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Extending Nonlinear Frequency Analysis to Flight Dynamics and Control Problems

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Thesis advisors: Prof. Mark H. Lowenberg and Prof. Simon A. Neild

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Unforced bifurcation analysis has proven to be a powerful method for nonlinear fixed-wing aircraft flight dynamics analysis in the past 40 years. On the other hand, the use of harmonically-forced bifurcation analysis in fixed-wing flight dynamics studies is almost unexplored. The advantage of this method is its ability to capture non-stationary nonlinearities, including but not limited to rate limiting, nonlinear damping, controller gain scheduling, and modal coupling. These phenomena are of great importance in flight dynamics analysis, especially as current and future aircraft continue to push the operating envelope and incorporate more nonlinear features in their designs.

In this thesis, it is proposed that harmonically-forced bifurcation analysis be exploited and the outputs visualised in the form of a nonlinear Bode plot. This approach provides the capability to assess the non-stationary nonlinearities, which cannot be reflected using conventional (unforced) bifurcation methods and linear frequency analysis. A wide range of promising applications is discussed in this thesis. In all cases, the nonlinear Bode plot not only accurately reflects aircraft’s transient dynamics in the time-domain, but also uncovers many interesting phenomena that may otherwise be undetected.

Nonlinear frequency response can reveal regions where the closed-loop performance deteriorates considerably comparing to the linear-based prediction. Furthermore, a dedicated study on the link between actuator saturation and pilot-induced oscillation reveals that rate saturation can create a pair of fold bifurcations, resulting in 180° jump in phase lag. It is also shown that harmonic forcing of the elevator can excite the aircraft’s natural resonance frequency, leading to a divergent oscillation that can be used to escape a locked-in deep stall. Additionally, the method verifies that unsteady aerodynamic effects have a negative impact on the aircraft flying qualities. The final study investigates the sub- and super-harmonic resonances, which presents a more theoretical analysis into the frequency-domain nonlinearities in a flight dynamics model.
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Last but not least, I would like to express my appreciation for the memes by Tom, Pete, Josh, Jamie, Alexis, and Jason. Special thanks are also reserved for Anna for the informal discussions throughout lockdown, and for Dr Trang for assisting with my transition to post-PhD life.
AUTHOR’S DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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DATE: ..............................................................
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NOMENCLATURE

\[ A = \text{forcing amplitude (deg or N)} \]
\[ CG = \text{centre of gravity position} \]
\[ f = \text{forcing frequency (Hz)} \]
\[ K = \text{gain} \]
\[ q = \text{pitch rate (deg/s)} \]
\[ r = \text{reference signal} \]
\[ r = \text{yaw rate (deg/s)} \]
\[ S = \text{maximum rate} \]
\[ t = \text{time (s)} \]
\[ u = \text{pilot input} \]
\[ V = \text{velocity (m/s)} \]
\[ x = \text{generic state} \]
\[ \alpha = \text{angle of attack (deg)} \]
\[ \delta_{e/s/a/r} = \text{elevator/stabilator/aileron/rudder deflection (deg)} \]
\[ \Delta = \text{incremental, relative to trim value (used as prefix)} \]
\[ \Lambda = \text{general scaling factor} \]
\[ \varepsilon = \text{structural state} \]
\[ \eta = \text{elevator/elevon deflection (deg)} \]
\[ \theta = \text{pitch angle (deg)} \]
\[ \tau = \text{time delay constant (s)} \]
\[ \phi = \text{bank angle (deg)} \]
\[ \omega = \text{forcing frequency (rad/s)} \]
\[ \Omega = \text{structural mode frequency (rad/s)} \]

Subscripts
\[ d = \text{demanded} \]
\[ o = \text{trim value or initial value} \]

Abbreviations
\[ MAC = \text{mean aerodynamic chord} \]
\[ PIO = \text{pilot-induced oscillation} \]
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I. BACKGROUND

I-1. INTRODUCTION

Nonlinearities have long been a challenging problem in flight dynamics – so much so that the even the highest performing fighter jets in operation are still subjected to many artificial restrictions due to our limited understanding of nonlinearities in flight dynamics [1]. This is further complicated by the increased use of fly-by-wire technology since the 1970s in both military and civil applications, which adds another nonlinear element to the already complex pilot-vehicle system. Nevertheless, the primary toolbox in the industry for flight dynamics and control analysis is still linear-based. This has led to many incidents of undesirable pilot-vehicle interactions, sometimes with catastrophic consequences, that are directly caused by overlooking the nonlinear effects during the design process. With the ongoing drive to build more fuel-efficient commercial aircraft and more complex multi-role military air vehicles that continuously push the operating envelope, these nonlinear elements can no longer be neglected. Therefore, there is an urgent need to develop new analysis techniques that can not only handle nonlinearity rigorously, but also be easily adapted by practicing engineers to complement the existing linear-based methods.

In the past 40 years, bifurcation analysis and numerical continuation have proven to be a powerful tool for flight dynamics and control studies. This is thanks to the method’s ability to characterise many nonlinear phenomena encountered at high angles of attack and sideslips or due to inertial coupling, which made bifurcation analysis a popular choice for investigating the dynamics of high-performance fighter jets. Bifurcation analysis has seen further expansion to civil applications in recent years following some high-profile accidents such as Air France Flight 447 that led to a global effort to address airliner upsets and loss-of-control. The insights provided by bifurcation analysis are extremely valuable and cannot be obtained using linear-based techniques.

Despite these capabilities, bifurcation analysis in its current implementation on fixed-wing aircraft cannot capture the non-stationary elements in flight dynamics. These include rate limiting, nonlinear damping, and modal coupling, all of which can have serious impact on the aircraft’s flying quality. Since these characteristics only manifest themselves when the system is non-stationary, an expansion to the manner in which bifurcations methods is applied to aircraft flight dynamics is necessary. We can achieve this by treating the aircraft as a harmonically-forced system and employing a frequency-domain approach. The continuation algorithm is then set up to find the aircraft’s periodic response to a harmonic forcing input and present the results in the form of a ‘nonlinear Bode plot’, which is more familiar to the practicing engineers comparing to the conventional bifurcation diagram. Unlike its linear counterpart, the nonlinear Bode plot does not involve any linearisation in the system’s equations of motion, thereby allowing the non-stationary nonlinear elements to be directly reflected in the diagram. The proposed method therefore promises to be a powerful complement to the existing nonlinear analysis toolbox, in addition to being in a form that can be easily adopted by the industry. This capability is invaluable, especially with the ever-increasing demand to achieve greater performance and higher level of safety in any design [1].
I-2. NON-STATIONARY NONLINEAR PROBLEMS IN FLIGHT DYNAMICS

Flying is an inherently nonlinear activity (both figuratively and literally). In order to simplify the problem for mathematical analysis, local linearisation is commonly employed to approximate the nonlinear aircraft as a linear plant. This process is done at a number of operating points and then combined into a nonlinear gain-scheduled scheduled controller. By nature, such a controller will perform as predicted when the flight condition is varied slowly and in a quasi-static manner, such as changes in altitude or Mach number. It is the fast dynamics, such as large variation in angle-of-attack and pitch rate during rapid manoeuvring, that degrades the controller’s performance and leads to poor handling qualities. Some of the documented problems involving nonlinear dynamics are discussed below to highlight the shortcomings of linear analysis and underlines the need for a more advanced technique.

I-2.1. Degraded performance of linear-based controllers

During a series of low-altitude and high-speed test flights outside the operational envelope, a B-2 encountered residual pitch oscillations in response to a doublet input on control surfaces [2]. The phenomenon was not predicted by analytical methods, wind tunnel tests, or previous flutter tests, and subsequent work determined the possible cause to be a complex interaction between transonic aerodynamics and aeroelasticity. The study concluded that a pilot should stay clear of that region in the envelope and made no mention of the B-2’s complex and highly classified stability augmentation system, which cannot be disabled in flight [2]. It is possible that the interaction between aerodynamic and structural dynamics had created a highly nonlinear oscillation situation, which coupled with the flight control system and led to the pitch oscillation in question.

A second widely studied example is the NASA Generic Transport Model (GTM) – a sub-scale remotely-piloted aircraft representing a typical airliner configuration. Reference [3] examined a number of different controllers for the GTM. As expected, the closed-loop dynamics matches the linear-based analysis when the aircraft operates locally. Exceeding the operating envelope leads to loss of stability and limit-cycle oscillation when using the fixed gain controller. The gain-scheduled controller does address the loss of stability in the examples considered in [3]. However, a subsequent study using bifurcation analysis has identified instances in which the gain-scheduled controller fails to maintain controlled-flight due to the aerodynamic nonlinearities at high angles-of-attack [4].

Even in the best-case-scenario where the linear controller achieves satisfactory (local) performance at every single operating point across the whole operating envelope by the use of gain scheduling, it can still suffer from degraded performance as the aircraft transitions between different operating points. For example, the controller can initially drive the aircraft in the wrong direction due to a reversal of elevator effectiveness at high angles-of-attack [5], or fails to take the aircraft to the commanded attitude due to the existence of a secondary attractor [6]. All of these behaviours are in stark contrast to the optimistic predictions made by linear-based analysis and seriously degrade the aircraft’s flying qualities.
I-2. Non-stationary nonlinear problems in flight dynamics

I-2.2. Rate limiting and handling qualities
Many modern military and civil aircraft are fly-by-wire, so there is no physical link between the pilot’s input device (stick/yoke and rudder pedals) and the hydraulically-driven control surfaces. As such, there exists a possibility of the pilot demanding movements that exceeds the maximum rate of the control surfaces’ actuators, especially in highly-augmented fly-by-wire aircraft that do not employ force feedback on the pilot input device (e.g., F-16, Space Shuttle, Airbus A320). These limits can be physical due to aerodynamic load and inadequate hydraulic power, or due to high gains in the control systems and digital delays. As a consequence, a number incidences of pilot-induced oscillations (PIO) and divergence has been directly attributed to neglecting actuator rate limiting during the design process, some of which are discussed below.

The flight control system of the Mach-6-capable X-15 experimental aircraft used an adaptive gain, which automatically increased the gain to the value just below a limit-cycle onset [7]. In this configuration, the maximum possible gain was always available to the pilot, which was deemed necessary to maintain adequate performance in such a demanding environment. The aircraft suffered a fatal in-flight breakup due to a divergent oscillation in 1967, and subsequent investigations (including one presented as recently as 2016 [8]) concluded that the limit-cycle is caused by a combination of structural coupling and more importantly actuator rate limiting [8]. At such a flight condition, the tailplane actuator has a very low maximum rate of 15 deg/s, which coupled with the high controller gain led to a limit-cycle-susceptible aircraft. In fact, it can be shown mathematically in simpler examples that a limit cycle may originate from a sub-critical Hopf bifurcation in feedback systems with rate limiting [9, 10]. This creates a situation in which the system can transition from equilibrium to limit-cycle oscillation solely due to an external disturbance (despite no change to the control input). The danger was not fully appreciated in the original design of the flight control system, which did not account for rate limiting and employed a very empirical approach as seen in the adaptive gain system. Rate limiting is therefore a very elusive threat that requires a fully nonlinear approach in order to fully understand its impacts.

Ten years later, the Space Shuttle Enterprise suffered a pilot-induced oscillation just before touchdown at one of its final test flights in 1977 [11-13]. Although the incident caused no damage or casualties, it attracted attention due to the media presence at the airport and the implications on subsequent space-bound missions. Investigations found that a combination of actuator rate limiting and time delay in the flight control system added considerable phase lag, which consequently degraded the flying qualities to a level far below the linear-based predictions. Although the problem was addressed in the end via the use of an input filter [11] and improved pilot training [13], the sudden degradation in flying qualities (referred to as the ‘flying qualities cliff’) experienced in the flight is not fully explained in the existing literature and therefore still present a hidden danger to modern aircraft.

An extreme example of the flying qualities cliff can be found in the two PIO incidents encountered on the Gripen fighter jet in the 90s [14, 15]. The Gripen is a highly unstable aircraft, which requires constant pitch control to maintain artificial stability. Its canard-delta configuration also means that the elevons are
I-2. Non-stationary nonlinear problems in flight dynamics

responsible for both pitch and roll control. In both incidents, the roll control channel was overloaded due to a combination of external disturbances and demanding pilot inputs, which led to reduced elevon margin and subsequently total loss of pitch stability. This agrees with the collected flight data, which showed that the elevon was heavily saturated. A similar conclusion to the Space Shuttle was reached: that the added phase lag due to actuator rate limiting contributed to the PIO by increasing the discrepancy between the pilot’s command and the aircraft’s response. This problem was addressed by a phase compensator system, which ensured that the control surfaces always travel in the same direction as the pilot’s input [14, 15]. This scheme reduces phase lag considerably in the presence of rate limiting by preventing the integrator from integrating the error signal when the pilot reverses the input direction. Since the implementation of the phase compensator system, no further PIO of a similar nature has been reported on the Gripen.

