\beta + a/q)|^4 \leq qN(q,P) \ll q^{1+\varepsilon}(MH)^4(\log P)^{-4}\big(q^{-1} + P^{-1}\big). \tag{3.1.39}$$

Combining (3.1.36) and (3.1.39) one has that

$$\begin{aligned}\int_{\mathfrak{N}} \Big|h(\alpha)^s - h^*(\alpha)^s\Big| |W(\alpha)|^t d\alpha &\ll (HM)^t P^{3s-3k-1} \sum_{q \leq H^{1/3}} q^{1+\varepsilon} w_k(q)^s \\ &\ll (HM)^t P^{3s-3k-\delta},\end{aligned}$$

where we used (3.1.10) and Lemma 3.1.4. Before introducing the auxiliary function $W^*(\alpha)$ to replace $W(\alpha)$ we must ensure that the contribution of the arcs with $M/4 < q \leq (6k)^{-1} H^{1/3}$ is small enough. By doing so we avoid having to approximate $W(\alpha)$ for the cases when $p \mid q$ for primes p appearing in the definition (3.1.3) of $W(\alpha)$. Combining Lemma 3.1.8 with (3.1.39) one finds that

$$\sum_{M/4<q\leq(6k)^{-1}H^{1/3}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_0^1 |h^*(\beta+a/q)|^s \big|W(\beta+a/q)\big|^t d\beta$$
$$\ll (HM)^t P^{3s-3k+\varepsilon} \sum_{M/4<q\leq(6k)^{-1}H^{1/3}} w_k(q)^s \ll (HM)^t P^{3s-3k-\delta},$$

where in the last step we applied the definition (3.1.21). For the range $q \leq M/4$ we always have $(p,q)=1$ for all primes $M/2 \leq p \leq M$, so we can use (3.1.37) and Lemma 3.1.8 to obtain

$$\sum_{q\leq M/4} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{N}(a,q)} |h^*(\alpha)|^s \Big|W(\alpha)^t - W^*(\alpha)^t\Big| d\alpha$$
$$\ll P^{3s-3k} M^t H^{t-1/3} \sum_{q\leq M/4} q^{2+\varepsilon} w_k(q)^{s+t} \ll (HM)^t P^{3s-3k-\delta},$$

where in the last line we used (3.1.10) and applied Lemma 3.1.4. By Lemmata 3.1.4 and 3.1.8 one has that

$$\sum_{q\leq M/4} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{|\alpha-a/q|>(6kq)^{-1}H^{1/3}n^{-1}} |h^*(\alpha)|^s |W^*(\alpha)|^t d\alpha$$
$$\ll H^{2t/3-s/3+1/3} M^t P^{3s-3k} \sum_{q\leq M/4} q^{s+t+\varepsilon} w_k(q)^{s+t} \ll (HM)^t P^{3s-3k-\delta}.$$

Therefore, using the previous bounds, making a change of variables and combining Lemmata 3.1.5 and 3.1.8 it follows that

$$\int_{\mathfrak{N}} h(\alpha)^s W(\alpha)^t e(-\alpha n) d\alpha = \sum_{\mathbf{Y},\mathbf{p}} \mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) J_{\mathbf{Y},\mathbf{p}}(n) + O\big((HM)^t P^{3s-3k-\delta}\big). \tag{3.1.40}$$

The rest of the subsection is devoted to ensure that the contribution of the remaining major arcs is smaller than the main term in the previous equation.

Let $R(q,P)$ be the number of solutions of the congruence

$$T(\mathbf{x}_1)^k + T(\mathbf{x}_2)^k \equiv T(\mathbf{x}_3)^k + T(\mathbf{x}_4)^k \pmod{q},$$

where $\mathbf{x}_i \in [1,P]^3$. Applying the same argument we used in (3.1.38) for bounding $N(q,P)$ we find that $R(q,P) \ll q^{\varepsilon} P^{12}(q^{-1}+P^{-1})$, and hence by orthogonality it follows that

$$\sum_{a=1}^{q} |h(\beta + a/q)|^4 \leq qR(q,P) \ll q^{1+\varepsilon} P^{12}(q^{-1}+P^{-1}). \tag{3.1.41}$$

Moreover, observe that by a similar argument for the case $k=2$ we get

$$\sum_{a=1}^{q} |h(\beta + a/q)|^2 \ll q^{1+\varepsilon} P^{6}(q^{-1}+P^{-1}). \tag{3.1.42}$$

We consider for convenience the mean value

$$I_M = \int_{\mathfrak{M}\setminus\mathfrak{N}} |h(\alpha)|^s \big|W(\alpha)\big|^t d\alpha.$$

Our strategy for the treatment of this integral will be to bound $W(\alpha)$ pointwise via Lemma 3.1.1 and use some major arc estimates. For such purposes, we define first $\Upsilon(\alpha)$ for $\alpha \in [0,1)$ by taking

$$\Upsilon(\alpha) = \tau_k(q)(1 + n|\alpha - a/q|)^{-1}$$

when $\alpha \in \mathfrak{M}(a,q) \subset \mathfrak{M}$ and $\Upsilon(\alpha) = 0$ otherwise. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $0 \leq a \leq q \leq M^k$ and $(a,q)=1$, consider the set of arcs

$$\mathfrak{M}'(a,q) = \Big\{\alpha \in [0,1) : \Big|\alpha - a/q\Big| \leq \frac{M}{q^{1/k} n}\Big\} \tag{3.1.43}$$

and take $\mathfrak{M}'$ to be the union of such arcs. Note that then one has $\mathfrak{M}' \subset \mathfrak{M}$. Observe that for $\alpha \in \mathfrak{M}\setminus\mathfrak{M}'$, the bound in the right handside of (3.1.7) corresponding to the diagonal contribution dominates over the one corresponding to the non-diagonal contribution. Therefore, we can apply the same argument that we applied in Proposition 3.1.1 to estimate the integral over this set. When $\alpha \in \mathfrak{M}'$ then it is the bound corresponding to the non-diagonal term the one which dominates. Let I_M' be the contribution of $\mathfrak{M}'\setminus\mathfrak{N}$ to the integral

I_M. By making use of Lemma 3.1.1 and (3.1.12) we obtain that

$$I'_M \ll H^{t+t/24-\delta} M^t \int_{\mathfrak{M}'\setminus\mathfrak{N}} |h(\alpha)|^s \Upsilon(\alpha)^{t/2} d\alpha \ll H^{t+t/24-\delta} M^t (I_1 + I_2),$$

where

$$I_i = \int_{\mathfrak{M}'\setminus\mathfrak{N}} |h(\alpha)|^{s-2} G_i(\alpha) \Upsilon(\alpha)^{t/2} d\alpha, \qquad i = 1, 2$$

with $G_1(\alpha) = |h^*(\alpha)|^2$ and $G_2(\alpha) = |h(\alpha) - h^*(\alpha)|^2$. In view of the definitions (3.1.14) and (3.1.43) for $\mathfrak{N}$ and $\mathfrak{M}'$ respectively, we make a distinction between the ranges $q \leq (6k)^{-1} H^{1/3}$ and $(6k)^{-1} H^{1/3} < q \leq M^k$. We also combine Lemmata 3.1.4 and 3.1.8 with equations (3.1.41) and (3.1.42) and the bound (3.1.5) to obtain

$$\begin{aligned} I_1 \ll & P^{3s-3k} H^{-t/6-1/3} \sum_{q \leq (6k)^{-1} H^{1/3}} w_k(q)^2 q^{t/2+1-t/2k+\varepsilon} \\ & + P^{3s-3k} \sum_{(6k)^{-1} H^{1/3} < q \leq M^k} w_k(q)^2 q^{1-t/2k+\varepsilon} \big(q^{-1} + P^{-1}\big) \ll P^{3s-3k+\varepsilon} H^{-t/6k}. \end{aligned}$$

Likewise, combining equations (3.1.41) and (3.1.42) with Lemmata 3.1.2 and 3.1.4 one finds that

$$\begin{aligned} I_2 \ll & P^{3s-3k-2+\varepsilon} H^{-t/6+2/3} \sum_{q \leq (6k)^{-1} H^{1/3}} q^{t/2-t/2k} w_k(q)^2 \\ & + P^{3s-3k-2+\varepsilon} \sum_{(6k)^{-1} H^{1/3} < q \leq M^k} w_k(q)^2 q^{3-t/2k} \big(q^{-1} + P^{-1}\big) \\ \ll & P^{3s-3k+\varepsilon} H^{-t/6k}, \end{aligned}$$

where we made use of (3.1.10). Therefore we obtain that

$$I'_M \ll (HM)^t P^{3s-3k-\delta},$$

whence the result of the proposition follows combining (3.1.40) with the previous estimates. □

Proof of Theorem 3.1.1 when $k = 2, 3$. Note first that Lemma 3.1.7 ensures positivity for $J_{\mathbf{Y},\mathbf{p}}(n)$ and guarantees that for $(\mathbf{Y}, \mathbf{p})$ in the range described in the lemma then $J_{\mathbf{Y},\mathbf{p}}(n) \gg P^s H^{t/3} n^{-1}$. Similarly, Lemma 3.1.6 ensures the positivity of $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)$ and implies that for $(\mathbf{Y}, \mathbf{p})$ satisfying the local conditions described after (3.1.32) then $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) \gg 1$. As observed at the beginning of the lemmas, the intersection of the sets of pairs $(\mathbf{Y}, \mathbf{p})$ satisfying those conditions has positive density. Therefore, we find that

$$\sum_{\mathbf{Y},\mathbf{p}} \mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) J_{\mathbf{Y},\mathbf{p}}(n) \gg (HM)^t P^{3s-3k} (\log P)^{-t}.$$

Propositions 3.1.1 and 3.1.2 then yield the bound $R(n) \gg (HM)^t P^{3s-3k} (\log P)^{-t}$, which proves the theorem for $k = 2, 3$.

3.1.7 The case $k = 4$.

In this subsection we discuss the proof of the theorem for fourth powers. For such purpose, it is convenient to introduce the exponential sum

$$f(\alpha) = \sum_{x \in \mathcal{A}(P,P^\eta)} e(\alpha x^{12}).$$

Let $R_4(n)$ be the number of solutions of the equation

$$n = \sum_{i=1}^{11} T(p_i \mathbf{x}_i)^4 + 81\big(y_1^{12} + \cdots + y_{46}^{12}\big),$$

where $\mathbf{x}_i \in \mathcal{W}$ with $M/2 \leq p_i \leq M$ for $1 \leq i \leq 11$ and $y_i \in \mathcal{A}(P, P^\eta)$ for $1 \leq i \leq 46$. Observe that the sums of three cubes on the right handside have been replaced by the specialization $3y^3$. Note as well that orthogonality yields the identity

$$R_4(n) = \int_0^1 W(\alpha)^{11} f(81\alpha)^{46} e(-\alpha n) d\alpha.$$

Our goal throughout the subsection is to obtain a lower bound for $R_4(n)$ for all sufficiently large n. Recalling (3.1.10) and (3.1.15) and using the table of permissible exponents for $k = 12$ in Vaughan and Wooley [143] we find that

$$\begin{aligned}\int_{\mathfrak{m}} |W(\alpha)|^{11} |f(81\alpha)|^{46} d\alpha &\ll H^{11+11/24-\delta} M^{11/2} \int_0^1 |f(\alpha)|^{46} d\alpha \\ &\ll (HM)^{11} P^{34+\Delta_{23}-1/2-\delta}, \end{aligned} \qquad (3.1.44)$$

where $\Delta_{23} = 0.4988383$, and hence it follows that the minor arc contribution is then $O\big((HM)^{11} P^{34-\delta}\big)$.

We define a set of narrow major arcs $\mathfrak{P}$ by taking the union of

$$\mathfrak{P}(a,q)=\Big\{\alpha\in[0,1):\Big|\alpha-a/q\Big|\leq\frac{R}{n}\Big\}$$

with $0\leq a\leq q\leq R$ and $(a,q)=1$, where $R=(\log P)^{1/5}$, and consider $\mathfrak{p}=[0,1)\setminus\mathfrak{P}$. In the next few lines we will combine all sort of major and minor arc techniques to prune back to the set of narrow arcs $\mathfrak{P}$. As observed after (3.1.43), whenever $\alpha\in\mathfrak{M}\setminus\mathfrak{M}'$ then the bound in the right handside of (3.1.7) corresponding to the diagonal contribution dominates over the one corresponding to the non-diagonal contribution. Therefore, we can apply the same argument that we applied in (3.1.44) to obtain that the integral over that set is $O\big((HM)^{11}P^{34-\delta}\big)$.

We next note for further purposes that Theorem 1.8 of Vaughan [139] yields

$$\sup_{\mathfrak{n}}|f(81\alpha)|\ll P^{1-\rho+\varepsilon}, \tag{3.1.45}$$

where $\rho=0.004259$. As experts will realise, one could obtain a slightly bigger ρ by applying the methods in [167]. For the sake of brevity though, we avoid that treatment and make use of the weaker version of the estimate. We also remark that such improvement in the exponent would make no impact in the argument. Observe that using the same procedure as in (3.1.39) and (3.1.41) we deduce that

$$\sum_{a=1}^{q}\big|f\big(81(\beta+a/q)\big)\big|^{12}\ll q^{1+\varepsilon}P^{12}(q^{-1}+P^{-1}). \tag{3.1.46}$$

Note as well that whenever $\alpha\in\mathfrak{M}'\setminus\mathfrak{N}$ then $(1+n|\beta|)^{3/2}\geq H^{1/3}q^{-1}$, and hence Lemmata 3.1.3 and 3.1.8 yield

$$W(\alpha)\ll MH^{2/3}q^{1+\varepsilon}w_4(q)(1+n|\beta|)^{1/2}.$$

By the preceding discussion together with Lemma 3.1.1 and equations (3.1.45) and (3.1.46) we obtain

$$\int_{\mathfrak{M}'\setminus\mathfrak{N}}|W(\alpha)|^{11}|f(81\alpha)|^{46}d\alpha\ll(HM)^{11}P^{34(1-\rho)}\sum_{q\leq M^4}q^2\tau_4(q)^4w_4(q)(q^{-1}+P^{-1}).$$

Here the reader may find useful to observe that we applied the estimates (3.1.7) and (3.1.12) to eight copies of $W(\alpha)$ and the bound for $W(\alpha)$ deduced above to just one of them. Likewise, we made use of the pointwise estimate (3.1.45) to bound 34 copies of $f(81\alpha)$ and we used the other 12 to exploit the congruence condition via (3.1.46). We get that the above sum when $q \leq P$ is $O\big((HM)^{11}P^{34-\delta}\big)$ via Lemma 3.1.4. Similarly, we use Lemma 3.1.4 and the bound $qP^{-1} \leq P^{1/11}$, which follows after (3.1.10), for the range $P \leq q \leq M^4$ to obtain that such contribution is also $O\big((HM)^{11}P^{34-\delta}\big)$. By the observation made before (3.1.36), which is still valid for $k = 4$, and Lemma 3.1.3 we find that whenever $\alpha \in \mathfrak{N}$ then

$$W(\alpha) \ll \frac{q^{\varepsilon} w_4(q) HM}{(1+n|\beta|)(\log P)}.$$

Therefore, the application of this bound and (3.1.39) yield

$$\begin{aligned}\int_{\mathfrak{N}\setminus\mathfrak{P}} |W(\alpha)|^{11}|f(81\alpha)|^{46}d\alpha \ll &(HM)^{11}P^{34}(\log P)^{-11}R^{-6}\sum_{q\leq R} q^{\varepsilon}w_4(q)^7 \\ &+ (HM)^{11}P^{34}(\log P)^{-11}\sum_{q>R} q^{\varepsilon}w_4(q)^7.\end{aligned}$$

Consequently, Lemma 3.1.4 and (3.1.21) imply that the above integral is then $O\big((HM)^{11}P^{34}(\log P)^{-11-\delta}\big)$.

In what follows, we will briefly describe the singular series associated to the problem. There might be other approaches that would lead to more precise asymptotic formulae, but for the sake of simplicity we avoid including the sums of three cubes in the singular series. On recalling (3.1.4), it is convenient to consider, for an integer $m \in \mathbb{N}$ and a prime p the sums

$$S_m(q) = q^{-46}\sum_{\substack{a=1\\(a,q)=1}}^{q} S_{12}(q, 81a)^{46} e_q\big(-a(n-m)\big), \qquad \sigma_m(p) = \sum_{h=0}^{\infty} S_m(p^h).$$

Observe that whenever $3 \nmid q$ then we can make a change of variables to rewrite $S_m(q)$ as

$$S_m(q) = q^{-46}\sum_{\substack{a=1\\(a,q)=1}}^{q} S_{12}(q, a)^{46} e_q\big(-a\overline{81}^{-1}(n-m)\big),$$

where $\overline{81}^{-1}$ denotes the inverse of 81 (mod q). Note as well that Lemma 3 of [137] yields the bound $S_m(q) \ll q\tau_{12}(q)^{46}$, which in turn further implies that $\sigma_m(p) = 1 + O(p^{-22})$ and delivers the convergence of the singular series

$$\mathfrak{S}_m(n) = \sum_{q=1}^{\infty} S_m(q) \tag{3.1.47}$$

and its upper bound $\mathfrak{S}_m(n) \ll 1$. Here the reader may find useful to observe that we implicitly used the multiplicativity of $S_m(q)$ and the expression of the singular series as the product

$$\mathfrak{S}_m(n) = \prod_p \sigma_m(p).$$

The estimate $S_m(q) \ll q^{-17/6}$, which follows trivially via an application of Vaughan [141, Theorem 4.2] also delivers, for $Q \geq 1$, the bound

$$\sum_{q>Q} |S_m(q)| \ll Q^{-\alpha} \tag{3.1.48}$$

for some $\alpha > 0$. Observe that by Lemmata 2.12, 2.13 and 2.15 of [141] one gets for every prime $p \neq 3$ the lower bound $\sigma_m(p) \geq p^{-45\gamma}$, where $\gamma = 3$ when $p = 2$ and $\gamma = 1$ otherwise. Likewise, note that when $m \equiv n \pmod{81}$ and $h \geq 5$ orthogonality yields

$$\sum_{l=0}^{h} S_m(3^l) = 3^{-45h} M_{n,m}(3^h),$$

where $M_{n,m}(3^h)$ denotes the number of solutions of the congruence

$$x_1^{12} + \cdots + x_{46}^{12} \equiv (n-m)/81 \pmod{3^{h-4}}$$

with $1 \leq x_i \leq 3^h$. Therefore, the application of Lemmata 2.13 and 2.15 of [141] gives $\sigma_m(3) \geq 3^{-86}$. Consequently, combining these lower bounds with the fact that $\sigma_m(p) - 1 = O(p^{-22})$ we obtain $\mathfrak{S}_m(n) \gg 1$. Observe as well that the preceding discussion yields $\mathfrak{S}_m(n) \geq 0$ for every $m \in \mathbb{N}$.

Before showing a lower bound of the expected size for the contribution of the set of narrow arcs, we introduce for convenience the weighted exponential sum

$$w(\beta) = \sum_{P^{12\eta} < x \le n} \frac{1}{12} x^{-11/12} \rho\Big(\frac{\log x}{12\eta \log P}\Big) e(\beta x),$$

where ρ denotes the Dickman's function, defined for real x by

$$\rho(x) = 0 \text{ when } x < 0,$$

$$\rho(x) = 1 \text{ when } 0 \le x \le 1,$$

$$\rho \text{ continuous for } x > 0,$$

$$\rho \text{ differentiable for } x > 1$$

$$x\rho'(x) = -\rho(x-1) \text{ when } x > 1.$$

For the sake of simplicity, we define the auxiliary function $f^*(\alpha)$ by putting

$$f^*(\alpha) = q^{-1} S(q, 81a) w\big(81(\alpha - a/q)\big)$$

when $\alpha \in \mathfrak{P}(a,q) \subset \mathfrak{P}$ and $f^*(\alpha) = 0$ for $\alpha \in \mathfrak{p}$. Then, it is a consequence of Vaughan[5] [139, Lemma 5.4] that for $\alpha \in \mathfrak{P}(a,q) \subset \mathfrak{P}$ one has

$$f(81\alpha) - f^*(\alpha) = O\big(PR^{-3}\big)$$

and

$$f^*(\alpha) \ll q^{-1/12} P (1 + n|\beta|)^{-1/12}.$$

Moreover, by the methods of Vaughan [141, Lemma 2.8] and the monotonicity of ρ it follows that

$$w(\beta) \ll \frac{P}{(1 + n|\beta|)^{1/12}}. \tag{3.1.49}$$

Finally, when $m \in \mathbb{N}$ it is convenient to introduce $K(m)$, defined as the number of solutions of the equation

$$m = T(p_1 \mathbf{x}_1)^4 + \cdots + T(p_{11} \mathbf{x}_{11})^4$$

for $\mathbf{x}_i \in \mathcal{W}$ and $M/2 \le p_i \le M$. Combining the estimates mentioned before

[5]Observe that the condition $(a,q) = 1$ in Vaughan [139, Lemma 5.4] can be relaxed to $(a,q) = C$ for some constant C.

(3.1.49) we obtain that

$$\int_{\mathfrak{P}} W(\alpha)^{11} f(81\alpha)^{46} e(-\alpha n) d\alpha = \sum_{m \leq 11n} K(m) \int_{\mathfrak{P}} f^*(\alpha)^{46} e\big(-\alpha(n-m)\big) d\alpha$$
$$+ O\big((HM)^{11} P^{34} (\log P)^{-11-\delta}\big).$$

Observe that the main term on the right can be written as

$$\sum_{m \leq 11n} K(m) \sum_{q \leq R} S_m(q) \int_{|\beta| \leq n^{-1}R} w(81\beta)^{46} e\big(-\beta(n-m)\big) d\beta. \tag{3.1.50}$$

By (3.1.49) we obtain that the integral on the above expression over the range $|\beta| > n^{-1}R$ is $O(P^{34}(\log P)^{-\delta})$. Therefore, an application of this observation and (3.1.48) gives that the contribution of the set of narrow arcs $\mathfrak{P}$ is

$$\sum_{m \leq 11n} K(m) \mathfrak{S}_m(n) \int_0^1 w(81\beta)^{46} e\big(-\beta(n-m)\big) d\beta + O\big((HM)^{11} P^{34} (\log P)^{-11-\delta}\big).$$

We further note that whenever $P^{12\eta} < x \leq n$ then

$$\rho\Big(\frac{\log x}{12\eta \log P}\Big) \gg 1,$$

so combining the positivity of $\mathfrak{S}_m(n)$, orthogonality and the lower bound $\mathfrak{S}_m(n) \gg 1$ when $m \equiv n \pmod{81}$ we obtain that (3.1.50) is bounded below by

$$\sum_{\substack{m \leq 11n/12 \\ m \equiv n \pmod{81}}} K(m)(n-m)^{17/6}.$$

One can check via an application of Hensel's Lemma[6] and Lemma 2.14 of [141] that the set of numbers of the shape $T(p_1\mathbf{x}_1)^4 + \cdots + T(p_{11}\mathbf{x}_{11})^4$ with $p_i, \mathbf{x}_i \leq 81$ covers all the residue classes modulo 81. Consequently, by the preceding discussion we find that

$$\int_{\mathfrak{P}} W(\alpha)^{11} f(81\alpha)^{46} e(-\alpha n) d\alpha \gg (HM)^{11} P^{34} (\log P)^{-11},$$

which combined with the estimates obtained through the pruning process

[6] Here the reader may find useful to observe that the set of sums of three cubes modulo 27 are the residue classes not congruent to 4 or 5 modulo 9.

yields $R_4(n) \gg (HM)^{11} P^{34} (\log P)^{-11}$.

3.2 On squares of sums of three cubes[7]

In this new section we make use of the above analysis when $k = 2$ to show that almost every positive integer can be expressed as a sum of four squares of integers represented as the sum of three positive cubes.

3.2.1 Introduction

It is often the case in additive number theory that there might be problems involving the representation of integers that remain open, yet it can be shown that almost all integers have a representation. Lagrange's celebrated theorem, proven in 1770, states that every positive integer n can be written as

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2, \tag{3.2.1}$$

where $x_i \in \mathbb{N} \cup \{0\}$. Let $\mathscr{C}$ denote the set of integers represented as sums of three positive cubes. In this memoir we will focus our attention on the problem of solving equation (3.2.1) where the set of variables lies on the set $\mathscr{C}$.

Not very much is known about $\mathscr{C}$. In fact, it isn't even known whether it has positive density or not, the best current lower bound on the cardinality of the set being

$$\mathcal{N}(X) = |\mathscr{C} \cap [1, X]| \gg X^{\beta - \varepsilon},$$

where $\beta = 0.91709477$, due to Wooley [174]. Under some unproved assumptions on the zeros of some Hasse-Weil L-functions, Hooley ([70], [71]) and Heath-Brown [63] showed using different procedures that

$$\sum_{n \leq X} r_3(n)^2 \ll X^{1+\varepsilon},$$

where $r_3(n)$ is the number of representations of n as a sum of three positive integral cubes, which implies by applying a standard Cauchy-Schwarz argument that $\mathcal{N}(X) \gg X^{1-\varepsilon}$. This lack of understanding of the cardinality of

[7]This section is based on a published paper by the author [111] in the Quarterly Journal of Mathematics.

the set also prevents us from understanding its distribution over arithmetic progressions, which often comes into play on the major arc analysis. In this memoir, though, we use the classical approach for dealing with exceptional sets involving Bessel's inequality and we make use of an estimate for the minor arcs obtained in the previous subsection to prove that for almost every positive integer n the equation (3.2.1) has a solution with $x_i \in \mathscr{C}$. More precisely, let $E(N)$ be the number of positive integers $n \leq N$ for which (3.2.1) fails to possess a solution with $x_i \in \mathscr{C}$.

Theorem 3.2.1. *For each $\varepsilon > 0$ one has*

$$E(N) \ll N(\log N)^{-4/31+\varepsilon}. \tag{3.2.2}$$

The reader might want to recall from the previous section that for the integers $n = 33 \cdot 2^{12j}$ for $j \geq 0$ then by taking the equation modulo powers of 2 one finds that for at least one of the variables, say x_1, one either has $x_1 \equiv 4 \pmod 9$ or $x_1 \equiv 5 \pmod 9$. Therefore, x_1 cannot lie on $\mathscr{C}$. In the above theorem we are far from obtaining an upper bound of the expected size, but as shown on the preceding discussion, the exceptional set of integers not represented as in (3.2.1) has infinite cardinality, and in fact

$$E(N) \gg \log N.$$

By using the natural polynomial structure given by $\mathscr{C}$ and an estimate for a mean value of some weighted exponential sums, we remind the reader that in the previous section we proved via an application of the Hardy-Littlewood method that every sufficiently large integer n can be represented as

$$n = \sum_{i=1}^{8} x_i^2$$

with $x_i \in \mathscr{C}$. In the setting of this paper, the constraint that prevents us from taking fewer variables is the analysis of the minor arcs. Such analysis is based on the use of non-optimal estimates of sums of the shape

$$\sum_{m \leq X} a_m^2, \qquad \text{where } a_m = \Big\{ \mathbf{x} \in \mathbb{N}^3 : \ m = x_1^3 + x_2^3 + x_3^3, \quad x_2, x_3 \in \mathcal{A}(P, P^{\eta}) \Big\}$$

with $\eta > 0$ a small enough parameter and

$$\mathcal{A}(Y, R) = \{n \in [1, Y] \cap \mathbb{N} : p \mid n \text{ and } p \text{ prime} \Rightarrow p \leq R\}.$$

Here, the reader may find it useful to observe that it is a consequence of Montgomery and Vaughan [102, Theorem 7.2] that

$$\operatorname{card}\big(\mathcal{A}(P, P^{\eta})\big) = c_{\eta} P + O\big(P/\log P\big) \tag{3.2.3}$$

for some constant $c_{\eta} > 0$ that only depends on η.

In order to prove Theorem 3.2.1 we show that for almost every integer the minor arc contribution is of smaller size than the expected main term. We also approximate the generating exponential sums of the problem by an auxiliary function over a set of narrower major arcs. Finally, we show via Bessel's inequality that for almost all integers the n-th Fourier coefficient of such exponential sums can also be approximated on the wider major arcs. As experts will realise, the power of $\log N$ saved in (3.2.2) comes from the choice of the narrower major arcs and the fact that the error term in (3.2.3) only saves a factor of $\log N$. Without severely complicating the argument, this choice seems inevitable for exploiting the information given by the variables x_2 and x_3 lying on $\mathcal{A}(P, P^{\eta})$ to ensure the convergence of the singular series and to obtain suitable properties for it. Therefore, the power saving for the bound of the cardinality of the exceptional set seems out of reach with these methods.

The application of Bessel's inequality for bounding exceptional sets has already been used by some authors before (see for instance Montgomery and Vaughan [101]). There is another approach by Wooley which instead uses an exponential sum over the exceptional set that often gives stronger upper bounds for the cardinality of those sets (see Wooley [162], [169]). However, in order to be able to use the latter method, one would need stronger minor arc bounds for auxiliary 8-th moments together with near optimal bounds for a_m, which are not available in the literature so far.

We devote the rest of the discussion to introduce a harder version of the problem studied here. It is well-known that the numbers that cannot be written as sums of three squares are the ones of the shape $4^{\nu} \cdot m$ for $m \equiv 7 \pmod 8$.

Let

$$\mathcal{N} = \big\{n \in \mathbb{N} : \; n \not\equiv 7 \pmod 9, \quad n \neq 4^{\nu} \cdot m \text{ for some } m \equiv 7 \pmod 8, \;\; \nu \geq 0\big\}.$$

Then one would hope to have for almost all integers $n \in \mathcal{N}$ a representation

$$n = x_1^2 + x_2^2 + x_3^2$$

for some $x_i \in \mathscr{C}$. If we seek to prove this statement using the circle method approach then one should be able to obtain good enough minor arc bounds of moments of exponential sums involving six variables. We remind the reader though that we are just able to deal with minor arc bounds when we have eight or more variables, and it seems out of reach to lower that number down to 6. Likewise, the analysis of the singular series with just three variables looks very challenging.

3.2.2 Preliminary definitions

Let N be a natural number and consider the parameters

$$P = \lfloor N^{1/6} \rfloor, \qquad M = P^{2/5}, \qquad H = P^{9/5}.$$

Observe that then one has $M^3 H = P^3$. Consider as well

$$H_1 = \Big(\frac{1}{2}\Big)^{1/3} H^{1/3}, \qquad H_2 = \Big(\frac{2}{3}\Big)^{1/3} H^{1/3}, \qquad H_3 = \Big(\frac{1}{6}\Big)^{1/3} H^{1/3}.$$

For any vector $\mathbf{x} \in \mathbb{R}^3$ set the function $T(\mathbf{x}) = x_1^3 + x_2^3 + x_3^3$, which will be used throughout the paper. Take the sets of triples

$$\mathcal{H} = \Big\{\mathbf{y} \in \mathbb{N}^3 : \quad P/2 < y_1 \leq P, \quad (y_2, y_3) \in \mathcal{A}(P, P^{\eta})^2\Big\},$$

$$\mathcal{W} = \Big\{\mathbf{y} \in \mathbb{N}^3 : \quad H_1 < y_1 \leq H_2, \quad (y_2, y_3) \in \mathcal{A}(H_3, P^{\eta})^2\Big\},$$

where η is a sufficiently small but positive parameter. Let $n \in \mathbb{N}$ such that $N/2 \leq n \leq N$. We define $R(n)$ as the number of solutions of the equation

$$n = T(p_1\mathbf{x}_1)^2 + T(p_2\mathbf{x}_2)^2 + T(\mathbf{x}_3)^2 + T(\mathbf{x}_4)^2,$$

where $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{W}$, $\mathbf{x}_3, \mathbf{x}_4 \in \mathcal{H}$ and $M/2 \leq p_1, p_2 \leq M$. Our goal in the next subsections will be to obtain a lower bound for $R(n)$ for almost all natural numbers. For such purpose, it is convenient to define the weights

$$a_x = \Big|\big\{\mathbf{y} \in \mathcal{H} : \quad x = T(\mathbf{y})\big\}\Big| \qquad \text{and} \qquad b_h = \Big|\big\{\mathbf{y} \in \mathcal{W} : \quad h = T(\mathbf{y})\big\}\Big|,$$

and consider the exponential sums

$$h(\alpha) = \sum_{x \leq 3P^3} a_x e(\alpha x^2) \qquad \text{and} \qquad W(\alpha) = \sum_{M/2 \leq p \leq M} \sum_{\frac{H}{2} \leq h \leq H} b_h e(\alpha p^6 h^2).$$

Observe that by orthogonality it follows that

$$R(n) = \int_0^1 h(\alpha)^2 W(\alpha)^2 e(-\alpha n) d\alpha.$$

We will make use of two Hardy-Littlewood dissections in our analysis, and these we now describe. Let $1 \leq X \leq P^{4/5}$. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $0 \leq a \leq q \leq X$ and $(a, q) = 1$, consider

$$\mathfrak{M}(a, q) = \Big\{\alpha \in [0, 1) : \Big|\alpha - a/q\Big| \leq \frac{X}{qn}\Big\}.$$

We take the major arcs $\mathfrak{M}(X)$ to be the union of the arcs $\mathfrak{M}(a, q)$. For the sake of simplicity we write

$$\mathfrak{M} = \mathfrak{M}(P^{4/5}), \qquad \mathfrak{N} = \mathfrak{M}\big((\log P)^{\tau}\big),$$

where $\tau = 18/31$. We also define the minor arcs as $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$.

Next we introduce the auxiliary functions that play a leading role in the discussion of Subsections 3.2.3 to 3.2.6. For $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, let $S(q, a)$ denote the complete exponential sum associated to the problem, which we define by

$$S(q, a) = \sum_{\mathbf{r} \leq q} e_q\big(aT(\mathbf{r})^2\big).$$

Consider the functions

$$v(\beta) = \int_{\mathbf{x} \in \mathcal{S}} e\big(F(\mathbf{x})\big) d\mathbf{x} \qquad \text{and} \qquad v_p(\beta) = \int_{\mathbf{x} \in \mathcal{S}_W} e\big(F_p(\mathbf{x})\big) d\mathbf{x},$$

where $F(\mathbf{x}) = \beta T(\mathbf{x})^2$ and $F_p(\mathbf{x}) = \beta T(p\mathbf{x})^2$, and the sets of integration taken are

$$\mathcal{S} = \Big\{\mathbf{x} \in [0,P]^3 : \quad P/2 \leq x_1 \leq P\Big\}$$

and

$$\mathcal{S}_W = \Big\{\mathbf{x} \in \mathbb{R}^3 : \quad H_1 \leq x_1 \leq H_2, \quad 0 \leq x_2, x_3 \leq H_3\Big\}.$$

Let $\alpha \in [0,1)$ and choose $\beta = \alpha - a/q$. Recalling the constant c_η mentioned in (3.2.3), define

$$V(\alpha,q,a) = q^{-3}S(q,a)c_\eta^2 v(\beta) \qquad \text{and} \qquad W(\alpha,q,a) = \sum_{M/2\leq p\leq M} V_p(\alpha,q,a), \tag{3.2.4}$$

where $V_p(\alpha,q,a) = q^{-3}S(q,a)c_\eta^2 v_p(\beta)$. For the sake of brevity, we define the auxiliary functions $h^*(\alpha)$ and $W^*(\alpha)$ by setting

$$h^*(\alpha) = V(\alpha,q,a) \qquad \text{and} \qquad W^*(\alpha) = W(\alpha,q,a)$$

when $\alpha \in \mathfrak{M}(a,q) \subset \mathfrak{M}$ and $h^*(\alpha) = W^*(\alpha) = 0$ for $\alpha \in \mathfrak{m}$. Before describing the outline of the memoir, it is convenient to introduce

$$\mathcal{F}(\alpha) = h(\alpha)^2 W(\alpha)^2 - h^*(\alpha)^2 W^*(\alpha)^2.$$

In Subsection 3.2.3 we approximate $h(\alpha)$ and $W(\alpha)$ by the functions $h^*(\alpha)$ and $W^*(\alpha)$ respectively when $\alpha \in \mathfrak{N}$. Making use of these approximations we bound the integral of $\mathcal{F}(\alpha)e(-\alpha n)$ over $\mathfrak{N}$ in Subsection 3.2.5. In the second part of that subsection and Subsection 3.2.6 we show that the upper bound for the integral of the same function over $\mathfrak{M} \setminus \mathfrak{N}$ still holds for almost all integers. We obtain such result by estimating the integral of $|\mathcal{F}(\alpha)|^2$ and applying Bessel's inequality. Subsection 3.2.4 is devoted to the study of the singular series. In such analysis we give a lower bound of the singular series for almost all integers, which combined with the lower bound for the singular integral computed in Subsection 3.2.6 provides a lower bound for the major arc contribution. We also combine the major arc estimates obtained throughout the memoir with the minor arc bound derived in the previous subsection to show in Subsection 3.2.6 that the minor arc contribution is smaller than the major arc one for almost all integers.

3.2.3 Approximation of exponential sums over the major arcs.

Based on previous work by the author we briefly provide some technical lemmas to approximate the exponential sums over the set of narrower major arcs.

Lemma 3.2.1. *Let $\alpha \in \mathfrak{N}(a,q)$ with $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a,q)=1$. Then one obtains the formula*

$$h(\alpha) = V(\alpha, q, a) + O\big(P^3 (\log P)^{\tau-1+\varepsilon}\big).$$

Proof. This is a consequence of Lemma 7.3 of [113]. The reader may check that the exponential sum $h(\alpha)$ here corresponds to $g_{Q,m}(\alpha)$ with the choices $Q=P$, $m=1$ and constants $C_1 = 1/2$, $C_2 = 1$ and $C_3 = 1$. □

Lemma 3.2.2. *Let $\alpha \in \mathfrak{N}(a,q)$ with $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a,q)=1$. Then,*

$$W(\alpha) = W(\alpha, q, a) + O\big(HM (\log P)^{\tau-2+\varepsilon}\big).$$

Proof. Observe that

$$W(\alpha) = \sum_{M/2 \leq p \leq M} W_p(\alpha), \qquad \text{where } \; W_p(\alpha) = \sum_{\mathbf{x} \in \mathcal{W}} e\big(\alpha T(p\mathbf{x})^2\big).$$

We also apply Lemma 7.3 of [113] to $W_p(\alpha)$. The reader may check that $W_p(\alpha)$ here corresponds to $g_{Q,m}(\alpha)$ with the choices $Q = H^{1/3}$ and $m = p$ and the constants $C_1 = \big(1/2\big)^{1/3}$, $C_2 = \big(2/3\big)^{1/3}$ and $C_3 = \big(1/6\big)^{1/3}$. Consequently, it transpires that

$$W_p(\alpha) = V_p(\alpha, q, a) + O\big(H\big(\log P)^{\tau-1+\varepsilon}\big),$$

which delivers the result. □

The following series of lemmas make use of the work done in the above section and the previous chapter to give upper bounds for the auxiliary functions that play a role in the main term of the contribution of the major arcs.

Lemma 3.2.3. *Let $\beta \in \mathbb{R}$. Then one has that*

$$v(\beta) \ll \frac{P^3}{1+n|\beta|} \qquad \text{and} \qquad v_p(\beta) \ll \frac{H}{1+n|\beta|}.$$

Proof. For each $\mathbf{y} \in \mathbb{R}^2$, consider $C_{\mathbf{y}} = y_1^3 + y_2^3$. Define the auxiliary functions

$$v_{\mathbf{y}}(\beta) = \int_{M_{\mathbf{y}}}^{N_{\mathbf{y}}} B_{\mathbf{y}}(\gamma) e(\beta\gamma) d\gamma \qquad \text{and} \qquad v_{\mathbf{y},p}(\beta) = \int_{M_{\mathbf{y},p}}^{N_{\mathbf{y},p}} B_{\mathbf{y},p}(\gamma) e(\beta\gamma) d\gamma$$

where $B_{\mathbf{y}}(\gamma)$, $B_{\mathbf{y},p}(\gamma)$ and the limits of integration taken are

$$B_{\mathbf{y}}(\gamma) = \frac{1}{6}\gamma^{-1/2}\big(\gamma^{1/2} - C_{\mathbf{y}}\big)^{-2/3}, \quad M_{\mathbf{y}} = \Big(\frac{P^3}{8} + C_{\mathbf{y}}\Big)^2, \quad N_{\mathbf{y}} = \big(P^3 + C_{\mathbf{y}}\big)^2$$

and

$$B_{\mathbf{y},p}(\gamma) = \frac{1}{6p}\gamma^{-1/2}\big(\gamma^{1/2} - C_{p\mathbf{y}}\big)^{-2/3}, \quad M_{\mathbf{y},p} = \Big(\frac{p^3 H}{2} + C_{p\mathbf{y}}\Big)^2$$

and

$$N_{\mathbf{y},p} = \Big(\frac{2p^3 H}{3} + C_{p\mathbf{y}}\Big)^2.$$

By a change of variables we find that

$$v(\beta) = \int_{\mathbf{y}\in[0,P]^2} v_{\mathbf{y}}(\beta) d\mathbf{y} \qquad \text{and} \qquad v_p(\beta) = \int_{\mathbf{y}\in[0,H_3]^2} v_{\mathbf{y},p}(\beta) d\mathbf{y}. \tag{3.2.5}$$

Observe that Lemma 3.1.8 yields the pointwise bounds $v_{\mathbf{y}}(\beta) \ll P(1+n|\beta|)^{-1}$ and $v_{\mathbf{y},p}(\beta) \ll H^{1/3}(1+n|\beta|)^{-1}$. The result then follows applying these estimates trivially to the above integrals. □

Lemma 3.2.4. *Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$. Then, one has*

$$S(q,a) \ll q^{5/2+\varepsilon}. \tag{3.2.6}$$

Moreover, when p is prime and $l \geq 3$ one finds that

$$S(p^l,a) \ll l p^{5l/2+\varepsilon}. \tag{3.2.7}$$

When $l = 2$ then $S(p^2,a) \ll p^5$ and for the case $l = 1$ we obtain the refinement

$$S(p,a) = p^2 S_2(p,a) + O(p^2), \qquad \textit{where } S_2(p,a) = \sum_{r=1}^{p} e_p(ar^2). \tag{3.2.8}$$

Proof. Equations (3.2.7) and (3.2.8) follow from Lemmata 2.3.1 and 2.3.2 of the previous chapter. The bound for the case $l = 2$ also follows from Lemma

2.3.1. We remind the reader of the estimates

$$d(q) \ll q^{\varepsilon}, \qquad \omega(q) \ll \log q/\log\log q,$$

where the functions $d(q)$ and $\omega(q)$ denote the number of divisors of q and the number of prime divisors of q, respectively. Combining such bounds with the multiplicative property of $S(q,a)$ and the estimates for $S(p^l,a)$ discussed before we obtain (3.2.6). □

The next lemma gathers the previous results to provide an upper bound for the auxiliary functions $h^*(\alpha)$ and $W^*(\alpha)$.

Lemma 3.2.5. *Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$. Take $\alpha \in \mathfrak{M}(a,q)$. Then,*

$$h^*(\alpha) \ll \frac{q^{-1/2+\varepsilon}P^3}{1+n|\beta|} \qquad \textit{and} \qquad W^*(\alpha) \ll \frac{q^{-1/2+\varepsilon}HM}{(\log P)(1+n|\beta|)}.$$

Proof. This follows from Lemmata 3.2.3 and 3.2.4 via equation (3.2.4). □

3.2.4 Treatment of the singular series

In this subsection we discuss some convergence properties of the singular series and we analyse the local solubility of the problem. As a consequence, we derive a lower bound for the singular series for almost all integers. For such purposes, it is convenient to define, for $q \in \mathbb{N}$, the sums

$$S_n(q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \big(q^{-3}S(q,a)\big)^4 e_q(-na), \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} S_n(q)$$

and for each prime p, the infinite series

$$\sigma(p) = \sum_{l=0}^{\infty} S_n(p^l).$$

Lemma 3.2.6. *One has that*

$$\mathfrak{S}(n) = \prod_p \sigma(p),$$

the singular series $\mathfrak{S}(n)$ converges absolutely and $\mathfrak{S}(n) \ll n^{\varepsilon}$. Also, when

$Q > 0$ one gets

$$\sum_{q \le Q} q^{1/2} |S_n(q)| \ll (nQ)^{\varepsilon} \qquad \textit{and} \qquad \sum_{q \ge Q} |S_n(q)| \ll n^{\varepsilon} Q^{\varepsilon - 1/2}. \tag{3.2.9}$$

Moreover, for any constant $\upsilon > 0$ there exists a set $A_\upsilon \subset [1, N]$ satisfying $|A_\upsilon| \ll N(\log N)^{-\upsilon}$ and such that for every $n \in [1, N] \setminus A_\upsilon$ one obtains

$$\mathfrak{S}(n) \gg (\log N)^{-\upsilon}.$$

Proof. Recalling (3.2.8), consider the exponential sum

$$S_{n,2}(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \big(q^{-1} S_2(q, a)\big)^4 e_q(-na).$$

By Lemma 3.2.4 one has that

$$S_n(p) = S_{n,2}(p) + O(p^{-3/2}). \tag{3.2.10}$$

Observe that an application of the same lemma yields $S_n(p^l) \ll l^4 p^{-l+\varepsilon}$ when $l \ge 3$ and $S_n(p^2) \ll p^{-2}$. One can also deduce from equation (4.27) of Vaughan [141, Theorem 4.3] that whenever $p \nmid n$ then $S_{n,2}(p) \ll p^{-3/2}$. Therefore, the combination of the previous estimates gives

$$\sum_{l=1}^{\infty} |S_n(p^l)| \ll p^{-3/2}.$$

Likewise, when $p \mid n$, then an application of the aforementioned bounds for $S_n(p^l)$ and the estimate $S_{n,2}(p) \ll p^{-1}$, which is an immediate consequence of Vaughan [141, Lemma 4.3], lead to

$$\sum_{l=1}^{\infty} |S_n(p^l)| \ll p^{-1}.$$

Therefore, by the preceeding discussion and the multiplicative property of $S_n(q)$, one gets the convergence for $\mathfrak{S}(n)$ and the upper bound

$$\mathfrak{S}(n) \ll \prod_{p \nmid n} \big(1 + C_1 p^{-3/2}\big) \prod_{p \mid n} \big(1 + C_2 p^{-1}\big) \ll n^{\varepsilon}$$

for some constants $C_1, C_2 > 0$. Similarly, one finds that

$$\sum_{l=1}^{\infty} p^{l/2}|S_n(p^l)| \ll p^{-\xi},$$

where $\xi = 1$ if $p \nmid n$ and $\xi = 1/2$ if $p \mid n$. Consequently, the combination of the above estimates yields the bound

$$\sum_{q\leq Q} q^{1/2}|S_n(q)| \ll \prod_{\substack{p\leq Q\\ p\nmid n}}(1+C_1p^{-1})\prod_{\substack{p\leq Q\\ p\mid n}}(1+C_2p^{-1/2}) \ll (nQ)^{\varepsilon}.$$

The second estimate in (3.2.9) follows observing that as a consequence of the above equation then

$$\sum_{Q\leq q\leq 2Q}|S_n(q)| \ll n^{\varepsilon}Q^{\varepsilon-1/2},$$

whence summing over dyadic intervals we obtain the desired result.

We will devote the rest of the subsection to prove the lower bound for the singular series. By equation (4.27) of Vaughan [141, Theorem 4.3] and (3.2.10) then whenever $p \nmid n$ one has $S_n(p) \ll p^{-3/2}$. We can also deduce from the proof[8] of [141, Theorem 4.5] that $S_{n,2}(p) \geq 0$ for $p \mid n$, which, combined with (3.2.10), yields $S_n(p) \geq -C_3p^{-3/2}$ for some $C_3 > 0$. Consequently, using the bound $S_n(p^l) \ll l^4p^{-l+\varepsilon}$ for $l \geq 2$ mentioned after (3.2.10) one gets that in both cases then $\sigma(p) \geq 1 - C_4p^{-3/2}$ for some $C_4 > 0$, and hence there exists a constant $C > 0$ for which

$$\mathfrak{S}(n) \gg \prod_{p\leq C}\sigma(p). \tag{3.2.11}$$

In order to give a more arithmetic description of $\sigma(p)$ we define for each $h \in \mathbb{N}$ the set

$$\mathcal{M}_n(p^h) = \Big\{\mathbf{Y} \in [1,p^h]^{12} : \sum_{i=1}^{4} T(\mathbf{y}_i)^2 \equiv n \pmod{p^h}\Big\},$$

and $M_n(p^h) = |\mathcal{M}_n(p^h)|$. Observe that orthogonality yields the identity

$$\sum_{l=0}^{h} S_n(p^l) = p^{-11h}M_n(p^h),$$

and hence $\sigma(p) = \lim_{h\to\infty} p^{-11h}M_n(p^h)$. For further discussion, it is relevant to

[8]See the argument just before [141, Theorem 4.6]

introduce the set

$$\mathcal{M}_n^*(p^h) = \Big\{ \mathbf{Y} \in \mathcal{M}_n(p^h) : \ \ p \nmid y_{1,1}, \ p \nmid T(\mathbf{y}_1) \Big\},$$

where $\mathbf{y}_1 = (y_{1,1}, y_{1,2}, y_{1,3})$, and $M_n^*(p^h) = |\mathcal{M}_n^*(p^h)|$. By Lemma 4.2 of [113] we have that whenever $p \neq 2, 3$ then $M_n^*(p) > 0$ for all n. Consequently, a standard application of Hensel's Lemma leads to $M_n(p^h) \geq p^{11(h-1)}$, which yields $\sigma(p) \geq p^{-11}$. For the cases $p = 2, 3$, it is convenient to define the set

$$\mathcal{M}_{3,3}(p^h) = \Big\{ T(\mathbf{x}) : \ \mathbf{x} \in \big(\mathbb{Z}/p^h\mathbb{Z}\big)^3, \ (x_1, p) = 1 \Big\}.$$

A slightly tedious computation reveals that $\mathcal{M}_{3,3}(27)$ is the set of residues not congruent to 4 or 5 modulo 9. Therefore, one has that

$$A = \Big\{ x^2 \ (\mathrm{mod}\ 27), \quad x \in \mathcal{M}_{3,3}(27) \Big\} = \Big\{ 0, 1, 4, 9, 10, 13, 19, 22 \Big\}.$$

Consider the set $B = \{ y \in A : \ (y, 3) = 1 \Big\}$. Observe that then

$$A + B = \Big\{ 1, 2, 4, 5, 8, 10, 11, 13, 14, 17, 19, 20, 22, 23, 26 \Big\}.$$

Consequently, by Cauchy-Davenport (see [141, Lemma 2.14]) we find that then $M_n^*(27) > 0$, whence another application of Hensel's Lemma gives the bound $M_n(3^h) \geq 3^{11(h-3)}$, and hence $\sigma(3) \geq 3^{-33}$.

The rest of the discussion will be devoted to the analysis for the prime $p = 2$. Take $\gamma \geq 0$ to be the exponent for which $2^\gamma || n$ and let $\theta = \lfloor (\gamma - 1)/2 \rfloor$. A routine application of Hensel's Lemma reveals that $\mathcal{M}_{3,3}(2^h)$ consists of all the residue classes modulo 2^h. Therefore, when $\gamma \leq 2$, one has that $M_n^*(8) > 0$, whence another application of Hensel's Lemma would yield $\sigma(2) \geq 2^{-33}$. For the case $\gamma \geq 3$ then whenever $h \geq \gamma + 2$ one can check that the congruence

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv n \ (\mathrm{mod}\ 2^h) \tag{3.2.12}$$

is soluble with solutions $x_i = 2^\theta y_i$, where y_i is defined modulo $2^{h-\theta}$ and

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 \equiv 2^{-2\theta} n \ (\mathrm{mod}\ 2^{h-2\theta}) \tag{3.2.13}$$

with $2 \nmid y_1$. Note that (3.2.13) has a solution modulo 8 with $2 \nmid y_1$, and hence by Lemma 2.13 of [141] there are at least $2^{3(h-2\theta-3)} \times 2^{4\theta}$ solutions to (3.2.12). By the same lemma, one has that the number of solutions to

$$z_1^3 + z_2^3 + z_3^3 \equiv x_i \pmod{2^h}$$

is bounded below by $2^{2(h-1)}$. Consequently, we obtain $M_n(2^h) \geq 2^{11h-\gamma-16}$, which delivers $\sigma(2) \gg 2^{-\gamma}$.

To finish the proof we take $A_v \subset [1, N]$ to be the set of numbers with $2^\gamma \geq (\log N)^v$. Observe that $|A_v| \leq N(\log N)^{-v}$. Then, by the preceding discussion and (3.2.11) it follows that whenever $n \notin A_v$ one has

$$\mathfrak{S}(n) \gg (\log N)^{-v}.$$

□

3.2.5 Mean values of the error term over the major arcs

Before proving an estimate for the first and second moment of $\mathcal{F}(\alpha)$ over $\mathfrak{N}$ and $\mathfrak{M} \setminus \mathfrak{N}$ respectively we will present some major arc type bounds that will be used later on the proof. For such matters, it is convenient to introduce the auxiliary multiplicative function $w_2(q)$, defined for prime powers by taking

$$w_2(p^{6u+v}) = \begin{cases} p^{-u-v/6} & \text{when } u \geq 1 \text{ and } 1 \leq v \leq 6 \\ p^{-1} & \text{when } u = 0 \text{ and } 2 \leq v \leq 6 \\ p^{-1/2} & \text{when } u = 0 \text{ and } v = 1. \end{cases} \tag{3.2.14}$$

Lemma 3.2.7. *Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and take $\alpha \in \mathfrak{M}(a, q)$. Denote $\beta = \alpha - a/q$. Then,*

$$h(\alpha) \ll \frac{q^\varepsilon w_2(q) P^3}{1 + n|\beta|},$$

and for $|\beta| \leq (12q)^{-1} H^{1/3} n^{-1}$ and $q \leq M/4$ we have that

$$W(\alpha) \ll \frac{q^\varepsilon w_2(q) HM}{(\log P)(1 + n|\beta|)}. \tag{3.2.15}$$

Proof. Lemmata 3.1.2 and 3.1.5 of the previous subsection yield the bounds

$$h(\alpha) \ll q^{\varepsilon} w_2(q) P^3 (1+n|\beta|)^{-1} + P^2 q^{1+\varepsilon} w_2(q)$$

whenever $\alpha \in \mathfrak{M}(a,q)$. Observe that $(1+n|\beta|)^{-1} \geq qP^{-1}$ when $\alpha \in \mathfrak{M}(a,q)$, and so the first term on the right side of the above equation dominates over the second one. Likewise, Lemmata 3.1.3 and 3.1.5 of the previous subsection deliver

$$W(\alpha) \ll q^{\varepsilon} w_2(q) MH (\log P)^{-1} (1+n|\beta|)^{-1} + MH^{2/3} q^{1+\varepsilon} w_2(q) (\log P)^{-1}$$

for the range described just before (3.2.15). Noting that $(1+n|\beta|)^{-1} \geq qH^{-1/3}$ we find that the first term also dominates over the second one in the above equation. The preceding discussion then provides the lemma. $\square$

Lemma 3.2.8. *For $q \in \mathbb{N}$ and every $Q > 0$ one finds that $w_2(q) \leq q^{-1/6}$. Moreover, one has*

$$\sum_{q \leq Q} w_2(q)^2 \ll Q^{\varepsilon} \qquad \textit{and} \qquad \sum_{q \leq Q} w_2(q)^{2+\delta} \ll 1$$

for any $\delta > 0$.

Proof. Both estimates follow from the definition (3.2.14) and the fact that $w_2(q)$ is multiplicative. $\square$

Combining the previous technical lemmas, we provide bounds for the L^1-norm of $\mathcal{F}(\alpha)$ over the set of arcs $\mathfrak{N}$ and the L^2-norm of the same function over $\mathfrak{M} \setminus \mathfrak{N}$ which are good enough for our purposes.

Proposition 3.2.1. *One has that*

$$\int_{\mathfrak{N}} \big|\mathcal{F}(\alpha)\big| d\alpha \ll (HM)^2 (\log P)^{3\tau/2-3+\varepsilon}. \tag{3.2.16}$$

Proof. Recalling Lemmata 3.2.1 and 3.2.5 it follows that for $\alpha \in \mathfrak{N}(a,q) \subset \mathfrak{N}$ then

$$h(\alpha)^2 - h^*(\alpha)^2 \ll P^6 (\log P)^{\tau-1+\varepsilon} \Big((\log P)^{\tau-1} + q^{-1/2} (1+n|\beta|)^{-1} \Big). \tag{3.2.17}$$

Likewise, by Lemmata 3.2.2 and 3.2.5 we have that for $\alpha \in \mathfrak{N}(a,q) \subset \mathfrak{N}$ then

$$W(\alpha)^2 - W^*(\alpha)^2 \ll (HM)^2(\log P)^{\tau-3+\varepsilon}\Big((\log P)^{\tau-1} + q^{-1/2}(1+n|\beta|)^{-1}\Big). \tag{3.2.18}$$

Denote by $N(q)$ the number of solutions of the congruence

$$T(p_1\mathbf{x}_1)^2 \equiv T(p_2\mathbf{x}_2)^2 \pmod{q},$$

where $\mathbf{x}_i \in [1, H^{1/3}]^3$ and $M/2 \le p_i \le M$. By expressing q as the product of prime powers, using the structure of the ring of integers of those prime powers and noting that the number of primes dividing q is bounded by q^{ε}, we obtain

$$N(q) \ll q^{\varepsilon-1}(HM)^2(\log P)^{-2},$$

and hence orthogonality delivers

$$\sum_{a=1}^{q}|W(\beta + a/q)|^2 \le qN(q) \ll q^{\varepsilon}(HM)^2(\log P)^{-2}. \tag{3.2.19}$$

Integrating over the major arcs and applying (3.2.17) and (3.2.19) one gets

$$\int_{\mathfrak{N}}\big|h(\alpha)^2 - h^*(\alpha)^2\big||W(\alpha)|^2 d\alpha \ll (HM)^2(\log P)^{3\tau-4+\varepsilon}\sum_{q\le(\log P)^{\tau}} q^{-1}$$
$$+ (HM)^2(\log P)^{\tau-3+\varepsilon}\sum_{q\le(\log P)^{\tau}} q^{-1/2} \ll (HM)^2(\log P)^{3\tau/2-3+\varepsilon}.$$

Similarly, using Lemma 3.2.5 and (3.2.18) we obtain

$$\int_{\mathfrak{N}}|h^*(\alpha)|^2\big|W(\alpha)^2 - W^*(\alpha)^2\big|d\alpha \ll (HM)^2(\log P)^{3\tau/2-3+\varepsilon}.$$

Equation (3.2.16) then holds combining the previous estimates and the triangle inequality. $\square$

Observe that the error term in Lemmata 3.2.1 and 3.2.2 when we approximate $h(\alpha)$ and $W(\alpha)$ by $h^*(\alpha)$ and $W^*(\alpha)$ respectively is non-trivial only for the set of small major arcs $\mathfrak{N}$. For the wider major arcs, we obtain instead an almost all result via Bessel's inequality.

Proposition 3.2.2. *One has that*

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} \big|\mathcal{F}(\alpha)\big|^2 d\alpha \ll P^6(HM)^4(\log P)^{-4-2\tau/3+\varepsilon}.$$

Proof. It is worth noting first that

$$|\mathcal{F}(\alpha)|^2 \ll |h(\alpha)|^4|W(\alpha)|^4 + |h^*(\alpha)|^4|W^*(\alpha)|^4.$$

For bounding the above integral we make use of standard major arc techniques and we exploit the extra number of variables that we get by taking squares. Before going into the proof, it is convenient to define $\Upsilon_\varepsilon(\alpha)$ for $\alpha \in [0,1)$ and $\varepsilon > 0$ by taking

$$\Upsilon_\varepsilon(\alpha) = q^\varepsilon w_2(q)(1+n|\alpha - a/q|)^{-1}$$

when $\alpha \in \mathfrak{M}(a,q) \subset \mathfrak{M}$ and $\Upsilon_\varepsilon(\alpha) = 0$ otherwise. Using Lemma 3.2.7 we find that

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |h(\alpha)|^4|W(\alpha)|^4 d\alpha \ll P^{12}\int_{\mathfrak{M}\setminus\mathfrak{N}} |W(\alpha)|^4\Upsilon_\varepsilon(\alpha)^4 d\alpha.$$

Observe that combining Lemma 3.2.8 with equation (3.2.19) we obtain that the contribution of the arcs with $q > M/4$ or $q \leq M/4$ and $|\beta| > (12q)^{-1}H^{1/3}n^{-1}$ is $O\big((HM)^4P^{6-\delta}\big)$. Let I' be the contribution to I of the arcs with $q \leq M/4$ and $|\beta| \leq (12q)^{-1}H^{1/3}n^{-1}$. Then Lemma 3.2.7 yields

$$I' \ll P^{12}(HM)^2(\log P)^{-2}\int_{\mathfrak{M}\setminus\mathfrak{N}} |W(\alpha)|^2\Upsilon_\varepsilon(\alpha)^6 d\alpha,$$

whence using Lemma 3.2.8 and (3.2.19) again we get that

$$I' \ll P^6(HM)^4(\log P)^{-4-2\tau/3+\varepsilon}.$$

On the other hand, an application of Lemma 3.2.5 gives the estimate

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |h^*(\alpha)|^4|W^*(\alpha)|^4 \ll P^6(HM)^4(\log P)^{-4-2\tau+\varepsilon},$$

which concludes the proof. □

3.2.6 Singular integral and Proof of Theorem 1.1

We briefly introduce the singular integral, give upper and lower bounds for it and discuss the size of the exceptional sets of the integers n with large minor arc contribution and for which the n-th Fourier coefficient of $\mathcal{F}(\alpha)$ over $\mathfrak{M}\setminus\mathfrak{N}$ is large as well. Consider

$$J(n)=\sum_{\mathbf{p}}\int_{\mathbf{Y}}J_{\mathbf{Y},\mathbf{p}}(n)d\mathbf{Y}, \tag{3.2.20}$$

where we define the collection $J_{\mathbf{Y},\mathbf{p}}(n)$ of singular integrals by

$$J_{\mathbf{Y},\mathbf{p}}(n)=\int_{-\infty}^{\infty}V_{\mathbf{Y},\mathbf{p}}(\beta)e(-n\beta)d\beta \quad \text{and} \quad V_{\mathbf{Y},\mathbf{p}}(\beta)=\prod_{i=1}^{2}v_{\mathbf{y}_i,p_i}(\beta)\prod_{i=3}^{4}v_{\mathbf{y}_i}(\beta).$$

The range of integration taken above is the set of tuples $\mathbf{Y}=(\mathbf{y}_1,\ldots,\mathbf{y}_4)$ with $\mathbf{y}_1,\mathbf{y}_2\in\mathcal{A}(H_3,P^{\eta})^2$ and $\mathbf{y}_3,\mathbf{y}_4\in\mathcal{A}(P,P^{\eta})^2$. Likewise, $\mathbf{p}$ runs over pairs of primes (p_1,p_2) with $M/2\leq p_i\leq M$. Here the reader might find useful to recall (3.2.5) and to observe that Lemma 3.2.3 guarantees the absolute convergence of the above integrals.