In all the examples discussed above, the rate limiting problem was only identified after the incident had taken place. Existing control theories can correct the problem very well, but the methods to predict pilot-induced oscillations due to rate limiting are still limited and mostly linear-based. In fact, it has been noted in [16] that ‘there are currently no existing metrics that account for these non-linear effects’. It is therefore essential that a method to quantify the effect of rate limiting on flying qualities and more specifically pilot-induced oscillation is developed, especially as more and more aircraft continue to make use of complex full-authority fly-by-wire systems.

I-2.3. Deep stall recovery

Deep stall (also known as super stall) is a dangerous phenomenon in which the aircraft is locked into a high angle-of-attack despite a full nose-down command. This problem is common among T-tail aircraft, although high performance fighter jets like the F-16 with centre of gravity far back may also be susceptible to deep stall. Despite its long history, research on the recovery procedure is still either empirical or based on reduced-order aircraft models, with both approaches having mixed results. A brief summary of these methods is provided in this section, along with a discussion on their effectiveness.

Nguyen et al suggested that upon entering a deep stall, the pilot would observe the transient oscillation as the aircraft settles into the stable high-α trim and pump the stick at the same frequency as that oscillation [17]. The idea is that by matching the forcing input with aircraft’s natural rigid-body frequency, it would be possible to build up some momentum with the little control authority available to push the nose down. An earlier study by NASA on a simplified T-tailed airliner model [18] used a somewhat similar idea (referred to as ‘dynamic recovery’): the elevator is excited in a square wave pattern, which reverses direction when the pitch rate reaches zero. This is essentially a bang-bang controller, which was also later adopted in a similar study on the F-16 fighter jet [19]. These studies require close observation of the ensuing transient oscillation in order to match the forcing input with the aircraft’s frequency. In fact, it was noted in [17] that slow or unsuccessful recoveries were attributed to the difficulty of matching the input frequency to the aircraft’s natural frequency. There is also the risk of the pilot not reacting fast enough to deep stall entry, meaning that the oscillation is already damped out by the time the pilot initiates the manoeuvre. Furthermore, it was not
highlighted in the studies above that the frequency-domain dynamics of the aircraft at such high angles-of-attack is highly nonlinear. Depending on how the aircraft entered the deep stall condition, the ensuing oscillation may have different and varying frequencies, making it more challenging to observe the motion and provide a forcing term. It also will be shown later that at such high angles-of-attack, the frequency separation between the short period and the phugoid mode is reduced considerably, suggesting that the use of reduced-order models that only consider the short-period dynamics as seen in [18-21] may give inaccurate results.

A different recovery procedure was proposed in [22]. Using an F-16 fighter jet model, it was found that at maximum thrust, the deep stall stable trim point at full nose-down stabilator becomes unstable. However, a time simulation in [22] shows that it takes 60 seconds of max thrust to take the aircraft out of the deep stall region (i.e., the unstable mode’s divergence is very slow). The method also demands that the engine produces maximum thrust throughout the entire manoeuvre. This is not necessarily a valid assumption as most engines will experience a noticeable performance reduction at such high angles-of-attack or even flame out. In addition, due to the lack of a publicly-available high-fidelity model of the F-16’s engine, it was decided not to further pursue this method.

I-2.4. Unsteady aerodynamic effects

Quasi-steady aerodynamic modelling remains the most popular method for representing aerodynamic forces and moments in flight dynamics studies. This method is traditionally based on the use of stability and control derivatives, which can adequately model both static and dynamics nonlinearities at low angles-of-attack during normal flights [23]. In the stall and post-stall regime, however, quasi-steady modelling cannot adequately capture many important phenomena due to the major influence of unsteady (time-dependent) effects in these regions, which only become noticeable when the aircraft is non-stationary. In fact, it has been shown that these unsteady effects can lead to strong discrepancies between the behaviours predicted using best-practice wind-tunnel tests and actual test flights [24]. Many important stall and post-stall flight characteristics cannot be adequately analysed as a result, which can affect further studies of highly manoeuvrable combat aircraft as well as of upsets and loss-of-control prevention in transport applications [25-27].

The need for adequate account for time-dependent flow phenomena is now recognised, and various methods to model these unsteady aerodynamics effects have been proposed as a result [28, 29]. However, most of the published studies on the topic only focus on accurate modelling of variations in the force and moment coefficients under wind-tunnel-like tests conditions, and no investigation has directly established the link between unsteady aerodynamics and flying characteristics in terms of stability and control. As observed in the real world, the presence of these time-dependent phenomena can seriously degrade the handling qualities of the aircraft in stall and post-stall regions [24]. A further study that constructs the basic framework to combine unsteady aerodynamics with stability and control analysis is therefore necessary.
I-2. Non-stationary nonlinear problems in flight dynamics

I-2.5. Discussion
All of the issues above share one common attribute: their influences on the dynamics only become noticeable when the aircraft is non-stationary. This means that neither linear-based analysis nor the existing implementation of unforced bifurcation analysis can detect these phenomena. A nonlinear frequency-based approach can solve this problem for two reasons:

- The periodic solutions of a harmonically forced system are non-stationary by definition. This means that the nonlinear effects discussed above can be directly examined in the frequency response.

- The method is based on a well-established mathematical background (bifurcation analysis and Floquet theory), which can facilitate its adoption by the research community and the industry.

A brief overview of bifurcation analysis with a focus on its applications in flight dynamics studies is provided in the next section.
I-3. BIFURCATION METHODS

Within academic literature, bifurcation analysis is a popular method for analysing nonlinear dynamical systems. It involves calculating the steady-state and limit-cycle (self-oscillation) solutions under the variation of one of the system parameters. This map of solutions is called a bifurcation diagram, which provides a global picture of the nonlinear dynamical system under consideration. In complex applications, bifurcation analysis is usually done numerically using continuation methods, which utilise a path-following algorithm to trace out the family of solutions and provide information on their stability – either using an eigen analysis for equilibrium or Floquet analysis for oscillatory solutions. The varying parameter is accordingly referred to as the continuation parameter. In the context of flight dynamics, this can be control surface deflections, controller gain, or centre of gravity positions, among others. Bifurcation diagrams provide the exact conditions required to observe many nonlinear phenomena that can exist in the nonlinear system considered, and more generally the driver behind these often counter-intuitive dynamics. Otherwise, it would be necessary to run large numbers of time simulations to detect these behaviours. This can be prohibitively expensive computationally for complex high order systems like an aircraft model whilst still missing some important dynamics. Accordingly, many different fields have benefitted from the time-saving capability of bifurcation analysis, ranging from epidemiology and medicine [30-33], economics and finance [34, 35], and mechanical engineering (such as railway vehicle dynamics [36] and internal combustion engine [37]). In this section, the history and state-of-the-art of bifurcation analysis in a flight dynamics context are discussed.

I-3.1. Unforced bifurcation analysis

Early works

The first application of bifurcation analysis in a flight dynamics context was done by Mehra and Carroll in the early 80s [38, 39]. Using an F-4 fighter jet model, a number of nonlinear and non-intuitive behaviour including jump phenomena, hysteresis, and spins observed in real aircraft were represented as equilibrium surfaces with stable and unstable solutions. Moreover, the inclusion of limit-cycle (self-oscillating) solutions explained the wing rock and post-stall oscillation motions – all of which were highly nonlinear regimes that are of great importance in flight dynamics studies. The results also showed the direct link between control surface movements and the formation of those undesirable flight regimes as stable solutions, which facilitated further analysis on recovery strategies and controller design.

The method was then quickly adopted by researchers around the world. For example, an analysis of the F-14 by Jahnke [40] found that stable spin solutions (both equilibrium and oscillatory) exist across the entire control surface deflection range, which explained the aircraft’s notorious flat spin behaviour (made popular by the Hollywood’s depiction in Top Gun) and corroborate the claim that the flat spin is unrecoverable. Further validation of bifurcation analysis was then done in a study by Guicheteau, where predictions of the Alpha-Jet aircraft’s spin characteristics made by bifurcation analysis were verified in actual test flights [41]. Chaotic motions were also predicted, which agreed with the pilot’s remarks from past flights although no recorded data was available due to the dangerous nature of such a test.
Expanding into feedback applications

Throughout the mid-90s and early-2000s, researchers started exploring the use of bifurcation analysis in closed-loop applications. The idea is to examine how a control system can extend the stable (desirable) solutions within the achievable parameter range, so as to prevent incursion to the regions of departed flight. Goman showed using two-parameter continuation technique that by setting the controller gain as one of the continuation parameters, we can directly track the movements of the Hopf bifurcations (which leads to limit cycles) as the controller gain varies [42]. This information can be used to determine the minimum stability-augmentation feedback gain required to prevent excursion into the spins and wing-rock regions.

Applications into more complex controller laws quickly followed. Analysis by Gibson demonstrated the effectiveness of an eigenstructure assignment controller in addressing the inertial coupling problem [43]. Specifically, the open-loop aircraft contained non-zero stable solutions in the lateral-directional states at low angles-of-attack, which reflected the transition to autorotation due to the loss of stability in the normal flight branch. When the controller was included in the analysis, all solutions associated with the autorotation branch were removed from the observed phase space. The lateral-directional states now contained stable solutions at the origin, indicating that the controller had successfully maintain lateral-directional stability. Although this study was done on a reduced-order aircraft model with linear aerodynamics, it has been demonstrated that bifurcation analysis provides a precise mechanism of how a controller removes the undesirable flight regimes from the phase space and modifies the aircraft’s dynamics. A more comprehensive study was done by Avanzini, in which the full-order longitudinally-unstable F-16 fighter jet was analysed whilst coupled with a realistic command-augmentation controller [22]. This study demonstrated that closed-loop bifurcation analysis can be done on a highly augmented aircraft model, albeit with some limitations since the rate-demand controller does not have equilibrium solutions in level flight. Since the F-16 is open-loop unstable, both open- and closed-loop bifurcation analysis are essential for understanding the underlying dynamics. One such example is how the controller could not remove a family of stable solutions at high angle-of-attack, which was linked to the deep stall branch in the open-loop aircraft. Bifurcation analysis therefore provides not only a picture of the aircraft’s steady-state dynamics but can also indicate where the controller may fail to perform as expected. These studies have demonstrated the effectiveness of bifurcation methods for validating existing controllers in highly nonlinear applications.

The next development involved expanding the method to be part of the control design process itself. One of the first attempts was done by Littleboy, who used the open-loop bifurcation diagram to guide the development of a nonlinear dynamic inversion controller in a fighter-type aircraft model [44]. Although the aircraft model was considered realistic, its control law was rather simple. The only objective was to stabilise the unstable solutions in the normal flight branch and remove the spin branch from the parameter space without any consideration in the closed-loop performance. This gap was later addressed by Richardson, where the eigenstructure assignment algorithm was embedded into the continuation software in order to directly calculate the controller gains using bifurcation analysis [5, 6, 45, 46]. In this setting, the continuation algorithm was set up to find the gains while keeping the derivatives of the eigenvalues constant, thereby
achieving a pole-placement objective using a smooth variation in gains as the aircraft moves across different operating points. Both studies scheduled the gains against the pilot input – an approach that encountered some performance issues at high angles-of-attack due to the aerodynamic nonlinearities. These were addressed by the dynamic gain scheduling technique, which transforms the input-scheduled gains into functions of the aircraft’s states [5, 46]. Performance was superior to all the previous controller schemes considered. Although the mathematical basis for calculating these gains is complex, continuation methods greatly reduced the process into a quick and simple numerical exercise. As a testament to its effectiveness, Baghdadi later used this method of continuation-based controller design on the multi-input (tailplane and canard) controller of a flexible aircraft model. This controller not only achieved desirable performance at high angles-of-attack, but also successfully addressed the structural coupling problem due to inadequate frequency separations between the rigid-body and the structural modes [47].

**Constrained bifurcation analysis**

A further extension to the method called ‘constrained bifurcation analysis’ was also developed in the mid-90s. First introduced in 1995 to study the inertial coupling problem [48], constrained bifurcation analysis involved swapping a state variable with a control input in the algorithm in order to determine the control input required to maintain some constraints on the selected state variable. Two common applications of this technique are calculating the thrust-elevator relationship to maintain zero flight path angle [49] and finding the elevator-aileron-rudder relationship to counteract aerodynamic asymmetries at high angles-of-attack [45]. In the context of flight control design, constrained bifurcation analysis can be used to find the feedforward signal in some controller inter-connect schemes to extend the stable solutions and push the undesirable bifurcations away from the operating region. This was done by Lowenberg to schedule the thrust vectoring angle against the elevator input in order to increase the maximum achievable angles-of-attack while also avoiding the unstable branch [50] – a technique known as bifurcation tailoring. Finally, a further development was done by Paranjape where the constraints were manipulated to output the maximum instantaneous and sustained turn rates on a full 8th-order model [51]. The study provided valuable insights on the aircraft’s performance and its link with physical limits like available thrust and load factors, which further demonstrate the versatility of bifurcation methods. Although no results presented in this thesis explicitly use constrained bifurcation analysis, it was routinely employed during preliminary analysis to find the suitable trim conditions for further studies.

**Increasing industrial adoption and recent developments**

Bifurcation analysis has received significant attention from the industry thanks to a partnership between Airbus UK and the University of Bristol in the mid-2000s. This led to the development of the Dynamical Systems Toolbox (DST), which is a MATLAB/Simulink implementation of the continuation software AUTO [52]. Creation of the DST allowed many industrial-standard models to be directly integrated into the bifurcation analysis framework in the MATLAB/Simulink environment without requiring a FORTRAN conversion [53], which facilitated many subsequent studies on industrial-focused topics. This includes a large body of works on airliner ground dynamics and landing gear mechanism summarised in [1], which has
recently been expanded to cover small unmanned vehicles [54]. The DST has since then been adopted by Airbus and is now ‘fully incorporated in the Airbus Methods and Tools portfolio as a supported tool for the evaluation of proposed works and new designs’ [55]. Another important project with Airbus involved studies of airliners upsets and loss-of-control, which led to a series of papers that were the first fully characterise the behaviour of an industrial-standard airliner model using bifurcation analysis in both open- and closed-loop applications [4, 56, 57]. These studies provided valuable contributions to the global effort to reduce airliner upsets and loss-of-control incidents [58-60] following some high-profile accidents like the Air France Flight 447. Lastly, the most recent developments lie in the field flutter analysis in nonlinear aeroelastic applications [61, 62]. This is a logical progression to meet the demands of future aircraft designs, which involve slender high aspect-ratio wings to improve fuel efficiency and performance.

I-3.2. Forced bifurcation analysis

Unforced bifurcation analysis has matured over the past 40 years and proven itself to be a powerful tool for nonlinear flight dynamics and control studies. Although there are many exciting future projects over the horizon, the technique has remained largely unchanged since its inception (at least in the field of flight dynamics). This means that the existing flight dynamics literature do not account for the non-stationary nonlinear elements since unforced bifurcation analysis primarily deals with equilibrium solutions. Accordingly, this thesis explores a different implementation of bifurcation analysis – referred to here as ‘nonlinear frequency response’ (or more accurately ‘harmonically-forced bifurcation analysis’, and in some instances ‘frequency-domain bifurcation analysis’). This approach involves examining the aircraft’s dynamics under an external oscillatory forcing. All resulting motions are therefore non-stationary, which promises to reflect the non-stationary effects and fill in the gap left by unforced bifurcation analysis. Harmonically-forced bifurcation analysis is already common in the fields of applied maths [63-68] and structural/mechanical engineering [69-74] (mostly for vibration analysis). However, little work has been done to study its applications in a flight dynamics context. At the time of writing, there are only four notable papers that feature harmonic forcing on fixed-wing aircraft (apart from the publications originating from this thesis):

- In 1996 and 1998, Sørensen et al. and Gránásy et al. applied harmonic forcing to the thrust vectoring angle in an F-18 model at a very high angle-of-attack [75, 76]. Although only small forcing amplitudes in the longitudinal plane were used, various period-doubling and torus bifurcations were detected, which led to complex period-2, period-4, quasi-periodic, and eventually chaotic motions.

- Also in 1998, Mehra (one of the two original authors of the first bifurcation analysis papers in flight dynamics [38, 39]) and Prasanth combined the harmonic forcing term with actuator rate limiting in a study on pilot-induced oscillation [77]. Strictly speaking, this was not ‘frequency analysis’ per se as only continuation in the pilot gain was studied. As Mehra and Prasanth were investigating the link between pilot gain and limit-cycle oscillation in a pilot-vehicle system, the harmonic forcing can be seen as redundant. Nevertheless, continuation in pilot gain is a useful starting point. Section IV-1 of this thesis adopts the idea to expand the analysis on the impact of pilot gain on pilot-induced oscillation.
Lowenberg and Menon presented in 2007 a preliminary study on the link between aircraft’s parameters uncertainties (e.g., variations in the pitching moment coefficient) and instabilities with a focus on flying qualities [78]. The harmonic forcing term is used here to generate the pilot’s forcing input, which is based on the ‘clonk’ manoeuvre used in pilot-induced oscillation investigations [79]. This paper therefore presents another novel application of the method that does not involve the frequency response concept.

The harmonic forcing approach can also be seen in some other theoretical applications that do not involve bifurcation analysis. For example, Hassan and Taha utilised the ‘Lie bracket’ methodology to devise a way to roll the aircraft via harmonic forcing of both aileron and elevator [80, 81]. It was shown that in stall regions where the aileron is less effective, the combined aileron and elevator forcing can induce large rolling motion that far exceeds what can be achieved using aileron alone. This approach, however, demands a forcing frequency as high as 48 Hz on the control surfaces at ±50 deg amplitude, which far exceeds any physical limits and therefore constrains the method to theoretical studies.

Harmonically-forced bifurcation analysis can also be seen in rotary-wing applications, albeit used to model the periodic variation in the aerodynamic forces as the blades in forward flight switch between advancing and retreating. This was first reported in a conference paper by Bedford and Lowenberg in 2006 on a full-order (25 states) helicopter flight simulator [82]. Compared with previous studies that approximated the helicopter dynamics as equilibrium solutions, the harmonic forcing method detected additional harmonics at high forward velocity, which has been recorded in real flights but had not been studied in depth at that point. These harmonics are caused by the nonlinear variations in the aerodynamic coefficients as the blade’s angle-of-attack varies as well as the complex coupling in the equations of motion and may lead to structural coupling if the frequency separation is not adequate. This novel approach to rotorcraft analysis was quickly adopted by researchers and widely discussed in subsequent conferences. One interesting example can be found in an analysis of a simple second-order main rotor blade model by Avanzini and De Matteis in 2007 [83]. It was found that for a parked helicopter on an aircraft carrier with high lateral wind (referred to as wind on deck), the blades are susceptible to high-amplitude flapping that can lead to a tail boom strike if the main rotor (most likely in idle power setting) does not rotate fast enough to counteract the side wind. The relationship between wind speed, flapping angle, and rotational speed relationship is very complex due to the coupling between pitching and flapping motions, which could only be uncovered using a nonlinear-based method like bifurcation analysis. These studies attracted considerable attention from the industry, leading to the first journal article that used the method in 2014 by Rezgui and Lowenberg in collaboration with AgustaWestland (now Leonardo Helicopters) [84]. This paper investigated the nonlinear coupling on a very high-fidelity model of one single blade (21st-order in the simplest case presented) due to the presence of an actively-controlled trailing-edge flap. Trailing edge flap is being investigated by many helicopter manufactures as a method to reduce noise and vibration, although its addition to the blade creates many complex phenomena that requires an advanced nonlinear-based analysis.
Contrary to the current state-of-the-art in rotary-wing analysis, it can be seen that all fixed-wing studies so far are few and of preliminary nature. As such, there is currently no practical use of harmonically-forced bifurcation analysis on fixed-wing aircraft. This thesis aims to fill in this gap by utilising the method in the form of nonlinear frequency analysis – thereby examining the oscillation amplitude and phase relationship of a nonlinear aircraft model under an external harmonic forcing. The impact of nonlinearities on the frequency-domain criteria has not been discussed in the literature since frequency analysis in a flight dynamics context has always been done using linear-based methods. It is suspected that in the presence of significant nonlinearities, the stringent loop shaping objectives of many linear-based controller design techniques could be violated. Past studies using unforced bifurcation analysis have highlighted that the dynamics of highly augmented aircraft depends more on the controller than the airframe, especially in high-performance unstable fighter jets. Therefore, incorrect prediction of the gain and phase relationship due to unaccounted nonlinear effects can rapidly degrade the flying qualities and potentially lead to instabilities. An extension of existing frequency-domain criteria in the presence of significant nonlinearities is necessary to ensure the controller achieves the desired performance in the most challenging situations.

I-3.3. Nonlinear frequency response
Most of the results in this thesis that use harmonically-forced bifurcation analysis are presented in the form of a nonlinear Bode plot, thereby providing the gain and phase relationship of the aircraft motion relative to the harmonic forcing input. Based on discussions with other academics and engineers, it has been noted that the term ‘frequency response’ instantly brings up the idea of stability margin. This is a logical but somewhat unfortunate link. Although linear and nonlinear frequency responses are conceptually equivalent, the additional nonlinear element opens up another dimension that demands a new approach to fully exploit the method. This difference in philosophy can be summarised as follows:

- Linear frequency analysis mainly focuses on examining the open-loop frequency response to derive the gain and phase margins, among other parameters (i.e., loop shaping). This information is then used to make indirect implications on the stability and performance of the closed-loop system.

- Nonlinear frequency analysis directly examines the closed-loop frequency response. The results should therefore be interpreted at face value. Any instability or performance issue is therefore directly reflected by the nonlinear closed-loop frequency response.

The point about interpreting the nonlinear results at face value is rather important because it goes against the existing convention. Most existing frequency-domain metrics are only applicable to the linear open-loop frequency response. This list includes the PIO prediction criteria in NATO RTO-TR-29 [85], SAAB’s guideline for a 45° phase margin in flight control systems [14, 15], and BAE Systems’ 9 dB gain margin requirement to reduce the risk of structural coupling [86]. In fact, the use of the open-loop stability margin is so common that many test flights have been done by NASA to measure the in-flight closed-loop frequency response as a means to estimate the open-loop result [87-89]. Another challenge is that there is no existing
guideline for interpreting nonlinear results [16]. Again, current linear frequency-domain criteria assume that the output always has the form $A \sin(\omega t + \varphi)$. This is not guaranteed in nonlinear systems, so as soon as the response ventures away from the simple harmonic assumption (e.g., containing more than one frequency or having a different period), then those criteria may no longer be applicable. Linear and nonlinear frequency analyses have their strengths and weaknesses and should be regarded as complementary to each other instead of being mutually exclusive. The former has been well-documented and is directly applicable to classical control design methods, which helped create many existing handling qualities criteria [85]. Nonlinear systems, on the other hand, do not permit the use of many common mathematical operations like the Laplace transform that are instrumental in classical control design. This shortcoming creates a significant gap in rigorous analysis capability for instances where linear-based predictions are no longer sufficiently accurate. Therefore, this thesis aims not only to study these nonlinear phenomena, but also to present the results in a way that makes sense to the engineering community, thereby taking a small step toward a future of nonlinear analysis in flight dynamics and control.
The objective of this thesis is to explore the use of nonlinear frequency analysis in dynamics and control problems of fixed-wing aircraft. Particularly, the method is applied to a number of problems in which the nonlinear elements only become significant when the aircraft is non-stationary (i.e., during transient motions). It is anticipated that this new approach will improve our understanding of nonlinearities beyond the time domain, at least in the current context of flight dynamics and control, and ultimately become another useful toolbox for the research community and the industry.

Following this introduction, chapter II presents some simple examples of both unforced and forced bifurcation analysis as background information before specific applications are considered. All the nonlinear phenomena encountered in the upcoming chapters are presented in chapter II.

The main body of the thesis is presented in chapters III-VII, which demonstrate the use of nonlinear frequency analysis in a number of different flight dynamics models. There are two reasons for using multiple aircraft models instead of focusing the study on one:

- Nonlinear frequency analysis has not been used in fixed-wing flight dynamics analysis. Therefore, it is desirable to demonstrate the method across a wide range of applications to convince researchers and the industry to adopt nonlinear frequency analysis.

- Apart from the GTT model in chapters V-4 and VI, all aircraft models considered in this thesis have been studied in depth elsewhere. These models can therefore function as ‘testbeds’ for nonlinear frequency analysis, specifically by allowing us to both demonstrate and verify the method against previous studies. This side-by-side comparison also makes it easier to highlight the advantages of nonlinear frequency analysis against existing methods.

Due to the large number of aircraft models used, the results are organised in order of increasing aerodynamic complexity as shown in Table 1. Each chapter presents an example of where nonlinear frequency analysis can provide additional insights that cannot be obtained using any other analysis methods. Specifically:

- Chapter III shows how nonlinear frequency analysis can identify regions where the closed-loop transient dynamics is severely degraded. Full-envelope manoeuvre-demand controllers are used throughout chapter III. Therefore, reduced-order flight dynamics models that only consider fast (short-period) longitudinal dynamics are used to reduce complexity. The examples in chapter III also show how the nonlinear Bode plot can be used to directly predict the closed-loop response, thereby extending the notion of ‘frequency analysis’ beyond the open-loop stability margin idea.

- Chapter IV discusses the link between actuator saturation and flying qualities with a focus on pilot-induced oscillations. The flight dynamics models used here are increased to fourth-order to include the
The only nonlinear elements considered are actuator rate and travel saturations. This was done to isolate the source of nonlinearities to the actuator, thereby providing an in-depth study on how such a simple nonlinear element can have a profound effect on handling qualities.