Lemma 3.2.9. *One has that*

$$J(n)\asymp(HM)^2(\log N)^{-2}.$$

Proof. An inspection of the proof of Lemma 3.1.7 of the above subsection reveals that the positivity and the upper bound for $J_{\mathbf{Y},\mathbf{p}}(n)$ deduced there remain valid subject only to the condition $s+t\geq 2$. Consequently, on making the choices $k=s=t=2$ here we obtain $0\leq J_{\mathbf{Y},\mathbf{p}}(n)\ll P^{-4}H^{2/3}$, whence applying this estimate trivially to (3.2.20) gives the required upper bound. Likewise, whenever one has $M/2\leq p_i\leq 51M/100$ for $i\leq t$ and $\mathbf{y}_i\leq P/2$ for $t+1\leq i\leq s+t$ then the lower bound obtained in that lemma also holds as long as $\big((3/8)^k s+(1/8)^k t\big)P^{3k}<n$. Therefore, on considering such range here we get that $J_{\mathbf{Y},\mathbf{p}}(n)\gg P^{-4}H^{2/3}$. Observe that the set of tuples on that range has positive density over the set without the restrictions. Consequently, the preceding remark and the positivity of $J_{\mathbf{Y},\mathbf{p}}(n)$ deliver the lower bound stated at the beginning. $\square$

We remind the reader of the definition (3.2.4). Note that then the combination of Lemma 3.2.3 and equation (3.2.9) with a change of variables yields

$$\int_{\mathfrak{M}} h^*(\alpha)^2 W^*(\alpha)^2 e(-\alpha n) d\alpha = \mathfrak{S}(n) J(n) + O\big((HM)^2 N^{-\delta}\big). \tag{3.2.21}$$

For the rest of the subsection we introduce the exceptional sets which arise in both the major and the minor arc analysis and we give bounds for the cardinality of them. Let $\delta > 0$ and let $\mathcal{E}_\delta(N)$ denote the set of integers satisfying $N/2 \leq n \leq N$ and with the property that

$$\int_{\mathfrak{m}} h(\alpha)^2 W(\alpha)^2 e(-n\alpha) d\alpha \gg (HM)^2 P^{-\delta/2}.$$

Likewise, define $\mathcal{E}(N)$ to be the set of integers $N/2 \leq n \leq N$ for which

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} \mathcal{F}(\alpha) e(-n\alpha) d\alpha \gg (HM)^2 (\log P)^{-2-2\tau/9}. \tag{3.2.22}$$

Proposition 3.2.3. *With the above notation, one has that*

$$|\mathcal{E}(N)| \ll N(\log N)^{-2\tau/9+\varepsilon},$$

and there exists some $\delta > 0$ for which $|\mathcal{E}_\delta(N)| \ll N^{1-\delta}$.

Proof. We obtain these two bounds via a routine application of Bessel's inequality. For such matters, observe first that Proposition 3.1.1 gives the estimate

$$\int_{\mathfrak{m}} |h(\alpha)W(\alpha)|^4 d\alpha \ll (HM)^4 P^{6-2\delta},$$

for some $\delta > 0$. Define the Fourier coefficient $c(n)$ of the product of the generating functions on the minor arcs by

$$c(n) = \int_{\mathfrak{m}} h(\alpha)^2 W(\alpha)^2 e(-n\alpha) d\alpha.$$

Note that Bessel's inequality yields

$$\sum_{N/2 \leq n \leq N} |c(n)|^2 \ll \int_{\mathfrak{m}} |h(\alpha)W(\alpha)|^4 d\alpha \ll (HM)^4 P^{6-2\delta},$$

whence the bound on $|\mathcal{E}_\delta(N)|$ follows from last equation. Similarly, we introduce the Fourier coefficient

$$a(n) = \int_{\mathfrak{M}\setminus\mathfrak{N}} \mathcal{F}(\alpha)e(-n\alpha)d\alpha.$$

Then by Proposition 3.2.2 and Bessel's inequality we get

$$\sum_{N/2\leq n\leq N} |a(n)|^2 \ll \int_{\mathfrak{M}\setminus\mathfrak{N}} |\mathcal{F}(\alpha)|^2 d\alpha \ll P^6(HM)^4(\log P)^{-4-2\tau/3+\varepsilon},$$

which yields the bound for $|\mathcal{E}(N)|$ stated at the beginning of the proposition. $\square$

Proof of Theorem 3.2.1. Take $n \in \mathbb{N}$ with $N/2 \leq n \leq N$ and $n \notin \mathcal{E}_\delta(N) \cup \mathcal{E}(N)$. Recalling the definitions before (3.2.22) and combining Propositions 3.2.1 and (3.2.21) we obtain

$$\begin{aligned} R(n) = &\int_{\mathfrak{m}} h(\alpha)^2 W(\alpha)^2 e(-\alpha n)d\alpha + \int_{\mathfrak{M}} h^*(\alpha)^2 W^*(\alpha)^2 e(-\alpha n)d\alpha \\ &+ \int_{\mathfrak{M}} \mathcal{F}(\alpha)e(-\alpha n)d\alpha = \mathfrak{S}(n)J(n) + O\big((HM)^2(\log N)^{-2-2\tau/9+\varepsilon}\big). \end{aligned}$$

Now fix a parameter $\upsilon < 2\tau/9$. Then applying Lemmata 3.2.6 and 3.2.9 we find that whenever n is as described above with the additional condition $n \notin A_\upsilon(N)$ then one has

$$R(n) \gg (HM)^2(\log N)^{-2-\upsilon}.$$

Observe that by Lemma 3.2.6 and Proposition 3.2.3 the cardinality of the set of integers $N/2 \leq n \leq N$ with $n \notin \mathcal{E}_\delta(N) \cup \mathcal{E}(N) \cup A_\upsilon(N)$ is $O\big(N(\log N)^{-\upsilon}\big)$. Consequently, summing over dyadic intervals and observing that we can take υ to be as close to $2\tau/9$ as possible we obtain the desired result.

Chapter 4

Uniform bounds in Waring's problem over diagonal forms

4.1 Introduction[1]

Waring's problem (first resolved by Hilbert) asserts that for every $k \in \mathbb{N}$ there exists $s = s_0(k)$ such that all positive natural numbers can be written as a sum of s positive integral k-th powers. Likewise, the problem of representing a sufficiently large natural number n in the shape

$$n = x_1^k + \cdots + x_s^k, \tag{4.1.1}$$

with $x_i \in \mathcal{S}$, where $\mathcal{S}$ is a given subset of the integers, has also been studied for particular cases. However, little has been written about Waring's problem when one considers specific sparse sets, and apart from the set of prime numbers, not much can be found on the literature. It is then rare to encounter examples of sparse sets with a structure fundamentally different in nature in which Waring's problem along the lines of equation (4.1.1) is solvable.

For a general set $\mathcal{A} \subset \mathbb{N}$, denote by $G_{\mathcal{A}}(k)$ the least positive integer s such that for all sufficiently large natural numbers n, the equation (4.1.1) possesses a solution with $x_i \in \mathcal{A}$. If such a number does not exist, define it as ∞. Let $\alpha > 0$ and consider sets $\mathcal{A}_\alpha \subset \mathbb{N}$ well distributed on arithmetic progressions

[1]This chapter is based on a submitted paper [112] by the author.

and with the property that $|\mathcal{A}_\alpha \cap [1, N]| \gg N^\alpha$. Then, on denoting

$$W(k, \alpha) = \sup_{\mathcal{A}_\alpha} \big\{ G_{\mathcal{A}_\alpha}(k) \big\},$$

one would hope to have $W(k, \alpha) < \infty$. In what follows we describe a particular family of sets satisfying the above property for which we expect the previous uniform bound to hold, but first we introduce, for convenience, some notation. For fixed $k, l, t \in \mathbb{N}$, let $\mathbf{x} = (x_1, \ldots, x_t) \in \mathbb{N}^t$, consider the function $T_t(\mathbf{x}) = x_1^l + \ldots + x_t^l$ and take the set

$$\mathcal{T}_t = \big\{ T_t(\mathbf{x}) : \ \mathbf{x} \in \mathbb{N}^t \big\}.$$

In this memoir we restrict our attention to the analysis of the solubility of (4.1.1) for the choice $\mathcal{S} = \mathcal{T}_t$ with t lying in the following two regimes:

(i) When $t = C(k)l$ for any fixed integer-valued function satisfying

$$C(k) \geq \tfrac{1}{2} \log(k(k+1)).$$

Work of Wooley [164] then yields the lower bound $|\mathcal{T}_t \cap [1, N]| \gg N^{1-\beta/k^2}$ for some constant $\beta > 0$. The reader may notice that once we fix k, the above bound is uniform over the family of sets $\mathcal{T}_t$, whence in view of the preceding discussion we expect to have $G_{\mathcal{T}_t}(k) < G_C(k)$ for all t in this regime with $G_C(k)$ being a constant depending on k and C. As will be discussed afterwards, the lower bound for the cardinality of the sets available is not strong enough to prove such a statement, and we end up showing something weaker.

(ii) When $t \geq \frac{l}{2}\big(\log l + \log(k(k+1)) + 2\big)$ then work of Wooley [164] yields the stronger lower bound $|\mathcal{T}_t \cap [1, N]| \gg N^{1-\gamma/lk^2}$ for some absolute constant $\gamma > 0$ at the cost of taking more variables. Were the sets $\mathcal{T}_t$ to have positive density, the argument would be considerably simplified and a pedestrian approach of the circle method would suffice. We use though the estimate available for the cardinality of these sets to derive a bound for $G_{\mathcal{T}_t}(k)$ that only depends on k.

For the rest of the introduction we discuss each of the two regimes described above and provide some motivation underlying their choice. As experts will realise, an application of the Hardy-Littlewood method delivers the solubility of (4.1.1) for $\mathcal{S} = \mathcal{T}_t$ when $t = C(k)l$ and s is large enough in terms of k and

l. We denote by $S_C(k,l)$ the minimum s with such a property and consider

$$P_C(k) = \sup_{l\geq 2} \big\{S_C(k,l)\big\},$$

which does not necessarily have to be finite. We also define the constant

$$\delta_r = \exp(1 - 2r/l) \tag{4.1.2}$$

for each $r \in \mathbb{N}$ and note that then combining the corollary to Theorem 2.1 of Wooley [164] and a standard argument involving Cauchy's inequality one obtains the lower bound

$$|\mathcal{T}_t \cap [1,N]| \gg N^{1-\delta_t}, \tag{4.1.3}$$

where $\delta_t = \exp\big(1 - 2C(k)\big)$ just depends on k. As previously mentioned, the estimate (4.1.3) is uniform once we fix k, whence the discussion made above motivates the following conjecture.

Conjecture 1. *Let $k \in \mathbb{N}$. There exists a positive integer-valued function $C : \mathbb{N} \to \mathbb{N}$ such that $P_C(k) < \infty$.*

This conjecture seems to be out of reach with the methods available in the literature for any value of k. However, in this paper we make some progress by using an argument which permits us to prove a weaker version which we describe next after introducing first some notation. For $s \in \mathbb{N}$, the choice of t described above and any $r \geq 0$, consider the equation

$$n = \sum_{i=1}^{s} T_t(\mathbf{x}_i)^k + \sum_{i=1}^{r} x_i^k, \tag{4.1.4}$$

where $\mathbf{x}_i \in \mathbb{N}^t$ and $x_i \in \mathbb{N}$. Let $S_C(k,l,r)$ denote the minimum number such that for $s \geq S_C(k,l,r)$, the equation (4.1.4) has a solution for all sufficiently large n and take

$$P_C(k,r) = \sup_{l\geq 2}\{S_C(k,l,r)\},$$

which, as before, does not necessarily have to be finite. We define $R_C(k)$ to be the minimum $r \geq 0$ such that $P_C(k,r)$ is finite. After the preceding discussion we are now equipped to state the main theorem of the paper.

Theorem 4.1.1. *Let $k \geq 2$ and consider any positive integer-valued function $C(k)$ with the property that*

$$C(k) \geq \max\big(4, \tfrac{1}{2}\log(k(k+1)) + 3/2\big).$$

Then one has the bound

$$R_C(k) \leq 4,$$

and for every $r \geq 4$ one finds that $P_C(k,r) \leq k^2 + O(k)$. Moreover, $R_C(2) \leq 2$.

We should emphasize that one could obtain the more precise bound

$$P_C(k,r) \leq k(k+1)$$

by introducing suitable weights in the exponential sums that we make use of and exploiting the information provided by such sums on the major arc analysis. We have omitted providing that discussion to make the exposition simpler. The reader might as well want to observe that $R_C(k) = 0$ is equivalent to Conjecture 1, whence the statement containing the relevant information in the above theorem is the upper bound on $R_C(k)$. As was foreshadowed earlier in the introduction, the approach taken herein further establishes a lower bound in the number of representations of the expected size, the diminishing ranges approach failing to achieve such an endeavour.

Let $G(k) = G_{\mathbb{N}}(k)$ be the smallest number such that for all $s \geq G(k)$, every large enough natural number can be written as a sum of s positive integral k-th powers. Vinogradov [153], Karatsuba [86] and Vaughan [139] made progress to achieve upper bounds for $G(k)$, the best current one for large k being

$$G(k) \leq k\Bigg(\log k + \log\log k + 2 + O\Big(\frac{\log\log k}{\log k}\Big)\Bigg) \tag{4.1.5}$$

due to Wooley [167]. Note that as a consequence of this bound one trivially has

$$R_C(k) \leq k\big(\log k + \log\log k + O(1)\big).$$

The reader then might want to observe that Theorem 4.1.1 improves this bound substantially. It is also worth noting that if Conjecture 1 were true for any fixed k, there would exist some $s = s(k)$ with the property that for any $l \geq 2$, every sufficiently big enough integer n would have a representation of the shape

$$n = \sum_{i=1}^{s} T_t(\mathbf{x}_i)^k$$

with $\mathbf{x}_i \in \mathbb{N}^t$. Observe that the right side of the above equation would consist of sums of Cl positive integral l-th powers gathered in groups and raised to the power k for some constant $C = C(k) > 0$ depending on k. This problem seems then even harder than the problem of proving that every sufficiently large integer can be written as the sum of Cl positive integral l-th powers, which would be a big breakthrough in view of (4.1.5).

Before describing the other regime for t analysed in the memoir, we note that as a consequence of the aforementioned work on $G(k)$, it follows that whenever $t \geq t_0(l)$ with

$$t_0(l) = \frac{l}{2}\big(\log l + \log\log l + 2 + o(1)\big) \tag{4.1.6}$$

then $\mathcal{T}_t$ has positive density, which greatly simplifies things (see, for example Brüdern, Kawada and Wooley [20, Theorem 1.5]). With the current state of knowledge, this turns out to be the threshold for which we can guarantee to have a lower bound of the shape $|\mathcal{T}_t \cap [1,N]| \gg N^{1-\varepsilon}$. Therefore, for fixed k and l large enough, the cardinality of the sets $\mathcal{A}_l = \mathcal{T}_{\xi_0(k,l)} \cap [1,N]$ with $\xi_0(k,l) = \lceil l/2\big(\log l + \log\big(k(k+1)\big) + 2\big)\rceil$ is not known to satisfy $|\mathcal{A}_l| \gg N^{1-\varepsilon}$, the best lower bound known being

$$|\mathcal{A}_l| \gg N^{1-1/k(k+1)le},$$

which is a consequence of (4.1.3).

Theorem 4.1.2. *Let $k, l \geq 2$ and take $\xi \geq \xi_0(k,l)$ and $s \geq s_0(k)$ with $s_0(k)$ satisfying $s_0(k) = k^2 + O(k)$. Then every sufficiently large n can be represented as*

$$n = \sum_{i=1}^{s} x_i^k,$$

where $x_i \in \mathcal{T}_\xi$.

The reader may want to observe that even if the sets $\mathcal{T}_\xi$ are not known to satisfy the estimate $|\mathcal{T}_\xi \cap [1,N]| \gg N^{1-\varepsilon}$, the bound on the number s of variables needed does not depend on l. This suggests that one should search for ideas which don't just make use of the polynomial structure of the sets $\mathcal{T}_\xi$ in order to prove such result. Experts in the area may also notice that one could prove a weaker version of the theorem by combining Corollary 1.4 of Wooley [177] with a pointwise bound over the minor arcs derived from Lemma 5.4 of Vaughan [141]. This strategy though would entail the restriction $s \geq (3/2)k^2 + O(k)$. We instead use a similar idea than the one we employ for the minor arc treatment in the proof of Theorem 4.1.1 that avoids relying on such pointwise bounds and enables us to win $k(k+1)/2$ variables. It is worth mentioning that one could also introduce suitable weights in the exponential sum that we make use of to obtain a more precise error term in the expression for $s_0(k)$.

Back to equation (4.1.1), the case when the set $\mathcal{S}$ is taken to be the prime numbers has been of interest to many mathematicians. Among others, Hua first ([73], [74]) and then Thanigasalam ([132], [133]), Kumchev [92], Kawada and Wooley [88], and Kumchev and Wooley ([93], [94]) have worked to give upper bounds for $H(k)$, where $H(k)$ is defined as the minimum number such that for every $s \geq H(k)$, the equation

$$n = p_1^k + \cdots + p_s^k$$

has a solution for all sufficiently large n satisfying $n \equiv s \pmod{K(k)}$, where $K(k)$ is a constant defined in terms of $K(k)$ to ensure appropiate local solubility conditions (see [93] for a more precise definition of $K(k)$). We note that the best current bound for large k is

$$H(k) \leq (4k-2)\log k - (2\log 2 - 1)k - 3$$

due to Kumchev and Wooley [94].

At the same time, some authors have been trying to find sparse sets with minimum density such that the problem of representing every sufficiently large positive number as a sum of k-th powers of elements of the set is still soluble with strong upper bounds for the number of variables needed. This other approach in Waring's problem has been studied by Nathanson [103], where he used the probabilistic method to prove that for all $s \geq G(k) + 1$, there exist sets A with $\mathrm{card}(A \cap [1, N]) \sim cN^{1-1/s+\varepsilon}$ such that (4.1.1) is soluble on A. The result was partially improved by Vu [154], when he showed under the condition $s \geq k^4 8^k$ the existence of a set $A \subset \mathbb{N}$ with the property that $\text{card}(A \cap [1, N]) = \Theta\big(N^{k/s}(\log N)^{1/s}\big)$ such that $R_A(n) \asymp \log n$, where $R_A(n)$ denotes the number of solutions of (4.1.1) with the variables lying in A. In a subsequent paper, Wooley [170] proved the same result for $s \geq T(k) + 2$, where $T(k)$ is bounded above by an explicit version of the right-hand side of (4.1.5). However, though the size of the sets is near optimal, the arguments used by the authors are probabilistic, so they don't give a description of those sets. Therefore, the approach of this paper might be the first one in which by giving an explicit family of sets with similar density, one tries to find a uniform bound for the number of variables needed to solve (4.1.1), as discussed at the beginning of the introduction.

Theorems 4.1.1 and 4.1.2 are proved via the circle method, and the exposition is organised as follows. We bound a mean value via restriction estimates in Section 4.2 and we expose the key argument which permits us to estimate mean values of a suitable exponential sum over the minor arcs uniformly on l. Section 4.3 is devoted to a brief study of the singular series. In Section 4.4 we approximate the generating function for the problem over the major arcs. We give an asymptotic formula for the integral of the product of some exponential sums over the major arcs in Section 4.5 and we use it to complete the proof of Theorem 4.1.1. We have included a small note in Section 4.6 that deals with the case $k = 2$. In Section 4.7 we slightly modify the exponential sum taken in Section 4.2 and use a similar argument to obtain a suitable estimate for the contribution of the minor arcs in the setting of Theorem 4.1.2. We combine such work with a standard major arc analysis to prove the theorem.

For the rest of the paper, we fix positive integers $l \geq 2$ and $k \geq 2$. For the sake of simplicity concerning local solubility, we assume that $t \geq 4l$, though most of the results throughout the paper don't require this restriction. For ease of notation we also write $T(\mathbf{x})$ instead of $T_t(\mathbf{x})$. The main objective of Theorem 4.1.1 is to prove a non-trivial uniformity bound for $S_C(k, l, r)$, and thus we just focus our attention in large values of l in terms of k. As mentioned above, even if we provide explicit bounds for $P_C(k, r)$ and $C(k)$, the relevant part of the result is the estimate on $R_C(k)$. For such purposes, we haven't included an investigation of the behaviour of $P_C(k, r)$ and $C(k)$ for small k.

4.2 Minor arc estimate

We will begin by displaying an upper bound for mean values of an exponential sum which will be of later use in the analysis of the minor arcs in the setting of both Theorems 4.1.1 and 4.1.2. This will be a straightfoward consequence of the work of Wooley [177] on Vinogradov's mean value theorem with weights. Let $r \in \mathbb{N}$, let $Y > 0$ be a real parameter and consider the set

$$\mathcal{S}_r(Y) = \Big\{ x_1^l + \ldots + x_r^l : \quad x_i \in \mathcal{A}(Y, Y^\eta), \ \ (1 \le i \le r) \Big\}, \tag{4.2.1}$$

where

$$\mathcal{A}(Y, R) = \{ n \in [1, Y] \cap \mathbb{N} : p \mid n \text{ and } p \text{ prime} \Rightarrow p \le R \}$$

and η is a sufficiently small but positive parameter. Note that then the corollary to Theorem 2.1 of Wooley [164] and a routine argument using Cauchy's inequality yield

$$|\mathcal{S}_r(Y)| \gg Y^{l - l\delta_r}, \tag{4.2.2}$$

where δ_r was defined in (4.1.2). In order to make further progress we need to introduce first some notation. Let n be a positive integer and take $X = n^{1/k}$ and $P = X^{1/l}$. Define for $\alpha \in [0,1)$ and $\boldsymbol{\alpha} \in [0,1)^k$ the exponential sums

$$f\big(\alpha, \mathcal{S}_r(Y)\big) = \sum_{x \in \mathcal{S}_r(Y)} e(\alpha x^k), \qquad f\big(\boldsymbol{\alpha}, \mathcal{S}_r(Y)\big) = \sum_{x \in \mathcal{S}_r(Y)} e(\alpha_1 x + \ldots + \alpha_k x^k). \tag{4.2.3}$$

For future purposes in the analysis, we consider the mean value

$$J_{s,r}^{(k)}(Y) = \int_{[0,1)^k} \big| f\big(\boldsymbol{\alpha}, \mathcal{S}_r(Y)\big) \big|^{2s} d\boldsymbol{\alpha}, \tag{4.2.4}$$

which by orthogonality counts the solutions to the system

$$x_1^j + \ldots + x_s^j = x_{s+1}^j + \ldots + x_{2s}^j \qquad (1 \le j \le k),$$

where $x_i \in \mathcal{S}_r(Y)$.

Proposition 4.2.1. *Let $s \ge k(k+1)/2$. Then one has that*

$$J_{s,r}^{(k)}(Y) \ll |\mathcal{S}_r(Y)|^{2s} Y^{-lk(k+1)/2 + l\Delta_r + \varepsilon},$$

where $\Delta_r = \delta_r k(k+1)/2$.

Proof. Define the weights $a_x = 1$ if $x \in \mathcal{S}_r(Y)$ and 0 otherwise. Then, we can rewrite $f(\boldsymbol{\alpha}, \mathcal{S}_r(Y))$ as

$$f(\boldsymbol{\alpha}, \mathcal{S}_r(Y)) = \sum_{x \leq rY^l} a_x e(\alpha_1 x + \ldots + \alpha_k x^k).$$

Therefore, combining Corollary 1.4 of Wooley [177] with (4.2.2) and the triangle inequality we obtain

$$J_{s,r}^{(k)}(Y) \ll |\mathcal{S}_r(Y)|^{2s} |\mathcal{S}_r(Y)|^{-k(k+1)/2} Y^{\varepsilon} \ll |\mathcal{S}_r(Y)|^{2s} Y^{-lk(k+1)/2 + l\Delta_r + \varepsilon},$$

which yields the desired result. $\square$

Before introducing the main ingredients for the minor arc analysis, we recall from the introduction that whenever $t \geq t_0(l)$, where $t_0(l)$ was defined in (4.1.6), then $\mathcal{T}_t$ has positive density. We deliberately avoid this situation by considering l sufficiently large in terms of k. The difficulty of the Conjecture 1 then lies on the fact that $\mathcal{T}_t$ is not known to have positive density, and the best lower bounds available on the cardinality of the set are not strong enough. Moreover, any approach making use of the fact that $T_t(\mathbf{x})^k$ is a polynomial of degree kl and applying a Weyl estimate for the corresponding exponential sum or any multivariable version of Vinogradov's Mean Value Theorem (see Theorem 1.3 and Theorem 2.1 of [108]) would entail a restriction in the number of variables that would depend on the degree of the polynomial.

We make though some progress by obtaining a uniform bound in l of a suitable exponential sum over the minor arcs. Our argument here is motivated by the treatment of Vaughan [141, Chapter 5], and it requires the estimate in Proposition 4.2.1. For such purposes, we introduce first some notation. Consider a positive integer-valued function $C(k)$ and set

$$t = C(k)l, \qquad t_1 = C(k)l - l.$$

We take the parameter

$$\varphi_{k,t_1} = 1 - k(k+1)\delta_{t_1}/2 - e^{-1}, \tag{4.2.5}$$

which for the sake of concision will be denoted by φ_k. Observe that whenever $C(k)$ satisfies the lower bound included in the hypothesis of Theorem 4.1.1 then one has $\varphi_k \geq 1/2 - e^{-1} > 0$. We also define the constants

$$C_1 = \big(k(k+1)2^{k+1}t_1^k\big)^{-1/lk}, \qquad C_2 = \min\Big((2lk)^{-1/l}, \big(k(k+1)2^{k+1}l^k\big)^{-1/lk}\Big). \tag{4.2.6}$$

and the natural numbers

$$P_1 = \lfloor C_1 P \rfloor, \quad P_2 = \lfloor C_2 P \rfloor. \tag{4.2.7}$$

For ease of notation, we denote $\mathcal{S}_{t_1}(P_1)$, and $\mathcal{S}_l(P_2)$ by $\mathcal{S}_1$, and $\mathcal{S}_2$ respectively. It is then convenient to define, for $m \in \mathcal{S}_2$, the exponential sums

$$f_m(\alpha) = \sum_{x \in \mathcal{S}_1} e\big(\alpha(x+m)^k\big) \qquad \text{and} \qquad \mathcal{F}(\alpha) = \sum_{m \in \mathcal{S}_2} f_m(\alpha).$$

In order to make further progress we make a Hardy-Littlewood dissection. When $1 \leq Q \leq X$ we define the major arcs $\mathfrak{M}(Q)$ to be the union of

$$\mathfrak{M}(a,q) = \Big\{\alpha \in [0,1) : |\alpha - a/q| \leq \frac{Q}{qn}\Big\} \tag{4.2.8}$$

with $0 \leq a \leq q \leq Q$ and $(a,q) = 1$. For the sake of brevity we write

$$\mathfrak{M} = \mathfrak{M}(X), \qquad \mathfrak{N} = \mathfrak{M}(P^{1/2}), \qquad \mathfrak{P} = \mathfrak{M}(\log P)$$

and we take $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$ and $\mathfrak{n} = [0,1) \setminus \mathfrak{N}$ to be the minor arcs.

Proposition 4.2.2. *Let $\alpha \in \mathfrak{m}$. Then one has*

$$\mathcal{F}(\alpha) \ll |\mathcal{S}_1||\mathcal{S}_2| X^{-\varphi_{k-1}/k(k-1)+\varepsilon},$$

where φ_{k-1} was defined in (4.2.5). Moreover, for $s \geq k(k+1)/2$ we obtain the mean value estimate

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^{2s} d\alpha \ll |\mathcal{S}_1|^{2s}|\mathcal{S}_2|^{2s} X^{-k-\varphi_k+\varepsilon}.$$

As experts will realise throughout the proof, one could obtain a similar result for the analogous Vinogradov generating function by using ideas of the proof of Theorem 5.2 of Vaughan [141]. We have ommited such analysis for the clarity of the exposition.

Proof. For every $m \in \mathcal{S}_2$ consider $\gamma(m) = (\gamma_1(m), \ldots, \gamma_{k-1}(m))$, where the entries taken are

$$\gamma_j(m) = \alpha \binom{k}{j} m^{k-j}, \qquad (1 \leq j \leq k-1). \tag{4.2.9}$$

Observe that then for every $x \in \mathcal{S}_1$ one obtains the relation

$$\alpha(x+m)^k = \boldsymbol{\nu}^{(k-1)}(x) \cdot \gamma(m) + \alpha x^k + \alpha m^k,$$

where we adopted the notation $\boldsymbol{\nu}^{(k-1)}(x) = (x, \ldots, x^{k-1})$. It is also convenient to define for $s \in \mathbb{N}$ the set of $(k-1)$-tuples of natural numbers

$$\mathcal{N} = \Big\{(n_1, \ldots, n_{k-1}) : \quad 1 \leq n_i \leq sX^i, \quad (1 \leq i \leq k-1)\Big\}.$$

By using the above relation we find that

$$\sum_{m \in \mathcal{S}_2} |f_m(\alpha)|^{2s} = \sum_{m \in \mathcal{S}_2} \Big| \sum_{\boldsymbol{n} \in \mathcal{N}} a(\boldsymbol{n}) e\big(\boldsymbol{n} \cdot \gamma(m)\big) \Big|^2, \tag{4.2.10}$$

where on denoting

$$\mathcal{X}(\boldsymbol{n}) = \Big\{\mathbf{x} \in \mathcal{S}_1^s : \ x_1^i + \ldots + x_s^i = n_i, \quad (1 \leq i \leq k-1)\Big\}$$

the coefficient $a(\boldsymbol{n})$ is defined as

$$a(\boldsymbol{n}) = \sum_{\mathbf{x} \in \mathcal{X}(\boldsymbol{n})} e\big(\alpha(x_1^k + \ldots + x_s^k)\big). \tag{4.2.11}$$

We devote the rest of the proof to apply a version of the large sieve inequality (Lemma 5.3 of Vaughan [141]) to the right side of (4.2.10). For such purpose, we shall consider the spacing modulo 1 of $\{\gamma(m)\}_m$. Take $x, y \in \mathcal{S}_2$ with $x \neq y$. Note that in view of (4.2.6) then one has

$$x, y \leq X/2k. \tag{4.2.12}$$

Observe that applying Dirichlet's approximation to each $\alpha \in \mathfrak{m}$ we obtain

$a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ such that $0 \leq a \leq q$ and

$$|\alpha - a/q| \leq q^{-1}X^{1-k},$$

where $X < q \leq X^{k-1}$. Note as well that by the choice of $\gamma_j(m)$ in (4.2.9) we find that

$$\|k\alpha(x-y)\| = \|\gamma_{k-1}(x) - \gamma_{k-1}(y)\|.$$

Then by the above discussion we obtain the lower bound

$$\|k\alpha(x-y)\| \geq \|ka(x-y)/q\| - (2q)^{-1}X^{2-k}.$$

Note that the only instance in which the first term on the right-hand side of the above equation can be 0 is when $y = x + nq(q,k)^{-1}$ for some $n \in \mathbb{N}$ with $n \neq 0$. However, $q(q,k)^{-1} \geq X/k$, which would contradict (4.2.12). Therefore, by the preceding discussion we get

$$\|k\alpha(x-y)\| \geq (2q)^{-1},$$

which delivers $\|\gamma_{k-1}(x) - \gamma_{k-1}(y)\| \gg X^{-k+1}$ and provides the spacing condition that we were seeking to prove.

Applying Lemma 5.3 of Vaughan [141] to (4.2.10) we obtain the upper bound

$$\sum_{m \in \mathcal{S}_2} |f_m(\alpha)|^{2s} \ll X^{k(k-1)/2} \sum_{\boldsymbol{n} \in \mathcal{N}} |a(\boldsymbol{n})|^2. \tag{4.2.13}$$

Note first that by bounding the coefficients $a(\boldsymbol{n})$ trivially one gets

$$\sum_{m \in \mathcal{S}_2} |f_m(\alpha)|^{2s} \ll X^{k(k-1)/2} J_{s,t_1}^{(k-1)}(P_1),$$

where $J_{s,t_1}^{(k-1)}(P_1)$ was defined in (4.2.4). Then combining the above equation with an application of Cauchy's inequality we obtain

$$\mathcal{F}(\alpha)^{2s} \ll |\mathcal{S}_2|^{2s-1} \sum_{m \in \mathcal{S}_2} |f_m(\alpha)|^{2s} \ll |\mathcal{S}_2|^{2s-1} X^{k(k-1)/2} J_{s,t_1}^{(k-1)}(P_1), \tag{4.2.14}$$

and hence for $s \geq k(k-1)/2$ then Proposition 4.2.1 delivers

$$\mathcal{F}(\alpha) \ll |\mathcal{S}_1||\mathcal{S}_2| \big(X^{\delta_{t_1} k(k-1)/2} |\mathcal{S}_2|^{-1}\big)^{1/2s} X^{\varepsilon}.$$

Therefore, fixing $s = k(k-1)/2$ and recalling (4.2.2) one gets

$$\mathcal{F}(\alpha) \ll |\mathcal{S}_1||\mathcal{S}_2|X^{-\varphi_{k-1}/k(k-1)+\varepsilon},$$

where φ_{k-1} was defined in (4.2.5).

For the second claim of the proposition we combine (4.2.13) and Cauchy's inequality in the same way as in (4.2.14) and we integrate over $\mathfrak{m}$ to get

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^{2s} d\alpha \ll |\mathcal{S}_2|^{2s-1} X^{k(k-1)/2} J_{s,t_1}^{(k)}(P_1).$$

An application of Proposition 4.2.1 and the estimate (4.2.2) to the above line then yields, for $s \geq k(k+1)/2$, the bound

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^{2s} d\alpha \ll |\mathcal{S}_1|^{2s}|\mathcal{S}_2|^{2s} X^{-k-\varphi_k+\varepsilon},$$

from where the second part of the proposition follows. □

Observe that the argument just makes use of the fact that $\mathcal{T}_t = \mathcal{T}_{t_1} + \mathcal{T}_l$, where $|\mathcal{T}_{t_1} \cap [1,N]|$ and $|\mathcal{T}_l \cap [1,N]|$ are appropriately large. Therefore, it could also be applied to other problems for sets with a similar property that don't necessarily have a polynomial structure. We conclude the investigation of the minor arcs by applying Weyl differencing to derive a bound for the exponential sum

$$f(\alpha) = \sum_{T_t(\mathbf{z}) \leq P^l} e(\alpha T_t(\mathbf{z})^k), \tag{4.2.15}$$

where in the above sum $z \in \mathbb{N}^t$. For ease of notation, we avoid writing the dependance on t. Note that then one can rewrite $f(\alpha)$ as

$$f(\alpha) = \sum_{\substack{\mathbf{x} \in \mathbb{N}^{t-1} \\ P_{\mathbf{x}} \geq 1}} f_{\mathbf{x}}(\alpha), \qquad \text{where } f_{\mathbf{x}}(\alpha) = \sum_{1 \leq x \leq P_{\mathbf{x}}} e\big(\alpha T(\mathbf{x}, x)^k\big) \tag{4.2.16}$$

and where we took the parameter $P_{\mathbf{x}} = \big(P^l - T_{t-1}(\mathbf{x})\big)^{1/l}$.

Lemma 4.2.1. *Let $\alpha \in [0,1)$ and suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $(a,q) = 1$ and $|\alpha - a/q| \leq q^{-2}$. Then*

$$f(\alpha) \ll P^{t+\varepsilon}(q^{-1} + P^{-1} + qP^{-kl})^{2^{1-kl}}.$$

Proof. Observe that the polynomial $T(\mathbf{x}, x)$ is monic and of degree kl on x. Note that the implicit constant in Weyl's inequality (Vaughan [141, Lemma 2.4]) does not depend on the coefficients which are not the leading one. Therefore, an application of such inequality to $f_{\mathbf{x}}(\alpha)$ delivers

$$f_{\mathbf{x}}(\alpha) \ll P_{\mathbf{x}}^{1+\varepsilon}\big(q^{-1} + P_{\mathbf{x}}^{-1} + qP_{\mathbf{x}}^{-kl}\big)^{2^{1-kl}},$$

which yields the above estimate by combining the bound $P_{\mathbf{x}} \leq P$ and (4.2.16). $\square$

4.3 Singular series

Throughout this section we will always assume that $t \geq 2l$. Define for $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ the complete exponential sums

$$S(q, a) = \sum_{1 \leq \mathbf{r} \leq q} e_q\big(aT(\mathbf{r})^k\big) \qquad \text{and} \qquad S_k(q, a) = \sum_{r=1}^{q} e_q(ar^k). \tag{4.3.1}$$

Note that by orthogonality we can express $S(q, a)$ as

$$S(q, a) = q^{-1} \sum_{u=1}^{q} S_l(q, u)^t S_k(q, a, -u),$$

where

$$S_k(q, a, -u) = \sum_{r=1}^{q} e_q(ar^k - ur).$$

Because of the quasi-multiplicative structure of the exponential sum, we will focus on the case $q = p^h$, where p is a prime number. Then, with the above notation one has that

$$S(p^h, a) = p^{(t-1)h} S_k(p^h, a) + E(p^h, a), \tag{4.3.2}$$

where

$$E(p^h, a) = p^{-h} \sum_{u=1}^{p^h - 1} S_l(p^h, u)^t S_k(p^h, a, -u).$$

We estimate $E(p^h,a)$ by using classical estimates for the sums $S_l(p^h,u)$ and $S_k(p^h,a,-u)$. Applying then Theorems 4.2 and 7.1 of Vaughan [141] we obtain

$$\begin{aligned} E(p^h,a) \ll p^{-h/k+\varepsilon} \sum_{u=1}^{p^h-1} p^{th(1-1/l)}(u,p^h)^{t/l} &\ll p^{h(t-1/k-t/l+1)+\varepsilon} \sum_{d=0}^{h-1} p^{d(t/l-1)} \\ &\ll p^{ht-h/k-(t/l-1)+\varepsilon}. \end{aligned} \tag{4.3.3}$$

In order to provide more explicit bounds for the exponential sum $S(q,a)$ it is convenient to introduce first the multiplicative function $w_k(q)$ defined as

$$w_k(p^{uk+v}) = p^{-u-1} \quad \text{when } u \geq 0 \text{ and } 2 \leq v \leq k,$$

$$w_k(p^{uk+v}) = kp^{-u-1/2} \quad \text{when } u \geq 0 \text{ and } v = 1.$$

Then, by Lemma 3 of Vaughan [137] we obtain

$$q^{-1}|S_k(q,a)| \ll w_k(q), \tag{4.3.4}$$

whence combining (4.3.2) and (4.3.3) with the quasi-multiplicative property of $S(q,a)$ we get

$$q^{-t}|S(q,a)| \ll w_k(q). \tag{4.3.5}$$

For future purposes in the memoir, we note that by applying the definition of $w_k(q)$ and multiplicativity then we obtain for $Q>0$ and $s \geq \max(4,k+1)$ the bounds

$$\sum_{q\leq Q} w_k(q)^2 \leq \prod_{p\leq Q} \big(1+C/p\big) \ll Q^{\varepsilon}, \qquad \sum_{q\leq Q} q w_k(q)^s \leq \prod_{p\leq Q} \big(1+C/p\big) \ll Q^{\varepsilon}. \tag{4.3.6}$$

Before defining the singular series, we need to consider first the exponential sum

$$W(q,a) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e_q(ar^k), \tag{4.3.7}$$

where $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$. Here the reader might want to observe that one can express the sum $W(p^h,a)$ in terms of $S_k(p^h,a)$ and $S_k(p^{h-k},a)$, and hence one can deduce the estimate

$$\varphi(q)^{-1}|W(q,a)| \ll w_k(q) \tag{4.3.8}$$

by just applying multiplicativity and the bound (4.3.4). In order to make further progress, it is worth defining

$$S_n(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \big(q^{-t}S(q,a)\big)^s \big(q^{-1}S_k(q,a)\big)^2 \big(\varphi(q)^{-1}W(q,a)\big)^2 e_q(-an).$$

We also consider for convenience the series

$$\mathfrak{S}(n) = \sum_{q=0}^{\infty} S_n(q), \qquad\qquad \sigma(p) = \sum_{h=0}^{\infty} S_n(p^h), \tag{4.3.9}$$

where p is a prime number. We can provide a more arithmetic description of $\sigma(p)$ by considering $\mathcal{C}_p = [1,p^h]^4 \times [1,p^h]^{st}$, defining the set

$$\mathcal{M}_n(p^h) = \Big\{(\mathbf{y},\mathbf{X}) \in \mathcal{C}_p:\ p \nmid y_1y_2,\ n \equiv \sum_{i=1}^{4} y_i^k + \sum_{i=1}^{s} T(\mathbf{x}_i)^k \pmod{p^h}\Big\}$$

and introducing the counting function $M_n(p^h) = |\mathcal{M}_n(p^h)|$. For each prime p take $\tau \geq 0$ such that $p^\tau \| k$ and

$$\gamma = \begin{cases} \tau+1, & \text{when } p > 2 \text{ or when } p = 2 \text{ and } \tau = 0, \\ \tau+2, & \text{when } p = 2 \text{ and } \tau > 0. \end{cases} \tag{4.3.10}$$

Lemma 4.3.1. *Suppose that $s+3 \geq \frac{p}{p-1}\big(k, p^\tau(p-1)\big)$ when $\gamma = \tau+1$, that $s+3 \geq 2^{\tau+2}$ when $\gamma = \tau+2$ and $k > 2$, and that $s \geq 2$ when $p = k = 2$. Suppose as well that $t \geq 4l$. Then one has $M_n(p^\gamma) > 0$.*

Proof. It is worth noting first that Lemma 2.15 of Vaughan [141] implies that

$$T(\mathbf{x}) = x_1^l + \cdots + x_t^l \equiv m \pmod{p^\gamma}$$

is soluble for all $m \in \mathbb{N}$. The result follows then using the previous remark and observing that under the conditions described above, the same lemma delivers a representation

$$\sum_{i=1}^{3} y_i^k + \sum_{i=1}^{s} z_i^k \equiv n \pmod{p^\gamma}$$

with $p \nmid y_1$. $\square$

Proposition 4.3.1. *Let $s \geq \max(1, k-2)$. Then one has that*

$$\mathfrak{S}(n) = \prod_p \sigma(p), \tag{4.3.11}$$

the singular series $\mathfrak{S}(n)$ converges and $\mathfrak{S}(n) \ll 1$. Also, for $Q > 0$ we obtain the estimate

$$\sum_{q=1}^{Q} q^{1/k} |S_n(q)| \ll Q^\varepsilon. \tag{4.3.12}$$

Moreover, if s satisfies the conditions of Lemma 4.3.1 then $\mathfrak{S}(n) \gg 1$.

Proof. The application of (4.3.4), (4.3.5) and (4.3.8) delivers

$$S_n(p^h) \ll p^h w_k(p^h)^{s+4},$$

whence combining such bound with the definition of $w_k(q)$ we obtain the estimates

$$\sum_{h=1}^{\infty} |S_n(p^h)| \ll p^{-3/2}, \qquad \sum_{h=1}^{\infty} p^{h/k} |S_n(p^h)| \ll p^{-1}. \tag{4.3.13}$$

Therefore, by the multiplicative property of $S_n(q)$ we get (4.3.11) and

$$\sum_{q=1}^{Q} |S_n(q)| \ll \prod_{p \leq Q} (1 + Cp^{-3/2}) \ll 1,$$

which delivers the upper bound for $\mathfrak{S}(n)$. The estimate (4.3.12) follows in a similar way.

Observe that expressing $S_n(p^h)$ as the difference of two complete exponential sums and using orthogonality we get

$$\sum_{j=0}^{h} S_n(p^j) = M_n(p^h) p^{-h(st+1)} \varphi(p^h)^{-2}. \tag{4.3.14}$$

We use Lemma 4.3.1 and the fact that if m with $(m,p)=1$ is a k-th power modulo p^γ then it is also a k-th power modulo p^h for $h\geq\gamma$ to obtain the lower bound $M_n(p^h)\geq p^{(st+3)(h-\gamma)}$. Combining such lower bound with the above expression and (4.3.9) we find that $\sigma(p)\geq p^{-\gamma(st+3)}$. Therefore, by the preceding discussion and equation (4.3.13) we get $\mathfrak{S}(n)\gg 1$. $\square$

We next define an analogous singular series that arises in the analysis of the major arc contribution in Theorem 4.1.2. This series will be in nature quite similar to $\mathfrak{S}(n)$, so we will skip some details for the sake of brevity. For such purposes, consider

$$S'_n(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\big(q^{-t}S(q,a)\big)^s e_q(-an). \tag{4.3.15}$$

Observe that using (4.3.5) then we have

$$S'_n(p^h)\ll p^h w_k(p^h)^s. \tag{4.3.16}$$

Moreover, for any prime p and $h\in\mathbb{N}$ and combining (4.3.2), (4.3.3), (4.3.4) and (4.3.5) we can rewrite $S'_n(p^h)$ as

$$S'_n(p^h)=\sum_{\substack{a=1\\(a,p)=1}}^{p^h}\big(p^{-h}S_k(p^h,a)\big)^s e_{p^h}(-an)+E(p^h), \tag{4.3.17}$$

where the first term is the analogous sum for the original Waring's problem and the error term satisfies

$$E(p^h)\ll p^{h-h/k-(t/l-1)+\varepsilon}w_k(p^h)^{s-1}.$$

We define next the aforementioned series

$$\mathfrak{S}'(n)=\sum_{q=0}^{\infty}S'_n(q),\qquad\qquad \sigma'(p)=\sum_{h=0}^{\infty}S'_n(p^h),$$

where p is a prime number. We can provide a more arithmetic description of

$\sigma'(p)$ by considering the set

$$\mathcal{M}_n^*(p^h) = \Big\{ \mathbf{X} \in [1, p^h]^{st} : \ p \nmid x_{1,1}, \ p \nmid T(\mathbf{x}_1), \quad n \equiv \sum_{i=1}^{s} T(\mathbf{x}_i)^k \pmod{p^h} \Big\},$$

where $\mathbf{x}_1 = (x_{1,1}, \ldots, x_{1,t})$, and taking the counting function $M_n^*(p^h) = |\mathcal{M}_n^*(p^h)|$. For each prime p take $\tau_1 \geq 0$ such that $p^{\tau_1} \| kl$ and $\nu = \nu(p) = 2\tau_1 + 1$. Before stating the following lemma, recall (4.3.10).

Lemma 4.3.2. *Suppose that* $s \geq \frac{p}{p-1}\big(k, p^{\tau}(p-1)\big)$ *when* $\gamma = \tau + 1$, *that* $s \geq 2^{\tau+2}$ *when* $\gamma = \tau + 2$ *and* $k > 2$, *and that* $s \geq 5$ *when* $p = k = 2$. *Suppose as well that* $t \geq 4l$. *Then one has* $M_n^*(p^{\nu}) > 0$.

Proof. It is worth noting first that since $\nu \geq \gamma$ then Lemma 2.15 of Vaughan [141] and the fact that if $b \in \mathbb{N}$ with $(b, p) = 1$ is a k-th power modulo p^{γ} then it is also a k-th power modulo p^{ν} imply that

$$T(\mathbf{x}) = x_1^l + \cdots + x_t^l \equiv m \pmod{p^{\nu}}$$

with $p \nmid x_1$ is soluble for all $m \in \mathbb{N}$. The lemma follows using the previous remarks and observing that under the conditions described above, Lemma 2.15 of Vaughan [141] delivers a representation

$$\sum_{i=1}^{s} y_i^k \equiv n \pmod{p^{\nu}}$$

with $p \nmid y_1$. $\square$

Proposition 4.3.2. *Let* $s \geq \max(5, k+2)$. *Then one has that*

$$\mathfrak{S}'(n) = \prod_p \sigma(p),$$

the singular series $\mathfrak{S}'(n)$ *converges and* $\mathfrak{S}'(n) \ll 1$. *Also, for* $Q > 0$ *we obtain the estimate*

$$\sum_{q=1}^{Q} q^{1/k} |S_n'(q)| \ll Q^{\varepsilon}. \tag{4.3.18}$$

Moreover, if s *satisfies the conditions of Lemma 4.3.2 then* $\mathfrak{S}'(n) \gg 1$.

Proof. Equation (4.3.16) and the multiplicativity of $w_k(q)$ yield

$$\sum_{h=1}^{\infty}|S_n'(p^h)| \ll p^{-3/2}, \qquad \sum_{h=1}^{\infty}p^{h/k}|S_n'(p^h)| \ll p^{-1}, \tag{4.3.19}$$

which imply the convergence, the upper bound for the singular series and (4.3.18). For obtaining the lower bound for $\mathfrak{S}'(n)$ we observe that Lemma 4.3.2, an application of Hensel's Lemma and the argument used to derive (4.3.14) allow one to obtain $\sigma'(p) \geq p^{-\nu(st-1)}$, whence combining such estimate with (4.3.19) we then get $\mathfrak{S}'(n) \gg 1$. $\square$

Scholars in the area will realise that one could use (4.3.17) and the bounds available in the literature for the singular series in the original Waring's problem (see [141, Theorem 4.5]) to obtain the estimate $\mathfrak{S}'(n) \gg 1$ for the range $s \geq \max(4, k+1)$. One could also deduce (4.3.18) but with an extra factor of n^{ε} in the right side of the bound for the same range (see [141, Lemma 4.8]).

4.4 Approximation of some exponential sum over the major arcs

In this section we approximate $f(\alpha)$ on the major arcs by some auxiliary function. For such purpose it is convenient to introduce first some notation. Let $\alpha \in [0,1)$ and $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q) = 1$. Denote $\beta = \alpha - a/q$ and consider the aforementioned function $U(\alpha, q, a) = c_{t,l}q^{-t}S(q,a)u(\beta)$, where

$$u(\beta) = k^{-1}\sum_{m=1}^{n} m^{t/kl-1}e(\beta m), \qquad c_{t,l} = \Gamma(1+1/l)^t\Gamma(t/l)^{-1}, \tag{4.4.1}$$

and $S(q,a)$ was defined in (4.3.1).

Proposition 4.4.1. *Let $q < P$ and α, a, q, β as above. Then one has*

$$f(\alpha) = U(\alpha, q, a) + O\big(qP^{t-1}(1 + n|\beta|)\big).$$

Proof. Before embarking on our task, it is convenient to consider for each

$\mathbf{r} \in \mathbb{N}^t$ and $r \in \mathbb{N}$ the sums

$$K_{\mathbf{r}}(\beta) = \sum_{\substack{T(\mathbf{x}) \leq P^l \\ \mathbf{x} \equiv \mathbf{r} \pmod{q}}} e\big(\beta T(\mathbf{x})^k\big), \qquad B_r(x) = \sum_{\substack{0 < z \leq x \\ z \equiv r \pmod{q}}} 1.$$

Observe that by sorting the summation into arithmetic progressions modulo q we find that

$$f(\alpha) = \sum_{\mathbf{r} \leq q} e_q\big(aT(\mathbf{r})^k\big) K_{\mathbf{r}}(\beta). \tag{4.4.2}$$

For each $\mathbf{r} \in \mathbb{N}^t$ write $\mathbf{r} = (\mathbf{r}_{t-1}, r_t)$, where $\mathbf{r}_{t-1} \in \mathbb{N}^{t-1}$. Then recalling the definition of $P_{\mathbf{x}}$ after (4.2.16) and using Abel's summation formula we find that

$$K_{\mathbf{r}}(\beta) = \sum_{\mathbf{x}} B_{r_t}(P_{\mathbf{x}}) e\big(\beta T(\mathbf{x}, P_{\mathbf{x}})^k\big) - \sum_{\mathbf{x}} \int_0^{P_{\mathbf{x}}} \frac{\partial}{\partial z} e\big(\beta T(\mathbf{x}, z)^k\big) B_{r_t}(z) dz,$$

where $\mathbf{x} \in \mathbb{N}^{t-1}$ runs over tuples satisfying $T_{t-1}(\mathbf{x}) \leq P^l - 1$ and

$$\mathbf{x} \equiv \mathbf{r}_{t-1} \pmod{q}.$$

Consequently, combining the formula $B_{r_t}(x) = x/q + O(1)$ and an application of integration by parts one gets

$$K_{\mathbf{r}}(\beta) = q^{-1} \sum_{\mathbf{x}} \int_0^{P_{\mathbf{x}}} e\big(\beta T(\mathbf{x}, z)^k\big) dz + O\big(q^{-t+1} P^{t-1}(1 + n|\beta|)\big).$$

We have included a brief discussion of the next step in the argument since it involves a technical detail which was not required before. Note first that Abel's summation formula combined with the above equation yields

$$\begin{aligned} K_{\mathbf{r}}(\beta) =& q^{-1} \sum_{\mathbf{x}} B_{r_{t-1}}(P_{(\mathbf{x},0)}) I(P_{(\mathbf{x},0)}) \\ &- q^{-1} \sum_{\mathbf{x}} \int_0^{P_{(\mathbf{x},0)}} \frac{\partial I}{\partial z_{t-1}}(z_{t-1}) B_{r_{t-1}}(z_{t-1}) dz_{t-1} \\ &+ O\big(q^{-t+1} P^{t-1}(1 + n|\beta|)\big), \end{aligned}$$

where $\mathbf{x} \in \mathbb{N}^{t-2}$ runs over tuples satisfying $T_{t-2}(\mathbf{x}) \leq P^l$ with $\mathbf{x} \equiv \mathbf{r}_{t-2} \pmod{q}$

and

$$I(z_{t-1})=\int_0^{P_{\mathbf{x},z_{t-1}}} e\big(\beta T(\mathbf{x},z_{t-1},z_t)^k\big)dz_t.$$

Observe that combining the Fundamental Theorem of Calculus and the exchangeability of derivation and integration we find that

$$\frac{\partial}{\partial z_{t-1}}I(z_{t-1})\ll\frac{\partial}{\partial z_{t-1}}P_{\mathbf{x},z_{t-1}}+n|\beta|,$$

whence another combination of the formula $B_{r_{t-1}}(x)=x/q+O(1)$ and integration by parts yields

$$K_{\mathbf{r}}(\beta)=q^{-2}\sum_{\mathbf{x}}\int_{\mathcal{C}}e\big(\beta T(\mathbf{x},z_{t-1},z_t)^k\big)dz_{t-1}dz_t+O\big(q^{-t+1}P^{t-1}(1+n|\beta|)\big),$$

where $\mathcal{C}$ is the set of pairs $(z_{t-1},z_t)\in\mathbb{R}_+^2$ satisfying $T(\mathbf{x},z_{t-1},z_t)\leq P^l$. We repeat a similar argument for the rest of the variables to obtain

$$K_{\mathbf{r}}(\beta)=q^{-t}u_1(\beta)+O\big(q^{-t+1}P^{t-1}(1+n|\beta|)\big),\tag{4.4.3}$$

where $u_1(\beta)$ here denotes the integral version of $u(\beta)$, which we define by

$$u_1(\beta)=\int_{T(\mathbf{x})\leq P^l}e\big(\beta T(\mathbf{x})^k\big)d\mathbf{x},$$

where $\mathbf{x}\in\mathbb{R}_+^t$. Observe that by several changes of variables, one can rewrite $u_1(\beta)$ as

$$u_1(\beta)=k^{-1}l^{-t}\int_0^n w^{1/k-1}e(\beta w)\int_{\mathbf{X}\in\mathcal{M}}B(w,\mathbf{X})d\mathbf{X}dw,$$

with

$$B(w,\mathbf{X})=x_1^{1/l-1}\cdots x_{t-1}^{1/l-1}(w^{1/k}-x_1-\ldots-x_{t-1})^{1/l-1}$$

and $\mathcal{M}\subset\mathbb{R}^{t-1}$ is the corresponding set determined by the underlying inequalities. Consequently, combining the formula for the Euler-Beta function and subsequent changes of variables we get

$$u_1(\beta)=c_{t,l}u_2(\beta),\qquad\text{where } u_2(\beta)=k^{-1}\int_0^n w^{t/kl-1}e(\beta w)dw.$$

We devote the rest of the proof to compute the error term when we approximate $u_2(\beta)$ by $u(\beta)$. We believe that working with $u(\beta)$ instead makes the analysis a bit more transparent and less tedious. Consider the function

$$G(\gamma) = \sum_{1\leq y\leq \gamma} y^{t/kl-1}$$

and note that the Euler-Maclaurin formula (see Vaughan [141, (4.8)]) yields

$$G(\gamma) = klt^{-1}\gamma^{t/kl} + O\big(Z(\gamma)\big),$$

where $Z(\gamma) = 1 + \gamma^{t/kl-1}$. Observe that then Abel's summation formula, integration by parts and the previous discussion yield

$$\begin{aligned} u(\beta) &= lt^{-1}n^{t/kl}e(\beta n) - 2\pi i\beta\int_0^n lt^{-1}\gamma^{t/kl}e(\beta\gamma)d\gamma + O\big(Z(n)(1+n|\beta|)\big) \\ &= u_2(\beta) + O\big(Z(n)(1+n|\beta|)\big). \end{aligned}$$

The lemma then follows combining the above approximation with (4.4.2) and (4.4.3). $\square$

Note that the error term in the above proposition differs from the trivial bound by a factor of $Pq^{-1}(1+n|\beta|)^{-1}$, and this saving is gained by fixing $t-1$ variables in the expression for $K_{\mathbf{r}}(\beta)$ and using a one-dimensional argument. The saving obtained in Proposition 4.4.1, however, is not enough for our choice of the major arcs when l is large enough. Likewise, the possible approaches involving the use of all of the variables don't seem to improve the error term substantially. We devote the rest of the section to provide an upper bound for $u(\beta)$.

Lemma 4.4.1. *Let $|\beta| \leq 1/2$. Then one has*

$$u(\beta) \ll \frac{P^t}{(1+n|\beta|)^{\gamma_{k,l}}},$$

where $\gamma_{k,l} = \min(1, t/kl)$.