- Chapter V includes aerodynamic nonlinearities in two fourth-order aircraft models to explore the topic of deep stall recovery. Specifically, it is possible to destabilise the statically-stable trim points at very high angles-of-attack via harmonic forcing of the pitch control devices (elevator/tailplane), thereby allowing us to rock the aircraft out of an otherwise unrecoverable deep stall. Of all the results presented this thesis, chapter V aligns the most with the conventional notion of ‘frequency analysis’ because the destabilisation process relies on knowing the resonance frequency. It will be shown that this frequency is not the same as the one obtained from the linear transfer function.

- Chapter VI presents an example of how nonlinear frequency analysis can be combined with unforced bifurcation analysis to uncover insights that cannot be obtained when one of the methods is used. This is done via a study of unsteady aerodynamic modelling, specifically to investigate whether such an advanced representation is necessary when compared with quasi-steady modelling.

- Chapter VII presents a more theoretical application of the method. Here, the nonlinear frequency-domain characteristics such as sub- and super-harmonic resonances are examined. The model dimensions include lateral-directional dynamics to reveal the modal coupling behaviours that are observed in highly-nonlinear harmonically-forced systems.

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Aerodynamic modelling</th>
<th>Lateral-directional dynamics</th>
<th>Other notable nonlinearities</th>
</tr>
</thead>
<tbody>
<tr>
<td>III</td>
<td>2nd order</td>
<td>No</td>
<td>Full-envelope controller</td>
</tr>
<tr>
<td>IV</td>
<td>4th order (linear)</td>
<td>No</td>
<td>Actuator saturation</td>
</tr>
<tr>
<td>V</td>
<td>4th and 8th orders</td>
<td>No</td>
<td>Locked-in deep stall</td>
</tr>
<tr>
<td>VI</td>
<td>4th and 6th order</td>
<td>No</td>
<td>Unsteady aerodynamics</td>
</tr>
<tr>
<td>VII-1</td>
<td>4th order</td>
<td>No</td>
<td>Modal coupling</td>
</tr>
<tr>
<td>VII-2</td>
<td>8th order</td>
<td>Yes</td>
<td>Modal coupling</td>
</tr>
</tbody>
</table>

The discussion throughout these chapters will focus not only on how to interpret the results from the nonlinear Bode plot, but also how to obtain them. Accordingly, this thesis can be treated as an introductory text to the use of nonlinear frequency analysis in a flight dynamics and control setting.

Finally, concluding remarks are presented in chapter VIII.
This chapter begins the discussion with a brief revision of unforced bifurcation analysis, followed by an introduction to nonlinear frequency analysis. In discussing the latter, a novel method to construct a nonlinear Bode plot from numerical periodic solutions obtained from harmonically-forced bifurcation analysis is also presented. The method will be demonstrated on weakly nonlinear second-order systems subjected to various types of nonlinearities in both open- and closed-loop settings so as to familiarise readers with the techniques employed in this thesis. Bifurcation analysis was performed using the Dynamical Systems Toolbox [53] – the MATLAB/Simulink implementation of the numerical continuation software AUTO [52].
II-1. A BRIEF INTRODUCTION TO UNFORCED BIFURCATION ANALYSIS

Consider a general autonomous dynamical system of the form:

\[ \dot{x} = f(x, u) \quad (a.1) \]

where \( f \) is a vector of \( n \) smooth (differentiable) functions, \( x \) is the state vector of dimension \((n \times 1)\) and \( u \) is the control input vector. In the context of open-loop flight dynamics, \( f \) is usually the equations of motion, \( x \) is the aircraft’s states like \( \alpha, V \), and \( u \) contains one of the control inputs (i.e., elevator, aileron, etc.). The system is in equilibrium when

\[ \dot{x} = 0 \quad (a.2) \]

A periodic solution of period \( T \) exists when

\[ x(t) = x(t + T) \quad (a.3) \]

By solving equation \((a.2)\) and/or equation \((a.3)\), a map of steady states (either equilibrium or periodic) as functions of one of the control inputs in \( u \) can be generated (assuming steady \( u \)). This map is referred to as a bifurcation diagram. The equations are solved numerically using continuation methods [90], which utilises a path-following algorithm to trace out a map of solutions as a parameter in \( u \) is varied. This varying parameter is referred to as the continuation parameter. Numerical continuation requires knowledge of at least one solution, which can be obtained by the user through time-integration method (simulating the system in equation \((a.1)\) long enough so that the states converge to their final values, assuming the system is stable) or Newton’s method. In many published works, the terms ‘bifurcation analysis’ and ‘numerical continuation’ are used interchangeably.

A bifurcation leads to nonlinear behaviour such as multiple solutions for the same input, or hysteresis, depending on the type of bifurcation encountered. The mathematical definition of a bifurcation is:

- For equilibrium solutions: when at least one eigenvalue of the system’s Jacobian matrix \( J = df/dx|_{x_0} \) (evaluated at the equilibrium point \( x_0 \)) crosses the imaginary axis.

- For oscillatory solutions: when a Floquet multiplier crosses the unit circle.

To demonstrate the concept, consider the following simple dynamical system:

\[ \dot{x} = -x^2 + r \quad (a.4) \]

When \( r = 4 \), there are two solutions \( x = 2 \) and \( x = -2 \) at equilibrium (\( \dot{x} = 0 \)). Fig. II-1 show the time simulations of this dynamical system from three different initial conditions \( x_0 \). In the first case, the dynamics
converges to the statically stable solution at \( x = 2 \). The second case starts from the unstable solution \( x = -2 \), which will diverge to infinity if the initial condition \( x_0 \) is reduced slightly like in Fig. II-1c.

![Fig. II-1 Time simulations of equation (a.4) at \( r = 4 \) with three different initial conditions](image)

We can use the equilibrium solution from Fig. II-1a or Fig. II-1b as the starting solution for the continuation algorithm to calculate the family of equilibrium solutions as \( r \) varies. Fig. II-2 shows the bifurcation diagram with \( r \) on the x-axis as the continuation parameter. Essentially, this is the map of \( x \) as a function of \( r \) that satisfies \( \dot{x} = 0 \). The stability of each solution is also identified. It can be seen that at \( r = 4 \), there are two solutions \( x = 2 \) (stable) and \( x = -2 \) (unstable) as mentioned previously. The fold bifurcation marks the point at which the solution in a branch changes direction, which is also accompanied by a change of stability.

![Fig. II-2 Bifurcation diagram of equation (a.4)](image)

Oscillatory solutions can also be encountered in a more complex system. Consider the following second-order system:

\[
\ddot{x} + (x^2 - m)\dot{x} + x = 0
\]  

(a.5)

This a variation of the Van der Pol oscillator. Nonlinearity comes from the damping term \( (x^2 - m) \). It can be seen that an equilibrium solution exists at the origin, and that increasing \( m \) beyond 0 destabilises this equilibrium point due to the negative damping. Indeed, Fig. II-3 shows the time simulation and phase plot at \( m = 1 \) with non-zero initial conditions. The system enters a stable self-sustained oscillation that is only
suppressed by the increased damping as $x$ moves away from the origin. Consequently, its phase plot shows a closed-trajectory (i.e., a limit cycle).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/figure1.png}
\caption{Fig. II-3 Time simulation (a) and phase plot (b) of equation (a.5)}
\end{figure}

Numerical continuation can identify both equilibrium and periodic solutions. Fig. II-4a shows the bifurcation diagrams with $m$ as the continuation parameter. Stable equilibrium solutions exist for all negative $m$. When $m$ exceeds 0, a Hopf bifurcation is detected, which led to a family of stable limit cycle that increases in amplitude with increasing $m$ (i.e., damping becomes increasingly negative). The link between the limit cycle amplitude and $m$ can be verified by running a time simulation with $m$ reducing linearly from a high value at a rate of $0.26t$. Data from this time simulation (plotted in terms of $m$ rather than $t$) are superimposed on Fig. II-4a, which matches the prediction made by bifurcation analysis. The 3D projection of the phase plot is shown in Fig. II-4b.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/figure2.png}
\caption{Fig. II-4 Bifurcation diagram of the modified Van der Pol oscillator with simulated response superimposed (a). 3D projection of the simulation data (b)}
\end{figure}

The limit cycle observed in this example is self-sustained. For nonlinear frequency analysis, an external forcing term is required, and all ensuing motions will be shown as periodic solutions in the bifurcation diagram. The method to implement harmonic forcing into the continuation solver is explained next.
II-2. FORCED BIFURCATION ANALYSIS: THE DUFFING EQUATION

The Duffing equation is a textbook example of a nonlinear harmonically forced system. It is physically equivalent to a mass-spring-damper model with a nonlinear spring. The equation of motion is:

\[ \ddot{x} + c\dot{x} + kx + ax^3 = A\cos \omega t \]  

(b.1)

where \( t \) is time in second, \( \omega \) is the forcing frequency rad/s, \( A \) is the forcing amplitude, \( c \) and \( k \) are the linear damping and stiffness terms, and \( a \) is the nonlinear stiffness term. In this setting, the restoring force is \( kx + ax^3 \), which is nonlinear when \( a \neq 0 \). There is no known analytical solution to equation (b.1), although various approximation methods can be found in the literature [63, 64].

In order to utilise the numerical continuation solver, which only works on first-order autonomous ODEs, we rewrite equation (b.1) using the following coordinate transformation:

\[
\begin{align*}
x_1 &= x \\
x_2 &= \dot{x} \\
x_3 &= \sin \omega t \\
x_4 &= \cos \omega t
\end{align*}
\]  

(b.2)

Giving us

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -kx_1 - ax_1^3 - cx_2 + Ax_4 \\
\dot{x}_3 &= x_3 + \omega x_4 - x_3(x_3^2 + x_4^2) \\
\dot{x}_4 &= -\omega x_3 + x_4 - x_4(x_3^2 + x_4^2)
\end{align*}
\]  

(b.3)

The third and fourth equations in (b.3) have an asymptotically stable pair of solutions \( x_3 = \sin \omega t \) and \( x_4 = \cos \omega t \) (see appendix A for proof). Therefore, the last term in \( \dot{x}_2 \) is actually \( Ax_4 = A\cos \omega t \), which is in the required autonomous form. States 3 and 4 are essentially ‘dummy states’ used to generate the harmonic forcing. This fourth-order system can now be used in the continuation solver for bifurcation analysis. By setting the forcing frequency \( \omega \) as the continuation parameter, the nonlinear frequency response can be generated.

Firstly, the effect of the nonlinear term is examined. If \( a = 0 \), equation (b.1) becomes a linear system with constant stiffness, while \( a > 0 \) results in a hardening spring and \( a < 0 \) results in a softening spring. Larger magnitude of \( a \) leads to a more nonlinear system. Consider a specific case:

\[ \ddot{x} + 0.2\dot{x} + x + ax^3 = 2.5 \cos \omega t \]  

(b.4)
Fig. II-5 shows the frequency responses of equation (b.4) for four different values of $\alpha$. When $\alpha$ is non-zero, the resonance bends either to the left (softening spring) or right (hardening spring). The nonlinear term therefore causes a shift the resonance frequency.

**Fig. II-5** Frequency responses of equation (b.4) for four different levels of nonlinearity

Fig. II-6 shows the magnified frequency response of $\alpha = 0.05$. Due to the effect of the nonlinear stiffening spring ($\alpha > 0$), the frequency response leans to the right side, which leads to two fold bifurcations and produces a region near the resonance with three solutions (two stable and one unstable). When forced at one of these frequencies, the system will converge to one of the two stable solutions depending on the initial conditions. If we run a frequency sweep, the response will follow the arrows shown in Fig. II-6a for increasing $\omega$ and Fig. II-6b for decreasing $\omega$. 

**Fig. II-6** Magnified view of the $\alpha = 0.05$ response. Arrows show the frequency sweep trajectory when $\omega$ is increasing (a) and decreasing (b)
The continuation solver only generates a one-cycle time history for each state at each value of the continuation parameter (\(\omega\) in this case). To construct the phase plot, it is necessary to process this data. Fig. II-7 shows the solutions to equation (b.4) at two values of \(\omega\) obtained from the output of the continuation solver. The gain in dB (for generation of a Bode plot in later parts) and phase relationships are devised as follows:

\[
\text{gain in dB} = 20 \log_{10} \left( \frac{Y_3 - Y_4}{Y_1 - Y_2} \right) \\
\text{phase in degree} = (X_1 - X_3) \times 360
\]

(b.5)

where \(X_i\) and \(Y_i\) refer to the x-axis and y-axis coordinates of point \(i\) in Fig. II-7. Note that \(Y_1 - Y_2 = 2A\).

This method of finding the phase is only valid when the response is simple harmonic. If additional frequencies are present like in Fig. II-7b, the horizontal distance between points 1 and 3 no longer gives the correct mathematical phase relationship between the input and the output. This way of defining phase, however, makes it very convenient to detect when nonlinear dynamics is present. Fig. II-7b is an example of subharmonic resonance, which is clearly visible at low frequencies in the gain and phase diagrams of Fig. II-6 using the current definition. This matches our main goal of determining regions where the nonlinear response differs from its linearised counterpart; the latter only produces simple harmonic sinusoidal outputs by definition. These additional harmonics are usually omitted in approximate analytical solutions of the Duffing equation, such as in [63], which further underlines the advantages of numerical methods.

The phase jump observed in Fig. II-6 is also directly linked to our definition of phase. This can be explained by simulating the system with a chirp signal that reduces \(\omega\) linearly at a rate of \(3.5 \times 10^{-4}\) rad/s\(^2\). The response shown in Fig. II-8 concentrates on when \(\omega\) crosses the subharmonic resonance region. Pass the \(t = 9540\)s mark, the highest point in the oscillation changes from the left peak to the right peak as \(\omega\) reduces, and this creates the apparent phase jump observed in the nonlinear Bode plot.

---

\(\text{Fig. II-7} \) Solutions to equation (b.4) at \(\omega = 2.509\) rad/s (a) and \(\omega = 0.298\) rad/s (b) obtained using numerical continuation. Points 1 and 3 are the peaks and points 2 and 4 are the troughs.
Another notable feature in a nonlinear harmonically-forced system is its amplitude dependency. As the nonlinear restoring force $\alpha x^3$ is proportional to $x^3$, increasing the forcing amplitude $A$ leads to larger $x^3$ and makes the system more nonlinear. Fig. II-9 shows the frequency responses for three different values of $A$. Since larger $A$ causes $x$ to increase, this leads to a more nonlinear system via cubic term $\alpha x^3$. It can be seen that nonlinearities increases with $A$, and the region with multiple solutions also widens as a result. Using a technique called two-parameter continuation, the locus of the fold bifurcations, which are the points where the response curve folds over and leads to the existence of multiple solutions, can be computed. This locus is shown as a thin line in Fig. II-9, and its projection onto the $\omega$-$A$ plane is shown in Fig. II-10. It can be seen that if the forcing amplitude is small enough, the fold bifurcations disappear and only one stable solution exists for each forcing frequency – similar to a linear system. Two-parameter continuation is a powerful technique that can be used to determine a nonlinear region’s sensitivity to multiple system’s parameter. The code to generate all analysis up to Fig. II-10 can be found in appendix 0.

Fig. II-9 Impact of increasing the forcing amplitude on the frequency response

Fig. II-10 Two-parameter continuation of the fold bifurcations in the $\omega$-$A$ plane

More complex behaviours can be detected using nonlinear frequency analysis. Consider another case of the Duffing equation:

$$\ddot{x} + 0.3\dot{x} - x + x^3 = A \cos 1.2t$$  \hspace{1cm} (b.6)
It is possible to set $A$ as the continuation parameter to directly assess the amplitude dependency. The bifurcation diagram of equation (b.6) is shown in Fig. II-11a. As the forcing amplitude increases beyond 0.266, a period-doubling bifurcation is detected, which changes the response from period-1 (Fig. II-11b) to period-2 (Fig. II-11c). This happens again as the forcing amplitude exceeds 0.287, and the motion becomes period-4 (Fig. II-11d). The ensuing period-doubling cascade results in chaotic motion, one of which is shown in Fig. II-12 at $A = 0.5$.

![Bifurcation diagram of equation (b.6) (a).](image-a)

**Fig. II-11** Bifurcation diagram of equation (b.6) (a).

Time histories for $A = 0.26$ (b), 0.28 (c) and 0.29 (d)

![Time histories and phase plot of equation (b.6) at $A = 0.5$.](image-b)

**Fig. II-12** Time histories (a) and phase plot (b) of equation (b.6) at $A = 0.5$, showing chaotic motion

A wide range of nonlinear dynamics have been uncovered by harmonically-forced bifurcation analysis. On the other hand, it is known that these behaviours do not exist in a linear system. The reason can be seen by examining the process to obtain the linear frequency response. Consider the linear ($\alpha = 0$) version of equation (b.4) with a general forcing amplitude $A$:

$$\ddot{x} + 0.2\dot{x} + x = A \cos \omega t$$  \hspace{1cm} (b.7)

Unlike in the case of the nonlinear Duffing equation, closed-form solution of equation (b.7) exists. Its frequency response is found by analysing the s-domain transfer function $G(s)$ of equation (b.7), which is:

$$G(s) = \frac{A}{s^2 + 0.2s + 1}$$  \hspace{1cm} (b.8)
Evaluate $G(s)$ at $s = j\omega$, we have:

$$G(j\omega) = \frac{A}{(1 - \omega^2) + 0.2\omega i} \quad (b.10)$$

The oscillation amplitude is the magnitude of $G(j\omega)$:

$$|G(j\omega)| = A \times \frac{\omega^4 - 1.96\omega^2 + 1}{\omega^8 - 3.92\omega^6 + 5.8416\omega^4 - 3.92\omega^2 + 1} \quad (b.10)$$

And the phase response is the angle of $G(j\omega)$:

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{0.2\omega}{1 - \omega^2}\right) \quad for \ \omega \leq 1$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{0.2\omega}{1 - \omega^2}\right) - \pi \quad for \ \omega > 1 \quad (b.11)$$

**Fig. II-13** Frequency response of equation (b.7) for $A = 2.5$ obtained using analytical method (a) and numerical continuation (b)

Fig. II-13a shows the analytical frequency response of equation (b.7) for $A = 2.5$. The same diagram obtained using numerical continuation is also shown in Fig. II-13 for reference. From equation (b.10), it is clear that increasing the forcing amplitude causes no change in the shape of the oscillation amplitude curve. This is contrary to the nonlinear Duffing system, which shows a strong amplitude dependency. The form of equation b.10 also does not permit the shape of folded solutions seen in the Duffing equation (leading to multiple possible responses), or the existence of additional peaks due to subharmonic and superharmonic resonances. Furthermore, equation (b.11) shows that the linear phase response is not a function of $A$, so the phase
relationship is unaffected by the forcing amplitude. Many important real-world limitations like actuator rate limiting can considerably affect the phase response as the forcing amplitude increases.

The basic features of a nonlinear harmonically-forced system have been presented using a simple example. Nonlinear frequency analysis can be used on closed-loop systems. This is discussed in the next section.
II-3. **ANALYSIS OF THE CLOSED-LOOP DUFFING SYSTEM**

This section presents the first example of how nonlinear frequency analysis can be used to assess the closed-loop performance. The analysis method proposed here exploits the concept that the more similar the linear and nonlinear frequency responses are, then the more similar their time-domain responses will be. Using this relationship, it is possible to draw conclusions about the time-domain responses based on a frequency-based analysis.

Equation (b.12) describes the open-loop system studied:

\[
\ddot{x} + 0.3\dot{x} - 0.5x + x^3 = 0.5 \cos \omega t
\]  
(b.12)

which is the Duffing system mentioned previously with parameters \(m = 1, c = 0.3, k = -0.5, \alpha = 1, \) and \(A = 0.5\). The parameters are chosen so that chaotic motion can be observed (Fig. II-14) as well as to somewhat resemble a real-world system that is unstable at the operating point (the origin) but never diverges to infinity due to some physical constraints. In this example, the nonlinear term \(x^3\) acts as that physical constraint as it provides the restoring force when the displacement is large enough but is mostly negligible when \(x\) is small.

![Fig. II-14 Time simulation (a) and phase plot (b) of equation (b.12) at \(\omega = 1.7\) rad/s, showing chaotic motion](image)

The system is now augmented with a standard position-demand PID controller using the scheme shown in Fig. II-15, where the reference signal \(r\) is the demanded position. The equation of motion is now:

\[
\ddot{x} + 0.3\dot{x} - 0.5x + x^3 = \left(K_p + K_I \int dt + K_D \frac{d}{dt}\right)(r - x)
\]  
(b.13)

![Fig. II-15 Block diagram of the closed-loop position-demand Duffing system](image)
The linearised equation of motion about the origin simply omits the nonlinear term:

\[ \ddot{x} + 0.3 \dot{x} - 0.5x = \left( K_P + K_I \int dt + K_D \frac{d}{dt} \right) (r - x) \]  

(b.14)

For nonlinear frequency response analysis, the reference signal is set to \( r = A \cos \omega t \). Using the following coordinate transformation:

\[
\begin{align*}
    x_1 &= x \\
    x_2 &= \dot{x} \\
    x_3 &= \ddot{x} \\
    x_4 &= \sin \omega t \\
    x_5 &= \cos \omega t
\end{align*}
\]  

(b.15)

equation (b.14) can now be expanded and rewritten into the following autonomous fifth-order system required for numerical continuation:

\[
\begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= x_3 \\
    \dot{x}_3 &= -K_I x_1 - (k + K_P + 3 \alpha x_4^2) x_2 - (c + K_D) x_3 - K_P A \omega x_4 + (K_I - K_D \omega^2) A x_5 \\
    \dot{x}_4 &= x_4 + \omega x_5 - x_4 (x_4^2 + x_5^2) \\
    \dot{x}_5 &= -\omega x_4 + x_5 - x_5 (x_4^2 + x_5^2)
\end{align*}
\]  

(b.16)

where \([c, k, \alpha, A] = [0.3, -0.5, 1, 0.5]\) as stated above.

The effect of the controller gains \([K_P, K_I, K_D]\) on the \( r \)-to-\( x \) frequency response and its relationship to the time-domain response are now discussed. Firstly, a bifurcation-free frequency response can be indicative of a less-nonlinear system. It was found that the bifurcation-free region in the \([K_P, K_I, K_D]\) space is bounded by a surface shown in Fig. II-16 in the form of contour plot in the \( K_I, K_P \) plane. The smallest \( K_D \) shown in Fig. II-16 is 0.145 and the largest value is 0.5. Further analysis shows that the bifurcation-free surface depends on maintaining an appropriate ratio between \( K_P, K_I, K_D \), but this is beyond the scope of the thesis.

Although the range of controller gains that guarantees no bifurcation in the frequency response has been determined, this does not guarantee that the time-domain response is similar to that of the linear system. Fig. II-17 shows the frequency responses and the step responses for four different combinations of controller gains, all of which lie inside the bifurcation-free surface shown in Fig. II-16. In all cases, the nonlinear frequency response has a lower peak and a higher resonant frequency, which is reflected in the step response as lower overshoot and shorter period, although this is much more prominent in Fig. II-17c and Fig. II-17d.
II-3. Analysis of the closed-loop Duffing system

**Fig. II-16** Bifurcation-free surface of the closed-loop Duffing system in the $K_p$-$K_I$-$K_D$ space

**Fig. II-17** Frequency and step responses (amplitude 0.5) of the closed-loop Duffing system at four different combinations of PID gains. Dotted lines show the dynamics of the plant linearised about the origin $[x, \dot{x}] = [0, 0]$. The gains used are also highlighted in Fig. II-18c

It can be seen that larger differences between the frequency responses lead to larger discrepancies in the step responses. Particularly, the following information can be inferred from the frequency responses:

- The damping ratio $\zeta$ is related to the ‘width’ of the resonant peak, which can be estimated from the frequency response using the half-power method.
- The overshoot is related to the (absolute) gain at resonance $G_R$.
- The damped frequency is related to the resonance frequency $\Omega$.

Using those three parameters, the author proposes a way to quantify the differences between the linear and nonlinear frequency responses so as to predict the extent of discrepancy to expect in the step responses. This is done using the metric called $E_3$ described by equation (b.17), which measures the percentage difference of the three parameters above:

$$E_3 = \frac{1}{3} \left[ \frac{|C_{\text{linear}} - C_{\text{nonlinear}}|}{C_{\text{nonlinear}}} + \frac{|G_{R,\text{linear}} - G_{R,\text{nonlinear}}|}{G_{R,\text{nonlinear}}} + \frac{|\Omega_{\text{linear}} - \Omega_{\text{nonlinear}}|}{\Omega_{\text{nonlinear}}} \right] \times 100 \quad (b.17)$$

Essentially, $E_3$ gives the average percentage error in the estimated damping ratio, overshoot, and resonance frequencies. The three terms have equal weighting in this example. However, it is entirely up to the designer to decide on the number of terms as well as the appropriate weighting for the system considered. Fig. II-18 shows the value of $E_3$ for a range of [$K_P$, $K_I$, $K_D$] combinations, all of which lie within the bifurcation-free region as shown in Fig. II-16.

By checking the time-domain responses to a step input of amplitude 0.5, it was found that for this system, $E_3 < 12.5$ provides a good match between the linear and nonlinear step responses. Two examples when this condition is satisfied have been shown in Fig. II-17a and Fig. II-17b; the condition is not satisfied in Fig. II-17c and Fig. II-17d. The gains used in Fig. II-17 are also marked as small circles in Fig. II-18c. We can see that this system is sensitive to the derivative gain, as increasing $K_D$ slightly will rapidly expand the $K_I$-
II-3. Analysis of the closed-loop Duffing system

$K_p$ envelope, and that when $K_D$ is not high enough (as seen in Fig. II-18a and Fig. II-18b), there is no combination of $K_I$ and $K_p$ that satisfies our criterion of $E_3 < 12.5$.