Proof. When $|\beta| \leq n^{-1}$ one finds that

$$u(\beta) \ll \sum_{m=1}^{n} m^{t/kl-1} \ll P^t,$$

which yields the required bound for that particular range. When $|\beta| > n^{-1}$ then denoting $M = \lfloor |\beta|^{-1} \rfloor$, we observe that

$$\sum_{m=1}^{M} m^{t/kl-1} e(\beta m) \ll |\beta|^{-t/kl}.$$

For the remaining range we combine partial summation and the monotonicity of $m^{t/kl-1}$ to obtain

$$\sum_{m>M}^{n} m^{t/kl-1} e(\beta m) \ll |\beta|^{-1}\big(|\beta|^{1-t/kl} + n^{t/kl-1}\big) = |\beta|^{-t/kl} + P^t |\beta|^{-1} n^{-1},$$

which delivers the required estimate. □

4.5 Treatment of the major arcs and proof of the main theorem

In this section we prune back to the narrower set $\mathfrak{P}$ of major arcs and deduce an asymptotic formula for the contribution of such set. In view of the weak bound obtained in Proposition 4.4.1 and the discussion made after it we are forced to introduce k-th powers of natural numbers and prime numbers, whose behaviour is much better understood, to provide strong enough estimates over $\mathfrak{M}$. For such purposes, it is convenient to present first some notation. Let

$$X_1 = 2^{-1}(2k)^{-1/(k-1)} X.$$

Consider the exponential sums

$$g(\alpha) = \sum_{X_1 < x \leq 2X_1} e(\alpha x^k), \qquad h(\alpha) = \sum_{p \leq X} e(\alpha p^k),$$

the weighted sums

$$v(\beta) = \sum_{X_1^k < x \leq (2X_1)^k} k^{-1} x^{1/k-1} e(\beta x), \quad w(\beta) = \sum_{2 \leq x \leq n} k^{-1} x^{1/k-1} (\log x)^{-1} e(\beta x),$$

and the functions

$$V(\alpha, q, a) = q^{-1} S_k(q, a) v(\beta) \qquad \text{and} \qquad W(\alpha, q, a) = \varphi(q)^{-1} W(q, a) w(\beta),$$

where $S_k(q, a)$ and $W(q, a)$ were defined in (4.3.1) and (4.3.7) respectively. For the sake of simplicity we further define the auxiliary functions

$$f^*(\alpha) = U(\alpha, q, a), \quad g^*(\alpha) = V(\alpha, q, a), \quad h^*(\alpha) = W(\alpha, q, a) \qquad (4.5.1)$$

when $\alpha \in \mathfrak{M}(a, q) \subset \mathfrak{M}$ and $f^*(\alpha) = g^*(\alpha) = h^*(\alpha) = 0$ for $\alpha \in \mathfrak{m}$. We recall for convenience that $U(\alpha, q, a)$ was defined just before (4.4.1). Before providing an asymptotic formula for the major arc contribution it is convenient to consider for any set $\mathfrak{B} \subset [0, 1)$ the integral

$$R_{\mathfrak{B}}(n) = \int_{\mathfrak{B}} f(\alpha)^s g(\alpha)^2 h(\alpha)^2 e(-\alpha n) d\alpha,$$

and to define the singular integral as

$$J(n) = \int_0^1 u(\beta)^s v(\beta)^2 w(\beta)^2 e(-\beta n) d\beta.$$

Here the reader might want to observe that as a consequence of orthogonality then $J(n)$ equals

$$\sum_{x_1, \ldots, x_4, y_1, \ldots, y_s} k^{-4-s} (x_1 x_2 x_3 x_4)^{1/k-1} \big(\log x_3 \log x_4 \big)^{-1} y_1^{t/kl-1} \cdots y_s^{t/kl-1},$$

where the sum is over $x_1, \ldots, x_4, y_1, \ldots, y_s$ satisfying

$$x_1 + \ldots + x_4 + y_1 + \ldots + y_s = n$$

with

$$X_1^k < x_1, x_2 \leq (2X_1)^k, \qquad 2 \leq x_3, x_4 \leq n \qquad \text{and} \qquad 1 \leq y_j \leq n \quad (1 \leq j \leq s).$$

It is worth observing that then one obtains the lower bound

$$J(n) \gg P^{st} X^4 n^{-1} (\log n)^{-2}. \qquad (4.5.2)$$

Proposition 4.5.1. *Let $s \geq \max(1, k-2)$. One has that*

$$R_{\mathfrak{M}}(n) = \mathfrak{S}(n)J(n) + O\big(P^{st}X^4 n^{-1}(\log n)^{-2-\delta}\big). \tag{4.5.3}$$

Moreover, if s satisfies the hypothesis of Lemma 4.3.1 then

$$R_{\mathfrak{M}}(n) \gg P^{st}X^4 n^{-1}(\log n)^{-2}.$$

Proof. Observe that Lemma 6.1 of Vaughan [141] for the choice of X_1 made yields that whenever $\alpha \in \mathfrak{M}$ then

$$g(\alpha) - g^*(\alpha) \ll q^{1/2+\varepsilon}. \tag{4.5.4}$$

Likewise, Lemma 6.2 of Vaughan [141] delivers the bound

$$v(\beta) \ll X(1 + n|\beta|)^{-1},$$

whence combining such estimate with (4.3.4) we get

$$g^*(\alpha) \ll w_k(q)X(1 + n|\beta|)^{-1}. \tag{4.5.5}$$

It is also worth noting that for any $q \leq X$, the number $N(q)$ of pairs of primes (p_1, p_2) with $p_1^k \equiv p_2^k \pmod q$ and $p_1, p_2 \leq X$ satisfies

$$N(q) \ll X^2(\log X)^{-2}q^{-1+\varepsilon}.$$

Consequently, by orthogonality we find that

$$\sum_{a=1}^{q} |h(a/q + \beta)|^2 \ll X^2(\log X)^{-2}q^{\varepsilon}. \tag{4.5.6}$$

Combining the previous discussion with Lemma 4.2.1 we obtain

$$R_{\mathfrak{M}\setminus\mathfrak{N}}(n) \ll P^{st-2^{-lk}+\varepsilon}X^4 n^{-1}\big(1 + \sum_{q\leq X} w_k(q)^2\big) \ll P^{st-\delta}X^4 n^{-1},$$

where in the last step we applied (4.3.6). The reader might want to observe that the bounds available for the exponential sums of k-th powers are robust enough to enable us to prune back to a set of narrower major arcs. It becomes transparent that in view of the weak estimates for $f(\alpha)$ available whenever $\alpha \in \mathfrak{M} \setminus \mathfrak{N}$, the use of such Weyl sums in this setting seems inevitable. Before moving on, it is convenient to observe that whenever $\alpha \in \mathfrak{N}$ then equations (4.3.5), (4.4.1) and Proposition 4.4.1 deliver $f(\alpha) \ll w_k(q)P^t$. Likewise, observe that (4.5.4) and (4.5.5) yield the estimate $g(\alpha) \ll w_k(q)X(1+n|\beta|)^{-1}$ for the same range. Consequently, combining the previous discussion with (4.3.6) and (4.5.6) we obtain

$$\begin{aligned} R_{\mathfrak{N}\setminus\mathfrak{P}}(n) &\ll P^{st}X^4n^{-1}(\log P)^{-2}\Big((\log P)^{-1+\varepsilon}\sum_{q\leq \log P} qw_k(q)^{s+2} + \sum_{q>\log P} w_k(q)^{s+2}\Big) \\ &\ll P^{st}X^4n^{-1}(\log P)^{-2-\delta}. \end{aligned}$$

In order to make further progress in the analysis, we note that

$$h(\alpha) = W(\alpha, q, a) + O(Xe^{-C_1\sqrt{\log X}})$$

for some $C_1 > 0$, which is an immediate consequence of Lemma 7.15 of Hua [73]. Observe as well that Proposition 4.4.1 delivers

$$f(\alpha)^s - f^*(\alpha)^s \ll P^{st-1+\varepsilon}$$

whenever $\alpha \in \mathfrak{P}$. Combining these estimates with (4.5.4) we find that

$$\int_{\mathfrak{P}} \big|f(\alpha)^s g(\alpha)^2 h(\alpha)^2 - f^*(\alpha)^s g^*(\alpha)^2 h^*(\alpha)^2\big| d\alpha \ll P^{st}X^4n^{-1}e^{-C\sqrt{\log X}}.$$

Observe as well that (4.3.12) and the estimate for $v(\beta)$ stated before (4.5.5) deliver the bounds

$$\sum_{q>Q} |S_n(q)| \ll Q^{\varepsilon-1/k}, \qquad \int_{|\beta|>(\log P)q^{-1}n^{-1}} |v(\beta)|^2 d\beta \ll X^2n^{-1}q(\log P)^{-1}$$

for any $Q > 0$ and $q \leq \log P$ respectively. Consequently, the above estimates and a change of variables yield

$$\int_{\mathfrak{P}} f^*(\alpha)^s g^*(\alpha)^2 h^*(\alpha)^2 e(-\alpha n) d\alpha = \mathfrak{S}(n)J(n) + O\big(P^{st}X^4n^{-1}(\log P)^{-2-\delta}\big),$$

whence the preceding discussion and the pruning bounds for $R_{\mathfrak{M}\setminus\mathfrak{N}}(n)$ and

$R_{\mathfrak{N}\setminus\mathfrak{P}}(n)$ deliver the main result of the proposition. The second part of the proposition follows combining (4.5.2) with (4.5.3) and Proposition 4.3.1. $\square$

We now gather all the work done previously to prove Theorem 4.1.1 by using the following quantitative version.

Proposition 4.5.2. *Let $s \geq 4k-3$ and $H = k(k+1)$. Then, one has the lower bound*

$$\int_0^1 \mathcal{F}(\alpha)^H f(\alpha)^s g(\alpha)^2 h(\alpha)^2 e(-\alpha n) d\alpha \gg |\mathcal{S}_1|^H |\mathcal{S}_2|^H P^{st} X^4 n^{-1} (\log n)^{-2}.$$

Proof. It is worth observing that the estimate over the minor arcs on Proposition 4.2.2 and the trivial bounds for $f(\alpha)$, $g(\alpha)$ and $h(\alpha)$ yield

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^H |f(\alpha)|^s |g(\alpha)|^2 |h(\alpha)|^2 d\alpha \ll |\mathcal{S}_1|^H |\mathcal{S}_2|^H P^{st} X^4 n^{-1-\delta} \tag{4.5.7}$$

for some $\delta > 0$. In order to compute the major arc contribution it is convenient to define for each $m \in \mathbb{N}$ the counting function

$$Q(m) = \Big| \Big\{ (\mathbf{x}, \mathbf{y}) \in \mathcal{S}_1^H \times \mathcal{S}_2^H : \quad m = \sum_{i=1}^{H} (y_i + z_i)^k \Big\} \Big|.$$

Observe that with the previous notation one finds that

$$\mathcal{F}(\alpha)^H = \sum_m Q(m) e(\alpha m).$$

Moreover, using (4.2.6), (4.2.7) and the definitions of $\mathcal{S}_1$ and $\mathcal{S}_2$ described after those equations we have $Q(m) = 0$ for $m > n/2$. Therefore, Proposition 4.5.1 yields

$$\begin{aligned} \int_{\mathfrak{M}} \mathcal{F}(\alpha)^H f(\alpha)^s g(\alpha)^2 h(\alpha)^2 e(-\alpha n) d\alpha &= \sum_{m \leq n/2} Q(m) R_{\mathfrak{M}}(n-m) \\ &\gg |\mathcal{S}_1|^H |\mathcal{S}_2|^H P^{st} X^4 n^{-1} (\log n)^{-2}. \end{aligned} \tag{4.5.8}$$

The combination of the equations (4.5.7) and (4.5.8) concludes the proof. Here the reader might want to observe that the choices for C_1 and C_2 guarantee that we get the expected lower bound. $\square$

Proof of Theorem 4.1.1. Note that the integral in Proposition 4.5.2 counts the number of solutions of equation (4.1.4) with certain multiplicities. Consequently, for all $r \geq 4$ we have

$$S_C(k, l, r) \leq k(k+1) + 4k - 3,$$

which delivers the same bound for $P_C(k, r)$ and yields $R_C(k) \leq 4$. As experts will realise, we could have avoided including the extra $4k-3$ copies of $f(\alpha)$ by introducing suitable weights for each of x and m in the definition of $\mathcal{F}(\alpha)$ to exploit the information given by such variables in the analysis of the singular series. However, we have prioritised the simplicity of the exposition over the preciseness of the upper bound for $P_C(k, r)$.

4.6 The case $k = 2$

We briefly sketch the proof for $R_4(2) \leq 2$. For the rest of the exposition then we take $t = 4l$. Let

$$g(\alpha) = \sum_{X/2 < x \leq X} e(\alpha x^2),$$

and on recalling (4.2.3) consider the mean value

$$\int_0^1 |g(\alpha)|^2 |f(\alpha, \mathcal{S}_t)|^4 d\alpha,$$

which by orthogonality counts the solutions to the equation

$$x_1^2 + y_1^2 + y_2^2 = x_2^2 + y_3^2 + y_4^2$$

with $X/2 \leq x_i \leq X$ and $y_i \in \mathcal{S}_t$. Observe that by a divisor function argument, the number of solutions of

$$y_1^2 + y_2^2 = y_3^2 + y_4^2$$

is $O(n^{\varepsilon}|\mathcal{S}_t|^2)$, and hence the contribution of the subset of solutions satisfying $x_1 = x_2$ is $O(n^{1/2+\varepsilon}|\mathcal{S}_t|^2)$. Likewise, the number of solutions of the above equation with $x_1 \neq x_2$ is $O(n^{\varepsilon}|\mathcal{S}_t|^4)$, whence equation (4.2.2) and the above estimates deliver

$$\int_0^1 |g(\alpha)|^2|f(\alpha,\mathcal{S}_t)|^4 d\alpha \ll n^{1/2+\varepsilon}|\mathcal{S}_t|^2 + n^{\varepsilon}|\mathcal{S}_t|^4 \ll n^{\varepsilon}|\mathcal{S}_t|^4.$$

Define $\mathfrak{M}_\tau = \mathfrak{M}(P^\tau)$ and $\mathfrak{m}_\tau = [0,1)\setminus\mathfrak{M}_\tau$ for some small enough $\tau > 0$. Then, combining the above estimate with Lemma 4.2.1 one gets

$$\int_{\mathfrak{m}_\tau} |f(\alpha,\mathcal{S}_t)|^4|g(\alpha)|^2|f(\alpha)|^3 d\alpha \ll |\mathcal{S}_t|^4 P^{3t-\delta}.$$

The reader might want to observe that in order to ensure local solubility and the convergence of the singular series, one should take 3 copies of $f(\alpha)$ instead of just 2 since we only have two Weyl sums of degree 2 available. The rest of the analysis of the major arcs is done using the estimates obtained throughout the memoir. This argument then yields $P_4(2,2) \leq 7$. As experts will realise, one could prove the bound $P_4(2,2) \leq 5$ by introducing suitable weights in the definition of $f(\alpha,\mathcal{S}_t)$ to simplify the singular series analysis and just use one copy of $f(\alpha)$. We have avoided discussing such refinement here for the sake of brevity.

4.7 Proof of Theorem 4.1.2

We combine the work of previous sections with slightly different ideas employed in the minor arc analysis to give a proof of Theorem 4.1.2. We first introduce an exponential sum restricted to elements in a convenient set that will provide the saving needed for the minor arc contribution. On recalling the parameter $\xi_0(k,l)$ defined before Theorem 4.1.2, we write $\xi = \xi_0(k,l)$ for ease of notation and consider

$$\xi_1 = \xi - 1, \qquad C_3 = \big(2^{k+1}k(k+1)\big)^{-1/kl}\xi_1^{-1/l}.$$

Recalling (4.2.1) as well, take $P_3 = C_3 P$ and set $\mathcal{S} = \mathcal{S}_{\xi_1}(P_3)$. It is then convenient to consider for $P/2 \leq p \leq P$ prime the exponential sums

$$f_p(\alpha) = \sum_{x\in\mathcal{S}} e\big(\alpha(x+p^l)^k\big) \qquad \text{and} \qquad \mathcal{G}(\alpha) = \sum_{P/2<p\leq P} f_p(\alpha).$$

Proposition 4.7.1. *Let $\alpha \in [0,1)$ and let $M > 0$ be a parameter with $M \leq P$. Denote by $\mathfrak{m}_M$ the set of α with the property that $\alpha = \beta + a/q$ with $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a,q) = 1$ satisfying $|\beta| \leq (2kqX)^{-1}$, $q \leq 2kX$ and such that whenever $q \leq M$ one has $|\beta| \geq Mq^{-1}n^{-1}$. Then for each $\alpha \in \mathfrak{m}_M$ one gets*

$$\mathcal{G}(\alpha) \ll |\mathcal{S}| P^{1+\varepsilon} M^{-1/k(k-1)} X^{\delta_{\xi_1}/2},$$

where δ_{ξ_1} was defined in (4.1.2). Moreover, for $s \geq k(k+1)/2$ we obtain the mean value

$$\int_{\mathfrak{m}_M} |\mathcal{G}(\alpha)|^{2s} d\alpha \ll |\mathcal{S}|^{2s} P^{2s} M^{-1} X^{-k+\Delta_{\xi_1}+\varepsilon},$$

where $\Delta_{\xi_1} = \delta_{\xi_1} k(k+1)/2$.

Proof. Recalling the notation used in the proof of Proposition 4.2.2 we find that

$$\sum_{P/2<p\leq P} |f_p(\alpha)|^{2s} = \sum_{P/2<p\leq P} \Big| \sum_{\boldsymbol{n}\in\mathcal{N}} a(\boldsymbol{n}) e\big(\boldsymbol{n}\cdot\gamma(p^l)\big)\Big|^2,$$

where $\mathcal{N}$ here denotes the set

$$\mathcal{X}(\boldsymbol{n}) = \Big\{\mathbf{x} \in \mathcal{S}^s : \ x_1^i + \ldots + x_s^i = n_i \ (1 \leq i \leq k-1)\Big\}$$

and the coefficient $a(\boldsymbol{n})$ is defined in the same way as in (4.2.11). We introduce for further convenience the number $q_1 = q(q,k)^{-1}$. Before going into the discussion for the spacing modulo 1 of $\{\gamma(p^l)\}_p$, the reader might find useful to observe that for fixed $h \in \mathbb{Z}$ with $(h, q_1) = 1$, the number of solutions L of the congruence

$$x^l \equiv h \ (\mathrm{mod}\ q_1)$$

satisfies $L \ll q_1^{\varepsilon}$. Therefore, we can partition the primes into L classes $\mathcal{P}_j$ such that for any pair of distinct primes $p_1, p_2 \in \mathcal{P}_j$ with $p_1^l \equiv p_2^l \ (\mathrm{mod}\ q_1)$ then $p_1 \equiv p_2 \ (\mathrm{mod}\ q_1)$.

Next observe that by the choice of $\gamma(p^l)$ made in (4.2.9) we find that

$$\|k\alpha(p_1^l - p_2^l)\| = \|\gamma_{k-1}(p_1^l) - \gamma_{k-1}(p_2^l)\|.$$

Then using the hypothesis on $|\beta|$ described above we obtain

$$\|k\alpha(p_1^l - p_2^l)\| \geq \|ka(p_1^l - p_2^l)/q\| - \frac{1}{2}q^{-1} \geq \frac{1}{2}q^{-1}$$

provided that $p_1 \not\equiv p_2 \pmod{q_1}$.

When $q_1 > P$ one cannot have pairs of distinct primes $p_1, p_2 \in \mathcal{P}_j$ with the property $p_1 \equiv p_2 \pmod{q_1}$, whence we always have

$$\|\gamma_{k-1}(p_1^l) - \gamma_{k-1}(p_2^l)\| \gg X^{-1}. \tag{4.7.1}$$

Whenever $M/k < q_1 \le P$ then we partition each of $\mathcal{P}_j$ into L_j classes $\mathcal{P}_{j,i}$ with the property that no pair of distinct primes belonging to $\mathcal{P}_{j,i}$ are congruent modulo q_1 and with L_j satisfying the bound $L_j \ll Pq_1^{-1}$. Consequently, the same argument leads to the estimate (4.7.1) for distinct $p_1, p_2 \in \mathcal{P}_{j,i}$. Finally, when $q_1 \le M/k$ one has $q \le M$, whence whenever $p_1 \equiv p_2 \pmod{q_1}$ then the condition on β described in the proposition yields

$$\|\gamma_{k-1}(p_1^l) - \gamma_{k-1}(p_2^l)\| = \|k\alpha(p_1^l - p_2^l)\| = |\beta||k(p_1^l - p_2^l)| \gg P^{l-1}Mn^{-1}.$$

We combine Lemma 5.3 of Vaughan [141] and the above discussion to obtain the upper bound

$$\sum_{\substack{P/2<p\le P\\ p\in\mathcal{P}_j}} |f_p(\alpha)|^{2s} \ll X^{k(k-1)/2}P(q+M)^{-1}\sum_{\boldsymbol{n}\in\mathcal{N}}|a(\boldsymbol{n})|^2 \tag{4.7.2}$$

for q in any of the ranges described above. Bounding the coefficients $a(\boldsymbol{n})$ trivially one gets

$$\sum_{\substack{P/2<p\le P\\ p\in\mathcal{P}_j}} |f_p(\alpha)|^{2s} \ll X^{k(k-1)/2}P(q+M)^{-1}J_{s,\xi_1}^{(k-1)}(P_3),$$

where $J_{s,\xi_1}^{(k-1)}(P_3)$ was defined in (4.2.4). Then combining the above equation with an application of Cauchy's inequality we get

$$|\mathcal{G}(\alpha)|^{2s} \ll P^{2s-1+\varepsilon}\sum_j\sum_{\substack{P/2<p\le P\\ p\in\mathcal{P}_j}} |f_p(\alpha)|^{2s} \ll P^{2s+\varepsilon}X^{k(k-1)/2}M^{-1}J_{s,\xi_1}^{(k-1)}(P_3),$$

and hence for the choice $s = k(k-1)/2$ then Proposition 4.2.1 delivers

$$\mathcal{G}(\alpha) \ll |\mathcal{S}|P^{1+\varepsilon}M^{-1/k(k-1)}X^{\delta_{\xi_1}/2}.$$

For the second claim of the proposition we combine (4.7.2) and Cauchy's inequality in the same way as above and we integrate over $\mathfrak{m}_M$ to obtain

$$\int_{\mathfrak{m}_M} |\mathcal{G}(\alpha)|^{2s} d\alpha \ll P^{2s+\varepsilon} M^{-1} X^{k(k-1)/2} J_{s,\xi_1}^{(k)}(P_3).$$

An application of Proposition 4.2.1 to the above line then yields

$$\int_{\mathfrak{m}_M} |\mathcal{G}(\alpha)|^{2s} d\alpha \ll |\mathcal{S}|^{2s} P^{2s+\varepsilon} M^{-1} X^{-k+\Delta_{\xi_1}},$$

from where the second statement follows. □

In the rest of the section we deliver a lower bound for the major arc contribution. We will work with the auxiliary functions $f(\alpha)$, $S'_n(q)$, $u(\beta)$ and $f^*(\alpha)$, defined in (4.2.15), (4.3.15), (4.4.1) and (4.5.1) respectively but replacing ξ by t whenever such parameters appear in any of the definitions. We have avoided making such distinction in the notation explicit for the sake of simplicity. For future purposes we define the singular integral as

$$J'_s(n) = \int_{-1/2}^{1/2} u(\beta)^s e(-\beta n) d\beta.$$

Lemma 4.7.1. *Suppose that $s \geq 2$. Then,*

$$J'_s(n) = n^{s\xi/kl-1}\Big(k^{-s}\Gamma(\xi/kl)^s\Gamma(s\xi/kl)^{-1} + O\big(B(n)\big)\Big),$$

where $B(n) = n^{-1} + n^{-\xi/kl}$.

Proof. We will proceed by induction. We consider for convenience the function

$$\phi(\gamma) = \gamma^{\xi/kl-1}(n-\gamma)^{\xi/kl-1}.$$

When $s = 2$ then orthogonality yields

$$\begin{aligned} J'_2(n) &= k^{-2}\sum_{m=1}^{n-1}\phi(m) = k^{-2}\int_0^n \phi(\gamma)d\gamma + O\big(n^{2\xi/kl-1}B(n)\big) \\ &= k^{-2}\Gamma(\xi/kl)^2\Gamma(2\xi/kl)^{-1}n^{2\xi/kl-1} + O\big(n^{2\xi/kl-1}B(n)\big), \end{aligned}$$

where we used the fact that $\phi(\gamma)$ has at most one stationary point on the interval $(0,n)$. By using the inductive hypothesis we obtain

$$\begin{aligned} J'_{s+1}(n) =& k^{-1}\sum_{m=1}^{n-1} m^{\xi/kl-1}J'_s(n-m) \\ &= k^{-s-1}\Gamma(\xi/lk)^s\Gamma(s\xi/kl)^{-1}\sum_{m=1}^{n-1} m^{\xi/kl-1}(n-m)^{s\xi/kl-1} \\ &\quad + O\big(n^{(s+1)\xi/kl-1}B(n)\big). \end{aligned}$$

Applying the same argument we used for the case $s=2$ we find that

$$\sum_{m=1}^{n-1} m^{\xi/kl-1}(n-m)^{s\xi/kl-1} = n^{(s+1)\xi/kl-1}\big(\lambda_s + O\big(B(n)\big)\big),$$

where $\lambda_s = \Gamma(s\xi/kl)\Gamma(\xi/kl)\Gamma\big((s+1)\xi/kl\big)^{-1}$, whence combining the above equations we obtain the desired result. □

In order to make further progress it is convenient to consider the set of major arcs $\mathfrak{N}_\iota = \mathfrak{M}(P^{1/2+\iota})$, where $\mathfrak{M}$ was defined in (4.2.8) and where we take $\iota = 1/1000$. Likewise, we define the minor arcs $\mathfrak{n}_\iota = [0,1)\setminus\mathfrak{N}_\iota$. Note that using equation (4.3.5) and Proposition 4.4.1 we obtain for $\alpha\in\mathfrak{N}_\iota$ the bound

$$f(\alpha)^s - f^*(\alpha)^s \ll P^{s\xi-s/2+s\iota} + P^{\xi-1/2+\iota}w_k(q)^{s-1}u(\beta)^{s-1}.$$

Consequently, whenever $s\geq k+2$ then Lemma 4.4.1 gives

$$\begin{aligned} \int_{\mathfrak{N}_\iota}|f(\alpha)^s - f^*(\alpha)^s|d\alpha &\ll P^{s\xi-s/2+(s+2)\iota+1}n^{-1} + P^{s\xi-1/2+\iota}n^{-1}\sum_{q\leq P^{1/2+\iota}} qw_k(q)^{s-1} \\ &\ll P^{s\xi-\delta}n^{-1}, \end{aligned}$$

where in the last step we used (4.3.6). Observe as well that (4.3.18) and Lemma 4.4.1 deliver the bounds

$$\sum_{q>Q}|S'_n(q)| \ll Q^{\varepsilon-1/k}, \qquad \int_{|\beta|>P^{1/2+\iota}q^{-1}n^{-1}}|u(\beta)|^s d\beta \ll P^{s\xi}n^{-1}q^{\delta}P^{-\delta(1/2+\iota)}$$

whenever $s\geq\max(5,k+2)$ for any $Q>0$ and $q\leq P^{1/2+\iota}$ respectively.

Therefore, Lemma 4.7.1, the above estimates and a change of variables give

$$\int_{\mathfrak{N}_\iota} f(\alpha)^s e(-\alpha n)d\alpha = C_{k,l,\xi} n^{s\xi/kl-1}\mathfrak{S}'(n) + O\big(n^{s\xi/kl-1-\delta}\big), \tag{4.7.3}$$

where $C_{k,l,\xi} = k^{-s}c_{\xi,l}^s\Gamma(\xi/lk)^s\Gamma(s\xi/kl)^{-1}$ and $c_{\xi,l}$ was defined in (4.4.1). It seems worth observing that when $s \geq 4k$ then Proposition 4.3.2 yields the lower bound $\mathfrak{S}'(n) \gg 1$. Likewise, Proposition 4.7.1 delivers

$$\begin{aligned}\int_{\mathfrak{n}_\iota} |\mathcal{G}(\alpha)|^{k(k+1)}d\alpha &\ll |\mathcal{S}|^{k(k+1)}P^{k(k+1)-1/2-\iota}X^{-k+\Delta_{\xi_1}+\varepsilon}\\ &\ll |\mathcal{S}|^{k(k+1)}P^{k(k+1)}X^{-k-\delta},\end{aligned}$$

whence using (4.7.3) and the ideas of the proof of Proposition 4.5.2 to derive a lower bound for the major arc contribution and combining such bound with the above minor arc estimate we obtain

$$\int_0^1 \mathcal{G}(\alpha)^{k(k+1)}f(\alpha)^s e(-\alpha n)d\alpha \gg |\mathcal{S}|^{k(k+1)}P^{k(k+1)+s\xi}(\log P)^{-k(k+1)}n^{-1},$$

which concludes the proof of Theorem 4.1.2.

Chapter 5

Mixed third moments of the Riemann zeta function

5.1 Introduction

Investigations concerning the asymptotic evaluation of moments of L-functions date back to the early work of Hardy-Littlewood (1918) establishing the second moment of the Riemann zeta function (see, for instance Titchmarsh [134, Theorem 7.3]), which was followed by the proof of an asymptotic formula for the fourth moment due to Ingham [80]. In view of the above historical note and for gaining preciseness in the upcoming discussion it seems worth defining, for $k \in \mathbb{N}$ the moment

$$M_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt.$$

Despite the efforts invested in the investigations pertaining to higher order moments to the end of establishing analogous evaluations, the extensive work done in this direction has been thus far conjectural, the articles earlier mentioned still comprising the only unconditionally proven formulae. Such examinations were initiatied by Conrey-Gosh [26] in their paper concerning the asymptotic evaluation of $M_3(T)$, followed by a memoir of Conrey-Gonek [27] accomplishing an analogous conjectural formula for the eighth moment, and were independently culminated with the incorporation of Random Matrix theory to the scene, which ultimately led to the asymptotic relation

$$M_k(T) \sim c_k T (\log T)^{k^2}$$

for precise explicit constants c_k by Keating-Snaith [89], such an especulation having been further refined in the work of Conrey et al. [25].

Little attention has been paid hirtherto to the problem of introducing fixed coefficients in the imaginary parts of the zeta factors comprising the moments, and we anticipate that such an endeavour will be the main object of study in the rest of the thesis. For the sake of transparency it seems desirable to present for positive real numbers $a, b, c > 0$ the mixed moment

$$I_{a,b,c}(T) = \int_0^T \zeta(1/2 + ait)\zeta(1/2 - bit)\zeta(1/2 - cit)dt. \tag{5.1.1}$$

Investigating the above is partially motivated by the desire to examine a broader set of examples in search of similar, or perhaps dissimilar but nonetheless interesting phenomena to that occuring in a recent paper of Conrey-Keating (see [28]) in connection with the arithmetic stratification of subvarieties examined by Manin [44]. We have deferred the fourth mixed moment analysis to a later occasion, it entailing some extra complications that have not been surmounted by the author hirtherto.

We shall primarily focus on the integer coefficient case, albeit a result concerning irrational coefficients will be presented. In the former case, it is worth anticipating that the error terms obtained in the main theorem may be substantially sharpened subjected to the validity of a weaker version of the abc conjecture that we shall now describe.

Conjecture 2. *Let $a, b, c \in \mathbb{N}$ be fixed natural numbers and $n_1, n_2, n_3 \in \mathbb{N}$. Denote*

$$D = n_1^a - n_2^b n_3^c.$$

Then, if $D \neq 0$ one has the lower bound

$$D \gg n_1^a (n_1 n_2 n_3)^{-1-\varepsilon},$$

where the implicit constant is universal and does not depend on the parameters a, b, c.

The reader shall rest assured that a statement of the abc conjecture in conjunction with a succint proof showing how to derive Conjecture 2 from the aforementioned conjecture will comprise a short separate section in this chapter. Equipped with the above considerations, we have reached a position from which to announce the main result of the present chapter.

Theorem 5.1.1. *Let $T>0$ and $a,b,c\in\mathbb{N}$ with the property that $(a,b,c)=1$. Then, whenever $a<c\leq b$ one has the asymptotic evaluation*

$$I_{a,b,c}(T)=\sigma_{a,b,c}T+E_{a,b,c}(T),$$

where $\sigma_{a,b,c}>1$ is a computable constant and

$$E_{a,b,c}(T)\ll T^{1-1/2a+1/2c}+T^{3/4+a/4c}.$$

If Conjecture 2 holds, the above error term may be refined to

$$\begin{aligned}E_{a,b,c}(T)\ll &T^{1/2+a/(a+c)+\varepsilon}+T^{5/4-c/4a}\\ &+(\log T)^2\big(T^{3/4}+T^{3/4-a/4c+(a,c)/2c}+T^{3/4-a/4b+(a,b)/2b}\big).\end{aligned}\tag{5.1.2}$$

Describing the prolix history of upper and lower bounds of moments is hardly the point of this introduction. Nonetheless, it has been thought pertinent to devote a few lines to such an endeavour for the sake of illustrating the exposition. Lower bounds of the shape $M_k(T)\gg T(\log T)^{k^2}$ were already established for integral k by Ramachandra [117], such a result being extended to positive rational numbers in the work of Heath-Brown [60] and accomplished for real $k\geq 0$ under the assumption of RH by Ramachandra [116]. The history of upper bounds is somewhat more recent, the best unconditional result improving earlier work of many others accomplishing sharp estimates of the shape $M_k(T)\ll T(\log T)^{k^2}$ for the range $0\leq k\leq 2$ by Heap-Radziwill-Soundararajan [57], whence in particular the preceding discussion yields $M_{3/2}(T)\asymp T(\log T)^{9/4}$ unconditionally.

The reader might observe that in view of the above considerations it transpires that under the above assumptions on the integer coefficients in Theorem 5.1.1 there is no power of $\log T$ factor in the main term and some cancelation is exhibited. It is apparent that the instance when the coefficients at hand are rational numbers may be easily reduced to the context of the above theorem via a simple change of variables. It is also widely known in the community that the functions $\zeta(1/2+ait)$, $\zeta(1/2+ibt)$ and $\zeta(1/2+ict)$ with $a,b,c\in\mathbb{R}$ having the property that a,b,c are linearly independent over $\mathbb{Q}$ are poorly correlated and barely see each other. Such a consideration may then lend credibility to the belief that the integral (5.1.1) should exhibit substantially more cancellation in the case when the corresponding coefficients satisfy the preceding condition. Confirming and quantifying this belief is, inter alia, the purpose of the upcoming theorem.

Theorem 5.1.2. *Let $T > 0$ and $a, b, c \in \mathbb{R}$ be algebraic numbers linearly independent over $\mathbb{Q}$. Then one has*

$$I_{a,b,c}(T) \sim T.$$

Considerations of space and time preclude us from providing an effective bound of the error term underlying the preceding formula or exploring the above problem whenever the coefficients are either linearly dependent but in irrational ratio or trascendental numbers, but we nonetheless announce our intention to return to these topics on a future occasion.

The approach which will be employed on this first chapter to prove Theorem 5.1.1 shall make use of the so-called approximate functional equation of the Riemman zeta function for each of the factors involved in the integral at hand. As experts may anticipate, such a procedure leaves one in the concomitant situation of having to analyse eight integrals of twisted Dirichlet polynomials, seven of which contain a twisting factor. In the investigation of the remaining one, one then makes a distinction between the diagonal and the off-diagonal contribution arising from such a term. It should be noted that the examination of the diagonal one pertaining to the rational case shall not present any major obstacles and will be of a more elementary number theoretic flavour.

On the contrary, the off-diagonal one will be somehow more problematic. As experts may notice, having a decent control of such a contribution amounts to understanding the number of solutions to the equation

$$n_2^b n_3^c - n_1^a = D \tag{5.1.3}$$

for a fixed (and possibly large) integer $D \in \mathbb{Z}$, such an endeavour being quite hard as shall be elucidated promptly. We should note first that as was previously observed in Conjecture 2, the assumption of the *abc* conjecture delivers the lower bound

$$D \gg n_1^{a-1-\varepsilon}(n_2 n_3)^{-1}.$$

Such a robust estimate already improves the unconditional error terms obtained herein, but does not suffice to enlarge the range of the parameters a, b, c for which one may assure the validity of the corresponding asymptotic formula with the set of tools employed in the present memoir.

As may become apparent shortly, it seems pertinent to draw the reader's attention to the work of scholars concerning gaps between perfect powers. We should mention that in a series of papers, Sprindzuk [129], [130] on improving upon work of Baker [2], showed that whenever $n, m \in \mathbb{N}$ are fixed then one has

$$|x^n - y^m| \gg (\log X)^{\delta(m,n)}, \quad \text{where } X = \max(x^n, y^m)$$

for some fixed $\delta(m, n) > 0$. The above inequality then lends credibility to the expectation that one may utilise the same circle of ideas involved in the proof of such theorems to deduce analogous unconditional estimates for the difference D in (5.1.3), but probably not better ones, such discussion being hardly the purpose of this memoir. In view of the previous comments, it transpires that the conditional bound (5.1.3) seems completely out of reach and illustrates the difficulty of the problem at hand. It should be noted though that even an improvement of the shape n_1^τ for some small $\tau > 0$ would hardly have an impact in the corresponding error term. As noted above, it would certainly not enlarge the range of the parameters a, b, c for which one can assure the validity of the asymptotic formula at hand.

We will surmount the difficulties associated with the analysis of the off-diagonal term by employing a trick, only valid for the range of parameters $a < c \leq b$ considered herein, at the cost of unconditionally obtaining a relatively weak error term which may be refined when one further assumes Conjecture 2.

The estimation process for rest of the integrals containing twisted factors will amount to the delivery of bounds for oscillatory integrals of sufficient strength. Those will present various different levels of complexity accordingly, most of them essentially requiring a straighforward application of a basic lemma only depending on the derivative of the argument of the integrand at hand (see Titchmarsh [134, Lemma 4.2]). It may be worth noting that there is a term whose analysis is slightly more elaborate and entails combining both the aforementioned lemma and an analogous one depending instead on the second derivative of the function at hand (see Titchmarsh [134, Lemma 4.4]).

Likewise, there is an additional integral whose examination further requires employing a stationary phase method type lemma of the strength of that of Graham and Kolesnik [46, Lemma 3.4] for the purpose of exploiting the extra cancellation stemming from some averaging process of the corresponding main term when assuming Conjecture 2, which would have otherwise been missed had we just applied the sequel of basic lemmata mentioned above.

A tedious intrincate analysis might have led one to obtain an asymptotic formula for the diagonal contribution comprising lower order terms of the shape of those at the end of (5.1.2) instead, such an approach hardly being the point herein in view of the error terms stemming from the off-diagonal analysis. It is nonetheless worth observing how the approach taken in this chapter ultimately delivers formulas encapturing divisibility relations between the coefficients a, b, c.

The proof of Theorem 5.1.2 shares a large intersection with that of Theorem 5.1.1 and primarily differs from that of the latter in the simplicity of the diagonal contribution, it being a consequence of a succint application of Baker's theorem on linear forms in logarithms. It should be noted that the analysis pertaining to the off-diagonal contribution will be the essentially the same and will ultimately deliver weaker error terms primarily because of the absence of a spacing condition, as is the case in the integer setting. The examination of the rest of the integrals will therefore be the same save for the instance in which Conjecture 2 is further assumed in the integer case.

On a different note, we shall also include a theorem concerning the asymptotic evalution of the above integral whenever $a = c < b$. We should remark that under such circumstances there is a slightly different behaviour underlying the anticipated formula.

Theorem 5.1.3. *Let $a < b$ be positive integers satisfying $(a, b) = 1$. Then one has the asymptotic formula*

$$I_{a,b,a}(T) \sim \zeta\big((a+b)/2\big)T \log T.$$

As would have probably been anticipated by experts, the above formula has an extra factor of $\log T$, as opposed to the corresponding one cognate to Theorem 5.1.1. In view of the difference in nature of both formulas, it transpires that the set of techniques employed in the course of the proof of the latter theorem may not be applicable herein, the arguments utilised in this new setting having their reliance on a more recondite framework of ideas.

It is worth mentioning nonetheless that many of the technical tedious arguments underpinning the proof of both Theorem 5.1.1 and the previous one will share common ground, whence the bulk of work that has to be done for this new result shall be considerably reduced. The analysis of the diagonal term appertaining to the corresponding integral without twisting factor, though slightly different in nature, shall be analysed as is customary via a conventional parametrization of the underlying diophantine equation and shall not present any major obstacle. It is then the examination of the off-diagonal one which departs from the analysis in the previous setting and exhibits some novelty in its treatment, the estimates ultimately obtained having their reliance on an astute application of Roth's theorem in diophantine approximation in conjunction with an intrincate analysis.

As may be apparent at first glance, the ineffectiveness of the error term in the asymptotic formula at hand comes from the ineffectiveness in Roth's theorem. Nonetheless, we feel obliged to remark that the corresponding exponent of $2+\varepsilon$ in the alluded theorem shall play a crucial role in the argument to the extent that analogous versions containing effective information about the corresponding constant thereof at the cost of increasing the aforementioned exponent at hand shall not find success when applied herein. It may be worth announcing that the assumption of Conjecture 2 would not have strengthen our result, and find it noteworthy that Roth's theorem ultimately delivers the same conclusion than that obtained on the aforementioned assumption of the *abc*-conjecture in this context.

For the sake of improving the exposition of ideas it has been thought pertinent to divide our line of argumentation into several lemmata. We begin our journey by presenting some preliminary manoeuvres which shall be employed throughout the entire chapter. Shortly after we succintly show how one may easily derive Conjecture 2 from the *abc*-conjecture. It seems convenient to mention that in Section 5.4 we reduce the problem at hand to that of computing a sum of eight integrals of products of various twisted Dirichlet polynomials. As we anticipated earlier, each of the integrals may exhibit a different behaviour. For the purpose of describing our ideas in a rather succinct manner, it seems desirable to organise the computations by gathering each of the integrals that present similar behaviour into groups and analysing those in various different lemmata accordingly. To the end of not providing the definitions of such integrals all at once and avoiding killing the reader's patience, it has been thought preferable to define those right before stating each of the lemma concerning their analysis. Section 5.5 is then devoted to the analysis of the diagonal and off-diagonal contribution arising from the term contaning no twisting factor, both for the integer case and for the instance when the coefficients are linearly independent over $\mathbb{Q}$. In Section 5.6 we essentially apply the most basic lemma for bounding oscillatory integrals to four of the terms involved in the formula at hand. In contrast, the analysis of the integral performed in Section 5.7 departs from the preceding one in that the stationary phase method is instead utilised in conjunction with a more sophisticated process to obtain conditional stronger bounds, the unconditonal examination of such an integral only requiring the application of basic lemmata. The investigation of the corresponding integrals in Section 5.8 only has its reliance on the aforementioned basic lemmata, but nonetheless ends up being somewhat more intrincate and elaborated. The section at hand concludes with the proof of Theorems 5.1.1 and 5.1.2, and Theorem 5.1.3 is discussed and proved in Section 5.9.

5.2 Preliminary lemmata

As a prelude to the analysis of integrals of unimodular functions, it has been thought convenient to include a sequel of lemmata concerning the estimation of such integrals which shall be employed henceforth throughout the memoir, and these we now describe.

Lemma 5.2.1. *Let $F(x)$ be a real differentiable function with the property that $F'(x)$ is monotonic and either $F'(x) > m > 0$ or $F'(x) < -m < 0$ in the interval $[\alpha, \beta]$. Then one has*

$$\int_{\alpha}^{\beta} e^{iF(x)} dx \leq \frac{4}{m}.$$

Proof. See Titchmarsh [134, Lemma 4.2]. □

Lemma 5.2.2. *Let $F(x)$ and $G(x)$ be real differentiable functions with the property that $G(x)/F'(x)$ is monotonic and either $F'(x)/G(x) \geq m > 0$ or*

$F'(x)/G(x) \leq -m < 0$ in the interval $[\alpha, \beta]$. Then one has

$$\int_{\alpha}^{\beta} G(x)e^{iF(x)}dx \leq \frac{4}{m}.$$

Proof. See Titchmarsh [134, Lemma 4.3]. $\square$

Lemma 5.2.3. *Let $F(x)$ be a real and twice differentiable function with the property that either $F''(x) \geq r > 0$ or $F''(x) \leq -r < 0$ in the interval $[\alpha, \beta]$. Then one has*

$$\int_{\alpha}^{\beta} e^{iF(x)}dx \leq \frac{8}{\sqrt{r}}.$$

Proof. See Titchmarsh [134, Lemma 4.4]. $\square$

Lemma 5.2.4. *Let $F(x)$ be a real function satisfying the conditions of the previous lemma and let $G(x)$ be a differentiable function with the property that $G(x)/F'(x)$ is monotonic and $|G(x)| \leq M$. Then,*

$$\int_{\alpha}^{\beta} G(x)e^{iF(x)}dx \leq \frac{8M}{\sqrt{r}}.$$

Proof. See Titchmarsh [134, Lemma 4.5]. $\square$

Lemma 5.2.5. *Let $g(x)$ be a real function with four derivatives in the interval $[\alpha, \beta]$. Suppose that in addition $|g''(x)| \geq \lambda_2 > 0$ and that there exists some $x_0 \in [\alpha, \beta]$ with the property that $g'(x_0) = 0$. Finally, assume that there exist constants λ_3 and λ_4 for which*

$$|g^{(3)}(x)| \leq \lambda_3, \qquad |g^{(4)}(x)| \leq \lambda_4$$

with $x \in [\alpha, \beta]$. Then

$$\int_{\alpha}^{\beta} e^{ig(x)}dx = \sqrt{2\pi}\frac{e^{i\pi/4+ig(x_0)}}{|g''(x_0)|^{1/2}} + O(R_1 + R_2),$$

where

$$R_1 = \min\Big(\frac{1}{\lambda_2(x_0 - \alpha)}, \lambda_2^{-1/2}\Big) + \min\Big(\frac{1}{\lambda_2(\beta - x_0)}, \lambda_2^{-1/2}\Big)$$

and

$$R_2 = (\beta - \alpha)\lambda_4\lambda_2^{-2} + (\beta - \alpha)\lambda_3^2\lambda_2^{-3}.$$

Proof. See Graham and Kolesnik [46, Lemma 3.4]. □

It also seems appropiate to consider the function

$$\chi(s) = 2^{s-1}\pi^s \sec(s\pi/2)\Gamma(s)^{-1}, \tag{5.2.1}$$

which as the reader may notice plays a role in the functional equation of $\zeta(s)$. For the integrals containing a twisting factor, the following basic standard technical lemma concerning the asymptotic expansion of the function $\chi(s)$ will be required.

Lemma 5.2.6. *Let $s \in \mathbb{C}$. Then, whenever $-\pi + \delta < \arg(s) < \pi - \delta$ for some fixed $\delta > 0$ one has*

$$\chi(s) = \Big(\frac{2\pi}{s}\Big)^{s-1/2} \frac{e^s}{2\cos(s\pi/2)}\Big(1 + O(|s|^{-1})\Big). \tag{5.2.2}$$

Consequently, for $t > 0$ one further has

$$\chi(1/2 + it) = \Big(\frac{2\pi}{t}\Big)^{it} e^{it+i\pi/4}\Big(1 + O\Big(\frac{1}{t}\Big)\Big)$$

and

$$\chi(1/2 - it) = \Big(\frac{2\pi}{t}\Big)^{-it} e^{-it-i\pi/4}\Big(1 + O\Big(\frac{1}{t}\Big)\Big)$$

as $t \to \infty$.

Proof. We recall Stirling's formula (see [158, Chaps. 12 and 13]), which establishes the asymptotic relation

$$\log\Gamma(s) = (s - 1/2)\log s - s + \frac{1}{2}\log 2\pi + O(|s|^{-1})$$

for the range described at the beginning of the statement of the lemma. It therefore transpires that

$$\Gamma(s) = \Big(\frac{s}{e}\Big)^s \Big(\frac{2\pi}{s}\Big)^{1/2}\Big(1 + O(|s|^{-1})\Big),$$

which then yields (5.2.2). In order to compute the second formula it might be convenient to observe first that

$$(1/2 + it)^{-it} = e^{-it\log t + \pi t/2 - 1/2}\big(1 + O(t^{-1})\big),$$

and that
$$\frac{1}{\cos\big((1/2+it)\pi/2\big)} = 2e^{i\pi/4-t\pi/2}\big(1+O(t^{-1})\big).$$
Therefore, the above equations in conjunction with (5.2.2) for the choice $s = 1/2+it$ yields the second statement at hand. The third one is deduced in a similar way.

□

5.3 A weak version of the *abc*-conjecture

As was foreshadowed in the former introduction, giving a short account of why the classical *abc*-conjecture implies Conjecture 2 is the purpose of the present note. To this end, it seems worth stating first such a well-known conjecture.

Conjecture 3. *[abc conjecture] Given $\varepsilon > 0$ there exists a constant C_ε with the property that for every triple of pairwise coprime integers (a, b, c) satisfying $a + b = c$, one has that*

$$\max(|a|, |b|, |c|) \leq C_\varepsilon \Big(\prod_{p|(abc)} p\Big)^{1+\varepsilon}.$$

Lemma 5.3.1. *Let $a, b, c \in \mathbb{N}$ be fixed. If Conjecture 3 holds then one has the inequality*

$$|n_1^a - n_2^b n_3^c| \gg n_1^{a-1-\varepsilon} n_2^{-1} n_3^{-1}, \qquad n_1, n_2, n_3 \in \mathbb{N}$$

whenever the absolute value in the left side of the above equation is non-zero.

Proof. We write
$$D = n_1^a - n_2^b n_3^c$$
for convenience and take

$$\lambda = \gcd(D, n_1^a, n_2^b n_3^c).$$

Observe that then the triple

$$(N_1, N_2, N_3) = (n_1^a \lambda^{-1}, n_2^b n_3^c \lambda^{-1}, D\lambda^{-1})$$

comprises pairwise disjoint integers, whence an application of Conjecture 3 delivers the inequality

$$N_1 \leq \max(N_1, N_2, |N_3|) \ll \Big(\prod_{p|(N_1N_2N_3)} p \Big)^{1+\varepsilon}.$$

It may be worth observing that

$$\prod_{p|(N_1N_2N_3)} p \ll \prod_{p|(n_1^a n_2^b n_3^c D\lambda^{-1})} p = \prod_{p|(n_1n_2n_3D\lambda^{-1})} p \ll n_1n_2n_3|D|\lambda^{-1},$$

whence a combination of the above equations yields the desired conclusion.

□

5.4 Initial manoeuvres

As was outlined above, we will demonstrate how the problem at hand shall be reduced to that of computing integrals of products of twisted Dirichlet polynomials, but before accomplishing such an endeavour it is desirable to define first

$$D(s) = \sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^s}, \tag{5.4.1}$$

where s is a complex variable, and

$$P(t) = D(1/2 + it) + \chi(1/2 + it)D(1/2 - it) \tag{5.4.2}$$

for $t \in \mathbb{R}$, where $\chi(s)$ was defined in (5.2.1).

Lemma 5.4.1. *With the above notation, one has*

$$I_{a,b,c}(T) = \int_0^T P(at)P(-bt)P(-ct)dt + O\big(T^{3/4}(\log T)\big). \tag{5.4.3}$$

Proof. We begin by using first the approximate functional equation for the Riemann Zeta function (see Titchmarsh [134, (4.12.4)]), namely

$$\zeta(1/2 + it) = P(t) + O(t^{-1/4}).$$

For the sake of simplicity we further define for $n \in \mathbb{Z}$ the function

$$\zeta_n(t) = \zeta(1/2 + nit).$$

By using the above approximation formula one readily sees that

$$I_{a,b,c}(T) = \int_0^T P(at)P(-bt)P(-ct)dt + E(T),$$

where the error term $E(T)$ in the above line satisfies

$$E(T) \ll T^{1/4} + E_1(T) + E_2(T),$$

and the terms $E_1(T)$ and $E_2(T)$ are defined by the relations

$$E_1(T) = \int_1^T t^{-1/2}\big(|\zeta_a(t)| + |\zeta_{-b}(t)| + |\zeta_{-c}(t)|\big)dt$$

and

$$E_2(T) = \int_1^T t^{-1/4}\Big(|\zeta_a(t)||\zeta_{-b}(t)| + |\zeta_a(t)||\zeta_{-c}(t)| + |\zeta_{-b}(t)||\zeta_{-c}(t)|\Big)dt.$$

We use Cauchy's inequality in conjunction with the asymptotic formula for the second moment of the Riemman Zeta function (see, for instance Titchmarsh [134, Theorem 7.3]) to obtain

$$E_1(T) \ll (\log T)^{1/2}\Big(\int_0^T |\zeta(1/2 + it)|^2\Big)^{1/2} \ll T^{1/2}\log T.$$

Likewise, integration by parts combined with another application of Cauchy's inequality and the aforementioned formula delivers

$$\begin{aligned} E_2(T) &\ll 1 + T^{-1/4}\int_0^T |\zeta(1/2 + it)|^2 + \int_1^T t^{-5/4}\int_0^t |\zeta(1/2 + is)|^2 ds dt \\ &\ll T^{3/4}\log T + \int_1^T t^{-1/4}\log t dt \ll T^{3/4}(\log T). \end{aligned}$$

The combination of the above estimates yields the desired result. □

In view of the above lemma it transpires that establishing the asymptotic evaluation which we seek to deliver amounts to compute eight integrals of products of twisted Dirichlet polynomials. To the end of not providing the definitions of such integrals all at once it has been thought preferable to define those right before stating each of the lemma concerning their analysis. Nonetheless, we content ourselves by mentioning that we shall write the integral in (5.4.3) as a sum of six terms, and thus obtain

$$I_{a,b,c}(T)=\sum_{j=1}^{6} I_j(T)+O\big(T^{3/4}(\log T)\big). \tag{5.4.4}$$

We also find it desirable anticipating that both $I_2(T)$ and $I_6(T)$ will be the sum of two integrals both of them symmetric on b and c, the analysis of which shall not utilise the fact that $b\geq c$. As will become apparent shortly, those terms will depend on a,b,c, but we won't make such a dependence explicit on the notation for the sake of concision.

5.5 Diagonal and off-diagonal contribution of the non-twisted term

As the heading suggests, we devote this section to the analysis of the integral that contains no twisted factor. It will become apparent that such a term will both provide the main contribution to the asymptotic formula at hand and will containt the recalcitrant character of the play earlier mentioned in the introduction. In order to make progress in such an endeavour, it has been thought pertinent to introduce first some notation which will be used henceforth. We write $\boldsymbol{n}$ to denote a triple $(n_1,n_2,n_3)\in\mathbb{N}^3$, and we consider the variables

$$n_1'=n_1/\sqrt{a},\qquad n_2'=n_2/\sqrt{b},\qquad n_3'=n_3/\sqrt{c}, \tag{5.5.1}$$

and the parameters

$$N_{\boldsymbol{n}}=2\pi\max(n_1'^2,n_2'^2,n_3'^2)\qquad\text{and}\qquad P_{\boldsymbol{n}}=n_1n_2n_3.$$

We also define for further convenience

$$T_1=T/2\pi$$

and write

$$\mathcal{B}_{a,b,c} = \Big\{ \boldsymbol{n} \in \mathbb{N}^3 : \quad n_1 \leq \sqrt{aT_1}, \quad n_2 \leq \sqrt{bT_1}, \quad n_3 \leq \sqrt{cT_1} \Big\}. \tag{5.5.2}$$

We consider the aforementioned integral

$$\begin{aligned} I_1(T) &= \int_0^T D(1/2 + ait)D(1/2 - bit)D(1/2 - cit)dt \\ &= \sum_{\boldsymbol{n} \in \mathcal{B}_{a,b,c}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^T \Big(\frac{n_2^b n_3^c}{n_1^a} \Big)^{it} dt. \end{aligned}$$

We make a distinction between the diagonal contribution, which will amount to the contribution of triples with the property that $n_1^a = n_2^b n_3^c$, and the off-diagonal one to obtain

$$I_1(T) = J_1(T) + J_2(T), \tag{5.5.3}$$

where in the above equation one has

$$J_1(T) = \sum_{\substack{\boldsymbol{n} \in \mathcal{B}_{a,b,c} \\ n_1^a = n_2^b n_3^c}} (T - N_{\boldsymbol{n}}) P_{\boldsymbol{n}}^{-1/2}, \qquad J_2(T) = \sum_{\substack{\boldsymbol{n} \in \mathcal{B}_{a,b,c} \\ n_1^a \neq n_2^b n_3^c}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^T \Big(\frac{n_2^b n_3^c}{n_1^a} \Big)^{it} dt. \tag{5.5.4}$$

For ease of notation we may omit writing $\boldsymbol{n} \in \mathcal{B}_{a,b,c}$ in the subscripts of the sums throughout the rest of the chapter. We further recall that the present section shall focus on the instance when

$$a < c \leq b. \tag{5.5.5}$$

The following proposition will be devoted to the estimation of $J_2(T)$ for the general case of positive real coefficients.

Lemma 5.5.1. *With the above notation one has that*

$$J_2(T) = o(T).$$

Proof. As will be apparent shortly, it may seem appropiate to define, for convenience, the number

$$N_{b,c} = \lfloor n_2^{b/a} n_3^{c/a} \rfloor$$

for each tuple (n_2, n_3). We split the corresponding sum into the case when n_1

differs from $N_{b,c}$ by at least 2, in which case we shall simply integrate and apply the triangle inequality, and the instance when n_1 is close to $N_{b,c}$ to obtain

$$J_2(T) \ll J_{2,1}(T) + J_{2,2}(T),$$

where

$$J_{2,1}(T) = \sum_{|n_1 - N_{b,c}|>1} P_{\boldsymbol{n}}^{-1/2} \left|\log\left(n_1^a/n_2^b n_3^c\right)\right|^{-1} \tag{5.5.6}$$

and

$$J_{2,2}(T) = \sum_{\substack{n_2,n_3 \\ |n_1 - N_{b,c}|\leq 1}} P_{\boldsymbol{n}}^{-1/2} \left| \int_{N_{\boldsymbol{n}}}^{T} e^{it\log(n_2^b n_3^c/n_1^a)} dt \right|. \tag{5.5.7}$$

We should remark that in the above summation we omitted writing the condition $n_1^a \neq n_2^b n_3^c$ for the sake of concission, and take this as an opportunity to announce that henceforth we shall avoid writing such a restriction in every sum cognate to the off-diagonal contribution. The reader may find it worth observing that whenever $n_1 \asymp N_{b,c}$ then

$$\left|\log\left(n_2^b n_3^c/n_1^a\right)\right| \asymp \frac{|n_2^b n_3^c - n_1^a|}{n_2^b n_3^c} \quad \text{and} \quad \left|\log\left(n_2^b n_3^c/n_1^a\right)\right| \asymp \frac{|n_2^{b/a} n_3^{c/a} - n_1|}{n_2^{b/a} n_3^{c/a}}. \tag{5.5.8}$$

It then transpires that in view of the above relation one may deduce the estimate

$$\begin{aligned}\sum_{\substack{N_{b,c}/2\leq n_1<2N_{b,c} \\ |n_1-N_{b,c}|>1}} \frac{n_1^{-1/2}}{\left|\log\left(n_2^b n_3^c/n_1^a\right)\right|} &\ll \sum_{1<|r|\leq N_{b,c}} \frac{N_{b,c}^{1/2}}{|n_2^{b/a} n_3^{c/a} - N_{b,c} - r|} \\ &\ll \sum_{1\leq r\leq N_{b,c}} \frac{N_{b,c}^{1/2}}{r} \ll N_{b,c}^{1/2} \log T. \end{aligned} \tag{5.5.9}$$

It seems appropiate to denote $J_{2,1,1}(T)$ the contribution to $J_{2,1}(T)$ of tuples in the range considered in the above line. Then, the preceding equation in conjuntion with (5.5.6) delivers

$$\begin{aligned}J_{2,1,1}(T) &\ll (\log T) \sum_{n_2^b n_3^c \ll T^{a/2}} n_2^{-1/2} n_3^{-1/2} N_{b,c}^{1/2} \\ &\ll (\log T) \sum_{n_2^b n_3^c \ll T^{a/2}} n_2^{-1/2+b/2a} n_3^{-1/2+c/2a} \\ &\ll T^{1/4+a/4b} (\log T) \sum_{n_3 \ll T^{a/2c}} n_3^{-1/2-c/2b} \ll (\log T) T^{1/4+a/4c} \ll T^{3/4},\end{aligned}$$

wherein we used (5.5.5). The reader may also find it worth noting that whenever n_1 is outside of the range considered above then it transpires that $|\log\left(n_1^a/n_2^b n_3^c\right)| \geq K > 0$ for some positive constant K, and hence the contribution of these cases will amount to $O(T^{3/4})$. Therefore, the preceding discussion yields

$$J_{2,1}(T) \ll T^{3/4}. \tag{5.5.10}$$

We next shift our focus to the analysis of $J_{2,2}(T)$, it being convenient to introduce first for each number $r \in \{-1, 0, 1\}$ the sums

$$J_{2,2,r}(T) = \sum_{\substack{n_2 \leq \sqrt{bT_1} \\ n_3 \leq \sqrt{cT_1}}} (n_2 n_3)^{-1/2} (N_{b,c} + r)^{-1/2} \Big| \int_{N_{\boldsymbol{n}}}^{T} e^{it \log\left(n_2^b n_3^c/(N_{b,c}+r)^a\right)} dt \Big|. \tag{5.5.11}$$

We introduce a large but fixed parameter $M > 0$ and divide the range of summation to obtain

$$J_{2,2,r}(T) \ll F_{1,r,M}(T) + F_{2,r,M}(T),$$

where the above terms are defined by means of the formulas

$$F_{1,r,M}(T) = \sum_{n_2, n_3 \leq M} n_2^{-1/2-b/2a} n_3^{-1/2-c/2a} \frac{1}{|\log\left(n_2^b n_3^c/(N_{b,c}+r)^a\right)|},$$

and

$$F_{2,r,M}(T) = T \sum_{\max(n_2,n_3) > M} n_2^{-1/2-b/2a} n_3^{-1/2-c/2a}.$$

The reader may find it useful to observe that by summing over n_2 and n_3

accordingly, one obtains

$$F_{2,r,M}(T) \ll TM^{1/2-c/2a}.$$

Moreover, it transpires that $F_{1,r,M}(T)$ does not depend on T, whence there exists a constant $C(M)$ with the property that

$$F_{1,r,M}(T) \leq C(M).$$

The preceding discussion then yields

$$J_{2,2}(T) \ll TM^{1/2-c/2a} + C(M),$$

whence on fixing M and letting $T \to \infty$ one readily sees that

$$\lim_{T\to\infty} \frac{|J_{2,2}(T)|}{T} \ll M^{1/2-c/2a}.$$

We recall the reader of the condition $a < c$ and let $M \to \infty$ to the end of deriving the expression $J_{2,2}(T) = o(T)$, which in conjunction with the above analysis yields the desired result. □

Lemma 5.5.2. *If $a, b, c \in \mathbb{N}$ the bound*

$$J_2(T) \ll T^{3/4} + T^{1+1/2c-1/2a}$$

holds unconditionally. If one assumes Conjecture 2 then one further has

$$J_2(T) \ll T^{1/4+3a/4c+\varepsilon}.$$

Proof. As a prelude to our discussion we begin by anticipating that we shall make use of the analysis pertaining to $J_{2,1}(T)$ in the above lemma and modify that of $J_{2,2}(T)$ to the end of deriving sharper estimates under the above assumptions. We divide the range of summation in (5.5.11) in accordance with (5.5.8) to obtain

$$J_{2,2,r}(T) \ll F_1(T) + F_2(T),$$

where the above terms are defined by means of the formulas

$$F_1(T) = \sum_{n_2^b n_3^c \leq T} n_2^{b-1/2-b/2a} n_3^{c-1/2-c/2a},$$

and

$$F_2(T) = T \sum_{n_2^b n_3^c > T} n_2^{-1/2-b/2a} n_3^{-1/2-c/2a}.$$

It may be worth clarifying that in the above analysis we bounded the integral in (5.5.11) by the inverse of the corresponding logarithm and applied (5.5.8) subsequently in conjunction with the fact that

$$|n_2^b n_3^c - n_1^a| \geq 1,$$

the same integral appertaining to the second range of summation being estimated by the length of the interval of integration.