We have seen that the differences between the linear and nonlinear frequency responses are reflected in the time-domain step responses. This is an important relationship that will be exploited in the upcoming chapters to assess the aircraft’s transient responses due to variation in the system’s parameters. The same concept can also be applied to variation in the forcing amplitudes $A$. In this instance, the controller gains can be set as functions of $A$ to accommodate more nonlinearities, leading to changes in the system dynamics due to larger $x$ (as a result of larger $A$). This is effectively an input-scheduled control system.
II-4. EFFECT OF SATURATION: OPEN-LOOP

We will now explore a different type of nonlinearity that can be studied using nonlinear frequency analysis: rate and travel saturations. Their impact on real-world system can be severe as discussed in chapter IV. A simple system shown in Fig. II-19 is used. The input signal is the demanded force in N, which goes through a first-order motor (the actuator). This actuator generates a force $f$ in N that drives the mass-spring-damper system to a distance $x$ in m away from the origin. The block diagram representations of the ideal and non-ideal actuator are shown in Fig. II-20, which is developed from the scheme presented in [91]. Specifically, rate saturation is achieved by limiting the magnitude of the signal just before the integrator, while output saturation is done by limiting the demanded signal that goes into the actuator. This setup effectively models hardware rate limiting and software travel limiting, which is the most numerically convenient setup to implement in AUTO. Specifically, we do not have to take any numerical derivative to model rate limiting, and the software travel limit will prevent the actuator from hitting its maximum position at speed, resulting in a sudden drop of the actuator state derivative to zero that could fail the continuation solver. The downside of software travel limit is that the actuator will asymptote to its final position. This issue can be solved by implementing a continuation algorithm with switching logic [92], but this beyond the scope of the thesis.

![Fig. II-19 Simple second-order system with a first-order actuator](image)

![Fig. II-20 Ideal actuator (a) and rate-and-output-limited actuator (b)](image)

To begin, let us first examine the dynamics of just the actuator rate limited to 1 N/s. Its frequency response in Fig. II-21 shows a strong amplitude dependency that highlights the nonlinear nature of the actuator. Intuitively, increasing the forcing amplitude leads to a lower roll-off frequency, which is marked by a reduction in both gain and phase since the rate-limited output cannot follow the demanded signal.

Fig. II-22a shows the linear and nonlinear frequency responses of the mass-spring-damper with rate-limited actuator system under a forcing amplitude of 2.5 N. A notable reduction in gain and a small phase lag is observed beyond the roll-off frequency at 0.5 rad/s. Time simulation at 1.25 rad/s (Fig. II-22b) confirms that the actuator is rate-saturated.
II-4. Effect of saturation: open-loop

Fig. II-21  Frequency response of the rate-limited actuator

Fig. II-22  Frequency (a) and simulated responses (b) at $\omega = 1.25$ rad/s – ideal vs rate limited

The effect of only a 1.7 N output limit in the actuator (with no rate limit) is now considered. Interestingly, the frequency response in Fig. II-23a suggests the existence of subharmonic-like resonance at low frequencies, which is accompanied by a small phase lead. Time simulations at 0.34 rad/s in Fig. II-23b shows that the phase lead is indeed due to the output limit of the actuator, which causes the mass to overshoot the steady-state value at the maximum output. The ensuing motion resembles a subharmonic resonance that is captured in the nonlinear frequency response.
It is possible to include both rate and output limits in the analysis using numerical continuation. Fig. II-24 shows the frequency response of the system with both rate and output limiting active. The effects of both constraints can be seen: a reduction in gain across the whole frequency range and additional phase lead/lag at low/high frequencies.

Although semi-analytical methods for studying rate limiting exist, the most common one being the describing function technique, most can only handle one nonlinear element. This means that in order to examine the impact of rate limiting, for example, the plant has to be linear. Numerical continuation is not subjected to this limitation. To demonstrate, replace the second-order linear plant above with the Duffing equation:

$$\ddot{x} + 0.2\dot{x} + x + 0.1x^3 = A \cos \omega t \quad (b.18)$$

Fig. II-25 compares the frequency response of the ensuing Duffing + ideal/nonlinear actuator system for $A = 2.5$ N. The impact of all three nonlinear elements can be seen in the frequency response: the nonlinear stiffness
term causes a leaning resonance curve, and rate and output saturations leads to an overall reduction in both gain and phase.

Fig. II-25  Frequency of equation (b.18) – ideal vs rate-and-travel-limited actuator. \( A = 2.5 \, N \)

Two-parameter continuation of the fold bifurcations in the \( \omega \)-\( A \) plane also reflects the impact of the two saturation elements. Fig. II-26 shows that:

- For \( A \leq 1.7 \, N \), rate limiting reduces the distance between the two fold bifurcations. i.e., the nonlinear resonance region becomes smaller comparing to the ideal case.

- Beyond 1.7 N, the impact of output limiting becomes significant and prevents a further widening of the multiple solution region.

Fig. II-26  Two-parameter continuation of the fold bifurcations

The important implication here is that rate and/or output saturation can sometimes ‘mask’ the impact of the nonlinear term in the plant. Assuming that nonlinearities are undesirable, then an improved actuator in this
example will actually result in a more nonlinear response. Since existing analytical techniques can only handle one nonlinear element at once, our proposed method can be a powerful alternative to help examine the impact of actuator saturations and provides a complete picture of the dynamics.
II-5. EFFECT OF SATURATION: CLOSED-LOOP

To conclude chapter II – the methodology chapter – it will be shown that rate saturation can lead to jump resonance in very simple feedback systems. This behaviour has been noted in real-world actuators with rate saturation and requires further analytical studies to explain the behaviours observed below.

Consider the second-order feedback system with rate limiting in the error signal as shown in Fig. II-27. Given a simple harmonic reference signal of the form \( r = A \sin \omega t \), the dynamics can be described by the following autonomous piecewise scheme:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \begin{cases} 
-kx_1 - cx_2 + S \times \text{sgn}(A\omega x_4 - x_1) & \text{if } |A\omega x_4 - x_1| > S \\
-(k + 1)x_1 - cx_2 + A\omega x_4 & \text{otherwise}
\end{cases} \\
\dot{x}_3 &= x_3 + \omega x_4 - x_3(x_3^2 + x_4^2) \\
\dot{x}_4 &= -\omega x_3 + x_4 - x_3(x_3^2 + x_4^2)
\end{align*}
\]

where \( x_1 = \dot{x} \) (not the usual \( x \)), \( x_2 = \ddot{x} \), \( x_3 = \sin \omega t \), \( x_4 = \cos \omega t \), \( k = 1 \), and \( c = 0.15 \). \( S \) is the maximum rate permitted in the error signal \( r - x \). It should be noted that if the exact scheme shown in Fig. II-27 is implemented in the Simulink environment using the built-in rate limiter block, the response might be inaccurate at heavy saturation level due to numerical issues involved in approximating the derivative of \( (r - x) \) (hence the use of the complicated piecewise system in equation (b.19)).

(continued on the next page)
For $A = 1$, the closed-loop frequency response generated using numerical continuation is shown in Fig. II-28 for a range of rate saturation levels. Fold bifurcations are detected in $S = 3$ and 1, leading to jump resonances as well as regions of two stable solutions that can be verified in time simulation shown in Fig. II-29. This phase jump behaviour has been noted in engineering examples through measurements and the describing function techniques, although the existence of multiple solutions was not revealed [93].

**Fig. II-28** $\dot{x}$-to-$r$ frequency responses for a range of rate limiting levels. At $S = 7$, rate saturation is not encountered, leading to a linear-like frequency response

**Fig. II-29** Time simulations at $[S, \omega] = [3, 1.316]$ with different initial conditions, leading to two different responses

A first glance at Fig. II-28 may suggest that the rate-limited system exhibits behaviour of a Duffing oscillator. This is not the case, however. Referring back to Fig. II-5, increasing nonlinearity via the $a$ term in the Duffing system will reduce the oscillation amplitude and cause the resonance curve to lean to the right, and vice versa. Neither behaviour is observed in our current rate limited example. Fig. II-28 shows that as $S$ gets smaller (i.e., increasing the nonlinear level) the resonance decreases in amplitude while also leaning further to the left (instead of to the right as seen in the Duffing oscillator). This suggests that rate limiting phenomena is not a Duffing-type system.

Now consider a slightly different example where rate saturation is inherent in the second-order plant instead of in the error signal as shown above. Fig. II-30a shows the block diagram of a second-order actuator model
from [93] with a 1200 deg/s² acceleration limit and no rate saturation. \( \eta \) is the output deflection in degree, and \( \eta_\text{dem} \) is the demanded deflection. Due to the 21.4 gain in the outer loop, the signal limit block in the inner loop has been scaled to represent the 1200 deg/s² acceleration limit. Fig. II-30b shows the closed-loop frequency responses for a range of forcing amplitudes \( A \) in degrees. Jump resonance appears at \( A = 5 \) and 7°. For low forcing amplitudes like 1°, the actuator is not acceleration-saturated in the frequency range considered, resulting in a linear-like frequency response with a natural frequency of 30 rad/s and damping ratio of 0.7. The jump resonance leads to a region of two stable solutions, which can be verified in time simulation (Fig. II-31). Increasing or reducing the forcing amplitude beyond the bistable region will lead to a cliff-like jump in phase lag.

![Block diagram and closed-loop frequency responses](image)

**Fig. II-30** Block diagram (a) and closed-loop \( \eta \)-to-\( \eta_\text{dem} \) frequency responses (b) of a second-order actuator model with 1200 deg/s² acceleration limit

![Simulated responses](image)

**Fig. II-31** Simulated responses under the same forcing input \([A, \omega] = [7, 13]\) but different initial conditions of \( \eta \)
The dangerous cliff-like jump in phase lag has been observed in both examples, which is solely caused by the rate limiting in the feedback loop – the only nonlinear element in an otherwise simple linear scheme. When a rate-limited actuator is placed inside another feedback loop as part of a control system, potentially with added time delay, jump behaviour can become more severe as will be shown in section IV-3. Numerical continuation can be employed in these instances to examine the system’s closed-loop frequency response.

To conclude, the nonlinear frequency response method has been presented. A range of interesting dynamics have been observed in weakly nonlinear systems – none of which cannot be reflected using equilibrium (unforced) bifurcation analysis. The basics of nonlinear frequency analysis have also been demonstrated. The rest of the thesis will apply this technique on a number of aircraft models with different types of nonlinearities.
II-6. NUMERICAL CONSIDERATIONS

This section will discuss some of the numerical considerations and limitations of bifurcation analysis from a ‘user-guide’ perspective. A detailed mathematical analysis of the topic is beyond the scope of this thesis.

II-6.1. Smoothness requirement

Numerical continuation requires the system to be sufficiently smooth and differentiable to the at least first order. On the other hand, most flight dynamics models in use store their aerodynamic data tabular form. The industrial practice to generate the time simulation of these models is to use linear interpolation, which can be insufficiently smooth for numerical continuation in certain cases. One of the common workarounds is to use spline interpolation, but this can cause overfitting. The ‘pchip’ interpolation method in MATLAB can prevent the overfitting issue, but this only works on 1D tables. To illustrate these two methods, Fig. II-32 and Fig. II-33 compare the result of spline and pchip interpolations of the GTT’s aerodynamic data – a model that will be studied in chapters V-4 and VI. It can be seen that pchip preserves the shape of the tabular data, whereas spline causes some notably unrealistic sections at high $\alpha$ in the $C_z$ elevator table. This is a trade-off that must be considered when implementing bifurcation analysis on models built upon tabular data.

![Fig. II-32 pchip interpolation and extrapolation of the GTT's 1D aerodynamic tables](image)

![Fig. II-33 Spline interpolation and extrapolation of the GTT's 2D aerodynamic tables. The second input parameter is $\delta_e$ between $-20^\circ$ and $20^\circ$](image)
The rest of this section will examine the impact of interpolating tabular data on the stability and control characteristics of the GTT for completeness. Fig. II-34 compares the unforced bifurcation diagrams of the original (linearly-interpolated) GTT and the smoothed one; the latter that uses a combination of spline and pchip interpolation as presented above. In both instances, the stabilator deflection is fixed at $-10^\circ$ deg (full nose up) and the centre of gravity is at 38% MAC. It can be seen that the differences between the original and the smoothed model are minimal. This is further corroborated by examining their step responses in Fig. II-35, again showing little differences.

![Fig. II-34 Comparison of the unforced bifurcation diagrams using different interpolation methods.](image)

**Stability information is not shown for clarity**

One area where linearly-interpolated data can be an issue is tracking the movements of the eigenvalues. This is shown Fig. II-36, where there is significant discontinuity in the original model that makes it difficult to
track the pole movements. The smoothed GTT, on the other hand, does not have this problem. Analyses that rely on examining the pole movements should therefore consider using smoothed aerodynamic data.

**Fig. II-36** Eigenvalues of equilibrium solutions in Fig. II-34 from 30° to 53° angle-of-attack

It can be concluded here that the use of spline and pchip interpolation is acceptable for the GTT. On the other hand, the linearly-interpolated data is still sufficiently smooth for the continuation algorithm to generate the full bifurcation diagram. Therefore, the rest of this thesis uses linearly-interpolated data for all GTT analysis.

**Fig. II-37** Adaptive mesh of all periodic solutions shown in Fig. II-6 on page 45 (a) and Fig. V-15 on page 128 (b)

### II-6.2. Adaptive mesh of periodic solutions

As discussed in section II-2 (particularly the part concerning gain and phase calculations shown in Fig. II-7 on page 46), our method of constructing the nonlinear Bode plot relies on AUTO providing an accurate one-cycle periodic solution at each step (i.e., each value of \(\omega\) or another continuation parameter of choice). The number of data points in each of these periodic solutions is determined by the mesh size – controlled by the two AUTO constants ‘Ntst’ and ‘Ncol’. This gives each periodic solution a resolution of Ntst x Ncol + 1 data points along the time axis; the default value in AUTO is 50 x 4 + 1 = 201. Generally speaking, higher mesh density gives more accurate solutions but reduces robustness and is more computationally expensive. It should be noted that this mesh is not uniformly spaced, but is instead automatically adapted by AUTO to maintain convergence. This is illustrated in Fig. II-37, which compares the mesh distribution in of the Duffing...
equation in the first plot (representing a weakly nonlinear system) and the GTT (representing a highly nonlinear system). If the mesh is perfectly uniform, the plot should show a single straight line with a slope of $1/201$. However, we can see that more mesh adaptation is required for more nonlinear systems. This may also sometimes distort the periodic solutions comparing to what is observed in time simulation. However, the author has empirically verified that the amplitude and positions of the peaks and troughs are conserved, which is sufficient for constructing the nonlinear Bode plot.
III. REDUCED-ORDER CLOSED-LOOP ANALYSIS

In this first result chapter, nonlinear frequency analysis will be used to assess the performance of a nonlinear second-order aircraft model coupled with a gain-scheduled manoeuvre demand controller, specifically to evaluate the negative impact of aerodynamic nonlinearities on the controller performance at high angles-of-attack. It will be shown that even when equilibrium bifurcation analysis indicates no potential issues, the underlying nonlinearities can still manifest themselves in the closed-loop frequency response, and that this information can be utilised to provide an indication of when then linear controller may fail to perform as expected. The objective is to inform the control designer of regions in which linear-based design is insufficient.

Each example in this chapter is reduced to second-order to include only the fast (short-period) dynamics, which is considered adequate for initial flight control design. The method is first demonstrated on a rigid aircraft model in sections III-1-III-5. Then, the final section III-6 expands the approach to include the impact of structural coupling on the closed-loop dynamics of a flexible aircraft. The result from section III-6 is not comprehensive but does further demonstrate the capability of nonlinear frequency analysis on another promising application.
### III-1. DESCRIPTION OF THE HHIRM

Nonlinear frequency analysis is demonstrated on the Hypothetical High Angle of Incidence Research Model (HHIRM), which was originally created for nonlinear flight dynamics studies in the Defence Research Agency in the UK (now QinetiQ) [94]. The model is made up of six aerodynamic force and moment coefficients represented as nonlinear spline functions of the angle of attack, sideslip angle, angular rates, and control surface deflections. Its dynamics are representative of a typical fighter aircraft. The use of spline functions rather than tabular data ensures that the system is smooth (differentiable), making the model highly suitable to be used as a testbed for bifurcation-based methods. Further description of the force and moment coefficients can be found in [42, 94]. In this thesis, the longitudinal 2nd-order version is used (the same approach adopted in [6]), which contains two states $\alpha$ (angle-of-attack in degrees) and $q$ (pitch rate in degree/s) to capture the short-period mode. The only control input is the elevator deflection $\eta$ in degrees. Using the reduced-order model restricts the case study to longitudinal dynamics, where only the fast mode is important, and allows for easier interpretation of the results. However, the approach developed here is not limited to such low-order systems.

The equations of motions for the open-loop aircraft are:

$$\ddot{\alpha} = \frac{\rho V S}{2m} \left[ C_Z(\alpha) + \eta C_{Z\eta}(\alpha) + \frac{q c}{2V} C_{Zq}(\alpha) \right] \cos(\alpha) + q$$

$$\ddot{q} = \frac{\rho V^2 S c}{2I_y} \left[ C_M(\alpha) + \eta C_{M\eta}(\alpha, \eta) + \frac{q c}{2V} C_{Mq}(\alpha) \right]$$

(c.1)

Table 2 lists the values of the physical parameters in equation (c.1), and Fig. III-1 shows the nonlinear aerodynamic coefficients.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>0.7358 kg/m$^3$</td>
</tr>
<tr>
<td>$V$</td>
<td>150 m/s</td>
</tr>
<tr>
<td>$S$</td>
<td>37.16 m$^2$</td>
</tr>
<tr>
<td>$c$</td>
<td>3.511 m</td>
</tr>
<tr>
<td>$m$</td>
<td>15,000 kg</td>
</tr>
<tr>
<td>$I_y$</td>
<td>163,280 kg m$^2$</td>
</tr>
</tbody>
</table>
The open-loop dynamics of the HHIRM has been studied using equilibrium bifurcation analysis [6] and is briefly reproduced here for completeness. Fig. III-2 shows the open-loop bifurcation diagrams of the two states $\alpha$ and $q$ with the elevator deflection $\eta$ on the x-axis as the continuation parameter. It can be seen that due to the fold bifurcations, the flight dynamics can be divided into three separate regions as illustrated in Fig. III-2a. From a practical perspective, this means that:

- For $-20^\circ \leq \eta \leq -10^\circ$, the aircraft has two stable equilibrium states: one at a lower and one at a higher (deep stall) angle of attack. Whichever solution the aircraft converges to will depend on the initial conditions in a time history simulation or on the magnitude of disturbances.

- It is not possible to manually trim the aircraft at angles-of-attack between $34^\circ$ and $46^\circ$ because the solutions in that range are unstable. Note that this corresponds to the static instability reflected in the region of positive pitching moment slope of $C_M(\alpha)$ in Fig. III-1.

Fig. III-1 HHIRM’s aerodynamic coefficients and derivatives

Fig. III-2 Open-loop bifurcation diagram – elevator continuation
III-2. ISSUES WITH INPUT GAIN SCHEDULING

III-2.1. Description of the command-augmentation controller

In the analysis to follow, the closed-loop HHIRM uses the longitudinal version of the gain-scheduled command-augmentation controller described in [45]. This is essentially a state-feedback controller with an integral term in the forward path to create a manoeuvre-demand system while also accounting for the dynamics of a first-order actuator. The block diagram and gain scheduling are shown in Fig. III-3. All three controller gains are scheduled against the pilot’s input $\alpha_d$ (demanded angle-of-attack), hence the term input gain scheduling. The objective of the controller is to place the short-period and integrator poles at fixed locations throughout the entire operating envelope to achieve consistent level 1 handling qualities. Fig. III-4a shows the pole positions in the complex plane at each operating point from $0^\circ$ to $60^\circ$ angle-of-attack at $1^\circ$ interval. These linear poles were obtained by linearising the open-loop aircraft at 61 different values of $\eta$ that give $\alpha = [0^\circ, 1^\circ, 2^\circ, …, 60^\circ]$, then adding the actuator and controller with the gains as shown in Fig. III-3b. It can be seen that the design objective of fixing the pole positions is achieved. Further verifications of linear-based analysis are shown in the linear step and frequency responses (Fig. III-4b and 4c). Since the responses are almost identical in all cases, it can be said that the aircraft exhibits consistent handling qualities across the entire operating envelope (in terms of pole positions). This is a best-case scenario as far as linear design goes, especially when gain scheduling in most real aircraft is done at much larger intervals.

![Fig. III-3](image1.png)  
**Fig. III-3** Block diagram (a) and gain schedules (b) – state feedback controller with integral action

![Fig. III-4](image2.png)  
**Fig. III-4** Linear analysis: closed-loop poles (a), unit step responses (b), and closed-loop $\alpha$-to-$\alpha_d$ frequency responses (c) at every operating point
Finally, Fig. III-5 shows the bifurcation diagram solutions of the closed-loop aircraft as the pilot input $\alpha_d$ varies. The slope of 1 in the $\alpha$ bifurcation diagram shows that in addition to achieving zero steady-state error, the controller has also ensured that there is no other attractor near the intended operating range. Thus far, the system looks promising as not only does the aircraft exhibit the desired characteristic of a manoeuvre-demand system, but also that conventional bifurcation analysis indicates no potential issue arising from the presence of nonlinearities.

![Bifurcation Diagram](image)

*Fig. III-5 Closed-loop bifurcation diagrams of angle-of-attack (a) and pitch rate (b) against the pilot input*

### III-2.2. Nonlinear frequency analysis

Fig. III-6 shows the simulation of the nonlinear HHIRM responding to a range of step inputs (i.e., stepping from $\alpha_0$ to $\alpha_d$). The normalised $\alpha$ response is defined as $(\alpha - \alpha_0)/(\alpha_d - \alpha_0)$. This was done to scale the initial and final values to 0 and 1, making it easier to compare the linear and nonlinear responses. Cases A and B in Fig. III-6 are almost identical, despite the large input of 15° in case B, and their dynamics resemble the linear responses in Fig. III-4b. However, as the initial value of $\alpha$ increases to 21° (case C), the step response starts to differ from those seen previously as well as the linearised responses. In case D, a small 1° step from 30° angle-of-attack results in an initial undershoot before the elevator movement reverts back to the correct direction. This behaviour is not detected using linear analysis and at first glance may resemble non-minimum phase behaviour. However, the cause is attributed to the aerodynamic nonlinearities of the model. Fig. III-1 shows that the slopes of $C_Z(\alpha)$ and $C_M(\alpha)$ reverse sign at around 30 degrees angle-of-attack, which is combined with a gradual loss of elevator effectiveness above $\alpha = 30^\circ$ as seen in the plots for $C_{Z\eta}(\alpha)$ and $C_{M\eta}(\alpha, \eta)$. Additionally, larger step inputs that cross this region will also be affected by the issues related to scheduling of the gains against the reference signal. A step input causes an instantaneous change in the
gain values, and Fig. III-3b shows that beyond $30^\circ$ angle-of-attack, the gains vary considerably (especially for $K_\alpha$). On the other hand, the aircraft states do not change as rapidly so at the beginning of the manoeuvre, there is a mismatch between the values of the current states and the optimal gain values.

The underlying cause is the nonlinearities in the aerodynamic coefficients shown in Fig. III-1. Since these elements can only be observed during the transient phase, existing approaches involving linear time/frequency-domain analysis and equilibrium bifurcation analysis cannot reflect these problems. The use of a non-stationary and nonlinear method like harmonically-forced bifurcation analysis therefore promises to bring new insights on the problem. This is implemented by augmenting the pilot input with a harmonic forcing term so that the former takes the form:

$$\alpha_d = \alpha_0 + A \sin \omega t$$

where $A$ (deg) is the forcing amplitude and $\alpha_0$ (deg) is the demanded angle-of-attack at trim, i.e., the pilot input when there is no harmonic forcing. From here, the nonlinear closed-loop Bode diagram can be generated. Fig. III-7 shows the frequency responses at $\alpha_0 = 30^\circ$ and $A = 1^\circ$. These values are chosen to match the step response from $30^\circ$ to $31^\circ$ seen in Fig. III-6. Similar to the step responses, there is a notable discrepancy between the linear and nonlinear frequency responses. As the forcing frequency $f = \omega/2\pi$ (in Hz) increases, the nonlinear frequency response has higher gain but lower phase compared to its linearised counterpart. This suggests that neither an additional pole nor a zero in the linear transfer function could correct their differences. It can therefore be concluded that the dynamics in this region cannot be captured by a linear transfer function, which matches the highly non-standard step responses from $30^\circ$ to $31^\circ$.

The linear analysis in Figs. 4b and 4c has shown that if the frequency responses at different operating points are similar, then their step responses are also similar. Based on this assumption, we can make predictions on the aircraft dynamics when it transitions between different operating points in response to a large step input. This is done by generating the nonlinear frequency response for a range of $\alpha_0$ and $A$ and comparing their
bandwidths – defined here as the frequency at which the gain drops to $-3$ dB. The result is shown as a scatterplot in Fig. III-8, which has the triangular shape because the controller is only designed for $\alpha_d$ between $0^o$ and $60^o$ (as a reminder, $\alpha_d = \alpha_0 + A \sin(2\pi ft)$ for the forced response). The lower-left section of Fig. III-8 has similar colouration and therefore suggests similar dynamics. This is expected, as points A ($1^o$, $1^o$) and B ($15^o$, $15^o$) are related to the step responses from $1^o$ to $2^o$ and $15^o$ to $30^o$, which are indeed similar as shown in Fig. III-6. On the other hand, points C and D (also labelled accordingly in both Fig. III-8 and Fig. III-6) exhibit different dynamics, and this is reflected by their differences in the frequency responses comparing to points A and B. Although the aircraft can have different responses to a range of inputs, the triangular plot in Fig. III-8 gives us an indication of where the dynamics may be comparable.