Summing over n_2 first yields

$$F_1(T) \ll T^{1+1/2b-1/2a} \sum_{n_3 \leq T^{1/c}} n_3^{-1/2-c/2b} \ll T^{1+1/2c-1/2a}.$$

Likewise, an analogous computation in the same spirit reveals that

$$F_2(T) \ll T^{1+1/2b-1/2a} \sum_{n_3 \leq T^{1/c}} n_3^{-1/2-c/2b} + T \sum_{n_3^c > T} n_3^{-1/2-c/2a} \ll T^{1+1/2c-1/2a},$$

thus yielding the bound

$$J_{2,2,r}(T) \ll T^{1+1/2c-1/2a},$$

whence the above estimate in conjunction with (5.5.7) delivers

$$J_{2,2}(T) \ll T^{1+1/2c-1/2a},$$

as desired.

Before making further progress in the proof, some observations shall be addressed. We find it convenient to draw the reader's attention to the above inequalities and point out that the assumption $a < \min(b, c)$ was crucially utilised therein, an analogue argument not being applicable in other circumstances. It is also worth noting that if $a = 1$ then we may obtain a sharper estimate which would ultimately deliver an error term $O(T^{1/4+1/4c})$ by making use of the corresponding inequality $n_2^b n_3^c \ll T^{1/2}$ in the above setting, such a refinement not having any impact whatsoever in the overall error term of the lemma at hand.

The reader may notice that if we further assume Conjecture 2 we can improve the error term arising from the estimation of $J_{2,2}(T)$ substantially. To this end it seems appropiate to note first that such a conjecture yields

$$\left|n_2^b n_3^c - (N_{b,c}+r)^a\right| \gg N_{b,c}^{a-1-\varepsilon} n_2^{-1} n_3^{-1}, \qquad r \in \{-1,0,1\}.$$

We use the previous estimate and (5.5.8) to obtain the bound

$$\left|\log\left((N_{b,c}+r)^a / n_2^b n_3^c\right)\right|^{-1} \ll (N_{b,c} n_2 n_3)^{1+\varepsilon}, \qquad r \in \{-1,0,1\},$$

and insert it in (5.5.7) to get

$$\begin{aligned} J_{2,2}(T) \ll T^{\varepsilon} \sum_{n_2^b n_3^c \ll T^{a/2}} n_2^{1/2+b/2a} n_3^{1/2+c/2a} &\ll T^{1/4+3a/4c+\varepsilon} \sum_{n_2 \ll T^{a/2b}} n_2^{1/2-3b/2c} \\ &\ll T^{1/4+3a/4c+\varepsilon}, \end{aligned}$$

as desired, and wherein the above line we used the inequality

$$1/2 - 3b/2c \leq -1$$

stemming from (5.5.5).

□

Equipped with the estimate provided by the above lemma we have ascended to a position from which to obtain an asymptotic formula for $I_1(T)$ in the integer case, and this we now provide. It should be noted that this will amount to computing the contribution stemming from the diagonal term, and thus requires performing an analysis of a different nature.

Proposition 5.5.1. *Let $a, b, c \in \mathbb{N}$ satisfying $(a, b, c) = 1$. On recalling (5.5.4), one has*

$$J_1(T) = \sigma_{a,b,c} T + O\big((\log T)^2 (T^{3/4} + T^{3/4-a/4c+(a,c)/2c} + T^{3/4-a/4b+(a,b)/2b})\big),$$

where $\sigma_{a,b,c} > 1$ is an explicit constant which shall be defined shortly. Therefore, the asymptotic formula

$$I_1(T) = \sigma_{a,b,c}T + R(T),$$

where

$$R(T) \ll T^{1+1/2c-1/2a} + (\log T)^2\big(T^{3/4} + T^{3/4-a/4c+(a,c)/2c} + T^{3/4-a/4b+(a,b)/2b}\big),$$

holds unconditionally . If one further assumes Conjecture 2 then the summand $T^{1+1/2c-1/2a}$ in the above error term may be replaced by $T^{1/4+3a/4c+\varepsilon}$.

Proof. It seems worth recalling (5.5.3) and writing

$$J_1(T) = \sigma_{a,b,c}T - J_3(T) - J_4(T),$$

where we define

$$J_3(T) = T \sum_{\substack{n_1^a = n_2^b n_3^c \\ \max(n_1', n_2', n_3') > \sqrt{T_1}}} P_{\boldsymbol{n}}^{-1/2}, \qquad J_4(T) = \sum_{\substack{\boldsymbol{n} \in \mathcal{B}_{a,b,c} \\ n_1^a = n_2^b n_3^c}} N_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2}, \tag{5.5.12}$$

wherein the constant $\sigma_{a,b,c}$ is defined by means of the formula

$$\sigma_{a,b,c} = \sum_{n_1^a = n_2^b n_3^c} P_{\boldsymbol{n}}^{-1/2}. \tag{5.5.13}$$

The reader shall rest assured that the convergence of the above series will be justified promptly. It appears at first glance that a straight substitution of the identity $n_1 = n_2^{b/a} n_3^{c/a}$ into the corresponding equations thereof already delivers error terms of the shape $O(T^{3/4+a/4c})$. Nonetheless, it seems appropiate to remark that such an approach does not further exploit the property of the product $n_2^b n_3^c$ at hand being a perfect a-th power. To remedy this situation and obtain an estimate superior to that stemming from the aforementioned cheap analysis, say of the shape $O\big(T^{3/4}(\log T)^2\big)$, it has been thought preferable to parametrize the equation at hand in a suitable manner, and this we now describe.

To this end, some notation will be required. We write

$$a_2 = a/(a,b), \qquad b_2 = b/(a,b), \tag{5.5.14}$$

$$a_3 = a/(a,c), \qquad c_3 = c/(a,c), \qquad A = \frac{a}{(a,b)(a,c)}.$$

It might be worth noting that the property of the parameter A being an integer stems from the coprimality condition $(a,b,c) = 1$. We begin by analysing first the instance when $a_2 \neq 1 \neq a_3$. The main idea underlying the parametrization process will have its reliance on the classification of the divisors $d|n_2$ (and similarly for n_3) according to whether the number d^b shall or not be a perfect a-th power. It may be appropiate to momentarily pause our explanation and clarify that such a condition amounts to the number d being a perfect a_2-th power. If instead d does not satisfy such a priviledge, some divisibility relation between both d and n_3 should then hold, and exploiting such a dependence among the divisors of both n_2 and n_3 shall ultimately lead to sharper estimates. In order to put these ideas into effect we begin by writing, as we may,

$$n_2 = r_2^{a_2} \prod_{u=1}^{a_2-1} d_u^u, \qquad n_3 = r_3^{a_3} \prod_{v=1}^{a_3-1} f_v^v,$$

where d_u and f_v denote squarefree numbers with the property that $(d_{u_1}, d_{u_2}) = 1$ whenever $u_1 \neq u_2$ and similarly for f_v. In view of the above definitions, we find it convenient to note that $a \nmid ub$ whenever $1 \leq u \leq a_2 - 1$. It might as well be noteworthy to observe that for fixed u satisfying the above line of inequalities, there is a unique solution to the congruence

$$ub + \alpha_u c \equiv 0 \pmod{a}, \qquad 1 \leq \alpha_u \leq a_3 - 1.$$

By the preceding discussion it transpires that for fixed u then there is some $1 \leq v \leq a_3 - 1$ satisfying $f_v = d_u$ with $v = \alpha_u$ as above, a concomitant aspect of the coprimality condition $(a,b,c) = 1$ being that $(a,c)|u$ and $(a,b)|\alpha_u$. The reader may find it desirable to observe that an analogous argument may be employed to deduce, for v, the existance of some $1 \leq u \leq a_2 - 1$ with the property that $d_u = f_v$. Therefore, we have thus far reached a position from which to assure that one may parametrize the triples satisfying the equation at hand by means of the relations

$$n_1 = r_2^{b_2} r_3^{c_3} P_{\boldsymbol{d}}, \qquad n_2 = r_2^{a_2} \prod_{j=1}^{A-1} d_j^{j(a,c)}, \qquad n_3 = r_3^{a_3} \prod_{j=1}^{A-1} d_j^{\alpha_j(a,b)}, \tag{5.5.15}$$

where we defined for convenience

$$P_{\boldsymbol{d}} = \prod_{j=1}^{A-1} d_j^{(bj(a,c)+c\alpha_j(a,b))/a}, \qquad M_{\boldsymbol{d}} = \prod_{j=1}^{A-1} d_j^{j(a,c)+\alpha_j(a,b)}, \tag{5.5.16}$$

the latter parameter being introduced for prompt convenience and wherein

$$jb_2 + \alpha_j c_3 \equiv 0 \ (\mathrm{mod}\ A), \qquad 1 \leq j \leq A-1, \ \ 1 \leq \alpha_j \leq A-1.$$

Combining the above equations one then finds that

$$\sigma_{a,b,c} = \sum_{r_2, r_3} r_2^{-(a_2+b_2)/2} r_3^{-(a_3+c_3)/2} \sum_{\boldsymbol{d}} P_{\boldsymbol{d}}^{-1/2} M_{\boldsymbol{d}}^{-1/2},$$

where $\boldsymbol{d}$ runs over the tuples described above, the convergence of the series being justified by the inequalities $a_2+b_2 \geq 3$ and $a_3+c_3 \geq 3$ in conjunction with the fact that the exponents cognate to the corresponding factors d_j involved in the above sum are smaller than or equal to -2.

We will first analyse the term $J_3(T)$. It might be worth noting that in view of (5.5.5) and the underlying equation satisfied by the triple $\boldsymbol{n}$ then one has $n_1' = \max(n_1', n_2', n_3')$, whence in the analysis of $J_3(T)$ it is always the case that $n_1 > \sqrt{aT_1}$. We then plug the above parametrization into (5.5.12) and sum over r_2 first to obtain

$$\begin{aligned} J_3(T) =& T \sum_{r_2^{b_2} r_3^{c_3} P_{\boldsymbol{d}} \geq \sqrt{aT_1}} r_2^{-(a_2+b_2)/2} r_3^{-(a_3+c_3)/2} P_{\boldsymbol{d}}^{-1/2} M_{\boldsymbol{d}}^{-1/2} \\ \ll& T^{3/4 - a_2/4b_2 + 1/2b_2} \sum_{r_3^{c_3} P_{\boldsymbol{d}} \leq \sqrt{aT_1}} P_{\boldsymbol{d}}^{-1/b_2 + a_2/2b_2} r_3^{-a_3/2 + c_3 a_2/2b_2 - c_3/b_2} M_{\boldsymbol{d}}^{-1/2} \\ &+ T \sum_{r_3^{c_3} P_{\boldsymbol{d}} \geq \sqrt{aT_1}} r_3^{-(a_3+c_3)/2} P_{\boldsymbol{d}}^{-1/2} M_{\boldsymbol{d}}^{-1/2}. \end{aligned} \tag{5.5.17}$$

In order to bound the first summand, which we denote by $J_{3,1}(T)$ for convenience, it might be worth considering first the case when

$$-a_3/2 + c_3 a_2/2b_2 - c_3/b_2 > -1. \tag{5.5.18}$$

Under such circumstances, it transpires that

$$J_{3,1}(T) \ll T^{3/4 - a_3/4c_3 + 1/2c_3} \sum_{P_{\boldsymbol{d}} \leq \sqrt{aT_1}} P_{\boldsymbol{d}}^{-1/c_3 + a_3/2c_3} M_{\boldsymbol{d}}^{-1/2}. \tag{5.5.19}$$

The reader may find it desirable to observe that the exponent of the factor d_1 involved in the term $M_{\boldsymbol{d}}^{-1/2}$ is at most -1, the exponents appertaining to the rest of the factors being at most $-3/2$. It also seems appropiate to note that in the above sum then

$$P_{\boldsymbol{d}}^{-1/c_3 + a_3/2c_3} \ll 1 + T^{-1/2c_3 + a_3/4c_3}.$$

It shall be noted that the previous estimate encompasses the cases when the exponent cognate to the term of the left side is both non-negative and negative.

The preceding discussion then yields

$$J_{3,1}(T) \ll T^{3/4 - a/4c + 1/2c_3} (\log T) + T^{3/4} (\log T). \tag{5.5.20}$$

If, on the contrary, condition (5.5.18) does not hold then

$$J_{3,1}(T) \ll T^{3/4 - a_2/4b_2 + 1/2b_2} (\log T) \sum_{P_{\boldsymbol{d}} \leq \sqrt{aT_1}} P_{\boldsymbol{d}}^{-1/b_2 + a_2/2b_2} M_{\boldsymbol{d}}^{-1/2},$$

and an argument reminiscent of the above yields

$$J_{3,1}(T) \ll T^{3/4} (\log T)^2 + T^{3/4 - a/4b + 1/2b_2} (\log T)^2. \tag{5.5.21}$$

It might be pertinent to clarify that the extra factor of $(\log T)$ has been added to encompass the case when the left side of (5.5.18) is -1 as well.

Likewise, we denote by $J_{3,2}(T)$ to the second summand of (5.5.17). An analogous argument to the one employed to bound $J_{3,1}(T)$ then reveals that

$$J_{3,2}(T) \ll T^{3/4-a_3/4c_3+1/2c_3} \sum_{P_{\boldsymbol{d}}\leq\sqrt{aT_1}} P_{\boldsymbol{d}}^{-1/c_3+a_3/2c_3} M_{\boldsymbol{d}}^{-1/2} + T \sum_{P_{\boldsymbol{d}}\geq\sqrt{aT_1}} P_{\boldsymbol{d}}^{-1/2} M_{\boldsymbol{d}}^{-1/2}.$$

It seems appropiate to note that in the first summand of the above equation one has

$$P_{\boldsymbol{d}}^{-1/c_3+a_3/2c_3} \ll 1 + T^{-1/2c_3+a_3/4c_3},$$

whilst in the second one it transpires that

$$P_{\boldsymbol{d}}^{-1/2} \ll T^{-1/4}.$$

Therefore, the above observations in conjunction with the discussion following (5.5.19) yield the bound

$$J_{3,2}(T) \ll T^{3/4}(\log T) + T^{3/4-a/4c+1/2c_3}(\log T),$$

a combination of the preceding estimates delivering

$$J_3(T) \ll T^{3/4}(\log T)^2+T^{3/4-a/4c+1/2c_3}(\log T)+T^{3/4-a/4b+1/2b_2}(\log T)^2 \quad (5.5.22)$$

as desired.

We next focus our attention on the term $J_4(T)$ in (5.5.12). We shall use the parametrization employed in the analysis of $J_3(T)$ herein as well, and announce that the argument on this occasion will be morally equivalent. Then, as was noted above, one necessarily has that the tuples involved in the sum in the definition of $J_4(T)$ have the property that $n_1 \geq \max(n_2, n_3)$, whence

$$J_4(T) \ll \sum_{r_2^{b_2} r_3^{c_3} P_{\boldsymbol{d}}\leq\sqrt{aT_1}} r_2^{(3b_2-a_2)/2} r_3^{(3c_3-a_3)/2} P_{\boldsymbol{d}}^{3/2} M_{\boldsymbol{d}}^{-1/2},$$

Summing over r_2 first we find that

$$J_4(T) \ll T^{3/4-a_2/4b_2+1/2b_2} \sum_{r_3^{c_3} P_{\boldsymbol{d}} \leq \sqrt{aT_1}} r_3^{a_2c_3/2b_2-c_3/b_2-a_3/2} P_{\boldsymbol{d}}^{a_2/2b_2-1/b_2} M_{\boldsymbol{d}}^{-1/2}$$
$$= J_{3,1}(T),$$

where the reader may find it useful to recall that $J_{3,1}(T)$ denoted the first summand in the equation (5.5.17). Consequently, (5.5.20) in conjunction with (5.5.21) and the above line of inequalities yields

$$J_4(T) \ll T^{3/4}(\log T)^2 + T^{3/4-a/4b+1/2b_2}(\log T)^2 + T^{3/4-a/4c+1/2c_3}(\log T),$$

as required, and concludes the proof thereof.

If instead either $a_2 = 1$ or $a_3 = 1$ then the parametrization of the underlying equation will then be

$$n_2 = r_2^{a_2}, \qquad n_3 = r_3^{a_3},$$

whence an insightful inspection of the above proof reveals that one may follow the same argument to reach an analogous conclusion.

$\square$

We complete this section with an application of Baker's theorem on linear forms in logarithms to discard the existence of non-trivial diagonal solutions of the underlying equation whenever the coefficients are linearly independent over $\mathbb{Q}$, thereby delivering the conclusion $I_1(T) \sim T$ for such cases.

Proposition 5.5.2. *Let $a, b, c \in \mathbb{R} \setminus \{0\}$ be algebraic numbers linearly independent over $\mathbb{Q}$. Then there are no solutions to the equation*

$$n_1^a = n_2^b n_3^c, \qquad n_1, n_2, n_3 \in \mathbb{N}, \quad (n_1, n_2, n_3) \neq (1,1,1). \tag{5.5.23}$$

Consequently, one has that

$$I_1(T) \sim T.$$

Proof. The second statement follows from the first by recalling (5.5.3) and (5.5.4) and noting that then

$$J_1(T) = T - 2\pi,$$

which in conjunction with Lemma 5.5.1 delivers the desired result.

We shift our focus to the first assertion then and begin by assuming the existence of a triple (n_1, n_2, n_3) with the above property. In particular, this entails the linearly dependence of $\log n_1, \log n_2, \log n_3$ and $2\pi i$ over the algebraic numbers. Therefore, an application of Baker's theorem already establishes the linearly dependence over $\mathbb{Q}$, which in turn yields the existence of rational numbers $r_1, r_2, r_3 \in \mathbb{Q}$ satisfying

$$n_1^{r_1} = n_2^{r_2} n_3^{r_3}.$$

We may assume without loss of generality that $r_1 \neq 0$. Moreover, an examination of (5.5.23) and the preceding equation reveals that

$$(n_2, n_3) \neq (1, 1). \tag{5.5.24}$$

Therefore, combining both of the equations then delivers the relation

$$n_2^{b/a - r_2/r_1} n_3^{c/a - r_3/r_1} = 1. \tag{5.5.25}$$

It seems pertinent to observe that in view of the linear independence over the rationals of the coefficients a, b, c then the exponents in the above line are non-zero. It therefore transpires by (5.5.24) and the preceding observation that then

$$n_2 \neq 1 \neq n_3.$$

We apply, as we may, Baker's theorem again to obtain a rational number $r_4 \in \mathbb{Q}$ having the property

$$n_2^{r_4} = n_3.$$

Combining the above line with that of (5.5.25) yields an equality between the corresponding exponents, namely,

$$b + c r_4 = \Big(\frac{r_4 r_3 + r_2}{r_1}\Big) a,$$

which contradicts the linear independence of the coefficients.

$\square$

5.6 Simple bounds for integrals of unimodular functions

The following lines will be devoted to provide estimates for some of the integrals involved in the main term of (5.4.4) and pertaining to both Theorems 5.1.1, 5.1.2 and 5.1.3 via a straightforward application of Lemma 5.2.1. The results in this section shall be obtained for positive real coefficients $a, b, c > 0$. For the purpose of making further progress it seems appropiate to define, for pairs of positive real numbers $r, s > 0$ the integral

$$Y_{2,r,s}(T) = \int_0^1 D(1/2 + ait)D(1/2 + irt)D(1/2 - ist)\chi(1/2 - irt)dt.$$

Equipped with this definition we consider

$$I_2(T) = Y_{2,b,c}(T) + Y_{2,c,b}(T). \tag{5.6.1}$$

Likewise, we further define the pairs of functions

$$f_3(t) = D(1/2 + ait)$$

and

$$f_4(t) = \chi(1/2 + ait)D(1/2 - ait),$$

and the integral

$$I_j(T) = \int_0^T D(1/2+bit)D(1/2+cit)\chi(1/2-bit)\chi(1/2-cit)f_j(t)dt, \quad j \in \{3, 4\}.$$

We should note that in the next sequel of lemmata we further assume that $a \leq c$ for future use despite not being required for the proof of Theorems 5.1.1 and 5.1.2.

Lemma 5.6.1. *Let $a, r, s > 0$ be real numbers with the property that $\min(r, s) \geq a$. Then*

$$Y_{2,r,s}(T) \ll T^{3/4}(\log T).$$

Moreover, whenever $a \leq c \leq b$ one has

$$\max\big(I_2(T), I_3(T), I_4(T)\big) \ll T^{3/4}(\log T).$$

Proof. We begin our proof by analysing first $Y_{2,r,s}(T)$. We use the approximation formula for $\chi(1/2 - rit)$ contained in Lemma 5.2.6 to express the above integral as

$$Y_{2,r,s}(T) = e^{-i\pi/4} \sum_{\boldsymbol{n}\in\mathcal{B}_{a,r,s}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^{T} e^{iF_2(t)} dt + O(T^{3/4}\log T),$$

where the function $F_2(t)$ is defined by

$$F_2(t) = rt\log rt - rt(\log 2\pi + 1) - t\log(n_1^a n_2^r / n_3^s).$$

It also seems appropiate to differentiate the above function and recall (5.5.1) to obtain

$$F_2'(t) = r\log t + \frac{s}{2}\log s - \frac{a}{2}\log a + \frac{r}{2}\log r - r\log 2\pi - \log(n_1'^a n_2'^r / n_3'^s).$$

The reader may find it useful to observe that then $F_2'(t)$ is an increasing function. In view of the fact that $r \geq a$, it then transpires that

$$F_2'(N_{\boldsymbol{n}}) \geq (r-a)\log\big(\max(n_1', n_2', n_3')\big) + \frac{r}{2}\log r - \frac{a}{2}\log a,$$

whence by monotonicity the same holds in the interval $[N_{\boldsymbol{n}}, T]$. Note that for triples (n_1, n_2, n_3) bounded by a fixed constant then the corresponding contribution to $Y_{2,r,s}(T)$ would then be $O(T^{1/2})$ by Lemma 5.2.3. For triples with one of the components being large enough then an application of Lemma 5.2.1 suffices to deduce that the cognate integral is $O(1)$. The preceding discussion yields

$$Y_{2,r,s}(T) \ll T^{3/4}\log T + \sum_{\boldsymbol{n}\in\mathcal{B}_{a,r,s}} P_{\boldsymbol{n}}^{-1/2} \ll T^{3/4}\log T,$$

which delivers the first part of the statement. It might be worth noting that recalling (5.6.1) and applying the estimates for $Y_{2,b,c}(T)$ and $Y_{2,c,b}(T)$ obtained herein one gets

$$I_2(T) \ll T^{3/4}(\log T).$$

As earlier mentioned, the term $I_3(T)$ will exhibit a similar behaviour, whence in the interest of not repeating ourselves we will try to be as expeditious as possible. We employ the approximation formula in Lemma 5.2.6 to obtain

$$I_3(T) = -i \sum_{\boldsymbol{n} \in \mathcal{B}_{a,b,c}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^{T} e^{iF_3(t)} dt + O(T^{3/4} \log T),$$

where the function $F_3(t)$ is defined by the equation

$$F_3(t) = bt \log bt + ct \log ct - (b + c)t(\log 2\pi + 1) - t \log(n_1^a n_2^b n_3^c).$$

As was noted above, it might be convenient to compute its derivative

$$F_3'(t) = (b + c) \log t + b \log b + c \log c - (b + c) \log 2\pi - \log(n_1^a n_2^b n_3^c).$$

We observe that then $F_3'(t)$ is monotonic and

$$F_3'(N_{\boldsymbol{n}}) \geq (b + c - a) \log \big(\max(n_1', n_2', n_3') \big) + \frac{b}{2} \log b + \frac{c}{2} \log c - \frac{a}{2} \log a$$

in the interval of integration at hand, wherein the reader may find it useful to recall (5.5.1), whence in a similar fashion as above, Lemmata 5.2.1 and 5.2.3 yield

$$I_3(T) \ll T^{3/4} \log T.$$

In order to estimate $I_4(T)$ we use as customary the corresponding approximation formulae in Lemma 5.2.6 to obtain

$$I_4(T) = e^{-i\pi/4} \sum_{\boldsymbol{n} \in \mathcal{B}_{a,b,c}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^{T} e^{iF_4(t)} dt + O(T^{3/4} \log T),$$

where the function $F_4(t)$ is defined by

$$F_4(t) = bt \log bt + ct \log ct - at \log at - (b + c - a)t(\log 2\pi + 1) - t \log(n_2^b n_3^c / n_1^a),$$

and its derivative,

$$F_4'(t) = (b+c-a) \log t + b \log b + c \log c - a \log a - (b+c-a) \log 2\pi - \log(n_2^b n_3^c / n_1^a).$$

We observe that then $F_4'(t)$ is monotonic and

$$F_4'(N_{\boldsymbol{n}}) \geq (b+c-2a) \log \big(\max(n_1', n_2', n_3') \big) + a \log n_1' + \frac{b}{2} \log b + \frac{c}{2} \log c - \frac{a}{2} \log a,$$

whence Lemmata 5.2.1 and 5.2.3 then yield

$$I_4(T) \ll T^{3/4} \log T,$$

as desired. $\square$

5.7 A refinement assuming Conjecture 2 via the stationary phase method

The investigation of the integral $I_5(T)$, which shall be defined promptly, is slightly more intrincate and different in nature from the previous ones. It seems appropiate to assure the reader that the analysis involved in this section will be both unconditional and conditional, as opposed to what the above heading may have probably suggested. By proceeding in a routinary manner we shall ascend to a position from which an application of Lemmata 5.2.1 and 5.2.3 will already suffice to obtain sufficiently strong unconditional bounds. Nonetheless, an estimate superior to that obtained in the unconditional analysis can be pursued via the stationary phase method if one further assumes Conjecture 2, and this we have included in the discussion. We consider then, for convenience

$$I_5(T) = \int_0^T D(1/2 - ait)D(1/2 - bit)D(1/2 - cit)\chi(1/2 + ait)dt,$$

and gather the unconditional work done concerning the estimation of such a term in the following lemma.

Lemma 5.7.1. *Let $a \leq c \leq b$ with $a, b, c \in \mathbb{R}_+$ and let $I_5(T)$ be defined as above. Then the estimate*

$$I_5(T) \ll T^{3/4 + a/4c}$$

holds unconditionally.

Proof. We begin the discussion by employing as customary Lemma 5.2.6 to approximate $\chi(1/2 + ait)$ and express the above term as

$$I_5(T) = e^{i\pi/4} \sum_{\boldsymbol{n} \in \mathcal{B}_{a,b,c}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^T e^{iF_5(t)} dt + O(T^{3/4} \log T), \tag{5.7.1}$$

where the function $F_5(t)$ is defined by the relation

$$F_5(t) = -at\log at + at(\log 2\pi + 1) + t\log(n_1^a n_2^b n_3^c).$$

We find it desirable to compute its derivative

$$F_5'(t) = -a\log t - a\log a + a\log 2\pi + \log(n_1^a n_2^b n_3^c), \tag{5.7.2}$$

whence on denoting $c_{\boldsymbol{n}} = 2\pi n_1 n_2^{b/a} n_3^{c/a}/a$ it transpires that $F_5'(c_{\boldsymbol{n}}) = 0$. We apply Lemmata 5.2.1 and 5.2.3 in conjunction with Lemma 5.2.5 to the integral in (5.7.1) to obtain

$$\begin{aligned} I_5(T) =& 2\pi i a^{-1} \sum_{N_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} \leq T} n_2^{b/2a-1/2} n_3^{c/2a-1/2} e(n_1 n_2^{b/a} n_3^{c/a}/a) \\ &+ O\Big(\sum_{N_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq 2N_{\boldsymbol{n}}} P_{\boldsymbol{n}}^{-1/2} \min\big(|F_5'(N_{\boldsymbol{n}})|^{-1}, N_{\boldsymbol{n}}^{1/2}\big)\Big) \\ &+ O\Big(\sum_{T/2 \leq c_{\boldsymbol{n}} \leq 2T} P_{\boldsymbol{n}}^{-1/2} \min\big(|F_5'(T)|^{-1}, T^{1/2}\big)\Big) + O(T^{3/4}). \end{aligned} \tag{5.7.3}$$

The reader shall rest assured that further details about such an application will be delivered promptly. It may first be useful to observe that in the preceding lines we implicitly applied Lemma 5.2.5 for the range $2N_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} \leq T/2$ to the integral

$$\int_{c_{\boldsymbol{n}}/2}^{2c_{\boldsymbol{n}}} e^{iF_5(t)} dt$$

and estimate the remaining parts of the integral in (5.7.1) employing Lemma 5.2.1. Observe that then the error term arising from such remaining parts is $O(1)$. Likewise, if $N_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} < 2N_{\boldsymbol{n}}$ and $c_{\boldsymbol{n}} \leq T/2$ then Lemma 5.2.5 is applied with the choices $\alpha = N_{\boldsymbol{n}}$ and $\beta = 2c_{\boldsymbol{n}}$. If instead $T/2 < c_{\boldsymbol{n}} \leq T$ and $2N_{\boldsymbol{n}} \leq c_{\boldsymbol{n}}$ then the latter lemma shall be employed by taking $\alpha = c_{\boldsymbol{n}}/2$ and $\beta = T$. If on the contrary one has $4N_{\boldsymbol{n}} > T$ then $\alpha = N_{\boldsymbol{n}}$ and $\beta = T$ will suffice to obtain the desired result. Finally, whenever either $N_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} < N_{\boldsymbol{n}}$ or $T < c_{\boldsymbol{n}} \leq 2T$ then a combination of both Lemmata 5.2.1 and 5.2.3 will provide a contribution which will be absorbed in the error term of the above equation.

We focus our attention on the main term of (5.7.3), which we denote by

$P(T)$. By bounding the exponential sum on $P(T)$ trivially we then find that

$$P(T) \ll \sum_{n_2^{b/a} n_3^{c/a} \ll T} n_2^{b/2a-1/2} n_3^{c/2a-1/2} \min\left(T^{1/2}, T n_2^{-b/a} n_3^{-c/a}\right)$$
$$\ll P_1(T) + P_2(T),$$

where

$$P_1(T) = T^{1/2} \sum_{n_2^{b/a} n_3^{c/a} \leq \sqrt{cT_1}} n_2^{b/2a-1/2} n_3^{c/2a-1/2},$$

and

$$P_2(T) = T \sum_{n_2^{b/a} n_3^{c/a} > \sqrt{cT_1}} n_2^{-b/2a-1/2} n_3^{-c/2a-1/2}.$$

We shall begin our investigation examining first $P_1(T)$. Summing over n_3 we obtain

$$P_1(T) \ll T^{3/4+a/4c} \sum_{n_2 \leq (cT_1)^{a/2b}} n_2^{-1/2-b/2c} \ll T^{3/4+a/4c}. \tag{5.7.4}$$

Likewise, by summing first over n_2 for convenience in the equation defining $P_2(T)$ we have

$$P_2(T) \ll T \sum_{n_3^{c/a} > \sqrt{cT_1}} n_3^{-c/2a-1/2} + T^{3/4+a/4b} \sum_{n_3^{c/a} \leq \sqrt{cT_1}} n_3^{-1/2-c/2b} \ll T^{3/4+a/4c}. \tag{5.7.5}$$

It is worth noting that whenever $a = c$ then one always has $n_3 \leq \sqrt{cT_1}$, whence in this particular instance there is no first summand on the right side of the above equation. We find it desirable to stress that this is the only point in the proof wherein the fact that $a = c$ could have added an extra factor of $\log T$ had we not made a suitable division of the sum pertaining to $P(T)$, the rest of the arguments in the proof being valid for both of the situations at hand.

The reader may observe that the above argument could have been employed after a straight application of Lemmata 5.2.1 and 5.2.3. We invoked Lemma 5.2.5 herein because one may obtain non-trivial cancellation when averaging over the triples $\boldsymbol{n}$ if one further assumes Conjecture 2, and this will be pursued in a subsequent lemma.

In order to bound the first error term in (5.7.3), we find it convenient to denote $E_i(T)$ to the contribution to the sum in the aforementioned error term of tuples satifying $n_i' = \max(n_1', n_2', n_3')$ respectively for each $1 \leq i \leq 3$. It seems appropiate to begin our discussion analysing $E_1(T)$. To this end we define, for each (n_2, n_3), the parameter $N_1 = n_2^{b/a} n_3^{c/a}$. We note for further use that using this notation then the range of summation $N_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq 2N_{\boldsymbol{n}}$ in the first error term of (5.7.3) is equivalent to $n_1/2 \leq N_1 \leq 2n_1$, and that on recalling (5.7.2) one has that

$$F_5'(N_{\boldsymbol{n}}) = a \log N_1 - a \log n_1.$$

We apply the same argument as the one deployed in (5.5.9) and the subsequent equations to deduce

$$\sum_{\substack{N_1/2 \leq n_1 \leq 2N_1 \\ |n_1 - N_1| > 1}} \frac{n_1^{-1/2}}{|\log\big(N_1/n_1\big)|} \ll \sum_{1 \leq r \leq N_1} \frac{N_1^{1/2}}{r} \ll N_1^{1/2} \log N_1.$$

We note that $\min\big(|F_5'(N_{\boldsymbol{n}})|^{-1}, N_{\boldsymbol{n}}^{1/2}\big) \ll N_1^{1/2}$ if $|n_1 - N_1| \leq 1$ and combine such an observation with the preceding discussion to obtain

$$\begin{aligned}
&\sum_{n_2, n_3} \sum_{N_1/2 \leq n_1 \leq 2N_1} P_{\boldsymbol{n}}^{-1/2} \min\big(|F_5'(N_{\boldsymbol{n}})|^{-1}, N_{\boldsymbol{n}}^{1/2}\big) \\
&\qquad \ll (\log T) \sum_{N_1 \leq 2\sqrt{aT_1}} n_2^{-1/2} n_3^{-1/2} N_1^{1/2} \ll T^{1/4} (\log T) \sum_{\substack{n_2 \leq \sqrt{bT_1} \\ n_3 \leq \sqrt{cT_1}}} n_2^{-1/2} n_3^{-1/2} \\
&\qquad \ll T^{3/4} (\log T).
\end{aligned}$$

The reader may note that for n_1 outside of the range considered above then $|F_5'(N_{\boldsymbol{n}})|^{-1} \ll 1$, whence the contribution to the above sum arising from such tuples is $O(T^{3/4})$, and hence

$$E_1(T) \ll T^{3/4} \log T.$$

We next focus on the term $E_2(T)$. It seems appropiate to observe first that when $b \geq 2a$ then in view of (5.7.2) one has

$$F_5'(N_{\boldsymbol{n}}) = \log(n_1^a n_2^{b-2a} n_3^c) + a \log(b/a).$$

It therefore transpires that whenever either n_1 or n_3 are sufficiently large then one may bound such a contribution to $E_2(T)$ by $O(T^{3/4})$, the instance when both of the entries are bounded being estimated by means of the trivial observation $N_{\boldsymbol{n}}^{1/2} \ll T^{1/2}$, which in turns yields the bound $O(T^{3/4})$ for such a contribution. Suffices it then to consider the case $b < 2a$. To this end it seems worth defining for each (n_1, n_3) the parameter

$$N_2 = n_1^{a/(2a-b)} n_3^{c/(2a-b)} (b/a)^{a/(2a-b)}.$$

We write $C = 2^{a/(2a-b)}$ and denote

$$I_{N_2} = [C^{-1}N_2, CN_2],$$

the assertion that $n_2 \in I_{N_2}$ being then equivalent to the inequalities

$$N_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq 2N_{\boldsymbol{n}}.$$

We now apply the same argument as above to get the estimate

$$\sum_{n_1,n_3} \sum_{\substack{n_2 \in I_{N_2} \\ |n_2 - N_2| > 1}} P_{\boldsymbol{n}}^{-1/2} \min\left(|F_5'(N_{\boldsymbol{n}})|^{-1}, N_{\boldsymbol{n}}^{1/2}\right)$$
$$\ll (\log T) \sum_{N_2 \ll \sqrt{T}} n_1^{-1/2} n_3^{-1/2} N_2^{1/2} \ll T^{3/4}(\log T). \tag{5.7.6}$$

The reader may find it worth observing that the restriction on N_2 in the above inequality stems from the fact that

$$N_2 \asymp n_2 \leq \sqrt{bT_1}.$$

Again, whenever n_2 is outside of this range then $|F_5'(N_{\boldsymbol{n}})|^{-1} \ll 1$, whence by the preceding discussion we get

$$E_2(T) \ll T^{3/4} \log T.$$

The term $E_3(T)$ is analogous to $E_2(T)$, whence a similar analysis delivers

$$E_3(T) \ll T^{3/4}(\log T).$$

We finally analyse the last error term in (5.7.3), and find it desirable to announce that the nature of the discussion will not be dissimilar to the preceding one. We begin by noting for convenience that whenever $T/2 \leq c_{\boldsymbol{n}} \leq 2T$ then $T^{1/2} \ll n_2^{b/a} n_3^{c/a} \ll T$. We consider $\Lambda_1 = aT_1 n_2^{-b/a} n_3^{-c/a}$ and observe as is customary that the inequality $T/2 \leq c_{\boldsymbol{n}} \leq 2T$ is equivalent to

$$\Lambda_1/2 \leq n_1 \leq 2\Lambda_1.$$

It transpires that for each pair (n_2, n_3) then the same arguments utilised on previous occasions deliver

$$\sum_{\substack{\Lambda_1/2 \leq n_1 \leq 2\Lambda_1 \\ |n_1 - \Lambda_1| > 1}} n_1^{-1/2} |F_5'(T)|^{-1} \ll \sum_{0 < r \leq \Lambda_1} \frac{\Lambda_1^{1/2}}{r} \ll \Lambda_1^{1/2} (\log T).$$

We also note, for convenience, that

$$n_1^{-1/2} \min\left(|F_5'(T)|^{-1}, T^{1/2}\right) \ll T^{1/2} \Lambda_1^{-1/2}$$

when $|n_1 - \Lambda_1| \leq 1$. Then, combining the above discussion with the observation that the constraint $n_1 \leq \sqrt{aT_1}$ and the condition $|n_1 - \Lambda_1| \leq 1$ yields $\Lambda_1 \ll T^{1/2}$ and $\Lambda_1^{1/2} \ll T^{1/2} \Lambda_1^{-1/2}$, we obtain

$$\begin{aligned}
&\sum_{n_2, n_3} n_2^{-1/2} n_3^{-1/2} \sum_{\Lambda_1/2 \leq n_1 \leq 2\Lambda_1} n_1^{-1/2} \min\left(|F_5'(T)|^{-1}, T^{1/2}\right) \\
&\quad \ll T^{1/2} (\log T) \sum_{n_2^b n_3^c \ll T^a} n_2^{-1/2} n_3^{-1/2} \Lambda_1^{-1/2} \\
&\quad \ll (\log T) \sum_{n_2^b n_3^c \ll T^a} n_2^{-1/2 + b/2a} n_3^{-1/2 + c/2a} \ll (\log T) T^{1/2 + a/2b} \sum_{n_3 \leq \sqrt{cT_1}} n_3^{-1/2 - c/2b} \\
&\quad \ll T^{3/4 + (2a - c)/4b} (\log T)^2 \ll T^{3/4 + a/4c}.
\end{aligned}$$

The preceding discussion in conjunction with (5.7.4) and (5.7.5) then delivers the desired conclusions. $\square$

Lemma 5.7.2. *Let $a, b, c \in \mathbb{N}$ with the property that $a < c \leq b$ and let $I_5(T)$ be defined as above. Assuming Conjecture 2 one further has*

$$\begin{aligned}
I_5(T) \ll & T^{3/4} (\log T)^2 + T^{3/4 + (2a - c)/4b} (\log T)^2 + T^{3/4 - a/4b + 1/2b_2} (\log T)^2 \\
& + T^{3/4 - a/4c + 1/2c_3} (\log T) + T^{1/2 + a/(a + c) + \varepsilon}.
\end{aligned}$$

Proof. We should note that the analysis of $I_5(T)$ under the assumption of Conjecture 2 will follow the same line of argumentation as in the above lemma, the only difference in the proof having its reliance on the analysis of the main term $P(T)$ in (5.7.3), and this we now pursue.

We denote for convenience by $M_1(T)$ to the contribution to the main term $P(T)$ of tuples with the property that $n_2^{b/a}n_3^{c/a}/a$ is not an integer. Likewise, let $K_1(T)$ be the contribution of tuples for which $n_2^{b/a}n_3^{c/a}/a$ is a natural number. Note first that the sum over n_1 is a sum of a geometric progression, whence

$$M_1(T) \ll \sum_{n_2^{b/a}n_3^{c/a}\ll T} n_2^{b/2a-1/2}n_3^{c/2a-1/2}\min\Big(\Big\Vert a^{-1}n_2^{b/a}n_3^{c/a}\Big\Vert^{-1}, \frac{T}{n_2^{b/a}n_3^{c/a}}\Big).$$

In order to progress in the estimation it might be worth considering a natural number $M_{b,c}$ with the property that

$$|n_2^b n_3^c - M_{b,c}^a|$$

is minimised. We denote for further convenience $D = n_2^b n_3^c - M_{b,c}^a$. Then one may apply the mean value theorem to obtain

$$\Big\Vert n_2^{b/a}n_3^{c/a}\Big\Vert = |(M_{b,c}^a + D)^{1/a} - M_{b,c}| \asymp DM_{b,c}^{1-a},$$

whence the inequality

$$|D| \gg M_{b,c}^{a-1-\varepsilon}(n_2n_3)^{-1},$$

which in turn is a consequence of Conjecture 2, yields the estimate

$$|(M_{b,c}^a + D)^{1/a} - M_{b,c}| \gg (n_2n_3)^{-1-\varepsilon},$$

which delivers

$$\Big\Vert a^{-1}n_2^{b/a}n_3^{c/a}\Big\Vert^{-1} \ll (n_2n_3)^{1+\varepsilon}.$$

Combining (5.7.3) and the preceding discussion one gets

$$M_1(T) \ll \sum_{n_2^{b/a}n_3^{c/a}\ll T_1} n_2^{b/2a-1/2}n_3^{c/2a-1/2}\min\Big((n_2n_3)^{1+\varepsilon}, \frac{T}{n_2^{b/a}n_3^{c/a}}\Big).$$

We shall divide the sum into parts for convenience. We denote by $M_{1,1}(T)$

to the contribution to $M_1(T)$ of tuples satisfying $n_2^{a+b}n_3^{a+c} \le T^a$. Then one has that

$$\begin{aligned} M_{1,1}(T) &\ll T^{\varepsilon} \sum_{n_2^{a+b}n_3^{a+c} \le T^a} n_2^{1/2+b/2a} n_3^{1/2+c/2a} \\ &\ll T^{(b+3a)/2(a+b)+\varepsilon} \sum_{n_3 \le T^{a/(a+c)}} n_3^{-(a+c)/(a+b)} \ll T^{1/2+a/(a+c)+\varepsilon}. \end{aligned}$$

Likewise, let $M_{1,2}(T)$ denote the contribution to $M_1(T)$ of tuples with the property that $T^a < n_2^{a+b}n_3^{a+c}$. Then one readily sees that

$$\begin{aligned} M_{1,2}(T) &\ll T \sum_{T^a < n_2^{a+b}n_3^{a+c}} n_2^{-1/2-b/2a} n_3^{-1/2-c/2a} \\ &\ll T^{1+(a-b)/2(a+b)} \sum_{n_3 \le T^{a/(a+c)}} n_3^{-(a+c)/(a+b)} + T \sum_{T^{a/(a+c)} < n_3} n_3^{-1/2-c/2a} \\ &\ll T^{1/2+a/(a+c)}. \end{aligned}$$

Next we turn to analyse the term $K_1(T)$ and write for convenience

$$K_1(T) = K_{1,1}(T) + K_{1,2}(T),$$

where $K_{1,1}(T)$ is defined as the corresponding sum over the tuples satisfying $n_2^{b/a}n_3^{c/a} \le aT_1^{1/2}$, and in $K_{1,2}(T)$ we sum over tuples with the property that $n_2^{b/a}n_3^{c/a} > aT_1^{1/2}$. Then in view of (5.7.3) one has

$$K_{1,1}(T) \ll T^{1/2} \sum_{\substack{(an_1)^a = n_2^b n_3^c \\ n_1 \le \sqrt{T_1}}} n_1^{1/2} n_2^{-1/2} n_3^{-1/2}.$$

The reader may find it useful to recall (5.5.14) and note that we have reached a position from which to utilise the parametrization (5.5.15), whence following an analogous argument of the flavour of those displayed therein enables one to obtain

$$K_{1,1}(T) \ll T^{1/2} \sum_{r_2^{b_2} r_3^{c_3} P_{\boldsymbol{d}} \le a\sqrt{T_1}} r_2^{(b_2-a_2)/2} r_3^{(c_3-a_3)/2} P_{\boldsymbol{d}}^{1/2} M_{\boldsymbol{d}}^{-1/2},$$

Summing over r_2 first we find that

$$\begin{aligned} K_{1,1}(T) &\ll T^{3/4-a_2/4b_2+1/2b_2} \sum_{r_3^{c_3} P_{\boldsymbol{d}} \leq a\sqrt{T_1}} r_3^{a_2c_3/2b_2-c_3/b_2-a_3/2} P_{\boldsymbol{d}}^{a_2/2b_2-1/b_2} M_{\boldsymbol{d}}^{-1/2} \\ &\ll J_{3,1}(a^3T), \end{aligned}$$

wherein we recall to the reader that $J_{3,1}(T)$ was defined as the first summand in the last line of inequalities in (5.5.17). Consequently, (5.5.20) in conjunction with (5.5.21) and the preceding equation yields

$$K_{1,1}(T) \ll T^{3/4}(\log T)^2 + T^{3/4-a/4b+1/2b_2}(\log T)^2 + T^{3/4-a/4c+1/2c_3}(\log T),$$

as required.

Likewise, one has that

$$K_{1,2}(T) \ll T \sum_{\substack{(an_1)^a = n_2^b n_3^c \\ n_1 > \sqrt{T_1}}} n_1^{-1/2} n_2^{-1/2} n_3^{-1/2} \ll J_3(T),$$

where $J_3(T)$ was defined in (5.5.12). Consequently, combining (5.5.22) and the preceding line delivers

$$K_{1,2}(T) \ll T^{3/4}(\log T)^2 + T^{3/4-a/4c+1/2c_3}(\log T) + T^{3/4-a/4b+1/2b_2}(\log T)^2$$

and

$$\begin{aligned} P(T) \ll &T^{1/2+a/(a+c)+\varepsilon} + T^{3/4}(\log T)^2 + T^{3/4-a/4c+1/2c_3}(\log T) \\ &+ T^{3/4-a/4b+1/2b_2}(\log T)^2. \end{aligned}$$

The desired result will follow combining the above estimates with the ones from Lemma 5.7.1.

$\square$

5.8 An intermediate estimate and proof of Theorem 5.1.1

We find it desirable to note, as was anticipated in the introduction of the chapter, that the difficulty of the analysis of the terms $I_6(T)$, which shall be defined shortly, may be regarded in between that of $I_5(T)$ and the one performed in Lemma 5.6.1. Whislt Lemma 5.6.1 essentially required a straighforward application of Lemma 5.2.1, the examination of $I_5(T)$ entailed employing a stationary phase method type lemma of the strength of that of Lemma 5.2.5 for the purpose of exploiting the extra cancellation stemming from the averaging of the main term when assuming Conjecture 2. In contrast, the application of Lemmata 5.2.1 and 5.2.3, as shall be shewn promptly, will already suffice to obtain a suitable bound for the term $I_6(T)$. Without further delay, we announce that the results in this section shall be obtained under the assumption $a, b, c \in \mathbb{R}_+$ and define $I_6(T)$ by means of the sum

$$I_6(T) = Y_{6,b,c}(T) + Y_{6,c,b}(T), \tag{5.8.1}$$

where for tuples $(r, s) \in \mathbb{R}_+^2$ the above summands are

$$Y_{6,r,s}(T) = \int_0^T D(1/2-ait)D(1/2+rit)D(1/2-sit)\chi(1/2+ait)\chi(1/2-rit)dt.$$

Lemma 5.8.1. *Let $r, s \in \mathbb{R}_+^2$ such that $r > a$ and $s \geq a$. Then one has*

$$Y_{6,r,s}(T) \ll T^{5/4-r/4a}(\log T)^\tau + T^{3/4}(\log T),$$

wherein $\tau = 1$ if $s = a$ and $\tau = 0$ if $s > a$. In particular, it transpires that

$$I_6(T) \ll T^{5/4-c/4a} + T^{3/4}(\log T)$$

whenever $a < c \leq b$.

Proof. The approximation formulae for $\chi(1/2-rit)$ and $\chi(1/2+ait)$ in Lemma 5.2.6 then yield

$$Y_{6,r,s}(T) = \sum_{\boldsymbol{n}\in\mathcal{B}_{a,b,c}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^T e^{iF_6(t)}dt + O(T^{3/4}\log T), \tag{5.8.2}$$

where the function $F_6(t)$ is defined by

$$F_6(t) = (r-a)t\log t + rt\log r - at\log a - (r-a)t(\log 2\pi + 1) + t\log(n_1^a n_3^s/n_2^r),$$

and its derivative,

$$F_6'(t) = (r-a)\log t + r\log r - a\log a - (r-a)\log 2\pi + \log(n_1^a n_3^s/n_2^r).$$

We may discard first the case $N_{\boldsymbol{n}} = 2\pi n_1^2/a$, since then

$$F_6'(N_{\boldsymbol{n}}) \geq (r-a)\log n_1' + s\log n_3'$$

and a customary application of Lemmata 5.2.1 and 5.2.3 would yield the conclusion that the contribution to (5.8.2) corresponding to tuples satisfying such a condition is $O(T^{3/4})$. If instead $N_{\boldsymbol{n}} = 2\pi n_3^2/s$ then

$$F_6'(N_{\boldsymbol{n}}) \geq (r+s-2a)\log n_3' + a\log n_1',$$

and an analogous application of Lemmata 5.2.1 and 5.2.3 would imply that that the contribution to (5.8.2) corresponding to tuples satisfying such a condition is $O(T^{3/4})$.

If $N_{\boldsymbol{n}} = 2\pi n_2^2/r$ and $r \geq 2a$ then an analogous argument reveals that

$$F_6'(N_{\boldsymbol{n}}) \geq \log(n_1^a n_3^s) + a\log(r/a).$$

The analysis when $a < r < 2a$ requires some extra reasoning. We define for convenience the parameter $c_{\boldsymbol{n}} = 2\pi\big(a^a r^{-r} n_2^r n_1^{-a} n_3^{-s}\big)^{1/(r-a)}$, which the reader may check that satisfies $F_6'(c_{\boldsymbol{n}}) = 0$. It seems worth noting that applying Lemma 5.2.3 one may deduce that the integral over $[N_{\boldsymbol{n}}, T] \cap [c_{\boldsymbol{n}}/2, 2c_{\boldsymbol{n}}]$ is $O(c_{\boldsymbol{n}}^{1/2})$. Likewise, the integral over the complement of the latter intersection in $[N_{\boldsymbol{n}}, T]$ would then be $O(1)$ by Lemma 5.2.1. It may also seems appropiate to remark that were the previous intersection non-empty then one would have

$$c_{\boldsymbol{n}} \ll T,$$

which would in turn imply the inequality

$$n_2 \leq C n_1^{a/r} n_3^{s/r} T^{1-a/r} \tag{5.8.3}$$

for some constant $C > 0$. We find it desirable to denote by $M_6(T)$ to the contribution of such intersections to $Y_{6,r,s}(T)$. We further divide such a contribution into the one corresponding to tuples with the property that

$$C_1 n_1^{a/r} n_3^{s/r} \leq T^{a/r-1/2}, \tag{5.8.4}$$

where $C_1 = C(2\pi)^{1/2}$, which will be denoted by $M_{6,1}(T)$, and $M_{6,2}(T)$, that shall denote the contribution stemming from the complement of such tuples, the set of which will be denoted by means of the letter $\mathcal{J}_2$. For the sake of concision we write $\mathcal{J}_1$ to denote the set of tuples satisfying the inequalities (5.8.3) and (5.8.4). One then has

$$\begin{aligned} M_{6,1}(T) &\ll \sum_{\boldsymbol{n}\in\mathcal{J}_1} P_{\boldsymbol{n}}^{-1/2} c_{\boldsymbol{n}}^{1/2} \ll \sum_{\boldsymbol{n}\in\mathcal{J}_1} n_1^{-1/2-a/2(r-a)} n_2^{-1/2+r/2(r-a)} n_3^{-1/2-s/2(r-a)} \\ &\ll T^{1-a/2r} \sum_{n_1^a n_3^s \ll T^{a-r/2}} n_1^{-1/2+a/2r} n_3^{-1/2+s/2r} \\ &\ll T^{1-a/2r+(2a-r)(a+r)/4ar} \sum_{n_3 \leq \sqrt{cT_1}} n_3^{-1/2-s/2a} \ll T^{5/4-r/4a} (\log T)^{\tau}, \end{aligned}$$

where τ was defined in the statement of the lemma. The reader should find it worth noting that in the second line we employed the inequality (5.8.3) when summing over n_2.

The analysis of $M_{6,2}(T)$, though similar in nature, will depart from the previous procedure in that we will instead utilise the bound $n_2 \leq \sqrt{bT_1}$ in due course. We thus obtain

$$\begin{aligned} M_{6,2}(T) &\ll \sum_{\boldsymbol{n}\in\mathcal{J}_2} P_{\boldsymbol{n}}^{-1/2} c_{\boldsymbol{n}}^{1/2} \ll \sum_{\boldsymbol{n}\in\mathcal{J}_2} n_1^{-1/2-a/2(r-a)} n_2^{-1/2+r/2(r-a)} n_3^{-1/2-s/2(r-a)} \\ &\ll T^{1/4+r/4(r-a)} \sum_{n_1^a n_3^s \gg T^{a-r/2}} n_1^{-1/2-a/2(r-a)} n_3^{-1/2-s/2(r-a)}. \end{aligned}$$

It seems appropiate to remark that in the above lines we utilised the fact that by definition the tuples in $\mathcal{J}_2$ satisfy (5.8.3) and the converse of the inequality (5.8.4). Therefore, summing over n_1 in the second line of inequalities and recalling the assumption $2a > r$ one gets

$$\begin{aligned} M_{6,2}(T) \ll & T^{5/4-r/4a} \sum_{n_3 \ll T^{(2a-r)/2s}} n_3^{-1/2-s/2a} \\ & + T^{1/4+r/4(r-a)} \sum_{n_3^s \gg T^{a-r/2}} n_3^{-1/2-s/2(r-a)} \\ \ll & T^{5/4-r/4a} (\log T)^{\tau} + T^{3/4+(2a-r)/4s}. \end{aligned}$$

The reader may find it worth observing that in view of the aforementioned assumption, it transpires that

$$r - a < a \leq s,$$

whence the exponent of n_3 in the above sum is smaller than -1, such an remark justifying the subsequent line of argumentation thereof. We pause our analysis to examine and compare the bounds already obtained. It may be worth noting that

$$\begin{aligned} 3/4 + (2a-r)/4s &= (2a-r)/4s + r/4a - 1/2 + 5/4 - r/4a \\ &= \frac{r(s-a) - 2a(s-a)}{4as} + 5/4 - r/4a \leq 5/4 - r/4a, \end{aligned}$$

where we used the fact that $r < 2a$ and $s \geq a$. The preceding estimates then yield the bounds

$$\max\big(M_{6,1}(T), M_{6,2}(T)\big) \ll T^{5/4-r/4a} (\log T)^{\tau},$$

as desired. The second statement follows by recalling (5.8.1) and applying the result obtained above for $Y_{6,b,c}(T)$ and $Y_{6,c,b}(T)$. $\square$

Proof of Theorem 5.1.1. After the prolix discussion held above, it just suffices to complete the proof by observing that the theorem at hand will follow combining Proposition 5.5.1 with Lemmata 5.4.1, 5.6.1, 5.7.1, 5.7.2 and 5.8.1 and equation (5.4.4). It might be worth noting that

$$\begin{aligned} 5/4 - c/4a &= 1/2 - c/4a - a/4c + 3/4 + a/4c \\ &= -(c-a)^2/4ac + 3/4 + a/4c < 3/4 + a/4c. \end{aligned}$$

Therefore, it transpires that the error term $T^{3/4+a/4c}$ stemming from Lemma 5.7.1 dominates over that of $T^{5/4-c/4a}$ arising after an application of Lemma 5.8.1. Likewise, the reader may find it useful to observe that

$$1/4 + 3a/4c < 1/2 + a/(a+c)$$

whenever $a < c$, whence the term $T^{1/2+a/(a+c)+\varepsilon}$ dominates over $T^{1/4+3a/4c+\varepsilon}$, the latter arising after an application of Proposition 5.5.1 under the assumption of Conjecture 2.

Proof of Theorem 5.1.2. We depart from the proof of the above theorem in that we employ Proposition 5.5.2 instead of Proposition 5.5.1, the rest of the argument being analogous save the absence of the necessity in the use of Lemma 5.7.2.

5.9 An application of Roth's theorem on diophantine approximation

As was previously mentioned, the prelude of the examination of $I_{a,b,a}(T)$ will comprise the same strategy of approximating each of the zeta factors individually. It should be noted that the majority of the integrals arising from that departure has already been investigated in previous sections. We thus shall devote this new section to succintly discuss the ones that exhibit a different behaviour and find it desirable to announce that the arguments employed herein are dissimilar to those appertaining to the analysis of Theorem 5.1.1. For the sake of concision we shall not give account of the details cognate to arguments already presented in the course of the proof of Theorem 5.1.1 and thus refer the reader to previous discussions involving those. We begin our journey as is customary with the formula

$$I_{a,b,a}(T) = \sum_{j=1}^{6} I_j(T) + O\big(T^{3/4}(\log T)\big) \tag{5.9.1}$$

in (5.4.4), wherein the $I_j(T)$ were defined at the beginning of each of the above corresponding sections. We recall (5.5.2) and obtain as in (5.5.3) the identity

$$I_1(T) = S_{a,b}(T)T + J_2(T) - J_4(T), \tag{5.9.2}$$

where

$$S_{a,b}(T) = \sum_{\substack{\boldsymbol{n}\in\mathcal{B}_{a,b,a}\\ n_1^a=n_2^bn_3^a}} P_{\boldsymbol{n}}^{-1/2}, \qquad J_4(T) = \sum_{\substack{\boldsymbol{n}\in\mathcal{B}_{a,b,a}\\ n_1^a=n_2^bn_3^a}} N_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2},$$

and

$$J_2(T) = \sum_{\substack{\boldsymbol{n}\in\mathcal{B}_{a,b,a}\\ n_1^a\neq n_2^bn_3^a}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^{T} \left(\frac{n_2^bn_3^a}{n_1^a}\right)^{it} dt.$$

Lemma 5.9.1. *With the above notation one has*

$$I_1(T) = \frac{\zeta\big((a+b)/2\big)}{2} T\log T + J_2(T) + O(T).$$

Proof. We begin the discussion by observing that the solutions of the underlying equation in both $S_{a,b}(T)$ and $J_4(T)$ can be parametrized by means of the expressions

$$n_1 = m_3m_2^b, \qquad n_2 = m_2^a, \qquad n_3 = m_3.$$

Making use of the above we obtain

$$\begin{aligned} S_{a,b}(T) = \sum_{m_2^bm_3\leq\sqrt{aT_1}} m_3^{-1}m_2^{-(a+b)/2} &= \frac{\log T}{2}\sum_{m_2\leq(aT_1)^{1/2b}} m_2^{-(a+b)/2} \\ &+ O(1) - b\sum_{m_2\leq(aT_1)^{1/2b}} (\log m_2)m_2^{-(a+b)/2}, \end{aligned}$$

which then yields

$$S_{a.b}(T) = \frac{\zeta\big((a+b)/2\big)}{2}\log T + O(1).$$

Likewise, we employ the parametrization at hand in the way alluded above to get

$$J_4(T) \ll \sum_{m_3m_2^b\leq\sqrt{aT_1}} m_2^{(3b-a)/2}m_3 \ll T^{3/4+1/2b-a/4b}\sum_{m_3\leq\sqrt{aT_1}} m_3^{-1/2-1/b+a/2b} \ll T.$$

Combining the preceding equations with (5.9.2) delivers the desired result and completes the proof. □

The reader may have noticed that one could have further refined the above analysis to obtain lower order terms in the asymptotic formula at hand. As may become apparent shortly, these improvements though would have been wrought in vain due to the poor understanding of $J_2(T)$ that we have.

Lemma 5.9.2. *One has that*

$$J_2(T) = o(T \log T).$$

Proof. It seems pertinent to define, for each tuple (n_2, n_3) for which $n_2^{b/a} n_3$ is not an integer, the number $N_{b,a} = \lfloor n_2^{b/a} n_3 \rfloor$, and on recalling (5.5.6) and (5.5.7) we write, as was done therein,

$$J_2(T) = J_{2,1}(T) + J_{2,2}(T).$$

We find it worth alluding to (5.5.10) to the end of obtaining the bound

$$J_{2,1}(T) \ll T^{3/4}.$$

In order to analyse $J_{2,2}(T)$ we observe first that

$$J_{2,2}(T) \ll \sum_{\substack{n_2, n_3 \\ |n_1 - N_{b,a}| \leq 1}} n_2^{-1/2 - b/2a} n_3^{-1} \Big| \int_{N_{\boldsymbol{n}}}^{T} e^{it \log(n_2^b n_3^a / n_1^a)} dt \Big|. \tag{5.9.3}$$

We also note that Roth's theorem on rational approximation [119] implies that for each pair (n_2, n_3) with the property that $n_2^{b/a}$ is not an integer and for every fixed $\varepsilon > 0$ then the inequality

$$\Big| n_2^{b/a} n_3 - N_{b,a} \Big| \geq \frac{C'(\varepsilon, n_2)}{n_3^{1+\varepsilon}} \tag{5.9.4}$$

holds, where $C'(\varepsilon, n_2)$ only depends on ε and n_2. Therefore, the above estimate in conjunction with (5.5.8) delivers the lower bound

$$\big| \log \big(n_2^b n_3^a / N_{b,a}^a \big) \big| \geq \frac{C(\varepsilon, n_2)}{n_3^{2+\varepsilon}}. \tag{5.9.5}$$

The reader may observe that the same inequalities hold whenever $N_{b,a} \pm 1$ is replaced by $N_{b,a}$ as well. For the purpose of organising our argument rather neatly it seems pertinent to denote $L_1(T)$ the contribution to $J_{2,2}(T)$ of tuples satisfying $C(\varepsilon, n_2)^{-1} n_3^{2+\varepsilon} \leq T$. Likewise, we write $L_2(T)$ for the contribution of tuples with the property that $C(\varepsilon, n_2)^{-1}n_3^{2+\varepsilon} > T$. Then by the preceding discussion it transpires that

$$L_1(T) \ll \sum_{C(\varepsilon,n_2)^{-1}n_3^{2+\varepsilon}\leq T} C(\varepsilon, n_2)^{-1} n_2^{-1/2-b/2a} n_3^{1+\varepsilon} \ll T \sum_{n_2\leq\sqrt{bT_1}} n_2^{-1/2-b/2a} \ll T,$$

where we estimated the integral in (5.9.3) by the inverse of the corresponding logarithm and (5.9.5).

In order to make further progress we find it desirable to introduce the parameter N, which the reader should think of as being large but fixed. We also write

$$C_\varepsilon(N) = \min_{1\leq n_2\leq N} C(\varepsilon, n_2)$$

for further convenience. We then estimate the integral on the right side of (5.9.3) by the trivial bound T and thus obtain

$$L_2(T) \ll T \sum_{C(\varepsilon,n_2)^{-1}n_3^{2+\varepsilon}>T} n_2^{-1/2-b/2a} n_3^{-1} \ll T\big(L_{2,1}(T) + L_{2,2}(T)\big),$$

where

$$L_{2,1}(T) = \sum_{\substack{n_2\geq N\\ n_3\leq\sqrt{aT_1}}} n_2^{-1/2-b/2a} n_3^{-1} \qquad \text{and} \qquad L_{2,2}(T) = \sum_{(C_\varepsilon(N)T)^{1/(2+\varepsilon)}\leq n_3\leq\sqrt{aT_1}} n_3^{-1}.$$

By summing over n_2 and n_3 we obtain the bound

$$L_{2,1}(T) \ll N^{1/2-b/2a} \log T \tag{5.9.6}$$

Likewise, for fixed $0 < \varepsilon \leq 1$ one finds that

$$L_{2,2}(T) \ll \varepsilon \log T + \log C_\varepsilon(N). \tag{5.9.7}$$

It is of great importance to emphasize that the implicit constants cognate to the above bounds for $L_{2,1}(T)$ and $L_{2,2}(T)$ do not depend on neither ε nor N. We also find it desirable to observe that in view of the estimation process appertaining to $L_{2,2}(T)$ in conjunction with a careful perusal of the underlying argument underpinning the choice of the above cutoff parameters, it transpires that the presence of the exponent $2+\varepsilon$ in (5.9.4) played a crucial roll, as was pointed out in the introduction. Therefore, by the preceding discussion we obtain for any fixed $\varepsilon > 0$ the estimate

$$\lim_{T\to\infty} \frac{|J_2(T)|}{T\log T} \ll \varepsilon + N^{1/2-b/2a}.$$

Consequently, letting $N \to \infty$ and $\varepsilon \to 0$ in the above line and recalling that $a < b$ we obtain the desired result. □

As was previously alluded to in the introduction, we take this as an opportunity to draw the reader's attention to the estimates (5.9.6) and (5.9.7) for the purpose of emphasizing on the fact that the ineffectiveness in Roth's theorem with respect to both ε and n_2 is then transferred to the ineffectiveness of the error term cognate to the asymptotic formula deduced herein.

For the purpose of ascending to a position from which to complete the proof, we find it desirable to combine the above lemmata to the end of obtaining the equation

$$I_1(T) = \frac{1}{2}\zeta\big((a+b)/2\big)T\log T + o(T\log T),$$

and remind the reader that the analysis of the terms $I_i(T)$ for $2 \le i \le 5$ in Lemmata 5.6.1 and 5.7.2 was performed under the assumption that $a \le c$. The application of those then yields

$$\sum_{i=2}^{5} |I_i(T)| \ll T.$$

It is worth noting that the largest contribution is stemming from the estimate cognate to $I_5(T)$. We finally use the observation that

$$\chi(1/2 + ait)\chi(1/2 - ait) = 1$$

to deduce that

$$Y_{6,a,b}(T) = \int_0^T D(1/2 - ait)D(1/2 - bit)D(1/2 + ait)dt = I_1(T),$$

where $Y_{6,a,b}(T)$ was defined right before Lemma 5.8.1. We also employ such a lemma to obtain the estimate

$$Y_{6,b,a}(T) \ll T^{5/4-b/4a}(\log T) + T^{3/4}(\log T),$$

and find it adecquate to recall for further purposes the equations (5.9.1) and

$$I_6(T) = Y_{6,a,b}(T) + Y_{6,b,a}(T)$$

presented in (5.8.1). The combination of the above estimates and identities then delivers the required asymptotic formula for $I_{a,b,a}(T)$ and completes the proof of Theorem 5.1.3.

Chapter 6

Mixed moments of the Riemann zeta function. A smooth approach

6.1 Introduction

The prolixity in the discussion appertaining to the previous chapter as a consequence of the approach chosen comprising the perusal of eight integrals, several of which exhibiting dissimilar behaviour, may have made the experienced reader wonder about the posibility of using an analogous approximation functional equation for the product

$$\zeta(1/2+iat)\zeta(1/2-ibt)\zeta(1/2-ict) \tag{6.1.1}$$

similar in vein to that utilised by Heath-Brown [59], such an avenue having the potential to possibly circumvent those recalcitrant computations. Exploring these circle of ideas is, inter alia, the main purpose of this chapter. For such purposes it seems convenient first to draw the reader's attention back to the definition of $I_{a,b,c}(T)$ stated in (5.1.1).

Theorem 6.1.1. *Let $T>0$. Then, whenever $a<c\leq b$ and $4a<b+c$ for fixed coefficients $a,b,c\in\mathbb{N}$ one has the formula*

$$I_{a,b,c}(T)=\sigma_{a,b,c}T+E_{a,b,c}(T),$$

where $\sigma_{a,b,c}$ was defined in (5.5.13) and the corresponding error term satisfies when $c < b$ the bound

$$E_{a,b,c}(T) \ll T^{1-1/2a+1/2c} + T^{3/4}(\log T)^4 + T^{3/4+(2a-c)/2(b-c)}, \tag{6.1.2}$$

the corresponding implicit constant only depending on a, b, c. If $b = c$, the term $T^{3/4+(2a-c)/2(b-c)}$ may be omitted from the above. If Conjecture 2 is assumed then one further has

$$E_{a,b,c}(T) \ll \min(T^{1/2+a/(a+c)+\varepsilon}, T^{1-1/2a+1/2c}) + T^{3/4}(\log T)^4 + T^{3/4+(2a-c)/2(b-c)}$$

whenever $c < b$, an analogous estimate holding without the term $T^{3/4+(2a-c)/2(b-c)}$ if $b = c$.

In view of the above statements and that of Theorem 5.1.1 it transpires that the novel error term introduced herein shall be inferior to the one pertaining to the latter theorem whenever the corresponding coefficients lie in certain ranges at the sacrifice of losing the validity of the asymptotic formula for other ranges, such an observation being the main reason for including this other approach in our memoir. It also seems worth announcing that the application of the approximate functional equation derived herein reduces the problem to computing only two integrals, as opposed to the situation in the previous chapter, it therefore diminishing the number of integrals to examine and partially alleviating the work at the cost of increasing the difficulty of such integrals. We draw the reader's attention back to (6.1.2) to the end of anticipating that the summand

$$T^{3/4+(2a-c)/2(b-c)} \tag{6.1.3}$$

therein stems from the application of the stationary phase method to the twisted integral in conjunction with an estimation of the main term arising from such an application by means of van der Corput's methods. As opposed to what one may have initially thought, it is a noteworthy feature that the application of van der Corput's estimate enables one to bound the main contribution by the term (6.1.3), such an estimate being esentially optimal to the effect that the error terms cognate to the asymptotic evaluation delivered by the stationary phase method are naturally bounded by an analogous quantity.

We remind the reader that in the preceding chapter the triples arising after the application of the approximate functional equation to each of the zeta functions lied in the cube $[1, \sqrt{T_1}]^3$. In contrast, we anticipate that the corresponding triples (n_1, n_2, n_3) when approximating the product of zeta functions will only be required to satisfy an inequality of the type

$$n_1 n_2 n_3 \ll T^{3/2}.$$

The excess of variables in comparison to the previous approach facilitates the examination pertaining to the residual terms stemming from the diagonal contribution, the corresponding bounds for such terms being of the requisite precision, at the cost of increasing the difficulty of the analysis of the integral containing a twisting factor and ultimately impairing the range pertaining to the coefficients a, b, c for which the asymptotic formula at hand holds.

We find it desirable to draw the reader's attention back to Section 5.9 of Chapter 5 to the end of remarking the necessary proviso that the variable n_2 underlying the analysis satisfied a bound of the shape

$$n_2 \ll \sqrt{T}$$

for the corresponding argument be applicable. It then transpires that the approach that shall be pursued in the upcoming chapter would lead one to a position from which to estimate an analogous sum to those arising in the aforementioned section with the corresponding variables satisfying the property that

$$n_2^2 n_3^{1+b/a} \ll T^{3/2},$$

the underlying ideas of the proof of Theorem 5.1.3 being of no longer utility.

In order to prepare the ground for the customary analysis pertaining to the perusal of the diagonal and off-diagonal contribution and that of the twisted integral, an intrincate process comprising the use of Cauchy residue theorem in conjunction with succesive applications of Stirling's formula has to be performed to the end of accomplishing the approximate functional equation. It seems desirable to draw the reader's attention to Heath-Brown's seminal paper [59] and anticipate that in the setting herein some extra terms in the corresponding auxiliary functions involved in the complex analysis arguments arise as a consequence of the loss of symmetry when considering arbitrary coefficients a, b, c, such complications being easily surmounted. We should nonetheless point out that a condition pertaining to the coefficients shall be imposed to the end of bounding the corresponding residues appropiately, such a constraint only excluding a small handful of cases which by no means affects the range considered in Theorem 6.1.1.

Once the approximate functional equation is established, there are some additional terms which arise in the analysis, the contribution of the corresponding integrals of such terms ultimately being of a small size. Nonetheless, those terms shall not be estimated pointwise, such a cheap approach leading otherwise to undesirably large bounds. Instead, we estimate them on average by means of oscillatory integral lemmata, and for the purpose of reaching a position from which to apply those, a verification of the monotonicity of some auxiliary function has to be performed. It is worth clarifying to the interested reader that our treatment to overcome such difficulties departs from that of Heath-Brown [59] in that we employ an estimate involving both the second derivative of the phase function comprising the twisting factor of the aforementioned additional term twisted by

$$\Big(\frac{n_2^b n_3^c}{n_1^a}\Big)^{it} \tag{6.1.4}$$

and a pointwise bound for the corresponding weight function. The treatment of Heath-Brown instead entailed integrating by parts the analogous product and utilised the pointwise bounds of the additional term, such an approach in our context diverting one to the undesirable position of encountering sums containing the factor

$$\Big|\log\Big(\frac{n_2^b n_3^c}{n_1^a}\Big)\Big|^{-1},$$

for which we have a poor understanding. It then transpires that the phase of (6.1.4) vanishes when taking the second derivative, such an observation comprising the genesis of the success in the approach taken herein.

It seems worth observing that the analysis of the contribution stemming from the main term in the approximate functional equation in the classical fourth moment setting (see [59]) makes an elegant use of the underlying symmetry to exhibit further cancellation when integrating such a term twisted by

$$\left(\frac{m}{n}\right)^{it}.$$

In the absence of such a property herein, our analysis will comprise a careful examination of the corresponding phases that will eventually lead to a division of the corresponding tuples depending on the size of the phases, such intrincate process being ultimately culminated with a routinary application of an oscillatory integral estimate.

The structure of the chapter is organised as follows: Section 6.2 is devoted to a prolix discussion concerning the approximate functional equation pertaining to (6.1.1). In Section 6.3 we analyse the monotonicity of a certain class of functions to the end of preparing the ground for an application of oscillatory integral estimates of a certain type at various points in the chapter. The diagonal contribution stemming from the non-twisted integral is computed in Section 6.4, the examination of which largely benefits from the earlier work done in the previous chapter. Section 6.5 is devoted to the analysis of the off-diagonal contribution and combines a diophantine approximation perspective, in the spirit to that of Lemma 5.5.2, with a more analytic one involving oscillatory integral estimates and complex analytic ideas in conjunction with simple combinatorial arguments. The discussion concerning various residual terms which arises from the contribution of the twisted integral in Section 6.6 utilises many of the ideas from its preceding section. The chapter concludes in Section 6.7 with an application of the stationary phase method to evaluate the twisted integral, a prolix intrincate examination of the error terms stemming from such an application and an estimate of the corresponding main term by means of van der Corput's methods.

6.2 The approximate functional equation

We begin by furnishing ourselves with a lemma which essentially follows Heath-Brown's approach [59] for computing the fourth moment of the Riemann Zeta function and shall ultimately provide the approximate functional equation to which we alluded in the introduction. As was earlier anticipated, the main result herein concerns the mixed third moment. Nevertheless, it has been thought preferable to present the lemma in wider generality for the purpose of preparing the ground for subsequent work. It should be noted then that the analysis embodied in the following discussion encompasses mixed k-th moments.

Before providing a rather precise formulation of such a result we find it convenient to define for each natural number $k \geq 2$ the subset $\mathcal{A}_k \subset (\mathbb{R} \setminus \{0\})^k$ of tuples $\boldsymbol{a} = (a_1, \ldots, a_k)$ satisfying the inequalities

$$a_j^2 > \frac{\pi}{4}\Big(- \xi_{\boldsymbol{a}} a_j + \sum_{l=1}^{k} |a_l| - \sum_{l} |a_l - a_j| \Big), \qquad (1 \leq j \leq k) \tag{6.2.1}$$

where in the above line the number $\xi_{\boldsymbol{a}}$ is defined by means of the formula

$$\xi_{\boldsymbol{a}} = \sum_{a_l > 0} 1 - \sum_{a_l < 0} 1. \tag{6.2.2}$$

For simplicity of the exposition it has been thought adequate to introduce the parameters

$$I_{\boldsymbol{a}} = \sum_{l=1}^{k} \frac{1}{a_l}, \qquad P_{\boldsymbol{a}} = \prod_{j=1}^{k} a_j, \tag{6.2.3}$$

which may not be of great theoretical relevance but might be worth considering nonetheless. For each tuple of natural numbers $\boldsymbol{n} = (n_1, \ldots, n_k)$ we further denote

$$P_{\boldsymbol{n}} = \prod_{j=1}^{k} n_j, \qquad L_{\boldsymbol{a}}(\boldsymbol{n}) = \prod_{j=1}^{k} n_j^{-a_j}.$$

We find it desirable to consider the product of gamma functions

$$P_{\boldsymbol{a}}(t) = \prod_{j=1}^{k} \Gamma\big(1/2(1/2 + i a_j t)\big),$$

which shall make its appearance in the course of the discussion concerning the approximate functional equation due to the concomitant aspect of the gamma function playing a role in the corresponding functional equation for the Riemman zeta function. Likewise, we further define

$$G_m(z, t) = \pi^{-k/4} P_{\boldsymbol{a}}(t)^{-1} \prod_{j=1}^{k} \Gamma\Big(\frac{1}{2}\Big(\frac{1}{2} + (-1)^{m+1} i a_j t + z\Big)\Big), \qquad m = 1, 2. \tag{6.2.4}$$

It may also seem appropiate to recall (6.2.2) and introduce the smoothing factor which gives rise to the title of this chapter

$$H(z,t) = e^{z^2/t - i\xi_{\boldsymbol{a}}\pi z/4}, \tag{6.2.5}$$

as shall be demonstrated promptly.

Lemma 6.2.1. *Let $\boldsymbol{a} \in \mathcal{A}_k$. Then one has that*

$$\prod_{j=1}^{k} \zeta(1/2 + ia_j t) = \sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2} I(P_{\boldsymbol{n}}, t) + O(t^{-2}), \tag{6.2.6}$$

wherein $\boldsymbol{n}$ runs over triples of natural numbers, the function $I(P_{\boldsymbol{n}}, t)$ at hand is defined by means of the equation

$$I(P_{\boldsymbol{n}}, t) = L_{\boldsymbol{a}}(\boldsymbol{n})^{it} I_1(P_{\boldsymbol{n}}, t) + L_{\boldsymbol{a}}(\boldsymbol{n})^{-it} I_2(P_{\boldsymbol{n}}, t),$$

the terms $I_1(x,t)$ and $I_2(x,t)$ for $x \in \mathbb{R}$ being

$$I_m(x,t) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} G_m(z,t)\big(\pi^{k/2}x\big)^{-z} H\big((-1)^{m+1}z, t\big)\frac{dz}{z}.$$

Proof. We define, for convenience, the meromorphic function

$$f_{\boldsymbol{a}}(w) = \pi^{-kw/2}\prod_{j=1}^{k}\Gamma\big(1/2(w + ia_j t)\big)\zeta(w + ia_j t)$$

with poles at $w = -ia_j t$ and $w = 1 - ia_j t$ in the region $\mathrm{Re}(w) \geq -3/2$. For ease of notation it has been thought preferable to denote henceforth $f_{\boldsymbol{a}}(w)$ by $f(w)$. As shall be elucidated promptly, we find it relevant to consider the integrals

$$M_m(t) = \frac{1}{2\pi i}\int_{(-1)^{m+1}-i\infty}^{(-1)^{m+1}+i\infty} f(1/2 + z) H(z,t)\frac{dz}{z}, \qquad m = 1, 2.$$

The reader may find it useful to observe that when $s \in \mathbb{R}$ then an application of the functional equation for the Riemann zeta function yields

$$f(-1/2 + is) = \pi^{-k(3/2 - is)/2}\prod_{j=1}^{k}\Gamma\big(1/2(3/2 - is - ia_j t)\big)\zeta(3/2 - is - ia_j t),$$

whence on defining the function

$$Y(z,t) = \pi^{-k/4}\pi^{-kz/2}\prod_{j=1}^{k}\Gamma\big(1/2(1/2 - ia_jt + z)\big)\zeta(1/2 - ia_jt + z)$$

and making a change of variables accordingly, it transpires that

$$M_2(t) = -\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} Y(z,t)H(-z,t)\frac{dz}{z}. \tag{6.2.7}$$

As a prelude to our discussion it seems pertinent to note that one may utilise the convergence of the series cognate to the Riemann zeta function at $\mathrm{Re}(z) = 3/2$ in conjunction with (6.2.4) to deduce

$$M_1(t) = \pi^{-k/4}P_{\boldsymbol{a}}(t)\sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2}L_{\boldsymbol{a}}(\boldsymbol{n})^{it}I_1(P_{\boldsymbol{n}},t) \tag{6.2.8}$$

and

$$M_2(t) = -\pi^{-k/4}P_{\boldsymbol{a}}(t)\sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2}L_{\boldsymbol{a}}(\boldsymbol{n})^{-it}I_2(P_{\boldsymbol{n}},t). \tag{6.2.9}$$

It then transpires that an application of Cauchy's residue theorem already delivers the formula

$$M_1(t) - M_2(t) = f(1/2) + \frac{1}{2\pi i}\sum_{m=1}^{2k}\int_{\mathcal{C}_m} f(1/2+z)H(z,t)\frac{dz}{z},$$

wherein the above equation $\mathcal{C}_m$ denotes a circular path of radius t^{-1} around each of the poles of $f(1/2+w)$. We have thus reach a position from which to derive the desired result subject to the estimation of the contribution of the remaining residues, and this we now perform. We employ Stirling's series (see Whittaker and Watson [158, §13.6]), namely

$$\log\Gamma(z) = (z-1/2)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{r=1}^{N} c_r z^{1-2r} + O(|z|^{-1-2N}), \tag{6.2.10}$$

where c_r are fixed coefficients and $N \in \mathbb{N}$, to the end of observing that whenever the pole at hand pertaining to the function $f(1/2+w)$ is either $w = 1/2 - ia_jt$

or $w=-1/2-ia_jt$ one may deduce the bound

$$f(1/2+w)\ll t^Ce^{-C_j\pi t/4}$$

in the corresponding contour cognate to the aforementioned poles, with C being a positive constant depending on the tuple $\boldsymbol{a}$ and

$$C_j=\sum_l|a_l-a_j|.$$

Likewise, under the same circumstances the estimate

$$H(z,t)\ll e^{-a_j^2t-\xi_{\boldsymbol{a}}\pi a_jt/4}$$

holds on the same contour, whence the preceding discussion then yields

$$f(1/2)=M_1(t)-M_2(t)+O(t^C\max_j e^{-a_j^2t-\xi_{\boldsymbol{a}}\pi a_jt/4-C_j\pi t/4}). \tag{6.2.11}$$

It therefore remains to divide the above equation by the product of gamma factors $\pi^{-k/4}P_{\boldsymbol{a}}(t)$, a concomitant requisite being the estimation of the inverse of such a product, and this we achieve by means of a routine application of Stirling's formula. To this end, it might be worth considering

$$C_{\boldsymbol{a}}=\sum_{l=1}^k|a_l|,$$

for convenience, and note that as was already anticipated, Stirling's formula then yields

$$P_{\boldsymbol{a}}(t)^{-1}\ll e^{\pi C_{\boldsymbol{a}}t/4}.$$

We then divide both sides of (6.2.11) by the product $P_{\boldsymbol{a}}(t)$ and combine it with equations (6.2.8) and (6.2.9) to the end of obtaining

$$\prod_{j=1}^k\zeta(1/2+ia_jt)=\sum_{\boldsymbol{n}}P_{\boldsymbol{n}}^{-1/2}I\big(P_{\boldsymbol{n}},t\big)+E(t),$$

where in the above equation the corresponding error term $E(t)$ satisfies

$$E(t)\ll t^C\max_j e^{-a_j^2t-\xi_{\boldsymbol{a}}\pi a_jt/4+(C_{\boldsymbol{a}}-C_j)\pi t/4}.$$

We find it desirable to note that we made use of (6.2.7) to derive the previous equation, and find it appropiate to remark that in view of the condition (6.2.1), it transpires that

$$E(t) \ll e^{-Kt}$$

for some constant $K > 0$. The preceding remark then, in conjunction with the above formula delivers the desired result.

$\square$

For the purpose of progressing in the proof, it seems pertinent to present the following technical lemma, the main idea latent in the corresponding analysis having its reliance on a routine application of Stirling's formula to prepare the ground for the integration over t. We find it worth anticipating that such a lemma will be reminiscent of the previous one, and note that we shall merely confine ourselves to decomposing the integrand involved in the expression for $I_m(P_{\boldsymbol{n}}, t)$ into a main term and a secondary term which shall not be treated as an error term but whose contribution after integrating over t shall be residual.

Before embarking ourselves in such an endeavour, it may be convenient to define

$$A(x,t) = P_{\boldsymbol{a}}^{1/2}(t/2\pi)^{k/2}/x \tag{6.2.12}$$

and to remind the reader of the definition of $\mathcal{A}_k$ right above (6.2.1) and $I_{\boldsymbol{a}}$ in (6.2.3).

Lemma 6.2.2. *Let $k \geq 2$ and $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathcal{A}_k$. Then there exist constants $c_i(u,v) \in \mathbb{C}$ with $i = 1, 2$ for which*

$$\prod_{j=1}^{k} \zeta(1/2 + ia_j t) = \sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2} K(\boldsymbol{n}, t) + O(t^{-1}),$$

where in the above equation the function $K(\boldsymbol{n}, t)$ is defined by means of the relation

$$K(\boldsymbol{n}, t) = L_{\boldsymbol{a}}(\boldsymbol{n})^{it} K_1(P_{\boldsymbol{n}}, t) + L_{\boldsymbol{a}}(\boldsymbol{n})^{-it} K_2(P_{\boldsymbol{n}}, t),$$

the alluded terms $K_1(x,t)$ and $K_2(x,t)$ for $x \in \mathbb{R}_+$ being

$$K_1(x,t) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+\infty} A(x,t)^z F_1(z,t) \Big(1 + \sum_{u,v} c_1(u,v) z^u t^{-v}\Big) \frac{dz}{z},$$

and

$$K_2(x,t) = \frac{\psi(t)}{2\pi i}\int_{1-i\infty}^{1+i\infty} A(x,t)^z F_2(z,t)\Big(1+\sum_{u,v} c_2(u,v)z^u t^{-v}\Big)\frac{dz}{z},$$

where the functions $F_m(z,t)$ *and* $\psi(t)$ *are defined as*

$$F_1(z,t) = e^{z^2/t - iI_{\boldsymbol{a}}z^2/4t}, \qquad F_2(z,t) = e^{z^2/t + iI_{\boldsymbol{a}}z^2/4t}, \qquad \psi(t) = e^{\xi_{\boldsymbol{a}}\pi i/4 - ig_{\boldsymbol{a}}(t)}, \tag{6.2.13}$$

and the function $g_{\boldsymbol{a}}(t)$ *is defined by means of the formula*

$$g_{\boldsymbol{a}}(t) = \sum_{j=1}^{k} a_j t\big(\log(|a_j|t/2) - 1\big). \tag{6.2.14}$$

Moreover, in the above sums the parameters (u,v) *run over the tuples satisfying* $1 \le u \le 3v/2$ *and* $1 \le v \le 2(k+5)$ *with the property that if* $u \ge v+1$ *then* $v \ge 2$.

Proof. We observe that in view of Lemma 6.2.1 it transpires that showing the validity of the above asymptotic evaluation amounts to proving that

$$I_m(x,t) - K_m(x,t) \ll x^{-1}t^{-2}, \qquad m = 1,2, \tag{6.2.15}$$

since then the corresponding error term that arises when substituting $I_m(P_{\boldsymbol{n}},t)$ by $K_m(P_{\boldsymbol{n}},t)$ in (6.2.6) will be bounded above by

$$t^{-2}\sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-3/2} \ll t^{-2}.$$

It may be worth analysing first $I_1(x,t)$. For such matters it seems convenient to introduce the parameters $\beta = z/2$, $\alpha_j = ia_jt/2$ and $\gamma_j = 1/4 + \alpha_j$ for each j. We shall henceforth assume that $\mathrm{Re}(z) = 1$ and confine ourselves first to the analysis of the function $G_1(z,t)$ when $|\mathrm{Im}(z)| \le t^{1/2}\log t$. It appears at first glance that a customary application of Stirling's formula (6.2.10) with the choice $N = \lceil k/4 + 3/4 \rceil$ delivers

$$\log \Gamma(\gamma_j + \beta) - \log \Gamma(\gamma_j) = \beta \log \gamma_j + (\gamma_j + \beta - 1/2) \log \big(1 + \beta/\gamma_j\big) \\ - \beta + \sum_{r=1}^{N} c_r \big((\gamma_j + \beta)^{1-2r} - \gamma_j^{1-2r}\big) + O(t^{-1-2N}).$$

The reader may notice that an application of the Taylor expansion of both $\log(1+w)$ and $(1+w)^{-1}$ reveals that the above equation equals

$$\log \Gamma(\gamma_j + \beta) - \log \Gamma(\gamma_j) = \beta \log \gamma_j + \frac{\beta^2}{2\gamma_j} + \sum_{u,v} c_1'''(u,v) \beta^u \gamma_j^{-v} + O(t^{-5/2-k/2}), \tag{6.2.16}$$

wherein the above sum (u, v) run over the tuples satisfying $1 \leq u \leq v + 1$ and $1 \leq v \leq k + 5$ with the property that if $u \geq v + 1$ then $v \geq 2$, and $c_1'''(u, v)$ are fixed coefficients. The reader may observe that the rest of the terms stemming from the application of the Taylor expansion thereof may be absorbed into the error term therein.

It also seems pertinent to note that a routine application of Taylor expansions yields

$$\gamma_j^{-n} = \alpha_j^{-n} + \alpha_j^{-n} \sum_{m=1}^{\infty} k_{1,m} \alpha_j^{-m}$$

and

$$\log \gamma_j = \log \alpha_j + \sum_{m=1}^{\infty} k_{2,m} \alpha_j^{-m},$$

wherein $k_{1,m}, k_{2,m} \in \mathbb{R}$ denote as is customary fixed coefficients. These expressions in conjunction with the above equation and the definitions for β and γ_j earlier described then enable one to express the right side of (6.2.16) as

$$\frac{z}{2}\big(\log(|a_j|t/2) + i\mathrm{sgn}(a_j)\pi/2\big) - i\frac{z^2}{4ta_j} + \sum_{u,v} c_1''(u,v) z^u \alpha_j^{-v} + O(t^{-5/2-k/2}),$$

where herein (u, v) runs over the collection of tuples earlier described, and $c_1''(u, v)$ denote fixed real coefficients. Therefore, by recalling (6.2.4) and summing over j we obtain

$$\log G_1(z,t) = \frac{z}{2}\log\Big((t/2)^k\prod_{j=1}^{k}|a_j|\Big) + zi\pi\xi_{\boldsymbol{a}}/4 - i\frac{I_{\boldsymbol{a}}}{4t}z^2 + \sum_{u,v}c_1'(u,v)z^u(it)^{-v} + O(t^{-5/2-k/2}),$$

whence raising the above equation to the power e then yields

$$G_1(z,t) = \Big((t/2)^{k/2}\prod_{j=1}^{k}|a_j|^{1/2}\Big)^z e^{i\pi\xi_{\boldsymbol{a}}z/4 - iI_{\boldsymbol{a}}z^2/(4t)} \times\Big(1+\sum_{u,v}c_1(u,v)z^u t^{-v} + O(t^{-5/2-k/2})\Big),$$

and where (u,v) lies in the range described right after (6.2.14). We should note that both $c_1'(u,v)$ and $c_1(u,v)$ in the above equations denote fixed coefficients. By the preceding discussion in conjunction with (6.2.13) we thus obtain

$$(\pi^{k/2}x)^{-z}G_1(z,t)H(z,t)z^{-1} = A(x,t)^z F_1(z,t)z^{-1}\Big(1+\sum_{u,v}c_1(u,v)z^u t^{-v}\Big) + O\Big(x^{-1}t^{-5/2}|z|^{-1}e^{|I_{\boldsymbol{a}}\mathrm{Im}(z)|/2t-(\mathrm{Im}(z))^2/t}\Big), \tag{6.2.17}$$

wherein the reader may find it desirable to recall the definition (6.2.5). By integrating the above equation over the segment $[1-it^{1/2}\log t, 1+it^{1/2}\log t]$, it transpires that the contribution $C_1(x,t)$ stemming from the error term will satisfy

$$C_1(x,t) \ll x^{-1}t^{-2}. \tag{6.2.18}$$

In order to ascend to a position from which to obtain the desired approximation, it seems pertinent to investigate the function $G_1(z,t)$ at hand whenever $|\mathrm{Im}(z)| > t^{1/2}\log t$. For ease of notation we denote $y = \mathrm{Im}(z)$, and apply (6.2.10) on the range $|y| > t^{1/2}\log t$ to obtain

$$\log\Gamma(\gamma_j+\beta) - \log\Gamma(\gamma_j) = (\gamma_j+\beta)\log(\gamma_j+\beta) - \gamma_j\log(\gamma_j) - \beta - 1/2\log(1+\beta/\gamma_j) + O(1),$$

whence taking real parts in the above expression yields

$$\log|\Gamma(\gamma_j+\beta)| - \log|\Gamma(\gamma_j)| = -\frac{\pi}{4}(a_jt+y)\mathrm{sgn}(a_jt+y) + \frac{\pi}{4}a_jt\cdot\mathrm{sgn}(a_j) + O\big(\log(t+|y|)\big).$$

It might be worth noting that on recalling (6.2.4) it follows that

$$\log|G_1(z,t)| = \sum_{j=1}^{k}\log|\Gamma(\gamma_j+\beta)| - \log|\Gamma(\gamma_j)|,$$

whence in the interest of deriving an estimate of an appropiate precision it seems pertinent to show the inequality

$$-\sum_{j=1}^{k}|a_jt+y| + \sum_{j=1}^{k}|a_jt| + \xi_{\boldsymbol{a}}y \leq 0.$$

The reader may observe that such a bound follows from the estimates

$$-\sum_{a_j>0}|a_jt+y| + \sum_{a_j>0}a_jt + y\sum_{a_j>0}1 \leq 0, \qquad -\sum_{a_j<0}|a_jt+y| - \sum_{a_j<0}a_jt - y\sum_{a_j<0}1 \leq 0,$$

which in turn are an immediate consequence of the triangle inequality. Therefore, combining the previous bounds we find that

$$|G_1(z,t)| \ll (yt)^C e^{-\pi\xi_{\boldsymbol{a}}y/4}$$

for some constant $C>0$. Such an estimate in conjunction with the definition (6.2.5) yields

$$(\pi^{k/2}x)^{-z}G_1(z,t)H(z,t)z^{-1} \ll x^{-1}(yt)^C e^{-y^2/t}.$$

It then transpires at first glance that whenever $|y| > t^{1/2}\log t$ the right side of the above equation is then $O(x^{-1}t^{-2})$. Likewise, by recalling (6.2.12) one may deduce under the same circumstances that

$$A(x,t)^z F_1(z,t)z^{-1}\big(1+\sum_{u,v}c_1(u,v)z^u t^{-v}\big) \ll x^{-1}(yt)^C e^{|yI_{\boldsymbol{a}}|/(2t)-y^2/t}.$$

We integrate (6.2.17) over the line $\mathrm{Re}(z)=1$ and utilise (6.2.18) in conjunction with the above inequalities to obtain (6.2.15) for the case $m=1$, as desired.

In order to make progress in our endeavour we shall next examine the term $I_2(x,t)$, and announce that its analysis, though not dissimilar, will be slightly more intrincate. We shall henceforth assume that $\mathrm{Re}(z)=1$ as is customary and investigate first the instance when $|\mathrm{Im}(z)|\leq t^{1/2}\log t$. A routine application of the formula (6.2.10) with the choice $N=\lceil k/4+3/4\rceil$ then yields

$$\begin{aligned}\log\Gamma\Big(\frac{1}{4}-\alpha_j+\beta\Big)-\log\Gamma\Big(\frac{1}{4}+\alpha_j\Big)=&\Big(-\frac{1}{4}-\alpha_j+\beta\Big)\log\Big(\frac{1}{4}-\alpha_j+\beta\Big)\\&-\Big(-\frac{1}{4}+\alpha_j\Big)\log\Big(\frac{1}{4}+\alpha_j\Big)+2\alpha_j-\beta\\&+\sum_{r=1}^{N}c_r\Big(\Big(\frac{1}{4}-\alpha_j+\beta\Big)^{1-2r}-\Big(\frac{1}{4}+\alpha_j\Big)^{1-2r}\Big)+O(t^{-1-2N}).\end{aligned}$$

Observe that then following an analogous argument we obtain that the above formula equals

$$h(\alpha_j)+z\log(-\alpha_j)/2+i\frac{z^2}{4a_jt}+\sum_{u,v}c_2'(u,v)\beta^u\alpha_j^{-v}+O(t^{-5/2-k/2}),\qquad(6.2.19)$$

where as above (u,v) runs over the range earlier described for the discussion pertaining to $G_1(x,t)$, the coefficients $c_2'(u,v)$ are some fixed complex numbers and the function $h(\alpha)$ is defined by means of the relation

$$h(\alpha)=-\Big(\frac{1}{4}+\alpha\Big)\log\Big(\frac{1}{4}-\alpha\Big)-\Big(\alpha-\frac{1}{4}\Big)\log\Big(\frac{1}{4}+\alpha\Big)+2\alpha.\qquad(6.2.20)$$

It seems pertinent to observe first that by definition one has

$$\log(-\alpha_j)=\log(|a_j|t/2)-i\mathrm{sgn}(a_j)\pi/2.$$

Consequently, it transpires that by recalling (6.2.4) and (6.2.12), summing the formula (6.2.19) over j and taking exponentials in the equation at hand then one obtains the approximation

$$\begin{aligned}(\pi^{k/2}x)^{-z}G_2(z,t)=&A(x,t)^ze^{\phi(t)-i\pi\xi_{\mathbf{a}}z/4+iI_{\mathbf{a}}z^2/(4t)}\times\Big(1+\sum_{u,v}c_2(u,v)z^ut^{-v}\\&+O(t^{-5/2-k/2})\Big),\end{aligned}$$

wherein we wrote

$$\phi(t) = \sum_{j=1}^{k} h(\alpha_j)$$

for the sake of concision. Therefore, multiplying both sides of the equation at hand by $H(-z,t)z^{-1}$ and recalling (6.2.13) we obtain

$$\begin{aligned}(\pi^{k/2}x)^{-z}G_2(z,t)H(-z,t)z^{-1} =& e^{\phi(t)}A(x,t)^z F_2(z,t)z^{-1}\big(1+\sum_{u,v} c_2(u,v)z^u t^{-v}\big) \\ &+ O\Big(x^{-1}t^{-5/2}|z|^{-1}e^{|I_{\boldsymbol{a}}y|/2t-y^2/t}\Big),\end{aligned} \tag{6.2.21}$$

where we remind the reader of the notation $y = \mathrm{Im}(z)$ and the definition (6.2.5). By integrating the above equation over the segment

$$[1 - it^{1/2}\log t, 1 + it^{1/2}\log t],$$

it transpires that the contribution $C_2(x,t)$ stemming from the error term will satisfy

$$C_2(x,t) \ll x^{-1}t^{-2}. \tag{6.2.22}$$

To the end of further progressing in the proof and before analysing the case $|\mathrm{Im}(z)| > t^{1/2}(\log t)$ it seems desirable to shew the estimate

$$|\psi(t) - e^{\phi(t)}| \ll t^{-1}, \tag{6.2.23}$$

and this we now describe. To this end it may as well be worth noting that a customary application of the Taylor expansion of $\log(1+w)$ in (6.2.20) then yields

$$h(\alpha) = -\Big(\frac{1}{4}+\alpha\Big)\log(-\alpha) - \Big(\alpha - \frac{1}{4}\Big)\log\alpha + 2\alpha + \sum_{v\geq 1} k_v\alpha^{-v}$$

for some real coefficients k_v. Therefore, on substituting α by α_j in the above formula we get

$$h(\alpha_j) = i\mathrm{sgn}(a_j)\pi/4 - ia_jt\big(\log(|a_j|t/2) - 1\big) + O(t^{-1}),$$

whence summing over j the above equation and taking exponentials delivers (6.2.23).

It might be worth shifting our focus to the case $|y| > t^{1/2}\log t$. We employ as is customary Stirling's formula (6.2.10) and subsequently take real parts to obtain

$$\log|\Gamma(1/4-\alpha_j+\beta)| - \log|\Gamma(1/4+\alpha_j)| = -\frac{\pi}{4}|y-a_jt| + \frac{\pi}{4}|a_j|t + O\big(\log(t+|y|)\big).$$

As was noted in the analogous analysis, it seems convenient to establish the inequality

$$-\sum_{j=1}^{k}|y-a_jt| + t\sum_{j=1}^{k}|a_j| - \xi_{\boldsymbol{a}}y \le 0$$

to the end of deriving suitable bounds for the function $G_2(z,t)$. The latter shall then follow in a similar manner as above by summing both sides of the inequalities

$$-\sum_{a_j>0}|y-a_jt| + t\sum_{a_j>0}a_j - \sum_{a_j>0}y \le 0, \qquad -\sum_{a_j<0}|y-a_jt| - t\sum_{a_j<0}a_j + \sum_{a_j<0}y \le 0,$$

that in turn hold via a routine application of the triangle inequality, an immediate consequence of which being that

$$|G_2(z,t)| \ll (yt)^C e^{\pi\xi_{\boldsymbol{a}}y/4},$$

wherein the above line $C>0$ denotes a fixed constant. Therefore, the previous bound in conjunction with the definition (6.2.5) delivers

$$(\pi^{k/2}x)^{-z}G_2(z,t)H(-z,t)z^{-1} \ll x^{-1}(yt)^C e^{-y^2/t},$$

whence for $|y| > t^{1/2}\log t$ the left side of the above equation is then $O(x^{-1}t^{-2})$. Likewise, an analogous argument reveals that

$$A(x,t)^z e^{z^2/t+iI_{\boldsymbol{a}}z^2/(4t)}z^{-1}\big(1+\sum_{u,v}c_2(u,v)z^ut^{-v}\big) \ll x^{-1}(yt)^C e^{|yI_{\boldsymbol{a}}|/(2t)-y^2/t}.$$

We integrate (6.2.21) over the line $\mathrm{Re}(z)=1$ and utilise (6.2.22) in conjunction with the above inequalities to obtain (6.2.15) for the case $m=2$, as desired. $\square$

As a prelude to the examination of the diagonal and off-diagonal solutions when integrating the approximate functional equation derived above, it seems desirable to prepare the ground for such an endeavour by succintly discussing basic estimates and approximations pertaining to the objects introduced in the previous analysis. To this end we recall the reader of (6.2.12), denote henceforth $\psi_1(t) = 1$ and $\psi_2(t) = \psi(t)$ and write

$$K_m(x,t) = \psi_m(t)\Big(J_m(x,t) + \sum_{u,v} c_m(u,v)K_{m,u,v}(x,t)\Big), \qquad m = 1, 2, \quad (6.2.24)$$

wherein

$$K_{m,u,v}(x,t) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} A(x,t)^z F_m(z,t) z^{u-1} t^{-v} dz, \qquad (6.2.25)$$

and

$$J_m(x,t) = K_{m,0,0}(x,t),$$

where on the above line the range of summation taken was earlier described right after (6.2.14). It may also seem worth anticipating the necessity of introducing beforehand the parameters

$$\alpha_{\boldsymbol{a}} = \frac{1}{8 + I_{\boldsymbol{a}}^2}, \qquad C_{\boldsymbol{a}} = \alpha_{\boldsymbol{a}}\big(1 - \alpha_{\boldsymbol{a}}(1 + I_{\boldsymbol{a}}^2/8)\big) = \frac{7\alpha_{\boldsymbol{a}}}{8}, \qquad (6.2.26)$$

wherein the reader might find it useful to recall (6.2.3).

Lemma 6.2.3. *Let (u, v) lie on the range described above. Then it follows that*

$$K_{m,u,v}(x,t) \ll t^{u/2-v} e^{-C_{\boldsymbol{a}} t(\log A(x,t))^2/2}, \qquad m = 1, 2$$

and

$$J_m(x,t) \ll \log t \qquad (6.2.27)$$

whenever $|\log A(x,t)| \ll t^{-1/2}\log t$. Likewise, one has

$$J_m(x,t) = H(x,t) + O\big(e^{-C_{\boldsymbol{a}} t(\log A(x,t))^2}\big)$$

if $|\log A(x,t)| \gg t^{-1/2}$, wherein the function $H(x,t)$ is defined by means of the relations $H(x,t) = 1$ if $A(x,t) > 1$ and $H(x,t) = 0$ if $A(x,t) < 1$.

The reader may note that we shall make use of the above lemma after integrating over t, whence an analogous formula whenever $A(x,t) = 1$ shall not be required, the latter condition corresponding to a single point once x is fixed, but nonetheless find it desirable to mention for the sake of completeness that an estimate of the shape (6.2.27) could have been deduced thereof.

Proof. For the sake of concision, it has been thought preferable to omit henceforth the dependence on x and t in $A(x,t)$ and just write A. We denote first for convenience $y = \mathrm{Im}(z)$, recall the definition (6.2.13) and observe that when $\mathrm{Re}(z) = -\alpha_{\boldsymbol{a}} t(\log A)$ one has

$$A^z F_m(z,t) z^{u-1} t^{-v} \ll t^{-v}\big(|t\log A|^{u-1} + y^{u-1}\big) e^{f(y)-y^2/2t},$$

where

$$f(y) = (-\alpha_{\boldsymbol{a}} + \alpha_{\boldsymbol{a}}^2) t(\log A)^2 + \alpha_{\boldsymbol{a}} |y(\log A)||I_{\boldsymbol{a}}|/2 - y^2/2t.$$

We find it pertinent to observe that the maximum value of the function $f(y)$ at hand is $-C_{\boldsymbol{a}} t(\log A)^2$, the constant $C_{\boldsymbol{a}} > 0$ defined above being positive, whence

$$A^z F_m(z,t) z^{u-1} t^{-v} \ll t^{-v}\big(|t\log A|^{u-1} + y^{u-1}\big) e^{-C_{\boldsymbol{a}} t(\log A)^2 - y^2/2t}. \tag{6.2.28}$$

It seems desirable to note first that by differentiating, if needed, one has

$$y^d e^{-y^2/2t} \ll t^{d/2} \quad \text{and} \quad |t\log A|^d e^{-C_{\boldsymbol{a}} t(\log A)^2/2} \ll t^{d/2} \tag{6.2.29}$$

for every $d > 0$. Consequently, integrating on both sides in the preceding inequality over the line $\mathrm{Re}(z) = -\alpha_{\boldsymbol{a}} t(\log A)$ and making use of the above bounds for the choice $d = u-1$ it follows that

$$\begin{aligned}\int_{-\alpha_{\boldsymbol{a}} t(\log A)-i\infty}^{-\alpha_{\boldsymbol{a}} t(\log A)+i\infty} A^z F_m(z,t) z^{u-1} t^{-v} dz \ll & t^{u/2-v-1/2} e^{-C_{\boldsymbol{a}} t(\log A)^2/2} \int_{-\infty}^{\infty} e^{-y^2/2t} dy \\ & + t^{-v} e^{-C_{\boldsymbol{a}} t(\log A)^2} \int_{-\infty}^{\infty} y^{u-1} e^{-y^2/2t} dy,\end{aligned}$$

whence a change of variables in the above integrals enables one to conclude that the integral on the left side is $O(t^{u/2-v} e^{-C_{\boldsymbol{a}} t(\log A)^2/2})$. It is convenient to observe as well that the integrand in the definition of $K_{m,u,v}(x,t)$ is an entire function, whence we can move the line of integration to $\mathrm{Re}(z) = -\alpha_{\boldsymbol{a}} t(\log A)$ and use the above estimate to obtain

$$K_{m,u,v}(x,t) \ll t^{u/2-v} e^{-C_{\boldsymbol{a}} t(\log A)^2/2}.$$

The analysis pertaining to $J_m(x,t)$ shall not be totally dissimilar to the previous one, whence in the interest of curtailing our exposition it has been thought preferable to avoid repeating ourselves. We observe first that when $\mathrm{Re}(z) = 1$ and $|\log A| \ll t^{-1/2}(\log t)$ then

$$A^z F_m(z,t) z^{-1} \ll \frac{A e^{|I_{\boldsymbol{a}} y|/(2t) - y^2/2t}}{1+|y|} \ll \frac{e^{|I_{\boldsymbol{a}} y|/(2t) - y^2/2t}}{1+|y|}, \tag{6.2.30}$$

whence utilising the above bound and integrating accordingly we deduce

$$J_m(x,t) \ll \log t,$$

as desired.

By following an analogous argument to that utilised above it transpires that whenever $|\log A| \gg t^{-1/2}$ then in the line $\mathrm{Re}(z) = -\alpha_{\boldsymbol{a}}(\log A)t$ one has the estimate

$$A^z F_m(z,t) z^{-1} \ll \frac{e^{-C_{\boldsymbol{a}} t(\log A)^2 - y^2/2t}}{|t \log A| + |y|}. \tag{6.2.31}$$

Therefore, integrating on both sides of the above estimate over the line at hand yields

$$\begin{aligned}\int_{-\alpha_{\boldsymbol{a}} t(\log A) - i\infty}^{-\alpha_{\boldsymbol{a}} t(\log A) + i\infty} A^z F_m(z,t) z^{-1} dz &\ll t^{-1/2} \int_{-\infty}^{\infty} e^{-C_{\boldsymbol{a}} t(\log A)^2 - y^2/2t} dy \\ &\ll e^{-C_{\boldsymbol{a}} t(\log A)^2}.\end{aligned}$$

It is worth noting that whenever $A > 1$ then $-\alpha_{\boldsymbol{a}} t(\log A) < 0$, whence under such circumstances the function on the left side of the above equation has a single pole at $z = 0$ in the region between the lines $\mathrm{Re}(z) = 1$ and $\mathrm{Re}(z) = -\alpha_{\boldsymbol{a}} t(\log A)$ of residue 1. If, on the contrary $A < 1$ then the aforementioned function does not possess a pole in such a region. Consequently, the preceding discussion in conjunction with the above estimate yields for $|\log A| \gg t^{-1/2}$

the expression

$$J_m(x,t) = H_m(x,t) + O\big(e^{-C_{\boldsymbol{a}}t(\log A)^2}\big),$$

which in turn completes the proof. □

6.3 Verification of the piecewise monotonicity

As shall be elucidated shortly afterwards, the analysis of both the off-diagonal contribution pertainig to the non-twisted integral and the one cognate to the twisted one shall comprise investigations of oscillatory integrals which shall ultimately be bounded by means of Lemma 5.2.4. To this end it has been thought pertinent to present beforehand a technical lemma that shall prepare the ground for such an application. We shall in particular be concerned herein about the perusal of the monotonicity of an auxiliary function cognate to the integrals at hand, it giving rise to the above heading. For such purposes it is convenient to introduce first

$$G_{\mathbf{C}}(t,y) = t^{-v}(c_1(\log A)t + iy)^{u-1}e^{c_2(\log A)^2t+c_3(\log A)y-y^2/t},$$

where we remind the reader of the definition (6.2.12) for $A = A(x,t)$, and what shall shortly play the role of the phase of the corresponding unimodular function, which we define by means of

$$F_{\mathbf{C},m}(t,y) = c_4y\log A+c_5(\log A)^2t+c_6y^2/t+\varepsilon_m t\log L_{\boldsymbol{a}}(\boldsymbol{n})-y_m(t), \qquad m=1,2$$

wherein the above coefficients $c_j \neq 0$ are non-zero real numbers, the parameter $\varepsilon_m = (-1)^{m+1}$ and $y_m(t)$ is defined, on recalling (6.2.14), by

$$y_1(t) \equiv 0, \qquad y_2(t) = g_{\boldsymbol{a}}(t) \tag{6.3.1}$$

As will become apparent to the reader promptly, we further impose the condition

$$c_5 \neq a-b-c. \tag{6.3.2}$$

Lemma 6.3.1. *With the above notation, we assume that $t \in [T/2, T]$, that the variable y satisfies $|y| \leq t^{1/2}(\log t)$ and that $|\log A| \ll t^{-1/2}(\log t)$. Then it* *transpires that for fixed y the numbers $N_1(y)$ and $N_2(y)$ of zeros of*

$$\frac{d}{dt}\left(\frac{Re\big(G_{\mathbf{C}}(t,y)\big)}{F'_{\mathbf{C},m}(t,y)}\right), \qquad \frac{d}{dt}\left(\frac{Im\big(G_{\mathbf{C}}(t,y)\big)}{F'_{\mathbf{C},m}(t,y)}\right) \tag{6.3.3}$$

respectively have the property that

$$\max(N_1(y), N_2(y)) \leq u+1.$$

It shall be noted that any analogous bound not depending on y or t would have been sufficient for our purposes. Nonetheless, the preciseness inherent in such an estimate was obtained with no extra effort, it being pertinent to provide such a refinement in our exposition.

Proof. For the sake of concision, it has been thought preferable to focus our analysis on the investigation of $N_1(y)$ and leave that of $N_2(y)$, since the underlying arguments are identical. We find it useful to observe first that the zeros of the derivative of the function at hand will also be solutions of the equation

$$\mathrm{Re}\Big(G'_{\mathbf{C}}(t,y)F'_{\mathbf{C},m}(t,y) - G_{\mathbf{C}}(t,y)F''_{\mathbf{C},m}(t,y)\Big) = 0. \tag{6.3.4}$$

We also denote for further convenience by

$$P(t,y) = \mathrm{Re}\big((c_1(\log A)t + iy)^{u-1}\big) = (c_1 t\log A)^{u-1} - (c_1 t \log A)^{u-3}y^2 + \dots$$

and

$$E(t) = e^{c_2(\log A)^2 t + c_3(\log A)y - y^2/t},$$

and note that

$$P'(t,y) = c_1(u-1)t^{u-2}(\log A)^{u-2}(\log A + 3/2) - \dots$$

if $u \neq 1$ and $P'(t,y) = 0$ otherwise. In order to present the computations in a concise manner, it may seem desirable to anticipate that

$$F'_{\mathbf{C},m}(t,y) = 3c_4 y/2t + 3c_5 \log A + c_5(\log A)^2 - c_6 y^2/t^2 + \varepsilon_m \log L_{\boldsymbol{a}}(\boldsymbol{n}) - y'_m(t).$$

Motivated by such remarks we denote

$$D_1(t,y) = t^v E(t)^{-1}\mathrm{Re}\big(G'_{\mathbf{C}}(t,y)F'_{\mathbf{C},m}(t,y)\big)$$

and

$$D_2(t,y) = t^v E(t)^{-1}\mathrm{Re}\big(G_{\mathbf{C}}(t,y)F''_{\mathbf{C},m}(t,y)\big),$$

and observe that with the above notation one has

$$\begin{aligned} D_1(t,y) = &\Big(P(t,y)\big(3c_2\log A + c_2(\log A)^2 + 3c_3 y/2t + y^2/t^2\big) \\ &- vt^{-1}P(t,y) + P'(t,y)\Big) \times F'_{\mathbf{C},m}(t,y) \end{aligned} \tag{6.3.5}$$

and

$$D_2(t,y) = P(t,y)\big(-3c_4 y/2t^2 + 9c_5/2t + 3c_5(\log A)/t + 2c_6 y^2/t^3 - y''_m(t)\big).$$

As may become apparent shortly, it seems worth defining

$$D(t,y) = D_1(t,y) - D_2(t,y). \tag{6.3.6}$$

It then transpires that the solutions of the equation (6.3.4) will satisfy

$$D(t,y) = 0.$$

The genesis of the principle underlying our approach will have its reliance on finding a positive integer n not depending on y having the property that the n-th derivative of $D(t,y)$ does not have any zeros. Succesive applications of Rolle's theorem will then enable us to conclude that the function at hand has at most n zeros, as desired. The rest of the discussion will be devoted to finding such an integer.

Consideration of space preclude us from exhibiting a more transparent exposition of the details latent in the ensuing analysis. We therefore content ourselves to merely note that an insightful inspection of the above equation in conjunction with the inequalities stated at the beginning of the lemma involving t and y reveals that in order to obtain the main term after differentiating u or $u+1$ times one should remove the power of $(\log A)$ factors of the summands having the largest power of t and not containing any power of y in such a process.

We begin our discussion by noting for further purposes that

$$\frac{\partial}{\partial^{u+1} t}\big(D_2(t,y)\big) \ll t^{-3}. \tag{6.3.7}$$

We have reached a position in the proof from which distinguishing between $m = 1$ and $m = 2$ becomes essential. If $m = 1$ then it seems pertinent to suppose first that $|\log\big(L_{\boldsymbol{a}}(\boldsymbol{n})\big)| \leq \delta$ for some small enough constant δ independent on t and y. On recalling that $|\log A| \gg t^{-1/2}(\log t)$ and $|y| \leq t^{1/2}(\log t)$ it then traspires that under such an assumption one has

$$\frac{\partial}{\partial^{u+1} t}\big(D_1(t,y)\big) = 9c_1^{u-1}c_2c_5\Big(\frac{3}{2}\Big)^{u+1}(u+1)!t^{-2} + O(\delta t^{-2}) + O\big(t^{-5/2}\log t\big).$$

The reader may find it worth observing that the main term thereof arises after differentiating the term $9c_1^{u-1}c_2c_5t^{u-1}(\log A)^{u+1}$ embeeded in (6.3.5), the first error term stemming from the contribution of the $(u+1)$-th derivative of the term $3c_1^{u-1}c_2t^{u-1}(\log A)^u \log\big(L_{\boldsymbol{a}}(\boldsymbol{n})\big)$. The preceding discussion thus enables one to conclude that under the assumptions earlier described, $D^{(u+1)}(t,y)$ does not vanish, as desired.

If instead, $|\log\big(L_{\boldsymbol{a}}(\boldsymbol{n})\big)| > \delta$ then we differentiate u times and obtain

$$\frac{\partial}{\partial^{u} t}\big(D_1(t,y)\big) = 3c_1^{u-1}c_2\Big(\frac{3}{2}\Big)^{u}u!t^{-1}\log\big(L_{\boldsymbol{a}}(\boldsymbol{n})\big) + O(t^{-3/2}\log t),$$

wherein the first summand above, which dominates over the second one in view of the preceding assumption, arises from the contribution of the u-th derivative of the term $3c_1^{u-1}c_2t^{u-1}(\log A)^u \log\big(L_{\boldsymbol{a}}(\boldsymbol{n})\big)$.

The framework when $m = 2$, though not dissimilar in treatment, shall exhibit a slight different behaviour, the corresponding approach employed herein thus necessitating a reappraisal of the underlying arguments. We define

$$g_2(t) = -\log L_{\boldsymbol{a}}(\boldsymbol{n}) - y_2'(t),$$

take $\delta > 0$ to be a sufficiently small fixed constant and introduce for further convenience the sets

$$\mathcal{D}_\delta = \Big\{t \in [T/2,T] : \quad |g_2(t)| \leq \delta\Big\}$$

and

$$\mathcal{E}_\delta = \Big\{t \in [T/2, T] : \quad |g_2(t)| > \delta\Big\}.$$

We find it desirable to note that in view of the monotonicity of $g_2(t)$ the set $\mathcal{D}_\delta$, if non-empty, is an interval, the analogous set $\mathcal{E}_\delta$ comprising at most two intervals. It then transpires that on recalling (6.3.1) then whenever $t \in \mathcal{D}_\delta$ one has

$$\begin{aligned}\frac{\partial}{\partial^{u+1} t}\big(D_1(t, y)\big) =& 3c_1^{u-1}c_2\Big(\frac{3}{2}\Big)^u (u+1)!(c_5 + b + c - a)t^{-2} + O(\delta t^{-2}) \\ &+ O(t^{-5/2}\log t).\end{aligned} \tag{6.3.8}$$

It should be noted that in the above equation the main term arises from the contribution of the $(u+1)$-th derivative of $9c_1^{u-1}c_2c_5t^{u-1}(\log A)^{u+1}$ plus the left side of the formula

$$\begin{aligned}\frac{\partial}{\partial^{u+1} t}\Big(3c_1^{u-1}c_2t^{u-1}(\log A)^u g_2(t)\Big) =& 3c_1^{u-1}c_2\Big(\frac{3}{2}\Big)^u (u+1)!y_2''(t)t^{-1} \\ &+ O(\delta t^{-2}) + O(t^{-5/2}\log t),\end{aligned}$$

which in turn is a consequence of an application of the product rule in conjunction with the assumptions on t and A earlier stated. The reader may find it worth observing that the first term on the right side of (6.3.8) is genuinely a main term as a consequence of the condition (6.3.2).

If on the contrary $t \in \mathcal{E}_\delta$ then on differentiating (6.3.5) u times it follows that

$$\frac{\partial}{\partial^u t}\big(D_1(t, y)\big) = 3c_1^{u-1}c_2\Big(\frac{3}{2}\Big)^u u!g_2(t)t^{-1} + O(t^{-3/2}\log t).$$

Combining (6.3.6) with (6.3.7) and the preceding discussion, the reader may note then that we have therefore dissected the interval $[T/2, T]$ into a union of at most three intervals in which either $D^{(u)}(t, y)$ or $D^{(u+1)}(t, y)$ does not vanish. Consequently, subsequent applications of Rolle's theorem accordingly enables one to deduce that under the assumptions made above, the function $D(t, y)$ vanishes at most $u+1$ times, as desired.

The same approach for the analysis pertaining to the second function in (6.3.3), which as was earlier anticipated shall not be deeply discussed herein, is still valid to deliver the same conclusions. The main term thereof will arise after differentiating either $u-1$ or u times accordingly the summand

$$c_1^{u-2}yt^{u-2}(\log A)^{u-1}\big(9c_2c_5\log A+3c_2(\varepsilon_m\log L_{\boldsymbol{a}}(\boldsymbol{n})-y'_m(t))\big),$$

the presence of the factor y above not giving rise to any recalcitrant situations whatsoever. $\square$

6.4 Asymptotic evaluation of the non-twisted term. The diagonal contribution

We have now ascended to a position from which to integrate the approximate functional equation obtained above, and this we shall describe shortly. We shall henceforth denote $\mathbf{a}=(a,-b,-c)$, wherein the above entries are those appertaining to Theorem 6.1.1 and satisfy $a<c\leq b$. It seems worth noting that under the assumptions of such a theorem, the tuple $\mathbf{a}$ satisfies the required inequalities (6.2.1) necessitated to ascend to a position from which to apply Lemma 6.2.2. We then utilise the aforementioned lemma and integrate over $[0,T]$ to obtain

$$I_{a,b,c}(T)=I_1(T)+I_2(T)+O\big(\log T\big), \tag{6.4.1}$$

where

$$I_m(T)=\sum_{\boldsymbol{n}}P_{\boldsymbol{n}}^{-1/2}\int_0^T L_{\boldsymbol{a}}(\boldsymbol{n})^{(-1)^{m+1}it}K_m\big(P_{\boldsymbol{n}},t\big)dt, \qquad m=1,2.$$

We feel the urge to anticipate that the above integrals are not related to the integals $I_1(T)$ and $I_2(T)$ in the previous chapter, the adoption of the same notation stemming from a lack of alphabetic symbols. It seems desirable to clarify that the error term in the above formula arises as a consequence of decomposing the interval $[0,T]$ into $[0,1)$ and $[1,T]$ and integrating accordingly. We also make a distinction, as is customary, between the diagonal and the off-diagonal contribution and write

$$I_1(T)=I_{1,1}(T)+I_{1,2}(T), \tag{6.4.2}$$

where

$$I_{1,1}(T) = \sum_{n_1^a = n_2^b n_3^c} (n_1 n_2 n_3)^{-1/2} \int_0^T L_{\boldsymbol{a}}(\boldsymbol{n})^{it} K_1\big(P_{\boldsymbol{n}}, t\big) dt$$

and

$$I_{1,2}(T) = \sum_{n_1^a \neq n_2^b n_3^c} (n_1 n_2 n_3)^{-1/2} \int_0^T L_{\boldsymbol{a}}(\boldsymbol{n})^{it} K_1\big(P_{\boldsymbol{n}}, t\big) dt.$$

The following lemma will convey the asymptotic evaluation of the diagonal contribution $I_{1,1}(T)$, and this we now describe, but not before recalling for the reader the definition (5.5.13).

Lemma 6.4.1. *With the above notation one has*

$$I_{1,1}(T) = \sigma_{a,b,c} T + O(T^{11/20+\varepsilon}).$$

It shall be noted that the above error term could have been refined, such an improvement having been wrought in vain in view of the error terms appertaining to both the off-diagonal contribution and the contribution stemming from the twisted integral which shall be made explicit shortly after the present discussion.

Proof. On recalling (6.2.24) and the bound $K_{m,u,v}(P_{\boldsymbol{n}}, t) \ll t^{-1/2}$ latent in the conclusions of Lemma 6.2.3 we first note that the contribution to the integral arising from the terms $K_{1,u,v}(P_{\boldsymbol{n}}, t)$ in the decomposition cognate to $K_1(P_{\boldsymbol{n}}, t)$ is bounded above by

$$\begin{aligned} \sum_{n_1^a = n_2^b n_3^c} (n_1 n_2 n_3)^{-1/2} \int_0^T t^{-1/2} dt &\ll \sum_{n_2, n_3} n_2^{-1/2-b/2a} n_3^{-1/2-c/2a} \int_0^T t^{-1/2} dt \\ &\ll T^{1/2}. \end{aligned} \tag{6.4.3}$$

It may also seem pertinent to define for further convenience in the memoir the parameters

$$\tau_{\boldsymbol{n}} = 2\pi (n_1 n_2 n_3)^{2/3} P_{\boldsymbol{a}}^{-1/3}, \qquad T_2 = \frac{P_{\boldsymbol{a}}^{1/3}}{2\pi} T. \tag{6.4.4}$$

It is then worth observing that employing this notation and making the choice

$x = n_1 n_2 n_3$ in (6.2.12) one has $A(x,t) = (t/\tau_{\boldsymbol{n}})^{3/2}$, whence whenever

$$(t - \tau_{\boldsymbol{n}})/t \leq t^{-1/2} \log t$$

it appears at first glance that

$$\frac{2}{3} \log A = (t - \tau_{\boldsymbol{n}})/t + O\big(t^{-1} (\log t)^2\big). \tag{6.4.5}$$

Then combining (6.2.24) and (6.4.3) one gets

$$\begin{aligned} I_{1,1}(T) = &\sum_{n_1^a = n_2^b n_3^c} P_{\boldsymbol{n}}^{-1/2} \int_{|t-\tau_{\boldsymbol{n}}| \geq t^{1/2}(\log t)} J_1(P_{\boldsymbol{n}}, t) dt \\ &+ \sum_{n_1^a = n_2^b n_3^c} P_{\boldsymbol{n}}^{-1/2} \int_{|t-\tau_{\boldsymbol{n}}| < t^{1/2}(\log t)} J_1(P_{\boldsymbol{n}}, t) dt + O(T^{1/2}), \end{aligned}$$

wherein we omitted writing the endpoints 0 and T in the above integrals for the sake of concision. The reader may note that an application of Lemma 6.2.3 in conjunction with the procedure employed to derive (6.4.3) enables one to infer that the second summand on the above equation is $O\big(T^{1/2}(\log T)^2\big)$. Likewise, it may be worth observing that whenever $|t - \tau_{\boldsymbol{n}}| \geq t^{1/2}(\log t)$ then $|\log A| \gg t^{-1/2}(\log t)$, a subsequent application of Lemma 6.2.3 thus delivering

$$I_{1,1}(T) = \sum_{n_1^a = n_2^b n_3^c} P_{\boldsymbol{n}}^{-1/2} \int_{|t-\tau_{\boldsymbol{n}}| \geq t^{1/2}(\log t)} H(P_{\boldsymbol{n}}, t) dt + O\big(T^{1/2}(\log T)^2\big),$$

whence the definitions of $H(x,t)$ and $\tau_{\boldsymbol{n}}$ combined with the aforementioned argument utilised to obtain (6.4.3) then yield

$$I_{1,1}(T) = \sum_{\substack{n_1^a = n_2^b n_3^c \\ n_1 n_2 n_3 \leq T_2^{3/2}}} P_{\boldsymbol{n}}^{-1/2} (T - \tau_{\boldsymbol{n}}) + O\big(T^{1/2}(\log T)^2\big).$$

We will then rewrite the above equation as

$$I_{1,1}(T) = \sigma_{a,b,c} T - A_1(T) - A_3(T) + O\big(T^{1/2}(\log T)^2\big),$$

where we denote

$$A_1(T)=\sum_{\substack{n_1n_2n_3\le T_2^{3/2}\\ n_1^a=n_2^bn_3^c}}\tau_{\boldsymbol{n}}P_{\boldsymbol{n}}^{-1/2},\qquad A_3(T)=T\sum_{\substack{n_1^a=n_2^bn_3^c\\ n_1n_2n_3>T_2^{3/2}}}P_{\boldsymbol{n}}^{-1/2},\tag{6.4.6}$$

and where the constant $\sigma_{a,b,c}$ was defined in (5.5.13). We find it pertinent to recall equation (5.5.14) for further convenience and to draw the reader's attention to the parametrization of the underlying equation at hand described in (5.5.15). It may also be desirable to bring the parameters $P_{\boldsymbol{d}}$ and $M_{\boldsymbol{d}}$, defined in (5.5.16), back to the discussion. In a similar fashion as therein and summing over r_2 first we find that

$$\begin{aligned}A_3(T)\ll& T\sum_{r_2^{a_2+b_2}r_3^{a_3+c_3}M_{\boldsymbol{d}}P_{\boldsymbol{d}}>T_2^{3/2}}r_2^{-(a_2+b_2)/2}r_3^{-(a_3+c_3)/2}(P_{\boldsymbol{d}}M_{\boldsymbol{d}})^{-1/2}\\ \ll& T^{1/4+3/2(a_2+b_2)}\sum_{r_3^{a_3+c_3}P_{\boldsymbol{d}}M_{\boldsymbol{d}}\le T_2^{3/2}}r_3^{-(a_3+c_3)/(a_2+b_2)}(P_{\boldsymbol{d}}M_{\boldsymbol{d}})^{-1/(a_2+b_2)}\\ &+T\sum_{r_3^{a_3+c_3}P_{\boldsymbol{d}}M_{\boldsymbol{d}}>T_2^{3/2}}r_3^{-(a_3+c_3)/2}(P_{\boldsymbol{d}}M_{\boldsymbol{d}})^{-1/2}.\end{aligned}\tag{6.4.7}$$

In order to bound the first term $A_{3,1}(T)$ in the above equation we sum over r_3 to get

$$\begin{aligned}A_{3,1}(T)\ll& T^{1/4+3/2(a_3+c_3)}\sum_{P_{\boldsymbol{d}}M_{\boldsymbol{d}}\le T_2^{3/2}}(P_{\boldsymbol{d}}M_{\boldsymbol{d}})^{-1/(a_3+c_3)}\\ &+T^{1/4+3/2(a_2+b_2)}(\log T)\sum_{P_{\boldsymbol{d}}M_{\boldsymbol{d}}\le T_2^{3/2}}(P_{\boldsymbol{d}}M_{\boldsymbol{d}})^{-1/(a_2+b_2)},\end{aligned}\tag{6.4.8}$$

where we encompassed both the instances when the coefficient of r_3 in $A_{3,1}(T)$ is smaller or greater than -1. The reader may find it desirable to observe that in view of (5.5.16) the exponents appertaining to the square-free factors d_j in the product $P_{\boldsymbol{d}}M_{\boldsymbol{d}}$ are at least 5. Consequently, the preceding discussion yields

$$\begin{aligned}A_{3,1}(T) \ll &T^{1/4+3/2(a_3+c_3)}\sum_{m\leq T_2^{3/10}}\tau_{A-1}(m)m^{-5/(a_3+c_3)}\\&+T^{1/4+3/2(a_2+b_2)}(\log T)\sum_{m\leq T_2^{3/10}}\tau_{A-1}(m)m^{-5/(a_2+b_2)}\ll T^{11/20+\varepsilon},\end{aligned} \tag{6.4.9}$$

wherein $\tau_{A-1}(m)$ denotes the number of representations of m as a product of $A-1$ factors. Likewise, we denote by $A_{3,2}(T)$ to the second summand of (6.4.7). An analogous argument reveals that

$$A_{3,2}(T)\ll T^{1/4+3/2(a_3+c_3)}\sum_{P_{\boldsymbol{d}}M_{\boldsymbol{d}}\leq T_2^{3/2}}(P_{\boldsymbol{d}}M_{\boldsymbol{d}})^{-1/(a_3+c_3)}+T\sum_{P_{\boldsymbol{d}}M_{\boldsymbol{d}}\geq T_2^{3/2}}(P_{\boldsymbol{d}}M_{\boldsymbol{d}})^{-1/2}.$$

The reader may note that the estimate obtained in the course of bounding $A_{3,1}(T)$ may be utilised herein to enable one to estimate the first term in the above line by $O(T^{11/20+\varepsilon})$. In the interest of analysing the second one at hand rather efficiently we find it pertinent to relabell the subindices of the factors cognate to the product in the above equation so that

$$P_{\boldsymbol{d}}M_{\boldsymbol{d}}=\prod_{j=1}^{A-1}d_j^{\beta_j},$$

wherein the corresponding exponents satisfy $\beta_j\leq\beta_{j+1}$ for $1\leq j\leq A-1$, the parameter A being defined in (5.5.14). It then transpires that by summing over d_1 first one has

$$\begin{aligned}T\sum_{P_{\boldsymbol{d}}M_{\boldsymbol{d}}\geq T_2^{3/2}}(P_{\boldsymbol{d}}M_{\boldsymbol{d}})^{-1/2}=&T\sum_{d_1^{\beta_1}\cdots d_{A-1}^{\beta_{A-1}}\geq T_2^{3/2}}d_1^{-\beta_1/2}\cdots d_{A-1}^{-\beta_{A-1}/2}\\ \ll&T^{1/4+3/2\beta_1}\sum_{d_2^{\beta_2}\cdots d_{A-1}^{\beta_{A-1}}\leq T_2^{3/2}}d_2^{-\beta_2/\beta_1}\cdots d_{A-1}^{-\beta_{A-1}/\beta_1}\\&+T\sum_{d_2^{\beta_2}\cdots d_{A-1}^{\beta_{A-1}}\geq T_2^{3/2}}d_2^{-\beta_2/2}\cdots d_{A-1}^{-\beta_{A-1}/2}.\end{aligned}$$

The reader may observe that an iteration of the above argument then delivers the estimate

$$T \sum_{P_{\boldsymbol{d}} M_{\boldsymbol{d}} \geq T_2^{3/2}} (P_{\boldsymbol{d}} M_{\boldsymbol{d}})^{-1/2} \ll T^{1/4+3/2\beta_1+\varepsilon} \ll T^{11/20+\varepsilon},$$

wherein we utilised the fact that $\beta_1 \geq 5$ mentioned above. It might be worth mentioning for the sake of transparency that the extra ε herein stems from the not so recalcitrant instance of some of the exponents at hand being equal.

In order to analyse the term $A_1(T)$ defined in (6.4.6) we allude to the parametrization (5.5.15) again to the end of obtaining

$$A_1(T) \ll \sum_{r_2^{a_2+b_2} r_3^{a_3+c_3} P_{\boldsymbol{d}} M_{\boldsymbol{d}} \leq T_2^{3/2}} r_2^{(a_2+b_2)/6} r_3^{(a_3+c_3)/6} (M_{\boldsymbol{d}} P_{\boldsymbol{d}})^{1/6}.$$

By summing subsequently over r_2 and r_3 we get

$$\begin{aligned} A_1(T) \ll & T^{1/4+3/2(a_2+b_2)} \sum_{r_3^{a_3+c_3} P_{\boldsymbol{d}} M_{\boldsymbol{d}} \leq T_2^{3/2}} r_3^{-(a_3+c_3)/(a_2+b_2)} (P_{\boldsymbol{d}} M_{\boldsymbol{d}})^{-1/(a_2+b_2)} \\ \ll & T^{1/4+3/2(a_3+c_3)} \sum_{P_{\boldsymbol{d}} M_{\boldsymbol{d}} \leq T_2^{3/2}} (P_{\boldsymbol{d}} M_{\boldsymbol{d}})^{-1/(a_3+c_3)} \\ & + T^{1/4+3/2(a_2+b_2)} (\log T) \sum_{P_{\boldsymbol{d}} M_{\boldsymbol{d}} \leq T_2^{3/2}} (P_{\boldsymbol{d}} M_{\boldsymbol{d}})^{-1/(a_2+b_2)}. \end{aligned}$$

The reader may observe that the terms in the last line of the above equation appeared in (6.4.8) and were already estimated in (6.4.9). Therefore, the same argument delivers

$$A_1(T) \ll T^{11/20+\varepsilon},$$

as desired. $\square$

6.5 Off-diagonal contribution of the non-twisted term

We now focus our attention on the term $I_{1,2}(T)$. We find it appropiate to consider

$$J_{1,u,v}(T) = \sum_{\substack{\boldsymbol{n} \\ n_1^a \neq n_2^b n_3^c}} P_{\boldsymbol{n}}^{-1/2} \int_0^T L_{\boldsymbol{a}}(\boldsymbol{n})^{it} K_{1,u,v}\big(P_{\boldsymbol{n}}, t\big) dt, \tag{6.5.1}$$

where we remind the reader that $K_{m,u,v}(P_{\boldsymbol{n}}, t)$ was defined right after (6.2.24). It may be worth introducing the analogous sum

$$J_{1,2}(T) = \sum_{\substack{\boldsymbol{n} \\ n_1^a \neq n_2^b n_3^c}} P_{\boldsymbol{n}}^{-1/2} \int_0^T L_{\boldsymbol{a}}(\boldsymbol{n})^{it} J_1(P_{\boldsymbol{n}}, t) dt$$

and observe that equipped with this notation we have reached a position from which to write $I_{1,2}(T)$ by making use of (6.2.24) in a rather concise manner, say

$$I_{1,2}(T) = J_{1,2}(T) + \sum_{u,v} c_1(u,v) J_{1,u,v}(T), \tag{6.5.2}$$

where (u,v) runs over the range described right after (6.2.14). As the reader may already anticipate in view of the conclusions derived in Lemma 6.2.3, the integrals $K_{1,u,v}(P_{\boldsymbol{n}}, t)$ satisfy a bound of the shape $O(t^{-1/2})$, such a pointwise estimate not being of the requisite precision if one ought to employ such a bound when integrating over t. Instead, we depart from that trivial approach in that further cancellation shall be exploited by means of oscillatory integral estimates in conjunction with the same framework of ideas underlying the proof of the aforementioned bound pertaining to $K_{1,u,v}(P_{\boldsymbol{n}}, T)$.

Lemma 6.5.1. *For (u,v) in the range to which we alluded above one has*

$$J_{1,u,v}(T) \ll T^{3/4} (\log T)^4.$$

Proof. As was previously done in the course of the proof of Lemma 6.2.3, we move the line of integration to $\mathrm{Re}(z) = -\alpha_{\boldsymbol{a}} t (\log A)$ and thus get

$$K_{1,u,v}(P_{\boldsymbol{n}}, t) = \int_{-\infty}^{\infty} G(t,y) e^{iF_1(t,y)} dy,$$

wherein the above line the corresponding function $G(t,y)$ is defined by means of the formula

$$G(t,y)=t^{-v}(-\alpha_{\boldsymbol{a}}(\log A)t+iy)^{u-1}e^{(\alpha_{\boldsymbol{a}}^2-\alpha_{\boldsymbol{a}})(\log A)^2t-\alpha_{\boldsymbol{a}}I_{\boldsymbol{a}}(\log A)y/2-y^2/t}$$

and the phase of the unimodular function is

$$F_1(t,y)=(1-2\alpha_{\boldsymbol{a}})(\log A)y-I_{\boldsymbol{a}}\alpha_{\boldsymbol{a}}^2(\log A)^2t/4+I_{\boldsymbol{a}}y^2/4t.$$

As may become apparent shortly, it has been thought preferable to make a dyadic dissection of the corresponding interval over which we shall integrate. The reader may find it useful to observe that in view of the above estimates and the argument preceding (6.2.28) if needed, it appears at first glimpse that the function $G(t,y)$ exhibits an exponential decay whenever $|y|\gg t^{1/2}(\log t)$, whence

$$\int_{T/2}^{T}L_{\boldsymbol{a}}(\boldsymbol{n})^{it}K_{1,u,v}(P_{\boldsymbol{n}},t)dt=\int_{T/2}^{T}\int_{-T^{1/2}(\log T)}^{T^{1/2}(\log T)}G(t,y)e^{iF(t,y)}dydt+O(1),$$

where

$$F(t,y)=F_1(t,y)+t\log(L_{\boldsymbol{a}}(\boldsymbol{n})).$$

In order to reach the position from which to make use of suitable oscillatory integral estimates it seems desirable noting first that an application of (6.2.28) in conjunction with (6.2.29) for the choice $d=u-1$ yields

$$\begin{aligned}G(t,y)&\ll t^{-v}(-\alpha|\log A|t+|y|)^{u-1}e^{-C_{\boldsymbol{a}}t(\log A)^2-y^2/2t}\\&\ll t^{u/2-v-1/2}e^{-C_{\boldsymbol{a}}t(\log A)^2/2}\ll t^{-1}e^{-C_{\boldsymbol{a}}t(\log A)^2/2}.\end{aligned}\tag{6.5.3}$$

In view of the ensuing discussion it transpires that one may assume the condition $|\log A|\ll t^{-1/2}(\log t)$, since otherwise the corresponding contribution to the integral at hand would be absorbed by the error term thereof, such an assumption further delivering the constraint

$$P_{\boldsymbol{n}}\asymp T^{3/2}.\tag{6.5.4}$$

We shift our focus to the analysis of the phase pertaining to the unimodular function comprising the integrand of the above integral. A routine computation in conjunction with the above condition and the assumption $|y| \ll t^{1/2}\log t$ then reveals that

$$\frac{\partial^2}{\partial^2 t}F(t,y) = -\frac{9}{8}I_{\boldsymbol{a}}\alpha_{\boldsymbol{a}}^2 t^{-1} + O\big(t^{-3/2}(\log t)\big),$$

which then yields

$$\Big|\frac{\partial^2}{\partial^2 t}F(t,y)\Big| \gg t^{-1}. \tag{6.5.5}$$

As the reader may have already anticipated, we are preparing the ground for an application of Lemma 5.2.4. To complete such an endeavour, it thus remains noting that on recalling (6.2.3) and (6.2.26) one has that

$$|I_{\boldsymbol{a}}|\alpha_{\boldsymbol{a}}^2/4 \leq \frac{3}{256a}, \tag{6.5.6}$$

whence it transpires that then $-I_{\boldsymbol{a}}\alpha_{\boldsymbol{a}}^2/4 \neq a-b-c$, which in turn ensures that condition (6.3.2) is satisfied. Therefore, the ensuing discussion in conjunction with an application of both Lemmata 5.2.4 and 6.3.1 and equations (6.5.3) and (6.5.5) delivers

$$\begin{aligned}\int_{-T^{1/2}(\log T)}^{T^{1/2}(\log T)}\int_{T/2}^{T} G(t,y)e^{iF(t,y)}dtdy &\ll T^{-1/2}\int_{-T^{1/2}(\log T)}^{T^{1/2}(\log T)} dy \\ &\ll \log T,\end{aligned}$$

whence an interchange of the order of integration combined with a dyadic argument yields

$$\int_0^T L_{\boldsymbol{a}}(\boldsymbol{n})^{it}K_{1,u,v}(P_{\boldsymbol{n}},t)dt \ll (\log T)^2.$$

The reader may also observe that

$$\sum_{P_{\boldsymbol{n}}\leq CT^{3/2}} P_{\boldsymbol{n}}^{-1/2} \ll T^{3/4}(\log T)^2, \tag{6.5.7}$$

for any constant $C>0$, whence recalling (6.5.4) and combining the above equations delivers

$$J_{1,u,v}(T) \ll T^{3/4}(\log T)^4.$$

$\square$

In order to make progress in the proof of the theorem at hand, it seems pertinent to shift our focus to the corresponding analysis of the contribution arising from $J_{1,2}(T)$. We find it worth anticipating that a dyadic argument shall be required henceforth. We also announce that in view of the conclusion of Lemma 6.2.3 it transpires that such an investigation shall necessitate a separate discussion for both the contribution stemming from the set of t that are close to $\tau_{\boldsymbol{n}}$ and the one comprising t which are not, the analysis of which shall have its reliance on a different framework of ideas. For such purposes we consider, for convenience, the sets

$$\mathcal{S}_{\boldsymbol{n}} = \Big\{ t \in [T/2, T] : \ |t - \tau_{\boldsymbol{n}}| \leq T^{1/2} \log T \Big\}, \tag{6.5.8}$$

$$\mathcal{S}'_{\boldsymbol{n}} = \Big\{ t \in [T/2, T] : \ |t - \tau_{\boldsymbol{n}}| > T^{1/2} \log T \Big\}. \tag{6.5.9}$$

We also find it worth writing

$$\sum_{\substack{\boldsymbol{n} \\ n_1^a \neq n_2^b n_3^c}} P_{\boldsymbol{n}}^{-1/2} \int_{T/2}^{T} L_{\boldsymbol{a}}(\boldsymbol{n})^{it} J_1(P_{\boldsymbol{n}}, t) dt = J_{\mathcal{S}_{\boldsymbol{n}}}(T) + J_{\mathcal{S}'_{\boldsymbol{n}}}(T), \tag{6.5.10}$$

where in the preceding line the summands involved therein are defined by means of the formulas

$$J_{\mathcal{S}}(T) = \sum_{\substack{\boldsymbol{n} \\ n_1^a \neq n_2^b n_3^c}} P_{\boldsymbol{n}}^{-1/2} I_{\mathcal{S}}(T) \tag{6.5.11}$$

with the term $I_{\mathcal{S}}(T)$ being

$$I_{\mathcal{S}}(T) = \int_{\mathcal{S}} L_{\boldsymbol{a}}(\boldsymbol{n})^{it} J_1(P_{\boldsymbol{n}}, t) dt \tag{6.5.12}$$

for the sets $\mathcal{S} = \mathcal{S}_{\boldsymbol{n}}, \mathcal{S}'_{\boldsymbol{n}}$. We shall focus our attention first on the term $J_{\mathcal{S}_{\boldsymbol{n}}}(T)$. As was discussed above, we find it worth warning the reader that an application of the trivial bound

$$I_{\mathcal{S}_{\boldsymbol{n}}}(T) \ll T^{1/2} (\log T)^2$$

shall not be of the sufficient strength required. Instead further cancellation shall be obtained as is customary by means of oscillatory integral estimates.

Lemma 6.5.2. *With the above notation, one has*

$$J_{\mathcal{S}_{\boldsymbol{n}}}(T) \ll T^{3/4}(\log T)^4.$$

Proof. As a prelude to our discussion we begin as is customary by furnishing ourselves with some notation. As was previously denoted in the preceding chapter, for tuples $\boldsymbol{n}_2 = (n_2, n_3)$ we consider

$$N_{b,c} = \lfloor n_2^{b/a} n_3^{c/a} \rfloor. \tag{6.5.13}$$

We find it worth writing for each triple $\boldsymbol{n} = (n_1, \boldsymbol{n}_2)$ the first entry by means of $n_1 = N_{b,c} + r$ for some $r \in \mathbb{Z}$. We shall henceforth write $\mathcal{S}_{\boldsymbol{n}_2,r}$ and $\tau_{\boldsymbol{n}_2,r}$ to denote $\mathcal{S}_{\boldsymbol{n}}$ and $\tau_{\boldsymbol{n}}$ respectively. It may also seem pertinent to introduce, as shall be elucidated promptly, the functions

$$G_1(t,y) = e^{(\log A)+1/t+I_{\boldsymbol{a}}y/2t-y^2/t}(1+iy)^{-1} \tag{6.5.14}$$

and

$$F_{\boldsymbol{n}_2,r}(t,y) = y(\log A) + 2y/t - I_{\boldsymbol{a}}/4t + I_{\boldsymbol{a}}y^2/4t + t\log\big(L_{\boldsymbol{a}}(\boldsymbol{n})\big).$$

We note first for further use that whenever $t \in \mathcal{S}_{\boldsymbol{n}}$ then $|\log A| \ll t^{-1/2}(\log t)$, whence

$$G_1(t,y) \ll \frac{1}{1+|y|}. \tag{6.5.15}$$

In view of the above equations, it is apparent that for fixed y, the zeros of the function

$$\frac{d}{dt}\Big(G_1(t,y)F'_{\boldsymbol{n}_2,r}(t,y)^{-1}\Big)$$

may also be zeros of a function

$$P_1\big(t, y, \log(L_{\boldsymbol{a}}(\boldsymbol{n}))\big),$$

wherein $P_1(z_1,z_2,z_3)$ is a polynomial of degree smaller than C for some universal constant $C>0$. It therefore transpires that when thinking of y and $\boldsymbol{n}$ as being fixed then subsequent applications of Rolle's theorem enables one to partition the set of integration into a bounded number of intervals (not depending on y) in which $G_1(t,y)F'_{\boldsymbol{n}_2,r}(t,y)^{-1}$ is monotonic.

By recalling (6.5.12) and in view of the decay exhibited by $G_1(t,y)$ with respect to y in (6.5.14), one has that

$$I_{\mathcal{S}_{\boldsymbol{n}}}(T) = \int_{\mathcal{S}_{\boldsymbol{n}}} \int_{-T^{1/2}\log T}^{T^{1/2}\log T} G_1(t,y) e^{iF_{\boldsymbol{n}_2,r}(t,y)} dy dt + O(T^{-2}).$$

We may suppose that $\mathcal{S}_{\boldsymbol{n}} \neq \emptyset$, since if not no further work would be required. It might be convenient to observe first that whenever y and t lie in the set of integration at hand then it follows that

$$F'_{\boldsymbol{n}_2,r}(t,y) = \frac{3}{2} y/t + \log(L_{\boldsymbol{a}}(\boldsymbol{n})) + O\big(t^{-1}(\log t)^2\big). \tag{6.5.16}$$

We also derive, but not before recalling first the definition of $L_{\boldsymbol{a}}(\boldsymbol{n})$, the formula

$$\begin{aligned} F'_{\boldsymbol{n}_2,r}(t,y) &= \frac{3}{2} y/t + \log\left(n_2^b n_3^c/(N_{b,c}+r)^a\right) + O\big(t^{-1}(\log t)^2\big) \\ &= \frac{3}{2} y/t + \log\left(n_2^b n_3^c/N_{b,c}^a\right) - a\log(1+r/N_{b,c}) + O\big(t^{-1}(\log t)^2\big). \end{aligned}$$

We further write, for convenience,

$$H_{\boldsymbol{n}_2}(t,y) = F'_{\boldsymbol{n}_2,r}(t,y) + a\log(1+r/N_{b,c}),$$

a careful examination of which reveals that it does not depend on r. The reader may find it useful to recall the definition of $\mathcal{S}_{\boldsymbol{n}_2,r}$ and $\tau_{\boldsymbol{n}_2,r}$ right after (6.5.13) and observe that for fixed $\boldsymbol{n}_2$, given $r_1, r_2 \in \mathbb{Z}$ satisfying $|r_1|, |r_2| \leq N_{b,c}/2$ and $t_1 \in \mathcal{S}_{\boldsymbol{n}_2,r_1}$ and $t_2 \in \mathcal{S}_{\boldsymbol{n}_2,r_2}$ then it transpires that

$$|H_{\boldsymbol{n}_2}(t_1,y) - H_{\boldsymbol{n}_2}(t_2,y)| \ll T^{-1/2}\log T,$$

the above implicit constant not depending on r_1, r_2, and in turn implies that the cardinality of the set $\mathcal{R}_1$ comprising integers $|r| \leq N_{b,c}/2$ with the property that $|F'_{\boldsymbol{n}_2,r}(t,y)| \leq N_{b,c}^{-1}$ for some $t \in \mathcal{S}_{\boldsymbol{n}_2,r}$ satisfies the bound

$$|\mathcal{R}_1| \ll N_{b,c} T^{-1/2}(\log T) + 1.$$

For these cases, a succint application of the trivial bound $T^{1/2}(\log T)^2$, it in turn stemming, inter alia, from the bound (6.5.15), to the integral at hand already suffices to bound the contribution arising from the aforementioned set by

$$\begin{aligned}\sum_{n_2^{a+b}n_3^{a+c}\ll T^{3a/2}}\sum_{r\in\mathcal{R}_1} & n_2^{-1/2}n_3^{-1/2}(N_{b,c}+r)^{-1/2}I_{\mathcal{S}_{\boldsymbol{n}}}(T)\\ &\ll(\log T)^3\sum_{n_2^{a+b}n_3^{a+c}\ll T^{3a/2}} n_2^{-1/2}n_3^{-1/2}N_{b,c}^{1/2}\\ &\quad+T^{1/2}(\log T)^2\sum_{n_2^{a+b}n_3^{a+c}\ll T^{3a/2}} n_2^{-1/2}n_3^{-1/2}N_{b,c}^{-1/2}\\ &\ll T^{3/4}(\log T)^3\sum_{n_3\ll T^{3a/(2(a+c))}} n_3^{-1}\ll T^{3/4}(\log T)^4.\end{aligned}$$

Moreover, on denoting $\mathcal{R}_2$ to the set of numbers r with the property that $|F'_{\boldsymbol{n}_2,r}(t,y)|>N_{b,c}^{-1}$ for each $t\in\mathcal{S}_{\boldsymbol{n}_2,r}$ one further has

$$\sum_{r\in\mathcal{R}_2}|F'_{\boldsymbol{n}_2,r}(t,y)|^{-1}\ll N_{b,c}\sum_{|r|\leq N_{b,c}/2}\frac{1}{r}\ll N_{b,c}\log T$$

for fixed t. Therefore, the preceding discussion in conjunction with Lemma 5.2.2 and equations (6.5.15) and the subsequent analysis delivers

$$\begin{aligned}&\sum_{n_2^{a+b}n_3^{a+c}\ll T^{3a/2}}\sum_{r\in\mathcal{R}_2} n_2^{-1/2}n_3^{-1/2}(N_{b,c}+r)^{-1/2}I_{\mathcal{S}_{\boldsymbol{n}}}(T)\\ &\ll(\log T)^2\sum_{n_2^{a+b}n_3^{a+c}\ll T^{3a/2}} n_2^{-1/2}n_3^{-1/2}N_{b,c}^{1/2}\ll T^{3/4}(\log T)^2\sum_{n_3\ll T^{3a/(2(a+c))}} n_3^{-1}\\ &\ll T^{3/4}(\log T)^3.\end{aligned}$$

We find it desirable to remark that for integers with the property that $|r|>N_{b,c}/2$ one then further has $|\log(L_{\boldsymbol{a}}(\boldsymbol{n}))|\gg 1$, an immediate consequence of which being when applied in conjunction with the observation that the rest of the summands in (6.5.16) are $O(T^{-1/2}\log T)$ that then

$$|F'_{\boldsymbol{n}_2,r}(t,y)|\gg 1.$$

Therefore, combining the previous discussion with another application of Lemma 5.2.2 and the analysis following (6.5.15) we derive that such a contribution is bounded above by a constant times

$$(\log T) \sum_{n_2,n_3} \sum_{|r|>N_{b,c}/2} n_2^{-1/2} n_3^{-1/2} (N_{b,c}+r)^{-1/2} \ll T^{3/4}(\log T)^3.$$

$\square$

We shift our focus to the analysis of the term $J_{\mathcal{S}'_{\boldsymbol{n}}}(T)$ and announce that its investigation shall have its reliance on another framework of ideas more aligned with that of the off-diagonal analysis of the non-twisted term pertaining to the approach taken in the previous chapter.

Lemma 6.5.3. *On recalling (6.5.11) one then has*

$$J_{\mathcal{S}'_{\boldsymbol{n}}}(T) \ll T^{1+1/2c-1/2a} + T^{3/4}(\log T)^2.$$

Moreover,

$$J_{1,2}(T) \ll T^{1+1/2c-1/2a} + T^{3/4}(\log T)^4$$

and

$$I_{1,2}(T) \ll T^{1+1/2c-1/2a} + T^{3/4}(\log T)^4.$$

Proof. As a prelude to the upcoming discussion, it should be noted, but not before recalling (6.5.10), that under the assumption of the first statement in conjunction with Lemma 6.5.2 one has

$$\sum_{\substack{\boldsymbol{n} \\ n_1^a \neq n_2^b n_3^c}} P_{\boldsymbol{n}}^{-1/2} \int_{T/2}^{T} L_{\boldsymbol{a}}(\boldsymbol{n})^{it} J_1(P_{\boldsymbol{n}}, t) dt \ll T^{1+1/2c-1/2a} + T^{3/4}(\log T)^4,$$

from where the second statement follows after an application of the latter and a dyadic decomposition argument. Moreover, the third one is an immediate consequence of a combination of the second one, equation (6.5.2) and Lemma 6.5.1.

It therefore transpires that fulfilling the commitment at hand will amount to accomplishing the first statement of the lemma, and this we now address. To this end, we first observe that an application of Lemma 6.2.3 to the integrand pertaining to $I_{\mathcal{S}'_{\boldsymbol{n}}}(T)$ delivers

$$I_{\mathcal{S}'_{\boldsymbol{n}}}(T) = \int_{\mathcal{S}'_{\boldsymbol{n}} \cap [\tau_{\boldsymbol{n}}, T]} L_{\boldsymbol{a}}(\boldsymbol{n})^{it} dt + O(T^{-2}) \ll \frac{1}{\left|\log\big(L_{\boldsymbol{a}}(\boldsymbol{n})\big)\right|} + T^{-2}.$$

As above, we split the corresponding sum into three parts. First, following the previous approach and recalling (6.5.13) we write $n_1 = N_{b,c} + r$ for $r \neq 0$ and observe that whenever $1 < |r| \leq N_{b,c}/2$ then

$$\big|\log(L_{\boldsymbol{a}}(\boldsymbol{n}))\big|^{-1} \asymp \frac{N_{b,c}^a}{|(N_{b,c}+r)^a - n_2^b n_3^c|} \asymp \frac{N_{b,c}}{|r|}. \tag{6.5.17}$$

It may be appropiate to denote $J_{\mathcal{S}'_{\boldsymbol{n}},1}(T)$ the contribution to $J_{\mathcal{S}'_{\boldsymbol{n}}}(T)$ stemming from tuples satisfying $|n_1 - N_{b,c}| > 1$, and thus write

$$J_{\mathcal{S}'_{\boldsymbol{n}}}(T) = J_{\mathcal{S}'_{\boldsymbol{n}},1}(T) + J_{\mathcal{S}'_{\boldsymbol{n}},2}(T), \tag{6.5.18}$$

wherein $J_{\mathcal{S}'_{\boldsymbol{n}},2}(T)$ denotes the corresponding contribution arising from the instances when $|n_1 - N_{b,c}| \leq 1$. Summing over $1 < |r| \leq N_{b,c}$, combining the above equations and utilising the same estimates as above then delivers

$$\begin{aligned} J_{\mathcal{S}'_{\boldsymbol{n}},1}(T) &\ll \sum_{n_2^{a+b} n_3^{a+c} \ll T^{3a/2}} \sum_{1<|r|\leq N_{b,c}/2} n_2^{-1/2} n_3^{-1/2} (N_{b,c}+r)^{-1/2} I_{\mathcal{S}'_{\boldsymbol{n}}}(T) \\ &\ll \sum_{n_2^{a+b} n_3^{a+c} \ll T^{3a/2}} n_2^{-1/2} n_3^{-1/2} N_{b,c}^{1/2} \sum_{1<|r|\leq N_{b,c}/2} \frac{1}{r} \ll T^{3/4} (\log T)^2. \end{aligned} \tag{6.5.19}$$

Likewise, it may be worth observing that whenever $|r| > N_{b,c}/2$ then one has $|\log\big(L_{\boldsymbol{a}}(\boldsymbol{n})\big)|^{-1} \ll 1$, the contribution stemming from triples satisfying such a property being bounded above by

$$\sum_{n_1 n_2 n_3 \ll T^{3/2}} n_1^{-1/2} n_2^{-1/2} n_3^{-1/2} \ll T^{3/4} (\log T)^2.$$

Thus it only remains to bound the contribution to $J_{\mathcal{S}'_{\boldsymbol{n}}}(T)$ stemming from triples satisfying $|n_1 - N_{b,c}| \leq 1$ and this we now discuss. To this end, it is worth noting that whenever $n_1 = N_{b,c}$ then one may assume that $n_2^{b/a} n_3^{c/a}$ is not an integer, since otherwise

$$n_1^a = n_2^b n_3^c,$$

a tuple with such a property then pertaining to the diagonal contribution already discussed.

It may also be desirable to observe in view of (6.5.17) that

$$\Big|\log\big(n_2^b n_3^c/(N_{b,c}+r)^a\big)\Big|^{-1}\asymp \frac{n_2^b n_3^c}{|(N_{b,c}+r)^a-n_2^b n_3^c|}, \tag{6.5.20}$$

and in fact the above line could be further stated by means of an asymptotic formula, such a refinement being of no utility herein. The reader may observe that the above discussion then assures that the denominator involved in the previous line shall not be zero. It also seems pertinent to observe for further purposes that

$$\frac{n_2^b n_3^c}{|(N_{b,c}+r)^a-n_2^b n_3^c|}\leq n_2^b n_3^c, \tag{6.5.21}$$

which in turn follows from the denominator in the above function being a non-zero integer. We further write for each $r\in\{-1,0,1\}$ the contribution

$$J_{\mathcal{S}'_{\boldsymbol{n}},2,r}(T)=\sum_{n_2^{a+b}n_3^{a+c}\ll T^{3a/2}}(n_2n_3)^{-1/2}(N_{b,c}+r)^{-1/2}\Big|\int_{\mathcal{S}'_{\boldsymbol{n}}}e^{it\log(n_2^b n_3^c/(N_{b,c}+r)^a)}dt\Big|. \tag{6.5.22}$$

We divide the range of summation in the above line in accordance with the previous ensuing discussion and utilise (6.5.21) to obtain

$$J_{\mathcal{S}'_{\boldsymbol{n}},2,r}(T)\ll F_{\mathcal{S}'_{\boldsymbol{n}},1}(T)+F_{\mathcal{S}'_{\boldsymbol{n}},2}(T),$$

where the above terms are defined by

$$F_{\mathcal{S}'_{\boldsymbol{n}},1}(T)=\sum_{n_2^b n_3^c\leq T} n_2^{b-1/2-b/2a}n_3^{c-1/2-c/2a},$$

and

$$F_{\mathcal{S}'_{\boldsymbol{n}},2}(T)=T\sum_{n_2^b n_3^c> T} n_2^{-1/2-b/2a}n_3^{-1/2-c/2a}.$$

It may be worth clarifying that in the above analysis we bounded the integral in (6.5.22) by means of the inverse of the corresponding logarithm in conjunction with an application of (6.5.21), the same integral appertaining to the second range of summation being estimated by the length of the interval of integration. Summing over n_2 first yields

$$F_{\mathcal{S}'_{\boldsymbol{n}},1}(T) \ll T^{1+1/2b-1/2a} \sum_{n_3 \leq T^{1/c}} n_3^{-1/2-c/2b} \ll T^{1+1/2c-1/2a}.$$

Likewise, an analogous computation in the same spirit reveals that

$$F_{\mathcal{S}'_{\boldsymbol{n}},2}(T) \ll T^{1+1/2b-1/2a} \sum_{n_3 \leq T^{1/c}} n_3^{-1/2-c/2b} + T \sum_{n_3^c > T} n_3^{-1/2-c/2a} \ll T^{1+1/2c-1/2a},$$

thus yielding the bound

$$J_{\mathcal{S}'_{\boldsymbol{n}},2}(T) \ll T^{1+1/2c-1/2a},$$

whence the above estimate in conjunction with that of (6.5.18) and (6.5.19) delivers the desired result. As was previously addressed in the above chapter, we draw the reader's attention to the above inequalities and point out that the assumption $a < \min(b, c)$ was crucially utilised therein, an analogous argument not being applicable in other circumstances.

□

Lemma 6.5.4. *In the above context, if one further assumes Conjecture 2 it then follows that*

$$J_{\mathcal{S}'_{\boldsymbol{n}}}(T) \ll T^{1/2+a/(a+c)+\varepsilon} + T^{3/4}(\log T)^2.$$

Consequently,

$$J_{1,2}(T) \ll T^{1/2+a/(a+c)+\varepsilon} + T^{3/4}(\log T)^4$$

and

$$I_{1,2}(T) \ll T^{1/2+a/(a+c)+\varepsilon} + T^{3/4}(\log T)^4.$$

Proof. We begin the discussion by noting that under the assumption of the first statement, then the second one follows as in the previous lemma by combining the latter with Lemma 6.5.2 and a dyadic decomposition process. Likewise, it transpires that the third one is an immediate consequence of a combination of the second one and Lemma 6.5.1.

It thus remains to accomplish the first statement of the lemma, and this we now address. The underlying idea of such a refinement with respect to the analogous unconditional lemma has its reliance on a substantial improvement of the error term pertaining to $J_{\mathcal{S}'_{\boldsymbol{n}},2}(T)$ defined in (6.5.18). It seems worth embarking on such an endeavour by first noting that for every $r \in \{-1,0,1\}$ the aforementioned conjecture yields

$$n_2^b n_3^c - (N_{b,c}+r)^a \gg (N_{b,c}+r)^{a-1} n_2^{-1-\varepsilon} n_3^{-1-\varepsilon}. \tag{6.5.23}$$

We use the previous estimates and (6.5.20) to obtain the bound

$$|\log((N_{b,c}+r)^a / n_2^b n_3^c)|^{-1} \ll (N_{b,c} n_2 n_3)^{1+\varepsilon},$$

such a bound being the genesis of the departure from the unconditional approach. Then, combining the above estimates, dividing the range of summation in concordance with the above analysis and recalling (6.5.22) one obtains

$$J_{\mathcal{S}'_{\boldsymbol{n}},2}(T) \ll T^{\varepsilon} M_1(T) + M_2(T),$$

where in the above estimate the terms $M_1(T)$ and $M_2(T)$ are defined by means of the sums

$$M_1(T) = \sum_{n_2^{a+b} n_3^{a+c} \leq T^a} n_2^{1/2+b/2a} n_3^{1/2+c/2a}$$

and

$$M_2(T) = T \sum_{n_2^{a+b} n_3^{a+c} > T^a} n_2^{-1/2-b/2a} n_3^{-1/2-c/2a}.$$

The reader may find it useful to observe that by summing first over n_2 one gets

$$M_1(T) \ll T^{1/2+a/(a+b)} \sum_{n_3 \leq T^{a/(a+c)}} n_3^{-(a+c)/(a+b)} \ll T^{1/2+a/(a+c)}.$$

Likewise, an analogous computation in a similar manner delivers

$$\begin{aligned} M_2(T) &\ll T^{1/2+a/(a+b)} \sum_{n_3 \leq T^{a/(a+c)}} n_3^{-(a+c)/(a+b)} + T \sum_{n_3 > T^{a/(a+c)}} n_3^{-1/2-c/2a} \\ &\ll T^{1/2+a/(a+c)}. \end{aligned}$$

The above discussion then yields

$$J_{\mathcal{S}'_{\boldsymbol{n}},2}(T) \ll T^{1/2+a/(a+c)+\varepsilon},$$

which combined with (6.5.18) and (6.5.19) then leads to the desired bound recorded in the first statement of the lemma. □

6.6 Residual terms arising from the twisted integral analysis

The prelude to the analysis of $I_2(T)$ will be very similar in nature to that of $I_{1,2}(T)$, the departing from the course of the discussion cognate to the latter term being discussed in the upcoming subsection. The investigations presented herein will not comprise novel ideas and shall be concerned with the estimation of the terms $J_{2,u,v}(T)$, the analysis of which shall be reminiscent to that of Lemma 6.5.1, and a succint discussion about the contribution stemming from the set $\mathcal{S}_{\boldsymbol{n}}$ in the spirit of that of Lemma 6.5.2. We find it appropiate to recall (6.2.25) and consider, as was done in (6.5.1), the sum

$$J_{2,u,v}(T) = \sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2} \int_0^T \psi(t) L_{\boldsymbol{a}}(\boldsymbol{n})^{-it} K_{2,u,v}\big(P_{\boldsymbol{n}}, t\big) dt.$$

It may be worth introducing the analogous sum

$$J_{2,1}(T) = \sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2} \int_0^T \psi(t) L_{\boldsymbol{a}}(\boldsymbol{n})^{-it} J_2(P_{\boldsymbol{n}}, t) dt \tag{6.6.1}$$

and observe that equipped with this notation we have reached a position from which to write $I_2(T)$ by making use of (6.2.24) in a rather concise manner, say

$$I_2(T) = J_{2,1}(T) + \sum_{u,v} c_2(u,v) J_{2,u,v}(T).$$

Lemma 6.6.1. *With the above notation, one has*

$$J_{2,u,v}(T) \ll T^{3/4} (\log T)^4.$$

Proof. As was previously noted in the preface of this section, we utilise the fact that the integrand in the definition of $K_{2,u,v}(T)$ is an entire function to move, in the same spirit as that of the proof of Lemma 6.2.3, the line of integration to $\mathrm{Re}(z) = -\alpha_{\boldsymbol{a}} t(\log A)$ and thus get

$$K_{2,u,v}(P_{\boldsymbol{n}}, t) = \int_{-\infty}^{\infty} G_2(t,y) e^{iF_{2,1}(t,y)} dy,$$

wherein the function $G_2(t,y)$ at hand is defined by means of the formula

$$G_2(t,y) = t^{-v}(-\alpha_{\boldsymbol{a}}(\log A)t + iy)^{u-1} e^{(\alpha_{\boldsymbol{a}}^2 - \alpha_{\boldsymbol{a}})(\log A)^2 t + \alpha_{\boldsymbol{a}} I_{\boldsymbol{a}} (\log A) y/2 - y^2/t}$$

and the phase of the corresponding unimodular function will then be

$$F_{2,1}(t,y) = y \log A + 2\alpha_{\boldsymbol{a}}(\log A) y + I_{\boldsymbol{a}} \alpha_{\boldsymbol{a}}^2 (\log A)^2 t/4 - I_{\boldsymbol{a}} y^2/4t.$$

The reader may find it useful to observe that

$$\begin{aligned}\int_{T/2}^{T} \psi(t) L_{\boldsymbol{a}}(\boldsymbol{n})^{-it} K_{2,u,v}(P_{\boldsymbol{n}}, t) dt = &\int_{T/2}^{T} \int_{-T^{1/2}(\log T)}^{T^{1/2}(\log T)} G_2(t,y) e^{iF_2(t,y)} dy dt \\ &+ O(T^{-2}),\end{aligned}$$

where

$$F_2(t,y) = F_{2,1}(t,y) - t \log(L_{\boldsymbol{a}}(\boldsymbol{n})) - g_{\boldsymbol{a}}(t) + \xi_{\boldsymbol{a}} \pi i/4,$$

where we remind the reader of (6.2.14). It is worth noting that then by employing (6.2.28) and (6.2.29) in a similar fashion as was previously done in Lemma 6.5.2, one obtains

$$\begin{aligned} G_2(t,y) &\ll t^{-v}(-\alpha_{\boldsymbol{a}} |\log A| t + |y|)^{u-1} e^{-C_{\boldsymbol{a}} t (\log A)^2 - y^2/2t} \\ &\ll t^{u/2 - v - 1/2} e^{-C_{\boldsymbol{a}} t (\log A)^2/2} \ll t^{-1} e^{-C_{\boldsymbol{a}} t (\log A)^2/2}. \end{aligned} \tag{6.6.2}$$

In view of the above estimates it transpires that we may assume $|y| \ll t^{1/2} \log t$ and $|\log A| \ll t^{-1/2}(\log t)$, since otherwise the contribution to the integral would be negligible. Observe that the latter condition further implies

$$P_{\boldsymbol{n}} \asymp T^{3/2}. \tag{6.6.3}$$

Similarly, as shall become apparent shortly, it may be worth observing that under the above constraints then one has

$$\frac{\partial^2}{\partial^2 t} F_2(t,y) = \frac{9}{8} I_{\boldsymbol{a}} \alpha_{\boldsymbol{a}}^2 t^{-1} + O(t^{-3/2} \log t).$$

We also find it convenient to observe that an analogous argument to that utilised in Lemma 6.5.1 transports us to a position from which to assure that the derivative of $G_2(t,y)/F_2'(t,y)$ with respect to t vanishes in at most $O(1)$ points. This can indeed be achieved by noting in view of (6.5.6) that condition (6.3.2) is satisfied, whence an application of Lemma 6.3.1 delivers the ensuing conclusion. Therefore, interchanging the order of integration and combining the above discussion with Lemma 5.2.4 and (6.6.2) we obtain

$$\begin{aligned}\int_{T/2}^{T}\int_{-T^{1/2}(\log T)}^{T^{1/2}(\log T)} G_2(t,y)e^{iF_2(t,y)}dtdy &\ll \int_{-T^{1/2}(\log T)}^{T^{1/2}(\log T)} T^{-1/2}dy \\ &\ll \log T.\end{aligned}$$

We conclude the lemma by interchanging the order of integration in conjunction with a customary dyadic argument that shall ultimately lead to the bound

$$\int_0^T \psi(t)L_{\boldsymbol{a}}(\boldsymbol{n})^{-it}K_{2,u,v}(P_{\boldsymbol{n}},t)dt \ll (\log T)^2.$$

Consequently, combining the above equations with (6.5.7) and (6.6.3) delivers

$$J_{2,u,v}(T) \ll T^{3/4}(\log T)^4.$$

□

In order to make progress in the proof, it seems pertinent to shift our focus to the contribution to $I_2(T)$ stemming from the term $J_{2,1}(T)$. We find it worth anticipating that a dyadic argument shall be required henceforth. To this end, we thus write

$$\sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2}\int_{T/2}^{T}\psi(t)L_{\boldsymbol{a}}(\boldsymbol{n})^{-it}J_2(P_{\boldsymbol{n}},t)dt = J^{\psi}_{\mathcal{S}_{\boldsymbol{n}}}(T) + J^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T), \tag{6.6.4}$$

wherein the above line the terms on the right side are defined by means of

$$J^{\psi}_{\mathcal{S}}(T) = \sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2} I^{\psi}_{\mathcal{S}}(T) \tag{6.6.5}$$

with

$$I^{\psi}_{\mathcal{S}}(T) = \int_{\mathcal{S}} \psi(t)L_{\boldsymbol{a}}(\boldsymbol{n})^{-it}J_2(P_{\boldsymbol{n}},t)dt$$

for the sets $\mathcal{S} = \mathcal{S}_{\boldsymbol{n}}, \mathcal{S}'_{\boldsymbol{n}}$, which the reader may find it helpful to recall that those were defined in (6.5.8) and (6.5.9). Before providing an explicit bound for the sum $J^{\psi}_{\mathcal{S}_{\boldsymbol{n}}}(T)$ it has been thought pertinent to present first a technical lemma that shall be used on several occasions in subsequent analysis. To this end, we find it worth introducing for further use, but not before recalling the definition of $g_{\boldsymbol{a}}(t)$ described in (6.2.14), the function

$$G_{\boldsymbol{n}}(t) = -t \log L_{\boldsymbol{a}}(\boldsymbol{n}) - g_{\boldsymbol{a}}(t). \tag{6.6.6}$$

Lemma 6.6.2. *Let $(t_{\boldsymbol{n}})_{\boldsymbol{n}}$ be any sequence of real numbers having the property that $t_{\boldsymbol{n}} \in \mathcal{S}_{\boldsymbol{n}}$, and assume that $4a < b+c$. Suppose that $(H_{\boldsymbol{n}}(t))_{\boldsymbol{n}}$ is a collection of functions with the property that*

$$H_{\boldsymbol{n}}(t) = G'_{\boldsymbol{n}}(t) + O(t^{-1/2} \log t) \tag{6.6.7}$$

for $t \in [T/2, T]$, the above implicit constant not depending on $\boldsymbol{n}$. Then one has that

$$\sum_{\tau_n \asymp T} P_{\boldsymbol{n}}^{-1/2} \min(|H_{\boldsymbol{n}}(t_{\boldsymbol{n}})|^{-1}, T^{1/2}) \ll T^{3/4} (\log T)^3$$

whenever $b > c$. If $b = c$ then an analogous estimate holds with $T^{3/4}(\log T)^2$ replacing the term in the right side of the bound.

Proof. We shall denote henceforth for convenience by $W(T)$ to the left side of the above equation. The reader may find it useful to note that then

$$G'_{\boldsymbol{n}}(t) = a \log n_1 - b \log n_2 - c \log n_3 + (b+c-a) \log t + \log\Big(\frac{b^b c^c}{a^a 2^{b+c-a}}\Big). \tag{6.6.8}$$

As shall be elucidated shortly, the evaluation of the above function at the point $\tau_{\boldsymbol{n}}$ shall play a not insignificant role in the course of the investigation cognate to this lemma. It has then been thought pertinent to recall (6.4.4) and compute such an evaluation beforehand, say

$$\begin{aligned} G'_{\boldsymbol{n}}(\tau_{\boldsymbol{n}}) =& \Big(\frac{2b+2c+a}{3}\Big) \log n_1 + \Big(\frac{2c-b-2a}{3}\Big) \log n_2 + \Big(\frac{2b-c-2a}{3}\Big) \log n_3 \\ & + \log K_{\boldsymbol{a}}, \end{aligned} \tag{6.6.9}$$

wherein

$$\log K_{\boldsymbol{a}} =(b+c-a)\log\pi-\Big(\frac{b+c+2a}{3}\Big)\log a+\Big(\frac{2b-c+a}{3}\Big)\log b$$
$$+\Big(\frac{2c-b+a}{3}\Big)\log c.$$

We also note for further use that it is apparent in view of (6.6.7) and (6.6.8) in conjunction with the fact that $t_{\boldsymbol{n}}\in\mathcal{S}_{\boldsymbol{n}}$ that

$$H_{\boldsymbol{n}}(t_{\boldsymbol{n}})=G'_{\boldsymbol{n}}(\tau_{\boldsymbol{n}})+O(T^{-1/2}\log T), \tag{6.6.10}$$

the above implicit constant being independent of $\boldsymbol{n}$.

It may be worth noting that in view of the assumptions on a,b,c earlier made in the statement of the lemma, whenever $2c>b+2a$ then the corresponding coefficients of the logarithmic summands in (6.6.9) are strictly positive. It then transpires that whenever either n_1,n_2 or n_3 are sufficiently large then $|G'_{\boldsymbol{n}}(\tau_{\boldsymbol{n}})|\gg 1$ in the interval of integration, the corresponding contribution to $W(T)$ appertaining to such tuples being $O\big(T^{3/4}(\log T)^2\big)$ by (6.5.7) and (6.6.10). If instead each of the entries are bounded, an application of the trivial observation $\min\big(|H_{\boldsymbol{n}}(t_{\boldsymbol{n}})|^{-1},T^{1/2}\big)\ll T^{1/2}$ enables one to deduce that such a contribution is $O(T^{1/2})$. The reader may notice that the instance $b=c$ is encompassed in this more general framework.

It has been thought pertinent to devote a few lines to the not entirely recalcitrant situation for which $b=2c-2a$. To this end, we draw the reader's attention to (6.6.9) and note that the coefficients in front of $\log n_1$ and $\log n_3$ are positive. In view of such an observation, it transpires that whenever either n_1 or n_3 are sufficiently big one has the bound $|G'_{\boldsymbol{n}}(\tau_{\boldsymbol{n}})|\gg 1$, an application of which, in conjunction with (6.5.7) and (6.6.10) enables one to estimate the corresponding contribution to $W(T)$ appertaining to such tuples by $O(T^{3/4}(\log T)^2)$. If instead both n_1 and n_3 are bounded then it is apparent that

$$\log K_{\boldsymbol{a}}=3(c-a)\log\pi-c\log a+(c-a)\log 2(c-a)+a\log c.$$

We find it desirable to focus our attention on the second part of the above equation pertaining to the summands not involving π and observe that in view of the fact that $c>2a$ as a consequence of both the assumptions on the statement of the lemma and the aforementioned equality between a, b, c, it is apparent that

$$\begin{aligned}-c\log a+(c-a)\log 2(c-a)+a\log c &\geq -c\log a+(c-a)\log 2a+a\log c\\ &= a(\log c-\log a)+(c-a)\log 2>0.\end{aligned}$$

Combining the above lines of inequalities with (6.6.9) delivers the bound $|G_{\boldsymbol{n}}'(\tau_{\boldsymbol{n}})|\gg 1$, which in conjunction with (6.5.7) and (6.6.10) and the ensuing discussion yields

$$W(T)\ll \sum_{\tau_{\boldsymbol{n}}\asymp T} P_{\boldsymbol{n}}^{-1/2}\ll T^{3/4}(\log T)^2$$

under such assumptions.

We shall henceforth assume $2c<b+2a$ throughout the rest of the discussion devoted to the investigation of $W(T)$. As a prelude to the aforementioned perusal, it seems worth anticipating that the manoeuvres which shall be deployed in due course shall not be dissimilar to those already presented throughout the rest of the chapter. To this end we introduce, for fixed (n_1,n_3), the parameter

$$N_2=\big(K_{\boldsymbol{a}}^3 n_1^{2b+2c+a}n_3^{2b-c-2a}\big)^{1/(b+2a-2c)}. \tag{6.6.11}$$

It has also been thought appropiate to define, for each triple (n_1,n_2,n_3) with $\boldsymbol{n}_1=(n_1,n_3)$ the number $r=n_2-N_2$, which may not be an integer, and write for ease of notation $H_{\boldsymbol{n}_1,r}(t)$, $G_{\boldsymbol{n}_1,r}(t)$, $t_{\boldsymbol{n}_1,r}$ and $\tau_{\boldsymbol{n}_1,r}$ to denote $H_{\boldsymbol{n}}(t)$, $G_{\boldsymbol{n}}(t)$, $t_{\boldsymbol{n}}$ and $\tau_{\boldsymbol{n}}$ respectively. By recalling (6.6.9) it then transpires that

$$\begin{aligned}G_{\boldsymbol{n}_1,r}'(\tau_{\boldsymbol{n}_1,r})=&\Big(\frac{2b+2c+a}{3}\Big)\log n_1+\Big(\frac{2c-b-2a}{3}\Big)\log(N_2+r)\\ &+\Big(\frac{2b-c-2a}{3}\Big)\log n_3+\log K_{\boldsymbol{a}},\end{aligned}$$

whence utilising the fact that (6.6.9) vanishes when substituting $n_2=N_2$ and combining it with (6.6.10) one may deduce

$$H_{\boldsymbol{n}_1,r}(t_{\boldsymbol{n}_1,r})=\frac{2c-b-2a}{3}\log(1+r/N_2)+O(T^{-1/2}\log T).$$

We denote as is customary by $\mathcal{G}_1$ to the set of integers $|r|\leq N_2/2$ having the property that $|H_{\boldsymbol{n}_1,r}(t_{\boldsymbol{n}_1,r})|\leq N_2^{-1}$. In view of the uniformity in the above error term with respect to r, as was assumed in the statement of the lemma, it then transpires that

$$|\mathcal{G}_1| \ll N_2 T^{-1/2}(\log T) + 1,$$

the contribution to $W(T)$ stemming from the corresponding tuples being bounded above by

$$\sum_{n_1 N_2 n_3 \asymp T^{3/2}} \sum_{r \in \mathcal{G}_1} n_1^{-1/2} N_2^{-1/2} n_3^{-1/2} \min\left(|H_{\boldsymbol{n}_1,r}(t_{\boldsymbol{n}_1,r})|^{-1}, T^{1/2}\right) \ll W_1(T) + W_2(T),$$

wherein

$$W_1(T) = (\log T) \sum_{n_1 N_2 n_3 \asymp T^{3/2}} n_1^{-1/2} N_2^{1/2} n_3^{-1/2}$$

and

$$W_2(T) = T^{1/2} \sum_{n_1 N_2 n_3 \asymp T^{3/2}} n_1^{-1/2} N_2^{-1/2} n_3^{-1/2}.$$

As a prelude to our analysis of the preceding terms, it seems worth noting that the tuples involved in the above sums satisfy

$$n_1^{1/2} N_2^{1/2} n_3^{1/2} \asymp T^{3/4}. \tag{6.6.12}$$

We utilise such an estimate to obtain

$$W_1(T) \ll T^{3/4}(\log T) \sum_{n_1 N_2 n_3 \asymp T^{3/2}} n_1^{-1} n_3^{-1} \ll T^{3/4}(\log T)^3.$$

In order to bound $W_2(T)$ we define first, for convenience, the exponents

$$\alpha_1 = \frac{3b + 3a}{b + 2a - 2c}, \qquad \alpha_3 = \frac{3b - 3c}{b + 2a - 2c},$$

we remind the reader of (6.6.11) and employ (6.6.12) to obtain

$$\begin{aligned} W_2(T) &\ll T^{-1/4} \sum_{n_1^{\alpha_1} n_3^{\alpha_3} \asymp T^{3/2}} 1 \ll T^{3/2\alpha_3 - 1/4} \sum_{n_1 \ll T^{3/2\alpha_1}} n_1^{-\alpha_1/\alpha_3} \\ &\ll T^{3/2\alpha_3 - 1/4} = T^{3/4 + (2a-b)/2(b-c)} \ll T^{3/4}, \end{aligned}$$

wherein we used the fact that $b > 2a$ stemming from the inequality $b + c > 4a$.

It thus remains to analyse the contribution of the set $\mathcal{G}_2$ comprising integers $|r| \leq N_2/2$ having the property that $|H_{\boldsymbol{n}_1,r}(t_{\boldsymbol{n}_1,r})| > N_2^{-1}$. Under such circumstances, it transpires that

$$\sum_{n_1N_2n_3\asymp T^{3/2}} \sum_{r\in\mathcal{G}_2} n_1^{-1/2}N_2^{-1/2}n_3^{-1/2}\min\big(|H_{\boldsymbol{n}_1,r}(t_{\boldsymbol{n}_1,r})|^{-1}, T^{1/2}\big)$$
$$\ll \sum_{n_1N_2n_3\asymp T^{3/2}} n_1^{-1/2}N_2^{1/2}n_3^{-1/2}\sum_{r\in\mathcal{G}_2}\frac{1}{r} \ll T^{3/4}(\log T)\sum_{n_1N_2n_3\asymp T^{3/2}} n_2^{-1}n_3^{-1}$$
$$\ll T^{3/4}(\log T)^3,$$

wherein we routinarily utilised (6.6.12), which completes the proof.

□

We are now equipped to expeditiously analyse $J^{\psi}_{\mathcal{S}_{\boldsymbol{n}}}(T)$, which we remind the reader that was defined in (6.6.5).

Lemma 6.6.3. *Assume that $4a < b + c$. Whenever $c < b$ one has that*

$$J^{\psi}_{\mathcal{S}_{\boldsymbol{n}}}(T) \ll T^{3/4}(\log T)^4.$$

If $b = c$ one instead has the bound $J^{\psi}_{\mathcal{S}_{\boldsymbol{n}}}(T) \ll T^{3/4}(\log T)^3$.

Proof. We find it convenient to prepare the ground for our analysis by denoting

$$G_3(t, y) = e^{(\log A)+1/t-I_{\boldsymbol{a}}y/2t-y^2/t}(1 + iy)^{-1},$$

it being convenient to note for further purposes that such a function satisfies

$$G_3(t, y) \ll \frac{1}{1 + |y|}. \tag{6.6.13}$$

It also seems reasonable to introduce for $\boldsymbol{n}$ the corresponding phase appertaining to the unimodular function involved in the integral at hand

$$F_{1,\boldsymbol{n}}(t, y) = y(\log A) + 2y/t + I_{\boldsymbol{a}}/4t - I_{\boldsymbol{a}}y^2/4t + G_{\boldsymbol{n}}(t), \tag{6.6.14}$$

wherein $G_{\boldsymbol{n}}(t)$ was defined in (6.6.6). In view of the decay exhibited by the function $G_3(t, y)$ in conjunction with (6.2.25) and (6.6.5) it then transpires that

$$I^{\psi}_{\mathcal{S}_{\boldsymbol{n}}}(T) = \int_{\mathcal{S}_{\boldsymbol{n}}}\int_{-T^{1/2}\log T}^{T^{1/2}\log T} G_3(t, y)e^{iF_{1,\boldsymbol{n}}(t,y)}dydt + O(T^{-2}).$$

The analysis of the piecewise monotonicity of the corresponding auxiliary function shall not be dissimilar to that described in Lemma 6.5.2. It seems worth noting for such purposes that it is apparent that for fixed y, the zeros of the function

$$\frac{d}{dt}\Big(G_3(t,y)/F'_{1,\boldsymbol{n}}(t,y)\Big)$$

are also zeros of a function

$$P_2(t, \log t, y, \log(L_{\boldsymbol{a}}(\boldsymbol{n}))),$$

wherein $P_2(z_1, z_2, z_3, z_4)$ is a polynomial of universally bounded degree linear in z_2. It therefore transpires that by differentiating the above a sufficiently large universally bounded number of times, the corresponding terms containing a factor of $(\log t)$ may eventually disappear, the zeros of the resulting expression further being zeros of a polynomial in t and y of bounded degree. Therefore, subsequent applications of Rolle's theorem enable one to partition the set of integration into a universally bounded number of intervals having the property that $G_3(t,y)/F'_{1,\boldsymbol{n}}(t,y)$ is monotonic on each of them.

We may suppose that $\mathcal{S}_{\boldsymbol{n}} \neq \emptyset$, since if not no further work would be required. We also find it desirable to recall (6.4.5) to the end of noting that whenever $t \in \mathcal{S}_{\boldsymbol{n}}$, as is the case herein, one has that $|\log A| \ll T^{-1/2}\log T$. It then seems worth recalling (6.6.14) and observing that if $|y| \leq T^{1/2}(\log T)$ one has

$$F'_{1,\boldsymbol{n}}(t,y) = G'_{\boldsymbol{n}}(t) + O\big(t^{-1/2}(\log t)\big),$$

the corresponding implicit constant not depending on $\boldsymbol{n}$. The reader may notice that we have merely prepared the ground for an application of both the above lemma and the oscillatory integral estimates ones. Before proceeding in such a way it is convenient to denote as we may by $t_{\boldsymbol{n}}$ to the real number $s_{\boldsymbol{n}} \in \mathcal{S}_{\boldsymbol{n}}$ having the property that $|F'_{1,\boldsymbol{n}}(s_{\boldsymbol{n}})|$ is minimum in $\mathcal{S}_{\boldsymbol{n}}$, the existence of such a number being assured by the compactness of the set $\mathcal{S}_{\boldsymbol{n}}$. Therefore, combining Lemmata 5.2.2 and 5.2.4 with (6.6.13) and Lemma 6.6.2 for the choice $H_{\boldsymbol{n}}(t) = F'_{1,\boldsymbol{n}}(t)$ one may deduce that

$$J^{\psi}_{\mathcal{S}_{\boldsymbol{n}}}(T) \ll (\log T)\sum_{\tau_n \asymp T} P_{\boldsymbol{n}}^{-1/2}\min(|H_{\boldsymbol{n}}(t_{\boldsymbol{n}})|^{-1}, T^{1/2}) \ll T^{3/4}(\log T)^4$$

if $c < b$, the bound

$$J^{\psi}_{\mathcal{S}_{\boldsymbol{n}}}(T) \ll T^{3/4}(\log T)^3$$

replacing the above for the case $b = c$.

$\square$

6.7 The stationary phase method and van der Corput's estimates

As was earlier anticipated, the departing in the analysis of $I_2(T)$ from the course of the discussion cognate to the term $I_{1,2}(T)$ presented above has its genesis on the necessity of an application of the stationary phase method as a consequence of the presence of the twisting factor to which we previously alluded. The remainder of the discussion in this chapter shall thus be devoted to the analysis of $J^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T)$. The latter shall essentially differ from that of previous sections in that the derivative of the phase function pertaining to the corresponding integral at hand may vanish. To this end, an application of Lemma 5.2.5 in conjunction with classical van der Corput's estimates and an intrincate analysis shall be required.

For such purposes we first apply Lemma 6.2.3 to obtain

$$\begin{aligned} I^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T) &= \int_{\mathcal{S}'_{\boldsymbol{n}}\cap[\tau_{\boldsymbol{n}},T]} \psi(t)L_{\boldsymbol{a}}(\boldsymbol{n})^{-it}dt + O(T^{-2}) \\ &= e^{i\pi\xi_{\boldsymbol{a}}/4}\int_{\mathcal{S}'_{\boldsymbol{n}}\cap[\tau_{\boldsymbol{n}},T]} e^{iG_{\boldsymbol{n}}(t)}dt + O(T^{-2}), \end{aligned} \tag{6.7.1}$$

wherein $G_{\boldsymbol{n}}(t)$ was defined in (6.6.6) and $\tau_{\boldsymbol{n}} \leq T$. As is customary, the zero of the derivative of the phase function at hand shall play a prominent role in the discussion, whence it seems worth recalling (6.6.8) and recording for further use that on writing

$$c_{\boldsymbol{n}} = \Big(\frac{n_2^b n_3^c}{n_1^a}\Big)^{1/(b+c-a)}C_{\boldsymbol{a}}, \qquad \text{with } C_{\boldsymbol{a}} = 2\Big(\frac{a^a}{b^b c^c}\Big)^{1/(b+c-a)}, \tag{6.7.2}$$

one then has $G'_{\boldsymbol{n}}(c_{\boldsymbol{n}}) = 0$. We also find it desirable to note, as may become apparent shortly, that

$$G_{\boldsymbol{n}}(c_{\boldsymbol{n}}) = -(b+c-a)c_{\boldsymbol{n}},$$

and define for convenience $G_1(c_{\boldsymbol{n}})$ by means of

$$G_1(c_{\boldsymbol{n}}) = G_{\boldsymbol{n}}(c_{\boldsymbol{n}}) + (\xi_{\boldsymbol{a}} + 1)\pi/4. \tag{6.7.3}$$

We shall provide an asymptotic evaluation of the term $J^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T)$ defined in (6.6.5), but before embarking in such an endeavour it seems desirable to write

$$T_{\boldsymbol{n}} = \max(T/2, \tau_{\boldsymbol{n}} + T^{1/2}\log T)$$

and observe that equipped with this notation we have reached a position from which to legitimately write

$$\mathcal{S}'_{\boldsymbol{n}} \cap [\tau_{\boldsymbol{n}}, T] = [T_{\boldsymbol{n}}, T],$$

it sufficing to assume that $\tau_{\boldsymbol{n}} + T^{1/2}\log T \leq T$ since otherwise $\mathcal{S}'_{\boldsymbol{n}} = \emptyset$ and we would be done. The following note shall succintly discard the case $b = c$ from our analysis.

Lemma 6.7.1. *Assume that $2a < b$ and $b = c$. Then it transpires that*

$$J^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T) \ll T^{3/4}(\log T).$$

Proof. We recall (6.6.8) to the end of observing that whenever $t \in [T/2, T]$ and $P_{\boldsymbol{n}} \ll T^{3/2}$ it follows that

$$\begin{aligned} G'_{\boldsymbol{n}}(t) &\geq (2b-a)\log t + \log\Big(\frac{b^{2b}}{a^a 2^{2b-a}}\Big) - b\log(n_2 n_3) \\ &\geq (b/2 - a)\log t - C \end{aligned}$$

for some constant $C > 0$. It therefore transpires that $G'_{\boldsymbol{n}}(t) \gg \log t$ in the interval at hand, whence by Lemma 5.2.1 one has

$$I^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T) \ll (\log T)^{-1},$$

which combined with (6.5.7) and (6.6.5) delivers

$$J^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T) \ll \sum_{P_{\boldsymbol{n}} \ll T^{3/2}} P_{\boldsymbol{n}}^{-1/2} I^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T) \ll (\log T)^{-1} \sum_{P_{\boldsymbol{n}} \ll T^{3/2}} P_{\boldsymbol{n}}^{-1/2} \ll T^{3/4}(\log T),$$

as desired. □

Lemma 6.7.2. *Assume that $4a < b+c$ and $b > c$. One then has that*

$$\begin{aligned} J^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T) =& (b+c-a)^{1/2}\sqrt{2\pi} \sum_{\substack{T_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} \leq T \\ \tau_{\boldsymbol{n}} \leq T}} P_{\boldsymbol{n}}^{-1/2} c_{\boldsymbol{n}}^{1/2} e^{iG_1(c_{\boldsymbol{n}})} \\ & + O\big(T^{3/4}(\log T)^3\big). \end{aligned}$$

Proof. We shall omit henceforth writing the condition $\tau_{\boldsymbol{n}} \leq T$ for the sake of simplicity. We begin our discussion by summing equation (6.7.1) over tuples $\boldsymbol{n}$ and applying Lemmata 5.2.1 and 5.2.3 in conjunction with Lemma 5.2.5 to obtain

$$\begin{aligned} J^{\psi}_{\mathcal{S}'_{\boldsymbol{n}}}(T) =& (b+c-a)^{1/2}\sqrt{2\pi} \sum_{T_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} \leq T} P_{\boldsymbol{n}}^{-1/2} c_{\boldsymbol{n}}^{1/2} e^{iG_1(c_{\boldsymbol{n}})} \\ & + O\Big(\sum_{T_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq 2T_{\boldsymbol{n}}} P_{\boldsymbol{n}}^{-1/2} \min\big(\lvert G'_{\boldsymbol{n}}(T_{\boldsymbol{n}})\rvert^{-1}, T_{\boldsymbol{n}}^{1/2}\big)\Big) \\ & + O\Big(\sum_{T/2 \leq c_{\boldsymbol{n}} \leq 2T} P_{\boldsymbol{n}}^{-1/2} \min\big(\lvert G'_{\boldsymbol{n}}(T)\rvert^{-1}, T^{1/2}\big)\Big) + O\big(T^{3/4}(\log T)^2\big), \end{aligned} \tag{6.7.4}$$

whence it transpires that proving the result at hand amounts to estimating the above error terms. The reader shall rest assured that further details about such an application will be delivered promptly, those nonetheless having been essentially delivered in Lemma 5.7.1 of the previous chapter. It may first be useful to observe that in the preceding lines we implicitly applied Lemma 5.2.5 for the tuples satisfying $T_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} \leq T$. Similarly, we further utilised Lemma 5.2.1 and 5.2.3 whenever either $T_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq T_{\boldsymbol{n}}$ or $T < c_{\boldsymbol{n}} \leq 2T$, the contribution stemming from such tuples being absorbed accordingly by the above error terms. If instead the parameter $c_{\boldsymbol{n}}$ does not belong to any of these ranges, we employed Lemma 5.2.1 to estimate the integrals appertaining to tuples satisfying such a condition by $O(1)$, an immediate application of (6.5.7) when summing over such tuples thus bounding the latter contribution by $O\big(T^{3/4}(\log T)^2\big)$.

We shift our attention first to the investigation of the first error term in the above formula, and find it convenient, as shall become transparent promptly, writing

$$\sum_{T_{\boldsymbol{n}}/2\leq c_{\boldsymbol{n}}\leq 2T_{\boldsymbol{n}}} P_{\boldsymbol{n}}^{-1/2}\min\big(|G'_{\boldsymbol{n}}(T_{\boldsymbol{n}})|^{-1},T_{\boldsymbol{n}}^{1/2}\big)=Y_1(T)+Y_2(T),$$

wherein

$$Y_1(T)=\sum_{\substack{T_{\boldsymbol{n}}/2\leq c_{\boldsymbol{n}}\leq 2T_{\boldsymbol{n}}\\ \tau_{\boldsymbol{n}}+T^{1/2}\log T\geq T/2}} P_{\boldsymbol{n}}^{-1/2}\min\big(|G'_{\boldsymbol{n}}(T_{\boldsymbol{n}})|^{-1},T_{\boldsymbol{n}}^{1/2}\big)$$

and

$$Y_2(T)=\sum_{\substack{T_{\boldsymbol{n}}/2\leq c_{\boldsymbol{n}}\leq 2T_{\boldsymbol{n}}\\ \tau_{\boldsymbol{n}}+T^{1/2}\log T< T/2}} P_{\boldsymbol{n}}^{-1/2}\min\big(|G'_{\boldsymbol{n}}(T_{\boldsymbol{n}})|^{-1},T_{\boldsymbol{n}}^{1/2}\big).$$

We begin our discussion by analysing first $Y_1(T)$. It seems worth noting that under the constraints imposed in the sum cognate to such a term and on recalling (6.5.8), one may infer that $T_{\boldsymbol{n}}=\tau_{\boldsymbol{n}}+T^{1/2}\log T\in\mathcal{S}_{\boldsymbol{n}}$. We have therefore reached a position from which to apply Lemma 6.6.2 for the choice $H_{\boldsymbol{n}}(t)=G'_{\boldsymbol{n}}(t)$, namely

$$Y_1(T)\ll\sum_{\tau_{\boldsymbol{n}}\asymp T} P_{\boldsymbol{n}}^{-1/2}\min\big(|G'_{\boldsymbol{n}}(T_{\boldsymbol{n}})|^{-1},T_{\boldsymbol{n}}^{1/2}\big)\ll T^{3/4}(\log T)^3$$

for both $b>c$ and $b=c$.

The analysis of $Y_2(T)$ shall be reminiscent in nature to the preceding one, the corresponding error terms derived being essentially equal to the earlier obtained ones. As was previously done, we recall (6.7.2) and define for fixed (n_1,n_3) and further convenience the parameter

$$N_T=(T/2)^{(b+c-a)/b}C_{\boldsymbol{a}}^{-(b+c-a)/b}n_1^{a/b}n_3^{-c/b}. \tag{6.7.5}$$

It may be worth observing that then

$$a\log n_1-b\log N_T-c\log n_3+(b+c-a)\log(T/2)+\log\Big(\frac{b^bc^c}{a^a2^{b+c-a}}\Big)=0.$$

Therefore, on introducing for each $n_2\in\mathbb{N}$ the real number $r=n_2-N_T$, recalling to the reader of (6.6.8) and using the above line, it transpires that then

$$G'_{\boldsymbol{n}}(T/2) = a\log n_1 - b\log(N_T+r) - c\log n_3 + (b+c-a)\log(T/2) + \log\Big(\frac{b^b c^c}{a^a 2^{b+c-a}}\Big) = -b\log(1+r/N_T). \tag{6.7.6}$$

We find it desirable to note for further use that in view of the constraints in the tuples pertaining to the sum involved in the definition of $Y_2(T)$ it is apparent that $T_{\boldsymbol{n}} = T/2$, the underlying inequality cognate to $c_{\boldsymbol{n}}$ thus being equivalent to

$$T/4 \le c_{\boldsymbol{n}} \le T,$$

which may in turn be rephrased by means of the bounds

$$2^{-(b+c-a)/b} N_T \le n_2 \le 2^{(b+c-a)/b} N_T.$$

We denote for simplicity by I_{N_T} to the above interval.

To the end of providing an exhaustive account of the whole picture, it has been thought pertinent to discuss first the instances for which $|n_2 - N_T| > 1$, and herein a simple application of (6.7.6) already delivers

$$\sum_{\substack{|n_2-N_T|>1\\ n_2\in I_{N_T}}} \frac{n_2^{-1/2}}{|G'_{\boldsymbol{n}}(T/2)|} \ll \sum_{0<r\le N_T} \frac{N_T^{1/2}}{r} \ll N_T^{1/2}\log N_T. \tag{6.7.7}$$

We use the trivial bound $\min\big(|G'_{\boldsymbol{n}}(T/2)|^{-1}, T^{1/2}\big) \ll T^{1/2}$ if $|n_2 - N_T| \le 1$ and combine such an observation with the preceding discussion to obtain

$$Y_2(T) \ll Y_{2,1}(T) + Y_{2,2}(T),$$

where

$$Y_{2,1}(T) = \sum_{n_1 N_T n_3 \ll T^{3/2}} n_1^{-1/2} n_3^{-1/2} \sum_{0<r\le N_T} \frac{N_T^{1/2}}{r}$$

and

$$Y_{2,2}(T) = T^{1/2} \sum_{n_1 N_T n_3 \ll T^{3/2}} (n_1 N_T n_3)^{-1/2}.$$

The line of attack to follow herein shall not be dissimilar to the one employed in the underlying analysis of $Y_1(T)$. In order to estimate $Y_{2,1}(T)$ it might be worth noting as is customary that $(n_1 N_T n_3)^{1/2} \ll T^{3/4}$, whence

$$\begin{aligned} Y_{2,1}(T) &\ll (\log T) \sum_{n_1 N_T n_3 \ll T^{3/2}} n_1^{-1/2} N_T^{1/2} n_3^{-1/2} \\ &\ll T^{3/4}(\log T) \sum_{n_1 N_T n_3 \ll T^{3/2}} n_1^{-1} n_3^{-1} \ll T^{3/4}(\log T)^3. \end{aligned}$$

We shift our focus to the perusal of $Y_{2,2}(T)$ and observe after recalling (6.7.5) to the reader that

$$Y_{2,2}(T) \ll T^{(a-c)/2b} \sum_{n_1^{a+b} n_3^{b-c} \ll T^{(b-2c+2a)/2}} n_1^{-1/2-a/2b} n_3^{-1/2+c/2b}.$$

It may be worth observing that if $b \leq 2c - 2a$ then the above term is trivial. For the rest of the instances we sum over n_1 first to obtain

$$Y_{2,2}(T) \ll T^{-1/4+(b-2c+2a)/2(a+b)} \sum_{n_3^{b-c} \ll T^{(b-2c+2a)/2}} n_3^{-(b-c)/(a+b)},$$

an analogous argument summing over n_3 thus yielding

$$Y_{2,2}(T) \ll T^{-1/4+(b-2c+2a)/2(b-c)} = T^{3/4+(2a-b)/2(b-c)},$$

as desired. The combination of the estimates obtained above for the terms $Y_1(T)$ and $Y_2(T)$ in conjunction with the observation that under the constraints in the statement of the lemma one has $b > 2a$ then enables one to deduce

$$Y(T) \ll T^{3/4}(\log T)^3.$$

The analysis of the second error term appertaining to (6.7.4) shall be completely identical to the one cognate to $Y_2(T)$ earlier exposed, whence in the interest of curtailing our discussion it has been thought preferable to succintly indicate that the proof of the analogous estimate for the term at hand would follow by replacing T by $T/2$ in (6.7.5), (6.7.6) and (6.7.7), such an observation in conjunction with the above conclusion thus completing the proof of the lemma at hand. $\square$

The rest of the discussion in this chapter shall be devoted to the investigation of the main term cognate to the formula earlier obtained in the above lemma, an application of van der Corput's second derivative test being sufficient to exploit further cancellation and derive satisfactory enough upper bounds for the exponential sum at hand.

Lemma 6.7.3. *Whenever $4a < b + c$ and $c < b$ one has that*

$$\sum_{\substack{T_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} \leq T \\ \tau_{\boldsymbol{n}} \leq T}} P_{\boldsymbol{n}}^{-1/2} c_{\boldsymbol{n}}^{1/2} e^{iG_1(c_{\boldsymbol{n}})} \ll T^{3/4+(2a-c)/2(b-c)} + T^{3/4}(\log T)^2,$$

Moreover, it follows that

$$J_{\mathcal{S}'_{\boldsymbol{n}}}^{\psi}(T) \ll T^{3/4+(2a-c)/2(b-c)} + T^{3/4}(\log T)^3$$

and

$$I_2(T) \ll T^{3/4+(2a-c)/2(b-c)} + T^{3/4}(\log T)^4.$$

If instead $b = c$ then

$$I_2(T) \ll T^{3/4}(\log T)^4.$$

Proof. We begin as customary by elucidating how to deduce the latter statements pertaining to the instance $c < b$ assuming the first one. We note first that one may obtain the second estimate after combining the one earlier assumed with Lemma 6.7.2, an immediate application of such an estimate in conjunction with equation (6.6.4) and Lemma 6.6.3 thereby delivering the bound

$$\sum_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2} \int_{T/2}^{T} \psi(t) L_{\boldsymbol{a}}(\boldsymbol{n})^{-it} J_2(P_{\boldsymbol{n}}, t) dt \ll T^{3/4+(2a-c)/2(b-c)} + T^{3/4}(\log T)^4.$$

Therefore, summing over dyadic intervals and recalling to the reader of (6.6.1) it transpires that then

$$J_{1,2}(T) \ll T^{3/4+(2a-c)/2(b-c)} + T^{3/4}(\log T)^4,$$

which, combined with Lemma 6.6.1 yields the third statement, as desired. If $b = c$ then the statement follows by combining Lemmata 6.6.1, 6.6.3 and 6.7.1 with a dyadic decomposition argument.

We shall shift therefore our focus to the analysis of the first estimate recorded above. To this end it seems adequate to note first that $T_{\boldsymbol{n}} \asymp T$, whence in view of the definition (6.7.2) it transpires that under the above constraints one has

$$n_2 \asymp T^{(b+c-a)/b} n_1^{a/b} n_3^{-c/b}.$$

We find it worth observing that the ensuing condition in conjunction with the inequality $\tau_{\boldsymbol{n}} \leq T$ delivers the restriction

$$n_1^{a+b} n_3^{b-c} \ll T^{(b-2c+2a)/2}.$$

It then transpires that we may assume $b \geq 2c - 2a$, since otherwise the above inequality would not hold for any tuple. In order to prepare the terrain for a succint application of the aforementioned van der Corput's estimates, some notation is required. We thus denote

$$\gamma_1 = a/(b+c-a), \qquad \gamma_2 = b/(b+c-a), \qquad \gamma_3 = c/(b+c-a).$$

We sum first over n_2 and apply van der Corput's second derivative test [134, Theorem 5.9], but not before recalling to the reader of (6.7.3), to obtain

$$\sum_{T_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} \leq T} P_{\boldsymbol{n}}^{-1/2} c_{\boldsymbol{n}}^{1/2} e^{iG_1(c_{\boldsymbol{n}})} \ll S_1(T) + S_2(T),$$

where

$$S_1(T) = \sum_{n_1^{a+b} n_3^{b-c} \ll T^{(b-2c+2a)/2}} \Big(T^{(b+c-a)/b} n_1^{a/b} n_3^{-c/b}\Big)^{-1/2+\gamma_2} n_1^{-1/2-\gamma_1} n_3^{-1/2+\gamma_3}$$

and

$$S_2(T) = \sum_{n_1^{a+b} n_3^{b-c} \ll T^{(b-2c+2a)/2}} n_1^{-1/2} n_3^{-1/2} \Big(T^{(b+c-a)/b} n_1^{a/b} n_3^{-c/b}\Big)^{1/2}.$$

It is apparent that the tuples pertaining to the above sums satisfy the inequality

$$n_1^{1/2} n_3^{1/2} \Big(T^{(b+c-a)/b} n_1^{a/b} n_3^{-c/b}\Big)^{1/2} \ll T^{3/4},$$

which in turn delivers, as is customary, the corresponding bounds

$$S_2(T) \ll T^{3/4} \sum_{n_1^{a+b} n_3^{b-c} \ll T^{(b-2c+2a)/2}} n_1^{-1} n_3^{-1} \ll T^{3/4} (\log T)^2.$$

We shall shift our focus next to the investigation of $S_1(T)$ and anticipate that its analysis shall be surprisingly reminiscent to that cognate to $Y_{2,2}(T)$ in the above lemma. By rearranging terms and summing over n_1 one readily sees that

$$\begin{aligned} S_1(T) &= T^{(b-c+a)/2b} \sum_{n_1^{a+b} n_3^{b-c} \ll T^{(b-2c+2a)/2}} n_1^{-1/2-a/2b} n_3^{-1/2+c/2b} \\ &\ll T^{1/4+(b-2c+2a)/2(a+b)} \sum_{n_3^{b-c} \ll T^{(b-2c+2a)/2}} n_3^{-(b-c)/(a+b)}. \end{aligned}$$

Proceeding in a similar manner by summing over n_3 then yields

$$S_1(T) \ll T^{1/4+(b-2c+2a)/2(b-c)} = T^{3/4+(2a-c)/2(b-c)},$$

as desired. □

It seems apparent that we have reached a position from which to easily combine the work done in the chapter to prove Theorem 6.1.1, and this we now describe.

Proof of Theorem 6.1.1. On combining equations (6.4.1) and (6.4.2) with Lemma 6.4.1 and both Lemmata 6.5.3 and 6.5.4 it transpires that the contribution stemming from the non-twisted integral satisfies

$$I_1(T) = \sigma_{a,b,c} T + O\big(T^{3/4} (\log T)^4\big) + B(T),$$

wherein the term $B(T)$ has the property that

$$B(T) \ll T^{1-1/2a+1/2c}$$

unconditionally and

$$B(T) \ll T^{1/2+a/(a+c)+\varepsilon}$$

under the assumption of Conjecture 2. Consequently, the above equation in conjunction with Lemma 6.7.3 delivers

$$I_{a,b,c}(T) = \sigma_{a,b,c}T + O\big(T^{3/4}(\log T)^4 + T^{3/4+(2a-c)/2(b-c)}\big) + B(T),$$

whenever $c < b$, an analogous formula holding by omitting the error term $T^{3/4+(2a-c)/2(b-c)}$ in the above equation if $b = c$, as desired.

Chapter 7

Lower order terms appertaining to mixed moments

7.1 Introduction

The extensive work appertaining to the sharpening of the asymptotic formula associated to both the second and the fourth moment of the Riemann zeta function, such examinations being initiated first by Hardy-Littlewood (see, for instance Titchmarsh [134, Theorem 7.3]) and Ingham [80] respectively, may belong to the not so numerous group of unconditional investigations concerning moments of L-functions which ultimately led to establishing the validity of the latter comprising extra lower order terms. In view of such an observation and for the sake of transparency it seems worth defining, for $k \in \mathbb{N}$ the moment

$$M_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt$$

and note first that after the work of many, the first one to pursue such an avenue being Titchmarsh, the formula

$$M_1(T) = T \log(T/2\pi) + (2\gamma - 1)T + E_1(T)$$

holds with $E_1(T)$ satisfying bounds of the shape $E_1(T) \ll T^{\Delta}$ for some fixed $\Delta > 0$, the sharpest of which follows after work of Bourgain and Watt and may be taken to be $\Delta = 1515/4816 + \varepsilon$.

In the same vein, investigations of Heath-Brown [59], which were eventually refined by those of Zavorotnyi [178] and Ivic-Motohashi [82], led to the analogous formula

$$M_2(T) = TP_4\big(\log T\big) + E_2(T),$$

where $P_4(x)$ denotes a quartic polynomial, and the best known bound for the error term is $E_2(T) \ll T^{2/3+\varepsilon}$. It should nonetheless be noted for the purpose of merely illustrating the present introduction that further conjectural examinations pertaining to higher order moments have been pursued by Conrey-Farmer-Keating-Rubinstein-Snaith [25] by bringing into use random-matrix theory utensils, thereby delivering analogous conclusions of the shape

$$M_k(T) = TP_{k^2}\big(\log T\big) + O(T^{1/2+\varepsilon})$$

for $k \geq 3$, wherein $P_{k^2}(x)$ is a degree-k^2 polynomial.

Hirtherto the asymptotic evaluations presented herein comprised lower order terms differing by a factor of a power of $\log T$ to the size of the main one, such a consideration raising the question of whether analogous formulas for moments of L-functions for which the sizes of the lower order terms differ from that of the main one by a power of T instead may be accomplished. To remedy the lack of such formulas in the literature and incorporate a new one to such a bizarre collection is, inter alia, the purpose of the present chapter. Stating the main result of our perusal rather promptly precludes us from providing a not at all prolix historical discussion concerning the appearance of such rare formulas in the literature, which shall be deferred to a later point in the introduction.

We define for convenience what will be the main object of study in the present chapter, namely

$$I(T) = \int_0^T \zeta(1/2 + 2it)\zeta(1/2 - it)^2 dt, \tag{7.1.1}$$

and anticipate the main theorem concerning its asymptotic evaluation.

Theorem 7.1.1. *For $T > 0$ one has the asymptotic formula*

$$I(T) = c_1T + c_2T^{7/8} + O\big(T^{3/4}(\log T)^3\big),$$

where the constants c_1, c_2 *are defined by means of the equations*

$$c_1 = \frac{\zeta(3/2)^3}{2\zeta(3)}(3-i),$$

and

$$c_2 = \frac{32(2\pi)^{1/8}}{7}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}i.$$

Investigations of moments leading to formulas containing such lower error terms date back to the seminal work of Diaconu, Goldfeld and Hoffstein [36] involving the use of multiple Dirichlet series in problems concerning moments of L-functions. To further describe rather precisely their achievements therein cognate to our considerations in the present memoir, it seems desirable to introduce for $r \in \mathbb{N}$ the moment of the family of quadratic L-functions

$$M_r(D) = \sum_{0<d<D} L\big(\frac{1}{2}, \chi_d\big)^r \tag{7.1.2}$$

over the real primitive Dirichlet characters.

Before accomplishing such an endeavour, it has been thought preferable to give account of the main results and conjectures akin to the above object. We should note first that succesful investigations were initiated by Jutila [84], who unconditionally provided an asymptotic evaluation for the cases $r = 1$ and $r = 2$, the formula for $r = 1$ further containing a main term of the shape $DP_1(\log D)$ with $P_1(x)$ being a linear polynomial. A few years later, precise conjectures about the asymptotics pertaining to (7.1.2) were provided by means of Random Matrix theory models (see [89]), such formulations being further refined by Conrey et al. [25] and taking the shape

$$M_r(D) = DP_{r(r+1)/2}(\log D) + E_r(D), \tag{7.1.3}$$

wherein $P_{r(r+1)/2}(x)$ is a completely explicit polynomial of degree $r(r+1)/2$ and $E_r(D) = o_r(D)$.

Soundararajan's influential paper [127] was the first instance in the literature for which an unconditional asymptotic formula for $r = 3$ was provided, it taking the shape (7.1.3) with $E_3(D) = O(D^{11/12+\varepsilon})$. Nonetheless, as earlier foreshadowed, the aforementioned work of Diaconu, Goldfeld and Hoffstein [36] in 2003, which in turn sharpened the above error term, and that of Zhang [179] further conjectured the presence of a lower order term of the shape $cD^{3/4}$ in the corresponding formula. This speculation was ultimately confirmed by the work of Diaconu and Whitehead [38] at the sacrifice of introducing a weight function, thereby providing such a lower term only for a smooth version of (7.1.2). It shall be noted for the purpose of illustrating the exposition that an analogous formula has been obtained by the first author earlier mentioned [35] in the function field setting.

Such considerations have been generalised and formally written down in recent work of Diaconu and Twiss [37], leading to the conjectural asymptotic formulae

$$M_r(D) = \sum_{n=1}^{N} D^{1/2+1/2n} Q_{n,r}(\log D) + O(D^{(1+\theta)/2}),$$

wherein $N \geq 1$, the parameter θ satisfies the property that $(N+1)^{-1} < \theta < N^{-1}$ and $Q_{n,r}(x)$ denotes a polynomial. Moreover, it should be noted that the aforementioned prediction and the corresponding results earlier mentioned whenever $r = 3$ concerning the lower order terms are delivered via multiple Dirichlet series arguments, the random matrix theoretic models failing to detect such terms.

We shift our focus to the main result of this chapter and anticipate that an approximate functional equation for both $\zeta(1/2 - it)$ and $\zeta(1/2 + 2it)$ will be utilised, thereby reducing the problem to that of computing integrals of twisted Dirichlet polynomials. Experts may recognise the necessity of approximating $\zeta(1/2 - it)$ rather than employing the approximate functional equation appertaining to $\zeta(1/2 - it)^2$ due to Hardy-Littlewood [53], the latter ultimately delivering an error term larger than the main term in the formula of the main theorem in the chapter.

The aforementioned lower order term $c_2 T^{7/8}$ that is obtained herein arises from the diagonal contribution appertaining to both the non-twisted integral and a twisted integral very similar in vein in conjunction with the contribution cognate to two twisted integrals that exhibit dissimilar behaviours. Whislt the former is derived by means of a parametrization of the underlying equation combined with a careful analysis, the latter necessitates an application of a version of the stationary phase method of the requisite precision in conjunction with a sharp perusal of the corresponding exponential sum stemming from such an application. We draw the reader's attention to Chapter 6 to the end of recalling that therein we were confronted with exponential sums that were only amenable to applications of van der Corput's type estimates, thus delivering only upper bounds for such sums. The treatment herein departs from that of that chapter in that explicit asymptotic control over the aforementioned sums may be achieved due to their inherent simpleness, such a gift stemming from the simplicity of the coefficients pertaining to the mixed moment at hand.

The structure of the chapter is organised as follows: In Section 7.2 we provide a somewhat novel approach to the stationary phase method at the cost of imposing certain restrictions, thereby deriving error terms of the requisite precision for our purposes but which by no means beat those stemming from the work of Graham and Kolesnik [46, Lemma 3.4]. Section 7.3 is devoted to the endeavour of the approximation of the mixed moment by a sum of integrals of twisted Dirichlet polynomials. The diagonal and off-diagonal analysis pertaining to the non-twisted integral in conjunction with the perusal of a twisted integral exhibiting a similar behaviour is performed in Section 7.4. In a different vein, Section 7.5 comprises routinary estimates of oscillatory integrals making use of standard lemmata which entail no difficulty. Section 7.6 is devoted to the examination of the integrals for which the stationary phase method is required in conjunction with an examination of the exponential sum stemming from such an application, it providing a more number theoretic flavour to the discussion, and ultimately leads to the proof of Theorem 7.1.1.

7.2 The stationary phase method

As was discussed above, we shall shortly provide the reader with yet another somewhat novel discussion pertaining to the stationary phase method. We feel the urge to anticipate that the error terms derived herein are not of a smaller size than those cognate to earlier versions of such a method when applied to the corresponding oscillatory integrals which we eventually encounter in the present memoir, the main reason behind the author's motivation to work on such endeavours stemming largely on his lack of awareness of the historic developments apposite to the problem when the latter encountered integrals of such a shape. Nonetheless, it has been thought pertinent to illustrate the exposition with the aforementioned version of the method for the purpose of both providing a not completely analogous approach to the problem at hand

and with the hope that the result derived herein may be applicable in other contexts, the bounds for the error terms appertaining to such applications potentially being superior to those stemming from earlier versions. We find it desirable to anticipate that Section 7.6 shall comprise an application of the version derived herein, the use of Lemma 5.2.5 having nonetheless been sufficient to obtain error terms of the required precision. Experts may recognise the resemblance between the beginning of the proofs of both Lemma 4.6 of Titchmarsh [134] and that of the present lemma, the departing in the course of it having its genesis on the use of a higher order Taylor approximation.

Lemma 7.2.1. *Let $F(x)$ be a real function with derivatives up to order k for $k \geq 4$ on an interval (α, β), and satisfying*

$$0 < \lambda_2 \leq F''(x) < A\lambda_2 \tag{7.2.1}$$

where $\lambda_2 > 0$ and $A > 0$ is some fixed constant, and

$$|F^{(k)}(x)| \leq \lambda_k$$

for $\lambda_k > 0$ on $x \in (\alpha, \beta)$. We denote for further convenience the parameter $\delta = \big(\lambda_2 \lambda_k\big)^{-1/(k+2)}$. We suppose that there exists some $c \in [\alpha, \beta]$ for which $F'(c) = 0$ and with the property that for every $3 \leq j \leq k$ then

$$\frac{\delta |F^{(j)}(c)|}{j} \leq \eta |F^{(j-1)}(c)|, \tag{7.2.2}$$

where $\eta > 0$ is a universally small enough constant. Then one obtains

$$\begin{aligned} \int_\alpha^\beta e^{iF(x)} dx =& \sqrt{2\pi} \frac{e^{i\pi/4 + iF(c)}}{|F''(c)|^{1/2}} + O\big(k\delta^{-1}\lambda_2^{-1}\big) \\ &+ O\Big(\min\big(|F'(\alpha)|^{-1}, \lambda_2^{-1/2}\big)\Big) + O\Big(\min\big(|F'(\beta)|^{-1}, \lambda_2^{-1/2}\big)\Big), \end{aligned} \tag{7.2.3}$$

wherein the implicit constants are universal. Moreover, suppose that either

$$F''(c)F^{(4)}(c) \leq C_1 F'''(c)^2 \qquad \text{or} \qquad F''(c)F^{(4)}(c) \geq C_2 F'''(c)^2 \tag{7.2.4}$$

for some parameters $C_1 < 5/3$ and $C_2 > 5/3$. Then one obtains (7.2.3) without the k factor in the first error term. If on the contrary $F'(x)$ does not vanish *on $[\alpha,\beta]$ then (7.2.3) holds without the leading term and the first error term.*

Proof. As a prelude to our discussion it shall be noted that under the latter circumstances the formula follows after an succint application of both Lemmata 5.2.1 and 5.2.3. If such a condition does not hold then there exists some $c \in [\alpha,\beta]$ satisfying $F'(c) = 0$, the positivity of the second derivative stated at the beginning of the proposition assuring the uniqueness of such a zero. It seems pertinent to consider first the case $\alpha+\delta \le c \le \beta-\delta$, the complementary situation being deferred in view of its simplicity, as shall be elucidated promptly. We then divide the range of integration of the above integral accordingly, and this we now describe.

It is worth observing first that the monotonicity of $F'(x)$ stemming from (7.2.1) and Lemma 5.2.1 yield

$$\int_{c+\delta}^{\beta} e^{iF(x)}dx \ll \frac{1}{|F'(c+\delta)|} \ll \frac{1}{\delta\lambda_2},$$

where in the last step we employed the mean value theorem in conjunction with (7.2.1) and the fact that $F'(c) = 0$. As the reader may anticipate, a not dissimilar argument enables one to deduce that the contribution pertaining to the interval $[\alpha, c-\delta]$ to the integral in (7.2.3) delivers the same error term to the formula at hand.

The remainder of the discussion shall be devoted to the contribution cognate to the remaining interval, the analysis of which being more intrincate, as may have already been anticipated by the reader, and relying on a more sophisticated framework of ideas. We find it worth beginning such an investigation by employing the Taylor expansion of $F(x)$ to obtain

$$\int_{c-\delta}^{c+\delta} e^{iF(x)}dx = \int_{c-\delta}^{c+\delta} e^{iP_{k-1}(x)}dx + \int_{c-\delta}^{c+\delta} O\big(\lambda_k(x-c)^k\big)dx,$$

where the polynomial $P_{k-1}(x)$ is defined by means of the formula

$$P_{k-1}(x) = F(c) + (x-c)F'(c) + \ldots + \frac{F^{(k-1)}(c)}{(k-1)!}(x-c)^{k-1}.$$

Utilising the fact that $F'(c) = 0$ and recalling the definition of δ established in the statement of the lemma we then find that

$$\int_{c-\delta}^{c+\delta} e^{iF(x)}dx = e^{iF(c)} \int_{c-\delta}^{c+\delta} e^{iu(x)}dx + O\big(\delta^{-1}\lambda_2^{-1}\big), \tag{7.2.5}$$

where in the above formula the function $u(x)$ denotes

$$u(x) = \frac{F''(c)}{2}(x-c)^2 + \ldots + \frac{F^{(k-1)}(c)}{(k-1)!}(x-c)^{k-1}. \tag{7.2.6}$$

It is convenient to observe for further purposes that in view of the positivity of $F''(c)$ and the bounds (7.2.2), the leading term in the corresponding expressions for $u(x)$ and $u'(x)$ dominates over the remaining ones, meaning

$$u(x) = \frac{F''(c)}{2}(x-c)^2 + E_{c,1}(x) \quad \text{and} \quad u'(x) = F''(c)(x-c) + E_{c,2}(x), \tag{7.2.7}$$

wherein the terms $E_{c,1}(x)$ and $E_{c,2}(x)$ satisfy the bounds

$$E_{c,1}(x) \leq \sum_{j=1}^{k-3} \eta^j |F''(c)|(x-c)^2 \ll \eta|F''(c)|(x-c)^2 \tag{7.2.8}$$

and

$$E_{c,2}(x) \leq \sum_{j=1}^{k-3} \eta^j |F''(c)(x-c)| \ll \eta|F''(c)(x-c)|,$$

whence it transpires that taking η small enough assures $u'(x) > 0$ for $x \in (c, c+\delta)$ and $u'(x) < 0$ whenever $x \in (c-\delta, c)$. Such a property, in conjunction with an application of the Inverse function theorem enables one to establish the existence of a differentiable function $x(u)$ whenever $u \in (0, u(c-\delta)) \cup \big(0, u(c+\delta)\big)$ with the property that

$$u = \frac{F''(c)}{2}(x(u)-c)^2 + \ldots + \frac{F^{(k-1)}(c)}{(k-1)!}(x(u)-c)^{k-1}, \quad x'(u) = \frac{1}{u'\big(x(u)\big)}, \tag{7.2.9}$$

the latter property further implying that $x'(u) > 0$ in $\big(0, u(c+\delta)\big)$ and $x'(u) < 0$ in $(0, u(c-\delta))$.

For the purpose of obtaining a main term in the desired formula it seems pertinent, as shall be ellucidated promptly, to make the change of variables

$u = u(x)$ in order to obtain

$$\int_c^{c+\delta} e^{iu(x)}dx = \int_0^{u(c+\delta)} \frac{e^{iu}}{u'(x)}du, \qquad \text{where } u'(x) = \frac{\partial u}{\partial x}(x(u)), \quad (7.2.10)$$

and similarly

$$\int_{c-\delta}^{c} e^{iu(x)}dx = -\int_0^{u(c-\delta)} \frac{e^{iu}}{u'(x)}du. \quad (7.2.11)$$

We shall henceforth consider u to be a variable lying on the interval $\big(0, u(c+\delta)\big)$ and think of $x(u)$ as being a function on u, it being desirable to abbreviate $x = x(u)$ at times for the clarity of the exposition. We also denote

$$u''(x) = \frac{\partial^2 u}{\partial x^2}\big(x(u)\big).$$

In the course of the approximation pertaining to the above integrals the function

$$G(u) = \frac{1}{u'(x)} - \frac{1}{\sqrt{2F''(c)u}} \quad (7.2.12)$$

shall play a prominent role. It then seems worth noting that by taking common denominator and multiplicating by the conjugate in the above expression one then has

$$G(u) = \frac{2F''(c)u - u'(x)^2}{u'(x)\sqrt{2F''(c)u}\big(\sqrt{2F''(c)u} + u'(x)\big)}$$

We draw the reader's attention to the bounds (7.2.2) and the identity (7.2.9) for the purpose of establishing the approximations

$$u(x) = \frac{F''(c)}{2}(x-c)^2 + \frac{F'''(c)}{6}(x-c)^3\big(1 + \Delta_1(u)\big),$$

$$u'(x) = F''(c)(x-c) + \frac{F'''(c)}{2}(x-c)^2\big(1 + \Delta_2(u)\big),$$

wherein $\Delta_i(u)$ are functions satisfying the bound $\Delta_i(u) \ll \eta$, the corrresponding derivative being taken only with respect to the variable x. Consequently, inserting the above formulas or those in (7.2.7) accordingly into the expression deduced for the function $G(u)$ delivers

$$G(u) = \frac{-\frac{2}{3}F''(c)F'''(c)(x-c)^3(1 + \Delta'(u))}{2F''(c)^3(x-c)^3(1 + \Delta''(u))},$$

for some functions $\Delta'(u), \Delta''(u) \ll \eta$, which then delivers

$$G(u) = -\frac{F'''(c)}{3F''(c)^2}(1+\Delta(u)), \tag{7.2.13}$$

where as is customary the function $\Delta(u)$ here has the property that $|\Delta(u)| \ll \eta$.

The above discussion implicitly establishes, when thinking of c as being fixed and u as a variable, that $G(u)$ behaves as a constant, it therefore being desirable to examine its monotonicity for the purpose of deriving suitable estimates for the integral of such a function twisted by the factor e^{iu}. To this end we compute $G'(u)$ by means of equation (7.2.9) and (7.2.12) to obtain

$$G'(u) = \frac{-u''(x)}{u'(x)^3} + \frac{1}{\sqrt{8F''(c)u^3}} = \frac{u'(x)^3 - u''(x)\sqrt{8F''(c)u^3}}{u'(x)^3\sqrt{8F''(c)u^3}}. \tag{7.2.14}$$

The reader might find it useful to observe that the zeros of $G'(u)$ satisfy

$$u'(x)^6 - 8u''(x)^2F''(c)u(x)^3 = 0, \tag{7.2.15}$$

which may be regarded as a polynomial equation in the variable $x(u)$ of degree at most $6k-12$. We write for convenience the list $u_1, \dots, u_N$ of zeros of $G'(u)$ and note that the preceding argument yields

$$N \leq 6k - 12.$$

Recalling to the reader of equations (7.2.1) and (7.2.2), integrating by parts and applying (7.2.13) delivers

$$\int_0^{u(c+\delta)} G(u)e^{iu}du = \sum_{j=1}^{N-1}\int_{u_j}^{u_{j+1}} G(u)e^{iu}du \ll k|F'''(c)||F''(c)|^{-2} \ll k\delta^{-1}\lambda_2^{-1},$$

wherein we employed the monotonicity of the function $G(u)$ inside the intervals of integration of each of the above integrals.

On the interval $(0, u(c-\delta))$ it is convenient to consider instead the function

$$G_1(u) = \frac{1}{u'(x)} + \frac{1}{\sqrt{2F''(c)u}},$$

since on this occasion $u'(x) < 0$. The analysis to deduce the formula

$$G_1(u) = -\frac{F'''(c)}{3F''(c)^2}\big(1 + \Delta(u)\big)$$

and the monotonicity of the function at hand is completely analogous to the one pertaining to the interval $(0, u(c+\delta))$, whence in the interest of curtailing the exposition it has been thought preferable not to provide further details to the end of avoiding unnecessary repetition. Therefore, recalling (7.2.12) and combining the previous discussion with (7.2.10) it follows that

$$\begin{aligned}\int_c^{c+\delta} e^{iu(x)}dx &= \frac{1}{\sqrt{2F''(c)}}\int_0^{u(c+\delta)} \frac{e^{iu}}{\sqrt{u}}du + \int_0^{u(c+\delta)} G(u)e^{iu}du \\ &= \frac{1}{\sqrt{2F''(c)}}\int_0^{u(c+\delta)} \frac{e^{iu}}{\sqrt{u}}du + O(k\delta^{-1}\lambda_2^{-1}),\end{aligned}$$

an analogous argument recalling insted (7.2.11) delivering

$$\begin{aligned}\int_{c-\delta}^{c} e^{iu(x)}dx &= \frac{1}{\sqrt{2F''(c)}}\int_0^{u(c-\delta)} \frac{e^{iu}}{\sqrt{u}}du - \int_0^{u(c-\delta)} G_1(u)e^{iu}du \\ &= \frac{1}{\sqrt{2F''(c)}}\int_0^{u(c-\delta)} \frac{e^{iu}}{\sqrt{u}}du + O(k\delta^{-1}\lambda_2^{-1}).\end{aligned}$$

As shall be elucidated promptly, it may also seem convenient to note that an application of (7.2.7) in conjunction with (7.2.8) and the monotonicity of the function $u^{-1/2}$ then yields

$$\int_0^{u(c+\delta)} \frac{e^{iu}}{\sqrt{u}}du = \int_0^{\infty} \frac{e^{iu}}{\sqrt{u}}du - \int_{u(c+\delta)}^{\infty} \frac{e^{iu}}{\sqrt{u}}du = e^{i\pi/4}\sqrt{\pi} + O(\delta^{-1}\lambda_2^{-1/2}),$$

and similarly

$$\int_0^{u(c-\delta)} \frac{e^{iu}}{\sqrt{u}}du = e^{i\pi/4}\sqrt{\pi} + O(\delta^{-1}\lambda_2^{-1/2}).$$

Combining the above equations it follows then that

$$\int_{c-\delta}^{c+\delta} e^{iu(x)}dx = \frac{(2\pi)^{1/2}e^{i\pi/4}}{|F''(c)|^{1/2}} + O(k\delta^{-1}\lambda_2^{-1}), \tag{7.2.16}$$

which delivers the desired result.

To conclude the proof it thus remains to briefly discuss the deferred case

$\beta - \delta < c$. Under such circumstances we utilise the same analysis as above with the peculiarity of necessitating to bound the remaining integral over the interval $[\beta, c+\delta]$, which we now describe. We write

$$\int_{\beta}^{c+\delta} e^{iF(x)}dx = e^{iF(c)} \int_{\beta}^{c+\delta} e^{iu(x)}dx + O(\delta^{-1}\lambda_2^{-1}),$$

as in (7.2.5), and observe that on recalling (7.2.7) and combining Lemmata 5.2.1 and 5.2.3 with the mean value theorem it transpires that

$$\int_{\beta}^{c+\delta} e^{iu(x)}dx \ll \min\Big(\frac{1}{(\beta - c)\lambda_2}, \lambda_2^{-1/2}\Big) \ll \min\Big(|F'(\beta)|^{-1}, \lambda_2^{-1/2}\Big).$$

The previous bound in conjunction with (7.2.16) then provides the desired conclusion. It shall be noted that an analogous computation for the case $c - \delta < a$ may lead one to obtain an analogous conclusion, thereby completing the proof of the lemma in the instance when only (7.2.2) is assumed.

We shall shift our focus to the analysis of the integral at hand under the assumption (7.2.4), and begin by noting first that an application of the bounds (7.2.2) then yields

$$u(x) = \frac{F''(c)}{2}(x-c)^2 + \frac{F'''(c)}{3!}(x-c)^3 + \frac{F^{(4)}(c)}{4!}(x-c)^4 + E_{c,4}(x),$$

wherein the error term in the above expression satisfies

$$E_{c,4}(x) \ll \eta |F^{(4)}(c)|(x-c)^4.$$

Likewise, on differentiating (7.2.6) and utilising the aforementioned bounds one has

$$u'(x) = F''(c)(x-c) + \frac{F'''(c)}{2}(x-c)^2 + \frac{F^{(4)}(c)}{3!}(x-c)^3 + E_{c,5}(x),$$

and similarly

$$u''(x) = F''(c) + F'''(c)(x-c) + \frac{F^{(4)}(c)}{2}(x-c)^2 + E_{c,6}(x),$$

wherein the above error terms satisfy the estimates

$$E_{c,5}(x) \ll \eta |F^{(4)}(c)|(x-c)^3, \qquad E_{c,6}(x) \ll \eta |F^{(4)}(c)|(x-c)^2.$$

We shall depart from the earlier approach in that a careful perusal of the equation (7.2.15) in conjunction with the assumption (7.2.4) will enable us to accomplish the monotonicity of $G(u)$ in the whole interval at hand, thereby saving the extra k factor in the corresponding error term. To this end, it seems worth noting that the above formula pertaining to $u'(x)$ and a rather tedious computation delivers the expression

$$\begin{aligned} u'(x)^6 =& F''(c)^6(x-c)^6 + 3F''(c)^5F'''(c)(x-c)^7 \\ &+ \big(F''(c)^5F^{(4)}(c) + \frac{15}{4}F''(c)^4F'''(c)^2\big)(x-c)^8 + G(x), \end{aligned}$$

where the function $G(x)$ satisfies the bound

$$G(x) \ll \eta F''(c)^4\big(|F''(c)||F^{(4)}(c)| + F'''(c)^2\big)(x-c)^8$$

for some small enough constant η. Similarly, one can find that

$$\begin{aligned} 8u''(x)^2F''(c)u(x)^3 =& F''(c)^6(x-c)^6 + 3F''(c)^5F'''(c)(x-c)^7 \\ &+ \big(\frac{5}{4}F''(c)^5F^{(4)}(c) + \frac{10}{3}F''(c)^4F'''(c)^2\big)(x-c)^8 + H(x), \end{aligned}$$

where the function $H(x)$ satisfies the above bound pertaining to $G(x)$. Consequently, combining the previous equations it transpires that

$$\begin{aligned} u'(x)^6 - 8u''(x)^2F''(c)u(x)^3 =& \frac{F''(c)^4}{4}\big(-F''(c)F^{(4)}(c) + \frac{5}{3}F'''(c)^2\big)(x-c)^8 \\ &+ G(x) - H(x). \end{aligned}$$

On recalling condition (7.2.4) one readily deduces that either

$$u'(x)^6 - 8u''(x)^2F''(c)u(x)^3 \geq c_1F''(c)^4F'''(c)^2(x-c)^8$$

or

$$u'(x)^6 - 8u''(x)^2F''(c)u(x)^3 \leq -c_2F''(c)^4F'''(c)^2(x-c)^8$$

for positive constants $c_1, c_2 > 0$. Therefore, by the preceding discussion in conjunction with (7.2.14) and a continuity argument we observe that either $G'(u) < 0$ for all $u \in (0, u(c+\delta))$ or $G'(u) > 0$, thereby concluding the monotonicity of $G(u)$ in the interval at hand. We find it desirable to note that an analogous process, up to a change of sign, can be applied to derive the monotonicity in the interval $(0, u(c-\delta))$ of the function $G_1(u)$. Thus combining (7.2.13) with integration by parts and (7.2.2) we deduce

$$\int_0^{u(c+\delta)} G(u)e^{iu}du \ll |F'''(c)||F''(c)|^{-2} \ll \delta^{-1}\lambda_2^{-1},$$

the analogous estimate holding when taking instead $G_1(u)$ and the interval $(0, u(c-\delta))$ in the above equation. Therefore, the same procedure delivers equation (7.2.16) with an error term of $O(\delta^{-1}\lambda_2^{-1})$ instead, and the discussion following the aforementioned equation yields the desired result. □

7.3 Initial manoeuvres

We begin our exposition by preparing the ground for the computation of the mixed moment at hand. As may become transparent shortly, it seems desirable to introduce minor changes in the ranges of the Dirichlet polynomials pertaining to the approximate functional equation cognate to the factor $\zeta(1/2+2it)$, such choices being in concordance with the underlying equation satisfied by the corresponding variables. To this end we define for convenience

$$D_j(s) = \sum_{n \le x_j} \frac{1}{n^s} \qquad s = 1/2+it, \qquad j = 0,1,2,$$

wherein

$$x_0 = \sqrt{t/2\pi}, \qquad x_1 = \sqrt{t/4\pi}, \qquad x_2 = \sqrt{t/\pi}.$$

We find it pertinent to anticipate the necessity of considering the approximate functional equation (see Titchmarsh [134, (4.12.4)])

$$\zeta(1/2+2it) = P_1(t) + O(t^{-1/4}),$$

wherein the main term $P_1(t)$ in the above expression is defined by means of the equation

$$P_1(t) = D_1(1/2 + 2it) + \chi(1/2 + 2it)D_2(1/2 - 2it).$$

It seems desirable to draw the reader's attention to the difference in length of the above polynomials, such a consideration being suitable to simplify the exposition. We also find it convenient to recall the reader of the definition

$$P(t) = D(1/2 + it) + \chi(1/2 + it)D(1/2 - it)$$

presented in (5.4.2), wherein $D(s)$ denotes $D_0(s)$, and remind that the aforementioned approximate functional equation yields

$$\zeta(1/2 + it) = P(t) + O(t^{-1/4}).$$

The bulk of the work pertaining to the mixed moment at hand shall comprise intrincate computations akin to diagonal and off-diagonal analysis in conjunction with estimates of oscillatory integrals, it being nonetheless worth starting the discussion by combining the above approximate functional equations to the end of reducing the problem to that of examining integrals of twisted Dirichlet polynomials.

Lemma 7.3.1. *With the above notation one then has that*

$$I(T) = \int_0^T P_1(t)P(-t)^2 dt + O(T^{3/4}\log T).$$

Proof. We find it worth beginning our analysis by noting that by inserting the above approximate functional equations into (7.1.1) one readily sees that

$$I(T) = \int_0^T P_1(t)P(-t)^2 dt + E(T), \qquad \text{wherein } E(T) \ll T^{1/4} + E_1(T) + E_2(T) \tag{7.3.1}$$

and the error terms $E_1(T)$ and $E_2(T)$ satisfy the bounds

$$E_1(T) \ll \int_0^T \big(t^{-1/2}|\zeta(1/2 + 2it)| + t^{-1/2}|\zeta(1/2 - it)|\big)dt$$

and

$$E_2(T) \ll \int_0^T \big(t^{-1/4}|\zeta(1/2+2it)|^2 + t^{-1/4}|\zeta(1/2-it)|^2\big)dt.$$

We use Cauchy's inequality and the asymptotic formula for the second moment of the Riemman Zeta function (see, for instance Titchmarsh [134, Theorem 7.3]) to obtain

$$E_1(T) \ll (\log T)^{1/2}\Big(\int_0^{2T} |\zeta(1/2+it)|^2\Big)^{1/2} \ll T^{1/2}\log T.$$

Likewise, it transpires that an application of integration by parts in conjunction with the aforementioned asymptotic evaluation for the second moment of the Riemman Zeta function delivers

$$E_2(T) \ll T^{3/4}\log T + \int_0^{2T} t^{-1/4}\log t \ll T^{3/4}\log T,$$

which completes the proof in view of (7.3.1). □

The rest of the investigation pertaining to the integral at hand shall be devoted to analyse the main term in (7.3.1). The reader may note that by expanding the product inside the integral and using the expressions for $P(t)$ and $P_1(t)$ we obtain

$$I(T) = \sum_{m=1}^{6} I_m(T), \tag{7.3.2}$$

wherein $I_m(T)$ are integrals of twisted Dirichlet polynomials which shall be introduced in the course of the discussion.

7.4 Diagonal and non-diagonal contribution of the non-twisted integrals

We introduce, for convenience, the parameter

$$T_1 = T/2\pi, \tag{7.4.1}$$

and anticipate its abundant presence throughout the chapter. We honour the heading of the section at hand and introduce the main character which shall be analysed herein, namely

$$I_1(T) = \int_0^T D_1(1/2+2it)D(1/2-it)^2 dt = \sum_{\boldsymbol{n}\le\sqrt{T_1}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^T \left(\frac{n_2n_3}{n_1^2}\right)^{it} dt,$$

wherein henceforth in considerations pertaining to such an analysis we write

$$N_{\boldsymbol{n}} = 2\pi \max(n_1^2, n_2^2, n_3^2) \qquad \text{and} \qquad P_{\boldsymbol{n}} = n_1n_2n_3. \tag{7.4.2}$$

The reader may also find it useful to recall the definition of $D(s)$ in (5.4.1). It also has been thought desirable to foreshadow that we might henceforth omit writing the restriction

$$\boldsymbol{n} \le \sqrt{T_1} \tag{7.4.3}$$

cognate to the corresponding sums for the sake of concission.

We shall make as is customary a distinction between triples satisfying the equation $n_1^2 = n_2n_3$ and the ones having the property that such an equation does not hold to obtain

$$I_1(T) = H_1(T) + O\big(H_2(T)\big),$$

wherein

$$H_1(T) = \sum_{\substack{\boldsymbol{n}\le\sqrt{T_1}\\ n_1^2=n_2n_3}} (T-N_{\boldsymbol{n}})P_{\boldsymbol{n}}^{-1/2}, \qquad H_2(T) = \sum_{\substack{\boldsymbol{n}\le\sqrt{T_1}\\ n_1^2\ne n_2n_3}} P_{\boldsymbol{n}}^{-1/2}|\log\big(n_1^2/n_2n_3\big)|^{-1}.$$

The context herein shall be dissimilar to the ones pertaining to the analysis in previous chapters in that the off-diagonal term does not present major complications, a routinary argument delivering an estimate of the shape $O\big(T^{3/4}(\log T)^2\big)$ for such a contribution, and this we now describe.

Lemma 7.4.1. *With the above notation one has*

$$H_2(T) \ll T^{3/4}(\log T)^2.$$

Proof. We write for convenience by $W_1(T)$ to the contribution to $H_2(T)$ arising from tuples with the property that $|n_2n_3 - n_1^2| \le n_1^2/2$, the corresponding term $W_2(T)$ denoting the contribution stemming from the complementary tuples. By observing that the logarithm in the formula pertaining to $W_2(T)$ satisfies

$|\log(n_2n_3/n_1^2)| \gg 1$, it then transpires that

$$W_2(T) \ll \sum_{\boldsymbol{n}\leq\sqrt{T_1}} P_{\boldsymbol{n}}^{-1/2} \ll T^{3/4}.$$

We shift our focus to the perusal of the term $W_1(T)$ and begin by noting that the change of variables $n_1^2 - n_2n_3 = r$ whenever $n_2n_3 < n_1^2$ or $n_2n_3 - n_1^2 = r$ when $n_2n_3 > n_1^2$ in conjunction with the observation that

$$|\log(n_2n_3/n_1^2)| \asymp \frac{r}{n_1^2}$$

for tuples cognate to $W_1(T)$ transports us to a position from which to derive the estimate

$$W_1(T) \ll \sum_{n_1\leq\sqrt{T_1}} n_1^{1/2} \sum_{0<|r|\leq n_1^2/2} d(n_1^2+r)/|r|.$$

It seems desirable to observe that an application of summation by parts in conjunction with classical asymptotic results concerning divisor sums enables one to deduce

$$\sum_{0<r\leq n_1^2/2} d(n_1^2+(-1)^m r)/r \ll (\log T)^2, \qquad m = 1, 2,$$

from where the bound

$$W_1(T) \ll T^{3/4}(\log T)^2$$

follows routinarily. $\square$

The remainder of the discussion shall be devoted to the investigation of the diagonal contribution.

Lemma 7.4.2. *With the above notation one has*

$$H_1(T) = \frac{\zeta(3/2)^3}{\zeta(3)}T - \frac{32\cdot(2\pi)^{1/8}}{7}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}T^{7/8} + O\big(T^{3/4}(\log T)\big).$$

Moreover,

$$I_1(T) = \frac{\zeta(3/2)^3}{\zeta(3)}T - \frac{32\cdot(2\pi)^{1/8}}{7}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}T^{7/8} + O\big(T^{3/4}(\log T)^2\big).$$

Proof. In view of the above lemma, it is apparent that showing the first asymptotic formula shall be sufficient to prove the lemma at hand. For such purposes it seems desirable to recall (7.4.1) and write

$$H_1(T) = \sigma_1 T - G_1(T) - G_2(T), \tag{7.4.4}$$

where we denote

$$G_1(T) = T \sum_{\substack{\max(n_1,n_2,n_3)>\sqrt{T_1} \\ n_1^2=n_2n_3}} P_{\boldsymbol{n}}^{-1/2}, \qquad G_2(T) = \sum_{\substack{\boldsymbol{n}\leq\sqrt{T_1} \\ n_1^2=n_2n_3}} N_{\boldsymbol{n}} P_{\boldsymbol{n}}^{-1/2} \tag{7.4.5}$$

and where the constant σ_1 is defined by means of the multidimensional series

$$\sigma_1 = \sum_{n_1^2=n_2n_3} P_{\boldsymbol{n}}^{-1/2}. \tag{7.4.6}$$

In order to progress in the analysis we begin by observing as customary that one may parametrize the underlying equation $n_1^2 = n_2n_3$ by means of

$$n_1 = \lambda m_2 m_3, \qquad n_2 = \lambda m_2^2, \qquad n_3 = \lambda m_3^2, \tag{7.4.7}$$

where $m_2, m_3 \in \mathbb{N}$ and λ is a square-free number. For ease of notation it seems desirable to clarify that henceforth whenever λ appears in any sum it will denote a square-free number in the corresponding range.

It seems worth noting first that for tuples satisfying the underlying equation it transpires then that if $n_1 \geq \max(n_2, n_3)$ then the equalities $n_1 = n_2 = n_3$ hold, the contribution to $G_1(T)$ stemming from this particular diagonal case being bounded above by

$$T \sum_{n_1>\sqrt{T_1}} n_1^{-3/2} \ll T^{3/4}.$$

The symmetry with respect to n_2 and n_3 in conjunction with the parametrization at hand and the fact that if $n_2 = n_3$ then $n_1 = n_2 = n_3$ enables one to deduce the equation

$$G_1(T) = 2T \sum_{\substack{\lambda m_2^2>\sqrt{T_1} \\ m_2>m_3}} \lambda^{-3/2} m_2^{-3/2} m_3^{-3/2} + O(T^{3/4}),$$

from where it follows by summing over m_3 that

$$G_1(T) = 2\zeta(3/2)T \sum_{\lambda m_2^2 > \sqrt{T_1}} \lambda^{-3/2} m_2^{-3/2} + O\Big(T \sum_{\lambda m_2^2 > \sqrt{T_1}} \lambda^{-3/2} m_2^{-2}\Big) + O(T^{3/4}).$$

The reader may find it useful to observe that by a routinary procedure it transpires that

$$\sum_{\lambda m_2^2 > \sqrt{T_1}} \lambda^{-3/2} m_2^{-2} \ll T^{-1/4} \sum_{\lambda \leq \sqrt{T_1}} \lambda^{-1} + \sum_{\lambda > \sqrt{T_1}} \lambda^{-3/2} \ll T^{-1/4} \log T. \quad (7.4.8)$$

Moreover, it seems appropiate to note that an analogous argument applied to the main term in the above equation pertaining to $G_1(T)$ enables one to deduce

$$\begin{aligned}\sum_{\lambda m_2^2 > \sqrt{T_1}} \lambda^{-3/2} m_2^{-3/2} =& 2T_1^{-1/8} \sum_{\lambda \leq \sqrt{T_1}} \lambda^{-5/4} + \zeta(3/2) \sum_{\lambda > \sqrt{T_1}} \lambda^{-3/2} \\ &+ O\Big(T^{-3/8} \sum_{\lambda \leq \sqrt{T_1}} \lambda^{-3/4}\Big) = 2T_1^{-1/8}\sigma_2 + O(T^{-1/4}),\end{aligned}$$

where we wrote

$$\sigma_2 = \sum_{\substack{\lambda=1 \\ \lambda \text{ square-free}}}^{\infty} \lambda^{-5/4}. \quad (7.4.9)$$

The reader may find it convenient to note that we used the fact that in the first equation the tail of the series pertaining to the main term is $O(T^{-1/8})$. Moreover, it transpires that

$$\sigma_2 = \sum_{\lambda=1}^{\infty} \sum_{d^2|\lambda} \lambda^{-5/4}\mu(d) = \sum_{m=1}^{\infty} m^{-5/4} \sum_{d=1}^{\infty} \mu(d) d^{-5/2} = \zeta(5/4)\zeta(5/2)^{-1}. \quad (7.4.10)$$

Inserting the above formulas into the equation cognate to $G_1(T)$ and combining the result obtained with (7.4.8) one readily sees that

$$G_1(T) = 4(2\pi)^{1/8}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}T^{7/8} + O(T^{3/4}\log T).$$

Before proceeding with the examination of the remaining term it seems worth computing the precise value of σ_1, defined in (7.4.6), by an analogue procedure to that deployed in the course of the evaluation of σ_2. To this end we employ the parametrization utilised above to show that

$$\sigma_1 = \sum_{\substack{m_2, m_3 \\ \lambda \text{ square-free}}} m_2^{-3/2} m_3^{-3/2} \lambda^{-3/2} = \sum_{m_2, m_3, \lambda} m_2^{-3/2} m_3^{-3/2} \lambda^{-3/2} \sum_{d^2 | \lambda} \mu(d)$$
$$= \sum_{m_2=1}^{\infty} m_2^{-3/2} \sum_{m_3=1}^{\infty} m_3^{-3/2} \sum_{d=1}^{\infty} \mu(d) d^{-3} \sum_{m=1}^{\infty} m^{-3/2} = \zeta(3/2)^3 \zeta(3)^{-1}.$$

We next shift our focus to the analysis of $G_2(T)$, and begin by noting as above that for tuples satisfying the underlying equation it transpires that if $n_1 \geq \max(n_2, n_3)$ one then has the equality $n_1 = n_2 = n_3$, the contribution to $G_2(T)$ stemming from this particular diagonal case being bounded above by

$$\sum_{n_1 \leq \sqrt{T_1}} n_1^{1/2} \ll T^{3/4}.$$

It is apparent that whenever $n_2 = n_3$ it also follows that $n_1 = n_2 = n_3$, whence recalling (7.4.5) and in view of the ensuing discussion and the symmetry with respect to n_2 and n_3 we deduce the formula

$$G_2(T) = 4\pi \sum_{\substack{m_2 < m_3 \\ \lambda m_3^2 \leq \sqrt{T_1}}} m_3^{5/2} \lambda^{1/2} m_2^{-3/2} + O(T^{3/4}),$$

where we recall to the reader that λ is square-free. Summing over m_2 then delivers the expression

$$G_2(T) = 4\pi\zeta(3/2) \sum_{\lambda m_3^2 \leq \sqrt{T_1}} m_3^{5/2} \lambda^{1/2} + O\Big(\sum_{\lambda m_3^2 \leq \sqrt{T_1}} m_3^2 \lambda^{1/2} \Big) + O(T^{3/4}). \quad (7.4.11)$$

It is worth noting that by omitting the square-free condition pertaining to the variable λ in the error term cognate to the above formula and summing accordingly it transpires that

$$\sum_{\lambda m_3^2 \leq \sqrt{T_1}} m_3^2 \lambda^{1/2} \ll T^{3/4} \sum_{m_3 \leq T_1^{1/4}} m_3^{-1} \ll T^{3/4} \log T.$$

For the purpose of making further progress in the proof, it seems desirable to denote $G_{2,1}(T)$ to the main term of (7.4.11), and observe that when summing over m_3 one finds that

$$\begin{aligned} G_{2,1}(T) &= \frac{8\pi}{7}\zeta(3/2)T_1^{7/8} \sum_{\substack{\lambda\le\sqrt{T_1}\\ \lambda \text{ square-free}}} \lambda^{-5/4} + O\Big(T^{5/8}\sum_{\lambda\le\sqrt{T_1}}\lambda^{-3/4}\Big).\\ &= \frac{4\cdot(2\pi)^{1/8}}{7}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}T^{7/8} + O(T^{3/4}), \end{aligned}$$

wherein we used (7.4.9) and (7.4.10). Therefore, by the preceding discussion and (7.4.4) we obtain the asymptotic formula

$$H_1(T) = \frac{\zeta(3/2)^3}{\zeta(3)}T - \frac{32\cdot(2\pi)^{1/8}}{7}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}T^{7/8} + O(T^{3/4}\log T),$$

which completes the first part of the analysis appertaining to the section at hand. □

In order to make further progress in the proof of the main theorem, it seems worth introducing the integral $I_2(T)$, namely

$$I_2(T) = \int_0^T \chi(1/2+2it)\chi(1/2-it)^2 D_2(1/2-2it)D(1/2+it)^2 dt,$$

and find it worth anticipating its resemblance to $I_1(T)$, it being possible to recycle some of the computations already presented.

Lemma 7.4.3. *With the above notation one has that*

$$\begin{aligned} I_2(T) =& \frac{\sqrt{2}\zeta(3/2)^3 e^{-i\pi/4}}{2\zeta(3)}T - \frac{16\sqrt{2}\cdot(2\pi)^{1/8}e^{-i\pi/4}}{7}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}T^{7/8}\\ &+ O\big(T^{3/4}(\log T)^2\big). \end{aligned}$$

Proof. We employ Lemma 5.2.6 as is customary to observe that one may express the above integral by means of the formula

$$I_2(T) = e^{-i\pi/4} \sum_{\substack{n_1\le\sqrt{2T/\pi}\\ n_2,n_3\le\sqrt{T_1}}} P_{\boldsymbol{n}}^{-1/2}\int_{N_{\boldsymbol{n},2}}^T e^{iF_2(t)}dt + O(T^{3/4}\log T),$$

wherein

$$N_{\boldsymbol{n},2} = 2\pi\max(n_1^2/4, n_2^2, n_3^2) \tag{7.4.12}$$

and the phase function $F_2(t)$ in the preceding equation is defined as

$$\begin{aligned}F_2(t) =& -2t\log 2t + 2t\big(\log(2\pi)+1\big) + 2t\log t - 2t\big(\log(2\pi)+1\big)\\ &+ t\log(n_1^2/n_2n_3) = t\log(n_1^2/4n_2n_3).\end{aligned}$$

Therefore, an analogous computation yields

$$I_2(T) = H_{2,1}(T) + H_{2,2}(T),$$

wherein

$$H_{2,1}(T) = e^{-i\pi/4}\sum_{\substack{n_1\leq\sqrt{2T/\pi}\\ n_1^2=4n_2n_3}} P_{\boldsymbol{n}}^{-1/2}(T-N_{\boldsymbol{n},2})$$

with $n_2, n_3 \leq \sqrt{T_1}$, and

$$H_{2,2}(T) \ll \sum_{\substack{n_1\leq\sqrt{2T/\pi}\\ n_1^2\neq 4n_2n_3}} \frac{P_{\boldsymbol{n}}^{-1/2}}{|\log(n_1^2/4n_2n_3)|}.$$

The reader may observe first that following an analogous procedure to that utilised in the course of the estimate pertaining to $H_2(T)$ enables one to deduce the bound $H_{2,2}(T) \ll T^{3/4}(\log T)^2$. Moreover, making the change of variables $n_1' = n_1/2$, as we may, in the sum cognate to $H_{2,1}(T)$ and applying Lemma 7.4.2 leads one to the expression

$$\begin{aligned}H_{2,1}(T) =& e^{-i\pi/4}\frac{H_1(T)}{\sqrt{2}}\\ =& e^{-i\pi/4}\frac{\sqrt{2}\zeta(3/2)^3}{2\zeta(3)}T - \frac{16\sqrt{2}\cdot(2\pi)^{1/8}e^{-i\pi/4}}{7}\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}T^{7/8}\\ &+ O(T^{3/4}\log T).\end{aligned}$$

Therefore, the above equation in conjunction with the previous discussion completes the proof of the lemma. □

7.5 Routine estimates for twisted integrals

The analysis of the integrals discussed in this section shall not present any major obstacle and will make use of both Lemmata 5.2.1 and 5.2.3. For the purpose of not elongating the exposition it seems desirable to introduce one of the main characters of the section promptly, namely

$$I_3(T) = \int_0^T \chi(1/2+2it)D_2(1/2-2it)D(1/2-it)^2 dt.$$

Lemma 7.5.1. *With the above notation one has*

$$I_3(T) \ll T^{3/4}\log T.$$

Proof. It seems desirable to begin our discussion by combining both the approximation formula stemming from Lemma 5.2.6 and the above definition for $I_3(T)$, but not before recalling (7.4.12), to the end of obtaining

$$I_3(T) = e^{i\pi/4} \sum_{\substack{n_1\leq\sqrt{2T/\pi}\\ n_2,n_3\leq\sqrt{T_1}}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n},2}}^T e^{iF_3(t)}dt + O(T^{3/4}\log T),$$

wherein the phase function $F_3(t)$ is defined by means of the expression

$$F_3(t) = -2t\log 2t + 2t(\log 2\pi + 1) + t(\log n_1^2 n_2 n_3).$$

The experience in previous chapters may convince the reader of the convenience of computing the derivative

$$F_3'(t) = -2\log t + \log(n_1^2 n_2 n_3 \pi^2).$$

It seems worth observing that whenever $n_1 > \max(2n_2, 2n_3)$ then it follows that $F_3'(N_{\boldsymbol{n},2}) = \log\left(4n_2n_3/n_1^2\right)$, and hence $|F_3'(t)| \gg \log(n_1/2n_2)$ in the interval at hand. In view of the previous remark it is apparent that whenever $n_1 > 4n_2$ then $|F_3'(t)| \gg 1$, whence an application of Lemma 5.2.1 enables one to deduce that the contribution stemming from such tuples to the above sum does not exceed, up to a constant,

$$\sum_{\boldsymbol{n}\leq\sqrt{T_1}} P_{\boldsymbol{n}}^{-1/2} \ll T^{3/4}.$$

For the remaining tuples, namely those satisfying $2n_2 < n_1 \leq 4n_2$ then the contribution is bounded above, up to a constant, by

$$\sum_{n_2,n_3\leq\sqrt{T_1}} n_3^{-1/2} \sum_{0<r\leq 2n_2} \frac{1}{r} \ll T^{3/4}(\log T),$$

as desired.

If instead $n_2 \geq \max(n_1/2, n_3)$ and $n_2 > n_1/2$ then it transpires that

$$F_3'(t) \leq -2\log(2n_2/n_1)$$

in the interval of integration. The contribution stemming from tuples satisfying $n_2 > n_1$ may be bounded via a routinary argument by $O(T^{3/4})$. If instead one has $n_1 \geq n_2$ then it is apparent that another application of Lemma 5.2.1 in conjunction with the change of variables $2n_2 = n_1 + r$ enables one to deduce the estimate

$$\sum_{\substack{n_2,n_3\leq\sqrt{T_1}\\ n_2\leq n_1<2n_2}} \frac{P_{\boldsymbol{n}}^{-1/2}}{\log(2n_2/n_1)} \ll \sum_{\substack{n_1,n_3\leq\sqrt{T_1}\\ 0<r\leq 2\sqrt{T_1}}} \frac{n_3^{-1/2}}{r} \ll T^{3/4}\log T.$$

Finally, for the remaining tuples satisfying $n_2 \geq n_3$ and $2n_2 = n_1$ we merely observe that it is always the case that $F''(t) \leq -2T^{-1}$ in the interval of integration, whence an application of Lemma 5.2.3 leads to the conclusion that the contribution of these triples does not exceed, up to a constant,

$$T^{1/2} \sum_{n_1,n_3\leq\sqrt{T_1}} n_1^{-1} n_3^{-1/2} \ll T^{3/4}\log T.$$

It is convenient to observe that the roles of n_2 and n_3 are symmetric, such an observation combined with the above analysis thus delivering a similar conclusion for tuples satisfying $n_3 \geq \max(n_1/2, n_2)$ and procluding us from devoting more space in such a nuance. Consequently, the preceding discussion leads to the bound

$$I_3(T) \ll T^{3/4} \log T,$$

as desired. $\square$

For the purpose of making progress towards the desired objective, it seems worth considering the integral

$$I_4(T) = \int_0^T \chi(1/2 - it)^2 D_1(1/2 + 2it) D(1/2 + it)^2 dt.$$

The reader shall rest assured that the analysis cognate to such an integral will largely make use of the work previously done pertaining to $I_3(T)$, and convincing the reader of such a statement shall become our main priority in the following lines.

Lemma 7.5.2. *With the above notation it transpires that*

$$I_4(T) \ll T^{3/4} \log T.$$

Proof. We begin as is customary by utilising Lemma 5.2.6 to derive

$$I_4(T) = -i \sum_{\boldsymbol{n} \leq \sqrt{T_1}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^T e^{iF_4(t)} dt + O(T^{3/4} \log T),$$

where we recall the reader of (7.4.2) and define the phase function $F_4(t)$ by means of the expression

$$F_4(t) = 2t \log t - t\big(2\log(2\pi) + 2 + \log(n_1^2 n_2 n_3)\big).$$

For further purposes that shall be ellucidated promptly, we find it convenient to compute its derivative

$$F_4'(t) = 2\log t - \log\big((2\pi)^2 n_1^2 n_2 n_3\big).$$

The analysis of this term is quite similar to the previous one, whence consideration of space and avoidance of repetition proclude us from providing all of the details pertaining to the computations deployed. It seems worth noting first that whenever $n_1 \geq \max(n_2, n_3)$ with $n_1 > n_2$ then it transpires that $F_4'(t) \geq \log(n_1/n_2)$ in the corresponding interval of integration, and the contribution of such triples to $I_4(T)$ will then be bounded above by

$$\sum_{n_2<n_1} \frac{P_{\boldsymbol{n}}^{-1/2}}{|\log(n_1/n_2)|} \ll T^{3/4} + \sum_{\substack{r\leq\sqrt{T_1}\\ n_1,n_3\leq\sqrt{T_1}}} \frac{n_3^{-1/2}}{r} \ll T^{3/4}\log T.$$

The reader may find it useful to note that in the above estimates we implicitly divided as is customary the range of summation into the tuples for which $|n_1-n_2|\geq n_1/2$, in which case $|\log(n_1/n_2)|\gg 1$ and the corresponding contribution is $O(T^{3/4})$, and the complementary tuples, for which making a change of variables $r=n_1-n_2$ leads one to the desired conclusion. It seems convenient to remark that the roles of n_2 and n_3 are symmetric in this context, whence an analogous argument may be applicable to derive a similar conclusion when interchanging n_2 and n_3 above.

If instead $n_2\geq\max(n_1,n_3)$ with $n_2>\min(n_1,n_3)$ then it transpires that $F_4'(t)\geq\log\big(n_2/\min(n_1,n_3)\big)$ on the interval of integration, and a similar argument applies, an analogous argument enabling one to derive the same conclusion when interchanging the roles of n_2 and n_3 in the preceding discussion. Finally, whenever $n_1=n_2=n_3$ then $F_4''(t)\geq 2T^{-1}$ and Lemma 5.2.3 allows one to bound, up to a constant, such a contribution by means of the term

$$T^{1/2}\sum_{n_1\leq\sqrt{T_1}} n_1^{-3/2} \ll T^{1/2}.$$

Combining the previous bounds yields the estimate

$$I_4(T)\ll T^{3/4}\log T,$$

as desired. □

7.6 Application of the stationary phase method

The upcoming discussion shall comprise the perusal of two oscillatory integrals, the computation of which shall have their reliance on an application of the stationary phase method in conjunction with an elementary but somewhat intrincate analysis of the exponential sum which arises after such an application. We also found it appropiate to illustrate the exposition herein with an explicit application of the version of the stationary phase method developed in earlier sections to the aforementioned oscillatory integrals. It shall further be noted as above that some of the calculations made in the course of the investigation pertaining to one of the integrals shall find their application in the analysis of the other one, thereby simplifying significantly the exposition. Without further delay, we present to the reader the term

$$I_5(T) = 2\int_0^T D_1(1/2+2it)D(1/2-it)D(1/2+it)\chi(1/2-it)dt,$$

and the phase function $F_5(t)$, which is defined by means of the formula

$$F_5(t) = t\log t - t(\log 2\pi + 1 + \log(n_1^2 n_2/n_3)), \tag{7.6.1}$$

and introduce for reasons that shall become apparent shortly the parameter

$$c_{\boldsymbol{n}} = 2\pi n_1^2 n_2 n_3^{-1} \tag{7.6.2}$$

Lemma 7.6.1. *With the above notation one has that*

$$I_5(T) = 4\pi \sum_{\substack{N_{\boldsymbol{n}}/2\pi \le n_1^2 n_2/n_3 \le T_1 \\ n_3 < \min(n_1,n_2)}} n_1^{1/2} n_3^{-1} e^{iF_5(c_{\boldsymbol{n}})} + O\big(T^{3/4}(\log T)^3\big).$$

Proof. A customary application of the approximation formula in Lemma 5.2.6 delivers

$$I_5(T) = 2e^{-i\pi/4} \sum_{n_i \le \sqrt{T_1}} P_{\boldsymbol{n}}^{-1/2} \int_{N_{\boldsymbol{n}}}^T e^{iF_5(t)}dt + O(T^{3/4}\log T). \tag{7.6.3}$$

It shall be useful for future purposes to consider the derivative of the phase function

$$F_5'(t) = \log t - \log(2\pi) - \log(n_1^2 n_2/n_3),$$

whence on recalling (7.6.2) it transpires that $F'(c_{\boldsymbol{n}}) = 0$.

It is worth observing first that in the not so recalcitrant situation in which the tuples satisfy the inequality $n_3 > \min(n_1, n_2)$ it is apparent that

$$F_5'(t) \ge \log\big(n_3/\min(n_1,n_2)\big)$$

in the corresponding interval of integration, whence an application of Lemma 5.2.1 in conjunction with such an observation enables one to bound that contribution above, up to a constant, by

$$\sum_{n_1\leq\sqrt{T_1}} n_1^{-1/2}\sum_{n_2<n_3}\frac{1}{n_2^{1/2}n_3^{1/2}\log(n_3/n_2)}.$$

The reader may observe that the contribution of tuples satisfying $n_1<n_3$ is bounded by the above sum with n_1 and n_2 interchanged. It seems convenient to note as is customary that the contribution to the above sum of tuples for which $2n_2\leq n_3$ is $O(T^{3/4})$. For the remaining part one has that

$$\begin{aligned}\sum_{n_1\leq\sqrt{T_1}} n_1^{-1/2}\sum_{n_3/2<n_2<n_3}\frac{1}{n_2^{1/2}n_3^{1/2}\log(n_3/n_2)}&\ll T^{1/4}\sum_{n_2\leq\sqrt{T}} n_2^{-1}\sum_{0<r\leq n_2}\frac{n_2}{r}\\ &\ll T^{3/4}\log T. \qquad (7.6.4)\end{aligned}$$

We shall devote the remainder of the analysis to discuss the treatment of tuples satisfying $n_3\leq\min(n_1,n_2)$. The purpose of the subsequent analysis will be to ensure that the conditions required to apply Lemma 7.2.1 are satisfied. To this end, it transpires that one may assume that

$$N_{\boldsymbol{n}}/2\leq c_{\boldsymbol{n}}\leq 2T, \qquad (7.6.5)$$

since otherwise a customary application of Lemma 5.2.1 would yield that the corresponding contribution to $I_5(T)$ is $O(T^{3/4})$. We first observe, for further convenience, that if $c_{\boldsymbol{n}}/2\leq t\leq 2c_{\boldsymbol{n}}$ then for any $k\geq 2$ one has

$$|F_5^{(k)}(t)|\leq\lambda_k,\qquad\text{where }\lambda_k=2^{k-1}c_{\boldsymbol{n}}^{1-k}(k-2)!$$

and

$$\lambda_2\leq|F_5''(t)|\leq 4\lambda_2,\qquad\text{where }\ \lambda_2=(2c_{\boldsymbol{n}})^{-1}.$$

We draw the reader's attention back to the notation of the statement of Lemma 7.2.1 and observe that then it transpires that

$$\delta=\Big(\frac{2^{2-k}c_{\boldsymbol{n}}^k}{(k-2)!}\Big)^{1/(k+2)}.$$

As shall be ellucidated shortly, it seems desirable to make the choice $k=\log T$

and observe that for every $2 \leq j \leq k-1$ one finds that

$$\delta|F_5^{(j+1)}(c_{\boldsymbol{n}})| = \delta c_{\boldsymbol{n}}^{-j}(j-1)! = c_{\boldsymbol{n}}^{-j+k/(k+2)} \frac{2^{(2-k)/(k+2)}(j-1)!}{\big((k-2)!\big)^{1/(k+2)}}$$

and

$$(j+1)|F_5^{(j)}(c_{\boldsymbol{n}})| = (j+1)(j-2)!c_{\boldsymbol{n}}^{1-j}.$$

Combining the above equations with Stirling's formula and (7.6.5) one readily sees that

$$\delta|F_5^{(j+1)}(c_{\boldsymbol{n}})|(j+1)^{-1}|F_5^{(j)}(c_{\boldsymbol{n}})|^{-1} \ll \frac{1}{k} = (\log T)^{-1},$$

whence (7.2.2) holds for T sufficiently large. Likewise, it transpires that $F_5''(c_{\boldsymbol{n}})F_5^{(4)}(c_{\boldsymbol{n}}) = 2F_5'''(c_{\boldsymbol{n}})^2$, which then delivers condition (7.2.4). The reader may find it useful to observe that a further application of Stirling's formula and (7.6.5) enables one to deduce the estimates

$$\lambda_2^{-1}\delta^{-1} \ll kc_{\boldsymbol{n}}^{2/(\log T+2)} \ll \log T.$$

Equipped with the above considerations we have reached a position from which to apply Lemma 7.2.1, which in turn delivers

$$\begin{aligned} I_5(T) =& 4\pi \sum_{\substack{N_{\boldsymbol{n}}/2\pi \leq n_1^2 n_2/n_3 \leq T_1 \\ n_3 \leq \min(n_1,n_2)}} n_1^{1/2} n_3^{-1} e^{iF_5(c_{\boldsymbol{n}})} \\ &+ O\Big(\sum_{N_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq 2N_{\boldsymbol{n}}} P_{\boldsymbol{n}}^{-1/2} \min\big(|F_5'(N_{\boldsymbol{n}})|^{-1}, N_{\boldsymbol{n}}^{1/2}\big)\Big) \\ &+ O\Big(\sum_{T/2 \leq c_{\boldsymbol{n}} \leq 2T} P_{\boldsymbol{n}}^{-1/2} \min\big(|F_5'(T)|^{-1}, T^{1/2}\big)\Big) + O\big(T^{3/4}\log T\big). \end{aligned} \tag{7.6.6}$$

The reader shall rest assured that further details about such an application will be delivered promptly. It may first be useful to observe that in the preceding lines we implicitly applied the aforementioned lemma for the range $2N_{\boldsymbol{n}} \leq c_{\boldsymbol{n}} \leq T/2$ to the integral

$$\int_{c_{\boldsymbol{n}}/2}^{2c_{\boldsymbol{n}}} e^{iF_5(t)}dt$$

and estimate the remaining parts of the integral in (7.6.3) via an application of Lemma 5.2.1. The same paragraph presented right after (5.7.3) to justify the aforementioned formula shall be applicable herein to discuss the validity of such an application in this context, whence in the interest of curtailing the discussion we refer the reader to that earlier explanation.

We shift our attention to the analysis of the error terms pertaining to the equation (7.6.6), and begin by examining first the tuples satisfying

$$n_1 \geq \max(n_2, n_3), \tag{7.6.7}$$

in which case $2\pi n_1^2 = N_{\boldsymbol{n}}$. It is worth noting that under such circumstances then

$$F_5'(N_{\boldsymbol{n}}) = \log(n_3/n_2),$$

whence

$$\sum_{n_1 \geq \max(n_2,n_3)} n_1^{-1/2} \sum_{n_3 < n_2 \leq 2n_3} n_2^{-1/2} n_3^{-1/2} \frac{1}{\log(n_2/n_3)} \ll T^{3/4}(\log T),$$

wherein we used the same procedure as the one employed in (7.6.4). It seems worth anticipating that we shall utilise the bound

$$\min\big(|F_5'(N_{\boldsymbol{n}})|^{-1}, N_{\boldsymbol{n}}^{1/2}\big) \leq N_{\boldsymbol{n}}^{1/2}$$

for the slightly more recalcitrant situation in which $n_2 = n_3$, and observe that

$$\sum_{n_1,n_2 \leq \sqrt{T_1}} n_1^{-1/2} n_2^{-1} N_{\boldsymbol{n}}^{1/2} \ll \sum_{n_1,n_2 \leq \sqrt{T_1}} n_1^{1/2} n_2^{-1} \ll T^{3/4}(\log T),$$

whence combining both estimates delivers

$$\sum_{\substack{N_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq 2N_{\boldsymbol{n}} \\ n_1 \geq \max(n_2,n_3)}} P_{\boldsymbol{n}}^{-1/2} \min\big(|F_5'(N_{\boldsymbol{n}})|^{-1}, N_{\boldsymbol{n}}^{1/2}\big) \ll T^{3/4}(\log T).$$

The reader may note as is customary that the contribution stemming from tuples outside of the ranges previously examined and satisfying (7.6.7) is $O(T^{3/4})$ in view of the fact that such tuples have the property that $|\log(n_3/n_2)| \gg 1$.

The rest of the investigation cognate to such an error term shall be devoted to the examination of the contribution arising from triples with the property that $n_2 \geq n_1 \geq n_3$. Under such circumstances, it transpires that

$$F_5'(N_{\boldsymbol{n}}) = \log(n_2 n_3/n_1^2).$$

We observe that the condition $N_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq 2N_{\boldsymbol{n}}$ is now equivalent to the inequalities

$$\frac{n_2 n_3}{2} \leq n_1^2 \leq 2 n_2 n_3.$$

As may become apparent shortly, it seems worth utilising a routinary argument to deduce

$$\sum_{\substack{n_2 n_3/2 \leq n_1^2 \leq 2 n_2 n_3 \\ n_1^2 \neq n_2 n_3}} P_{\boldsymbol{n}}^{-1/2} \frac{1}{|\log(n_2 n_3/n_1^2)|} \ll \sum_{n_1 \leq \sqrt{T_1}} n_1^{-3/2} \sum_{\substack{-n_1^2/2 \leq r \leq n_1^2 \\ r \neq 0}} \frac{d(n_1^2 + r)}{\log((n_1^2 + r)/n_1^2)}$$
$$\ll \sum_{n_1 \leq \sqrt{T_1}} n_1^{1/2} \sum_{0 < |r| \leq n_1^2} \frac{d(n_1^2 + r)}{|r|} \ll T^{3/4} (\log T)^2.$$

Likewise, whenever $n_1^2 = n_2 n_3$ then on observing that $N_{\boldsymbol{n}} = 2\pi n_2^2$ it then transpires that

$$\sum_{n_2, n_3 \leq \sqrt{T_1}} n_2^{-3/4} n_3^{-3/4} N_{\boldsymbol{n}}^{1/2} \ll \sum_{n_2, n_3 \leq \sqrt{T_1}} n_2^{1/4} n_3^{-3/4} \ll T^{3/4},$$

whence combining both estimates delivers

$$\sum_{\substack{N_{\boldsymbol{n}}/2 \leq c_{\boldsymbol{n}} \leq 2N_{\boldsymbol{n}} \\ n_3 \leq n_1 \leq n_2}} P_{\boldsymbol{n}}^{-1/2} \min\left(|F_5'(N_{\boldsymbol{n}})|^{-1}, N_{\boldsymbol{n}}^{1/2}\right) \ll T^{3/4} (\log T)^2,$$

which completes the perusal of the first error term cognate to (7.6.6).

In order to make further progress in the lemma we shall examine the second error term pertaining to the equation (7.6.6). We define for further convenience the parameter $\Lambda_1 = \big(T_1 n_3/n_2\big)^{1/2}$ for each (n_2, n_3), and find it worth observing for further use that on recalling (7.6.2) then the range of summation described by means of the inequalities $T/2 \leq c_{\boldsymbol{n}} \leq 2T$ is equivalent to

$$\Lambda_1/\sqrt{2} \leq n_1 \leq \sqrt{2}\Lambda_1.$$

We apply the same argument as above to deduce that

$$\sum_{n_2,n_3\le\sqrt{T_1}}\sum_{\substack{|n_1-\Lambda_1|>1\\ \Lambda_1/\sqrt{2}\le n_1\le\sqrt{2}\Lambda_1}}\frac{P_{\boldsymbol{n}}^{-1/2}}{|F_5'(T)|}\ll\sum_{n_2,n_3\le\sqrt{T_1}}n_2^{-1/2}n_3^{-1/2}\sum_{0<r\le\Lambda_1}\frac{\Lambda_1^{1/2}}{r}$$
$$\ll T^{1/4}(\log T)\sum_{n_2,n_3\le\sqrt{T_1}}n_2^{-3/4}n_3^{-1/4}$$
$$\ll T^{3/4}\log T.$$

Likewise, if $|n_1-\Lambda_1|\le 1$ it seems worth anticipating that the trivial bound

$$\min\big(|F_5'(T)|^{-1},T^{1/2}\big)\ll T^{1/2}$$

will already suffice to obtain the estimate

$$\sum_{n_2,n_3\le\sqrt{T_1}}\sum_{|n_1-\Lambda_1|\le 1}P_{\boldsymbol{n}}^{-1/2}\min(|F_5'(T)|^{-1},T^{1/2})$$
$$\ll T^{1/2}\sum_{n_2,n_3\le\sqrt{T_1}}\sum_{|n_1-\Lambda_1|\le 1}n_2^{-1/2}n_3^{-1/2}\Lambda_1^{-1/2}$$
$$\ll T^{1/4}\sum_{n_2,n_3\le\sqrt{T_1}}n_2^{-1/4}n_3^{-3/4}\ll T^{3/4}.$$

The combination of the above bounds then yields

$$\sum_{T/2\le c_{\boldsymbol{n}}\le 2T}P_{\boldsymbol{n}}^{-1/2}\min\big(|F_5'(T)|^{-1},T^{1/2}\big)\ll T^{3/4}\log T$$

and completes the examination of the error terms cognate to the equation (7.6.6). □

In order to make further progress in the section we find convenient to shift our focus to the investigation of the main term in (7.6.6), the analysis of which shall have a more number theoretic flavour. As was earlier outlined in the introduction, the simplicity of the exponential sum comprising such a main term, which in turn stems from the simplicity in the coefficients involved in the mixed moment at hand, shall enable us to depart from earlier treatments in previous chapters in that an explicit asymptotic evaluation shall be achieved, thereby precluding us from the necessity of applying van der Corput's estimates. Such an advantage therefore enables one to incorporate lower order terms to the main asymptotic formula of the chapter, as opposed to the situations previously encountered in the memoir.

Lemma 7.6.2. *One has that*

$$\sum_{\substack{N_{\boldsymbol{n}}/2\pi\leq n_1^2n_2/n_3\leq T_1\\ n_3\leq\min(n_1,n_2)}} n_1^{1/2}n_3^{-1}e^{iF_5(c_{\boldsymbol{n}})}=\frac{16\zeta(3/2)\zeta(5/4)}{7\zeta(5/2)}T_1^{7/8}+O\big(T^{3/4}(\log T)^3\big).$$

Moreover,

$$I_5(T)=\frac{32\cdot(2\pi)^{1/8}\zeta(3/2)\zeta(5/4)}{7\zeta(5/2)}T^{7/8}+O\big(T^{3/4}(\log T)^3\big).$$

Proof. It is apparent at first glance that under the assumption of the first equation then the second follows after an application of Lemma 7.6.1. We find it worth putting ideas into effect by observing first that

$$F_5(c_{\boldsymbol{n}})=-2\pi n_1^2n_2n_3^{-1}.$$

Therefore, by summing over n_2 and applying orthogonality in the main term of the equation (7.6.6) it transpires that

$$\sum_{\substack{N_{\boldsymbol{n}}/2\pi\leq n_1^2n_2/n_3\leq T_1\\ n_3\leq\min(n_1,n_2)}} n_1^{1/2}n_3^{-1}e^{iF_5(c_{\boldsymbol{n}})}=S_1(T)+S_2(T),$$

where we divided the range of summation according to the divisibility of n_1^2 by n_3 to obtain

$$S_1(T)=\sum_{\substack{N_{\boldsymbol{n}}/2\pi\leq n_1^2n_2/n_3\leq T_1\\ n_3\mid n_1^2}} n_1^{1/2}n_3^{-1} \tag{7.6.8}$$

and

$$S_2(T)=\sum_{\substack{N_{\boldsymbol{n}}/2\pi\leq n_1^2n_2/n_3\leq T_1\\ n_3\nmid n_1^2}} n_1^{1/2}n_3^{-1}e^{iF_5(c_{\boldsymbol{n}})},$$

wherein we omitted for ease of notation writing $n_3\leq\min(n_1,n_2)$.

It seems worth noting first that summing over n_2 in the above equation then yields

$$S_2(T)\ll\sum_{n_3\nmid n_1^2} n_1^{1/2}n_3^{-1}\left\|\frac{n_1^2}{n_3}\right\|^{-1}.$$

As shall be ellucidated shortly, it transpires that for the purpose of analysing the above sum rather precisely, another parametrization encoding the divisibility relations between n_1^2 and n_3 shall be required. To this end we take the tuple n_1, n_3 and denote by q the largest natural number with the property that $q \mid n_1$ and $q^2 \mid n_3$. The reader may note that on writing $n_1 = qn_1'$ and $n_3 = q^2 n_3'$ it is apparent at first glance that $\lambda = (n_1', n_3')$ is square-free, since otherwise we would be contradicting the maximality of q, an analogous reason further delivering the coprimality condition $(\lambda, n_3'\lambda^{-1}) = 1$. In view of the ensuing discussion and on writing $n_1' = \lambda m_1$ and $n_3' = \lambda m_3$ we find that then

$$n_1 = q\lambda m_1, \qquad n_3 = q^2 \lambda m_3,$$

wherein $(m_1, m_3) = (\lambda, m_3) = 1$. We make use of the parametrization deduced in the preceding discussion to obtain the bound

$$\begin{aligned} S_2(T) &\ll \sum_{\substack{\lambda m_3 \leq \sqrt{T_1} \\ (\lambda, m_3)=1}} \lambda^{-1/2} m_3^{-1} \sum_{\substack{(m_3, m_1)=1 \\ m_1 \lambda \leq \sqrt{T_1}}} m_1^{1/2} \left\| \frac{\lambda m_1^2}{m_3} \right\|^{-1} \sum_{q=1}^{\infty} q^{-3/2} \\ &\ll \sum_{\substack{m_3 \lambda \leq \sqrt{T_1} \\ (\lambda, m_3)=1}} \lambda^{-1/2} m_3^{-1} \sum_{\substack{m_1 \lambda \leq \sqrt{T_1} \\ (m_3, m_1)=1}} m_1^{1/2} \left\| \frac{\lambda m_1^2}{m_3} \right\|^{-1}. \end{aligned} \tag{7.6.9}$$

Under such circumstances, the reader may find it useful to note that for fixed m_1, m_3 there are at most $d(m_3)$ solutions to the congruence

$$m_1^2 \equiv m_1'^2 \pmod{m_3}$$

with $(m_1, m_3) = (m_1', m_3) = 1$, such a bound arising as a consequence of the remainder theorem in conjunction with the fact that there are at most 2 solutions of such a congruence modulo an odd prime power. In view of the above observation it seems convenient to divide for fixed λ and m_3 the range of summation for m_1 into intervals of length m_3, namely

$$[0, \lambda^{-1}\sqrt{T_1}] = \Big(\bigcup_{j=1}^{J(m_3,\lambda)} [(j-1)m_3, jm_3] \Big) \cup I_{J(m_3,\lambda)+1},$$

wherein the cardinality of the last interval satisfies $|I_{J(m_3,\lambda)+1}| \leq m_3$ and

$$J(m_3, \lambda) = \left\lfloor (m_3\lambda)^{-1}\sqrt{T_1} \right\rfloor.$$

Equipped with such a dissection it is worth noting first that for fixed j it is apparent that

$$\sum_{m_1 \in [(j-1)m_3, jm_3]} m_1^{1/2} \left\| \frac{\lambda m_1^2}{m_3} \right\|^{-1} \ll d(m_3) m_3^{3/2} j^{1/2} \sum_{r=1}^{m_3} \frac{1}{r} \ll j^{1/2} d(m_3) m_3^{3/2} \log T.$$

We have thus reached a position from which to succintly deduce an estimate for $S_2(T)$ of the requisite precision by simply summing the above equation over $j \leq J(m_3, \lambda) + 1$ and inserting such a line of estimates in (7.6.9) to obtain

$$\begin{aligned} S_2(T) &\ll (\log T) \sum_{m_3\lambda \leq \sqrt{T_1}} \lambda^{-1/2} d(m_3) m_3^{1/2} \sum_{j=1}^{J(m_3,\lambda)+1} j^{1/2} \\ &\ll (\log T) \sum_{m_3\lambda \leq \sqrt{T_1}} \lambda^{-1/2} d(m_3) m_3^{1/2} J(m_3, \lambda)^{3/2} \\ &\ll T^{3/4} (\log T) \sum_{m_3\lambda \leq \sqrt{T_1}} \lambda^{-2} d(m_3) m_3^{-1} \ll T^{3/4} (\log T)^3, \end{aligned} \tag{7.6.10}$$

as desired.

We shall shift our attention to the term $S_1(T)$, but not before anticipating that for the sake of simplicity, we will avoid henceforth writing the condition $n_3 \mid n_1^2$ in the corresponding sums. To this end, it seems worth noting that the underlying restrictions on the tuples pertaining to the sum in $S_1(T)$ are equivalent to the conditions

$$n_2 n_3 \leq n_1^2, \quad n_1^2 n_2 \leq T_1 n_3, \quad n_3 \leq \min(n_1, n_2), \quad n_1, n_2 \leq \sqrt{T_1}. \tag{7.6.11}$$

We divide the sum accordingly into tuples depending on whether $n_1^2 < n_3\sqrt{T_1}$ holds or does not hold to find that

$$S_1(T) = U_1(T) + U_2(T) - U_3(T) + O\big(U_4(T)\big),$$

where

$$U_1(T) = \sum_{\substack{n_1^2 < n_3\sqrt{T_1} \\ n_3 \le n_1}} n_1^{5/2} n_3^{-2}, \qquad U_2(T) = T_1 \sum_{\substack{n_3\sqrt{T_1} \le n_1^2 \\ n_3 \le n_1}} n_1^{-3/2}$$

and

$$U_3(T) = \sum_{n_2 < n_3} n_1^{1/2} n_3^{-1}, \qquad U_4(T) = \sum_{n_1, n_3 \le \sqrt{T_1}} n_1^{1/2} n_3^{-1},$$

wherein $U_3(T)$ the triples also satisfy all but the third condition in (7.6.11). We employ the parametrization introduced in (7.4.7) to obtain the estimate

$$U_3(T) \ll \sum_{n_1, n_3 \le \sqrt{T_1}} n_1^{1/2} \ll \sum_{\lambda m_1 m_2 \le \sqrt{T_1}} \lambda^{1/2} m_2^{1/2} m_3^{1/2} \ll T^{3/4} \sum_{m_2 m_3 \le \sqrt{T_1}} (m_2 m_3)^{-1}$$
$$\ll T^{3/4} (\log T)^2. \qquad (7.6.12)$$

Likewise, it transpires that

$$U_4(T) \ll \sum_{\lambda m_2 m_3 \le \sqrt{T_1}} \lambda^{-1/2} m_2^{1/2} m_3^{-3/2} \ll T^{3/4} \sum_{\lambda \le \sqrt{T_1}} \lambda^{-2} \ll T^{3/4}.$$

The reader may observe that the same parametrization inserted on this occasion in the expression for $U_1(T)$ enables one to deduce

$$U_1(T) = \sum_{\substack{m_2^2 \lambda < \sqrt{T_1} \\ m_3 \le m_2}} \lambda^{1/2} m_2^{5/2} m_3^{-3/2},$$

wherein we recall to the reader that λ runs over square-free numbers. We sum over m_3 in the above expression to obtain

$$\sum_{\substack{m_2^2 \lambda < \sqrt{T_1} \\ m_3 \le m_2}} \lambda^{1/2} m_2^{5/2} m_3^{-3/2} = \zeta(3/2) \sum_{m_2^2 \lambda < \sqrt{T_1}} \lambda^{1/2} m_2^{5/2} + O\Big(\sum_{m_2^2 \lambda \le \sqrt{T_1}} \lambda^{1/2} m_2^2 \Big).$$

It seems desirable to note that the error term in the preceding equation is $O(T^{3/4}(\log T))$. Likewise, summing over m_2 in the main term of the above formula yields

$$\sum_{m_2^2 \lambda < \sqrt{T_1}} \lambda^{1/2} m_2^{5/2} = \frac{2}{7} T_1^{7/8} \sum_{\lambda < \sqrt{T_1}} \lambda^{-5/4} + O\Big(\sum_{\lambda \le \sqrt{T_1}} \lambda^{1/2} \big(T^{1/4} \lambda^{-1/2}\big)^{5/2} \Big).$$

We find it worth remarking that in the above equation the error term is $O(T^{3/4})$, a routinary argument enabling one to bound the tail of the series cognate to the main term of such an equation by $O(T^{-1/8})$. By recalling that λ runs over the square-free numbers in conjunction with the preceding discussion and (7.4.10) it then transpires that

$$\sum_{m_2^2\lambda<\sqrt{T_1}} \lambda^{1/2}m_2^{5/2} = \frac{2\zeta(5/4)}{7\zeta(5/2)}T_1^{7/8} + O(T^{3/4}),$$

which in turn yields

$$U_1(T) = \frac{2\zeta(5/4)\zeta(3/2)}{7\zeta(5/2)}T_1^{7/8} + O\big(T^{3/4}(\log T)\big). \tag{7.6.13}$$

In order to make progress in the proof of the lemma we find it convenient to focus our attention in the analysis of $U_2(T)$. To this end it seems desirable to observe that the use of the aforementioned parametrization delivers

$$U_2(T) = T_1 \sum_{m_2,m_3,\lambda} \lambda^{-3/2}m_2^{-3/2}m_3^{-3/2},$$

wherein the above triples satisfy the conditions

$$\sqrt{T_1} \leq \lambda m_2^2, \quad m_3 \leq m_2 \quad \lambda m_3^2 \leq \sqrt{T_1}, \quad \lambda m_2 m_3 \leq \sqrt{T_1},$$

the last two stemming from the customary inequality (7.4.3). We note that the underlying conditions on the parameters earlier described are equivalent to the collection of inequalities

$$\frac{T_1^{1/4}}{\lambda^{1/2}} \leq m_2 \leq \frac{\sqrt{T_1}}{\lambda m_3}, \qquad \lambda m_3^2 \leq \sqrt{T_1}.$$

Equipped with such an observation one readily deduces that

$$\sum_{\substack{\sqrt{T_1}\le\lambda m_2^2\\ \lambda m_2m_3\le\sqrt{T_1}\\ m_3^2\lambda\le\sqrt{T_1}}}\lambda^{-3/2}m_2^{-3/2}m_3^{-3/2}=\sum_{\substack{T_1^{1/4}\lambda^{-1/2}\le m_2\\ \lambda m_3^2\le\sqrt{T_1}}}\lambda^{-3/2}m_2^{-3/2}m_3^{-3/2}$$
$$-\sum_{\substack{\sqrt{T_1}\lambda^{-1}m_3^{-1}<m_2\\ \lambda m_3^2\le\sqrt{T_1}}}\lambda^{-3/2}m_2^{-3/2}m_3^{-3/2}. \qquad (7.6.14)$$

We sum over m_2 in the second summand of the above equation to get

$$\sum_{\substack{\sqrt{T_1}\lambda^{-1}m_3^{-1}<m_2\\ \lambda m_3^2\le\sqrt{T_1}}}\lambda^{-3/2}m_2^{-3/2}m_3^{-3/2}\ll T^{-1/4}\sum_{\lambda m_3^2\le\sqrt{T_1}}\lambda^{-1}m_3^{-1}\ll T^{-1/4}(\log T)^2.$$

It might also seem desirable to note that when summing over m_2 in the main term of (7.6.14) we obtain the asymptotic relation

$$\sum_{\substack{T_1^{1/4}\lambda^{-1/2}\le m_2\\ \lambda m_3^2\le\sqrt{T_1}}}\lambda^{-3/2}m_2^{-3/2}m_3^{-3/2}=2T_1^{-1/8}\sum_{\lambda m_3^2\le\sqrt{T_1}}\lambda^{-5/4}m_3^{-3/2}$$
$$+O\Big(T^{-3/8}\sum_{\lambda m_3^2\le\sqrt{T_1}}\lambda^{-3/4}m_3^{-3/2}\Big).$$

By summing over both m_3 and λ one may find that the error term in the above equation is $O(T^{-1/4})$. To complete the evaluation of $U_2(T)$ it thus remains to analyse the sum cognate to the above main term, say

$$\sum_{\lambda m_3^2\le\sqrt{T_1}}\lambda^{-5/4}m_3^{-3/2}=\zeta(3/2)\sum_{\substack{\lambda=1\\ \lambda\ \text{square-free}}}^{\infty}\lambda^{-5/4}+O\Big(\sum_{\lambda m_3^2>\sqrt{T_1}}\lambda^{-5/4}m_3^{-3/2}\Big).$$

One then clearly deduces that the error term is bounded above by a constant times

$$T^{-1/8}\sum_{\lambda\le\sqrt{T_1}}\lambda^{-1}+\sum_{\lambda>\sqrt{T_1}}\lambda^{-5/4}\ll T^{-1/8}(\log T),$$

which in conjunction with a customary evaluation of the above series yields

$$\sum_{\lambda m_3^2\le\sqrt{T_1}}\lambda^{-5/4}m_3^{-3/2}=\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}+T^{-1/8}(\log T).$$

Consequently, the preceding discussion delivers

$$U_2(T) = 2\zeta(3/2)\zeta(5/4)\zeta(5/2)^{-1}T_1^{7/8} + O(T^{3/4}(\log T)^2),$$

whence combining such an equation with (7.6.12) and (7.6.13) yields

$$S_1(T) = \frac{16\zeta(3/2)\zeta(5/4)}{7\zeta(5/2)}T_1^{7/8} + O(T^{3/4}(\log T)^2). \tag{7.6.15}$$

In order to finish the proof it suffices to combine the above formula with (7.6.10). □

For the purpose of completing the discussion it seems desirable to consider the final term

$$I_6(T) = 2\int_0^T D(1/2-it)D(1/2+it)D_2(1/2-2it)\chi(1/2-it)\chi(1/2+2it)dt.$$

Lemma 7.6.3. *One has that*

$$I_6(T) = \frac{16\cdot(2\pi)^{1/8}\sqrt{2}}{7\zeta(5/2)}e^{i\pi/4}\zeta(3/2)\zeta(5/4)T^{7/8} + O\big(T^{3/4}(\log T)^3\big).$$

Proof. We begin the proof by recalling (7.4.12) and employing the approximation formula found in Lemma 5.2.6 to obtain

$$I_6(T) = 2\sum_{\substack{n_1\leq 2\sqrt{T_1}\\ n_2,n_3\leq\sqrt{T_1}}} P_{\boldsymbol{n}}^{-1/2}\int_{N_{\boldsymbol{n},2}}^T e^{iF_6(t)}dt + O(T^{3/4}\log T), \tag{7.6.16}$$

wherein the phase function is defined by means of the expression

$$F_6(t) = -2t\log 2t + t\log t + t(\log 2\pi + 1 + \log(n_1^2n_2/n_3)).$$

The reader may observe that then on recalling (7.6.1) it transpires that

$$F_6(t) = -F_5(t) - t\log 4.$$

Thus on writing $d_{\boldsymbol{n}} = 2\pi n_1^2n_2/4n_3$ then one readily sees that $F_6'(d_{\boldsymbol{n}}) = 0$ and that

$$F_6(d_{\boldsymbol{n}}) = 2\pi n_1^2n_2/4n_3. \tag{7.6.17}$$

An insightful inspection of the proof of Lemma 7.6.1 in the course of the estimation of the error terms reveals that the only property utilised therein pertaining to the number n_1 being an integer was the spacing condition in $\mathbb{N}$. It therefore transpires that upon making the change of variables $n_1' = n_1/2$ in the analysis of the corresponding error terms herein we can reproduce the same proof to the end of deriving the equation

$$I_6(T) = 2\pi e^{i\pi/4} \sum_{\substack{n_3 \leq \min(n_1/2, n_2) \\ n_1^2 n_2/4n_3 \leq T_1}} n_1^{1/2} n_3^{-1} e^{iF_6(d_{\mathbf{n}})} + O\big(T^{3/4}(\log T)^2\big), \qquad (7.6.18)$$

wherein the tuples in the above sum are subjected to the same constraints as in the sum in (7.6.16). It seems convenient to write, as was previously done, the sum $M_6(T)$ in the above expression by means of the formula

$$M_6(T) = V_1(T) + V_2(T),$$

wherein

$$V_1(T) = \sum_{\substack{n_1^2 n_2/4n_3 \leq T_1 \\ 4n_3 \mid n_1^2}} n_1^{1/2} n_3^{-1}$$

and

$$V_2(T) = \sum_{\substack{n_1^2 n_2/4n_3 \leq T_1 \\ 4n_3 \nmid n_1^2}} n_1^{1/2} n_3^{-1} e^{iF_6(d_{\mathbf{n}})},$$

where we omitted for ease of notation writing $n_3 \leq \min(n_1/2, n_2)$.

We find it desirable to observe first that on recalling (7.6.17) one has

$$V_2(T) \ll \sum_{4n_3 \nmid n_1^2} n_1^{1/2} n_3^{-1} \left\| \frac{n_1^2}{4n_3} \right\|^{-1}.$$

The reader may note that another insightful perusal of the analysis pertaining to $S_2(T)$ found right after (7.6.8) reveals that the same argument deployed therein may be still valid when replacing n_3 by $4n_3$, thus delivering

$$V_2(T) \ll T^{3/4}(\log T)^3.$$

Yet another insightful examination of the term $V_1(T)$ might suggest that on making the change of variables $n_1' = n_1/2$, as we may, and recalling (7.6.8) to the reader, it transpires that

$$V_1(T) = \sqrt{2} S_1(T).$$

Therefore, on combining the above identity with the asymptotic evaluation (7.6.15) one obtains

$$V_1(T) = \frac{16\sqrt{2}\zeta(3/2)\zeta(5/4)}{7\zeta(5/2)} T_1^{7/8} + O(T^{3/4}(\log T)^2),$$

which in conjunction with the preceding discussion and (7.6.18) delivers

$$I_6(T) = \frac{16\sqrt{2}(2\pi)^{1/8}}{7\zeta(5/2)} e^{i\pi/4}\zeta(3/2)\zeta(5/4)T^{7/8} + O\big(T^{3/4}(\log T)^3\big),$$

as desired.

□

Proof of Theorem 7.1.1 We have now reached a position from which to expeditiously complete the proof of Theorem 7.1.1 by means of the combination of equation (7.3.2) with Lemmata 7.3.1, 7.4.2, 7.4.3, 7.5.1, 7.5.2, 7.6.2 and 7.6.3.

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