![Fig. III-8 Closed-loop bandwidth variation – feedback with integral action controller](image)

In addition to the undershoot behaviour as discussed, there is an apparent separate nonlinear region around point E in Fig. III-8. Fig. III-9a shows that at point E ($48^o$, $1^o$), the nonlinear frequency response has a notably higher resonance gain than the linear prediction, suggesting that the nonlinear step response will have a much higher overshoot, a fact confirmed in Fig. III-9b. The higher resonance leads to an increase in bandwidth, which is reflected by the area around point E in Fig. III-8. Based on the shape of the nonlinear frequency response, it is inferred that the linearised response needs an additional zero in its transfer function in order to capture the dynamics correctly.

![Fig. III-9 Frequency response at $\alpha_0 = 48^o$ and $A = 1^o$ (a) and step response from $48^o$ to $49^o$ (b)](image)
A physical explanation for the undesirable dynamics observed in this section can be found by linking the triangular envelope with the open-loop $\alpha$ bifurcation diagram. It should be noted that the region surrounding points D and E in Fig. III-8 involves the aircraft traversing near or past the fold bifurcations at 34° and 46° in Fig. III-2a. Each time a fold bifurcation is crossed, the relationship between $\alpha$ and $\eta$ is reversed. The direct consequence is that in order to follow the sinusoidal input of $\alpha_d$, the elevator movement must change direction during the manoeuvre. This element is not accounted for in linear pole-placement design, which contributed to the undesirable responses observed. Fig. III-10 shows one example of a simple sinusoidal $\alpha_d$ input that results in very complex variations of both $\alpha$ and $\eta$. The nonlinear Bode plot reflects these complex dynamics in a way that is consistent with the observed step response, such as in Fig. III-9, where both the frequency- and time-domain analyses indicated an increase in overshoot of the short-period response.

![Nonlinear forced response (f = 0.05 Hz)](image)

It has been shown that nonlinear frequency response analysis can verify whether consistent handling qualities are achieved across different operating regions. The result has highlighted the issues associated with input gain scheduling, especially when strong nonlinearities are present. It is important to point out that the purpose of the triangular envelope is not to compare the closed-loop bandwidth in responses to different values of $\alpha_0$ and $A$, but to indicate that bandwidth might be used as a metric to quantify the differences in the frequency response at different operating points. For a more complex system with multiple peaks, the engineer might apply a different metric, such as resonance peak or estimated damping of each mode, depending on the application considered.

### III-2.3. Validating the gain margin predictions from linear analysis

To further highlight how conventional linear-based design methods can be deleteriously affected by the nonlinear phenomena discussed above, let us verify the gain margin predictions of the open-loop frequency response by comparing it with an equivalent analysis using unforced bifurcation analysis. In this case, the open-loop system is defined by removing the outer loop in Fig. III-3a while keeping the inner stability-
augmentation loops with $K_a(\alpha_d)$ and $K_q(\alpha_d)$ unchanged. The gain margins of the open-loop transfer functions at each operating point are then plotted as a solid thin line in Fig. III-11a. By definition, this is the maximum value of the proportional gain $\Lambda$ shown in Fig. III-12 before the closed-loop system becomes unstable. The same calculation can be done on the nonlinear system using continuation by doing equilibrium bifurcation analysis (without the harmonic oscillator) with $\Lambda$ as the continuation parameter. An example of $\Lambda$ continuation is shown in Fig. III-13 at 30° angle-of-attack. As $\Lambda$ increases, the system becomes unstable at $\Lambda = 4.48$ via a Hopf bifurcation. Beyond this value, any perturbation from the trim point will send the aircraft into an unstable (divergent) limit cycle. In this example, $\Lambda = 4.48$ is the ‘nonlinear gain margin’. $\Lambda$ is calculated at each operating point of the nonlinear system (which can be done manually or by using two-parameter continuation of the Hopf bifurcation – the latter is much more computationally efficient) and plotted in Fig. III-11a as a thick dashed line. In this instance, the gain margin predictions using continuation matches its linear counterpart, indicating that linear analysis is correct.

![Fig. III-11 Gain margin variation of the CAS controller (a) and the simple pilot-vehicle system (b)](image)

Fig. III-11 Gain margin variation of the CAS controller (a) and the simple pilot-vehicle system (b)

![Fig. III-12 Block diagrams for gain margin calculation using continuation: CAS controller (a) and simple pilot-vehicle system (b)](image)

Fig. III-12 Block diagrams for gain margin calculation using continuation: CAS controller (a) and simple pilot-vehicle system (b)
Although the linear gain margin obtained from the open-loop transfer function appears accurate, it has been shown in the previous section that there are cases in which the linear and nonlinear closed-loop frequency responses differ significantly. This suggests that the gain margin predictions from these closed-loop frequency responses can be incorrect. To assess the accuracy of these predictions, let us remove $\Lambda$ and add an additional outer loop to the controller along with a proportional gain $K_P$, resulting in the system shown in Fig. III-12b. This setup allows us to assess the impact of incorrect gain margin predictions if the linear frequency responses like the ones in Fig. III-7 and Fig. III-9 are used in stability and control analysis. From a physical perspective, the proportional gain $K_P$ can be considered a simple pilot model and the input signal will be the target angle-of-attack for the pilot to track.

The process is now repeated by trimming the aircraft, conducting unforced bifurcation analysis with $K_P$ as the continuation parameter, finding the locus of the Hopf bifurcation as $\alpha_0$ varies, then comparing the result with the linear analysis. Fig. III-11b shows that the linear and nonlinear gain margins now differ significantly beyond $\alpha_0 = 5^\circ$, which is a very low angle-of-attack, and suggests that each time a loop is closed, the effect of nonlinearities increases considerably. This is despite the fact that the additional elements, the outer loop and $K_P$, are all linear components. The physical meaning behind this discrepancy requires further investigation. However, it can be said that the differences in the linear and nonlinear frequency responses as observed in Fig. III-7 to Fig. III-9 provides a useful indication of changes in stability. It can also be inferred that the addition of more complex components to the loop, such as a more realistic nonlinear pilot model or a rate and travel-limited actuator, will further degrade stability.

By using nonlinear frequency analysis, it has been shown that aerodynamic nonlinearities can negatively affect the aircraft’s responses in both the time and the frequency domains, especially in a way that linear-based design (including input gain scheduling) cannot anticipate. The use of the triangular envelope reveals regions where degraded performance in the time domain can be expected, thereby eliminating the need to check the standard time-stepping simulation at every operating point. The next section will further utilise the nonlinear frequency response method to analyse a different controller scheme, which promises to address some of the challenges observed here due to input gain scheduling.
A potential solution to the performance issues discussed above is through the use of dynamic gain scheduling. This method involves scheduling the gains against the fast-varying states rather than the slow-varying ones or the input while accounting for the effect of this on the local stability (Jacobian matrix). Dynamic gain scheduling shows superior performance to conventional input gain scheduling and conventional scheduling with fast-varying states [5, 46, 95, 96]. Therefore, a controller of this type is examined here to ensure that our proposed method does not generate false-positive results (i.e., indicating a problem when there is none). The controller schematic is similar to Fig. III-3a, except that the inner-loop gains are now scheduled against their respective states (i.e., $K_\alpha(\alpha_d)$ and $K_q(\alpha_d)$ become $K_\alpha(\alpha)$ and $K_q(q)$). Fig. III-14 plots these gains, which were taken directly from the data in [5]. It has been noted in [5] that although $K_\alpha(\alpha)$ is available for the entire operating envelope from 0° to 60° angle-of-attack, $K_q(q)$ was only calculated up to $q = 12.75°$ (the value at $\alpha_d = 36°$) due to a fold bifurcation that leads to a non 1:1 mapping between $K_q(q)$ and $q$. In Fig. III-14b, the relationship between $q$ and $\alpha_d$ can be seen in the secondary x-axis, which maps $q$ on the main x-axis to the pilot input required to achieve that value of $q$ at steady state. Therefore, full dynamic gain scheduling is only available for $\alpha_d$ below 36°. Above this value, $K_\alpha$ is still dynamically scheduled whereas $K_q$ is fixed at its maximum calculated value of 1.702.

![Fig. III-14 Dynamic gain schedules](image-url)
As before, nonlinear frequency response is used to compare the closed-loop bandwidths at different operating points. The result is shown in Fig. III-15a, which uses the same colour mapping as Fig. III-8. A boundary is defined to separate regions with full and partial dynamic gain scheduling. There is a region of missing data points in Fig. III-15a at very high angles-of-attack. In these instances, the continuation algorithm failed to solve, potentially because larger $K_q$ is required to prevent the aircraft from diverging. Despite the lack of scheduling data, it can be seen that the performance is more consistent across the entire envelope than in Fig. III-8. Specifically, points F (15°, 15°) and G (30°, 1°) now have similar bandwidth, and the undershooting previously seen at point D in Fig. III-6 and Fig. III-8 is no longer an issue based on the time simulation of point G in Fig. III-15b. The triangular envelope also indicates that points H (a very demanding input) and I have slightly different dynamics comparing to points F and G. Their step responses in Fig. III-15b show that although this is the case, the effect is minimal, and that the overall performance is much more consistent across the entire envelope when compared to input gain scheduling. This confirms the observation made in [5] that dynamic gain scheduling gives improved transient performance and can address the problems encountered in input gain scheduling with fast-varying state variables, especially when the aircraft manoeuvres rapidly across different operating points.

Fig. III-15  Dynamic gain scheduled controller: closed-loop bandwidth variation (a) and step responses (b)
III-4. DETECTING THE SECONDARY ATTRACTOR

III-4.1. Description of the feedforward controller

The aerodynamic nonlinearities of the HHIRM can lead to the formation of a secondary attractor that significantly degrades flying qualities. This problem was first detected in [6] and is further investigated here using nonlinear frequency analysis. For this study, the HHIRM is equipped with a gain-scheduled stability-augmentation controller from [6]. Fig. III-16(a) shows the schematic block diagram. In essence, this is a state-feedback controller with a feedforward path that modifies the reference signal (the demanded angle-of-attack $\alpha_d$) to create a manoeuvre-demand system. The feedforward signals are shown in Fig. III-16b and were calculated by inverting the open-loop bifurcation diagrams in Fig. III-2 so that given the demanded angle-of-attack $\alpha_d$, the steady-state values of the elevator deflection $\eta_{trim}$ and pitch rate $q_{trim}$ at that angle-of-attack are obtained. This feedforward scheme eliminates the need for an integrator, which provides a faster response, although the issues of robustness and parameter variation need to be considered in a real-world application.

The feedback gains in Fig. III-16c are scheduled against the pilot’s input $\alpha_d$ as before, and were calculated using eigenstructure assignment so that the short-period poles are fixed at $-2 \pm 2i$ throughout the intended operating range of $0^\circ \leq \alpha_d \leq 60^\circ$. The use of gain-scheduled feedforward and feedback paths with eigenstructure assignments to ensure consistent handling qualities (i.e., fixed pole positions) can be found in many highly augmented aircraft flight control systems, including the F-22 [16]. Examining the closed-loop poles at each operating point from $0^\circ$ to $60^\circ$ in $1^\circ$ intervals shows that this objective is achieved (Fig. III-17), indicating that the linear step responses are similar throughout the entire operating envelope.

![Fig. III-16 Gain-scheduled controller block diagram (a), feedforward signals (b), and scheduled feedback gains (c). Fig. III-16b was constructed using data from Fig. III-2](image-url)
III-4. Detecting the secondary attractor

It has been shown in [6] that the local performances in the form of small-amplitude step responses are desirable and similar across the entire operating envelope. However, some large step inputs cause the aircraft to respond in an undesirable manner. This can be explained by examining the closed-loop bifurcation diagram (Fig. III-18). The purpose of scaling the input signal using the feedforward path is to ensure that there is a 1:1 mapping between the pilot input $\alpha_d$ and the resulting angle of attack $\alpha$, which is reflected by the main stable branch in Fig. III-18a: a straight line with slope 1 indicating the intended 1:1 mapping between $\alpha_d$ and $\alpha$. This scheme, however, does not guarantee the uniqueness of solutions due to the lack of an integrator, and an isola with stable solutions around $\alpha_d = 50^\circ$ was detected [6]. This isola is a second and undesirable attractor that will influence the aircraft dynamics. For instance, a reference signal of $\alpha_d = 50^\circ$ will converge to one of the two angles of attack: the demanded $\alpha = 50^\circ$ or the undesirable $\alpha = 32^\circ$ that originated from the isola. The initial conditions will determine which solution the aircraft converges to.

The rest of this section will further investigate the isola’s influence on the closed-loop dynamics before using nonlinear frequency methods to indicate the presence of this secondary attractor. Firstly, three step responses are examined in Fig. III-19a along with their phase plots in Fig. III-19b. The first case ($30^\circ$ and $35^\circ$ step input) resembles the ideal response. In the second case ($\alpha_d$ steps from $30^\circ$ to $50^\circ$), the trajectory lands on the isola at $\alpha = 32^\circ$ instead of the commanded position at $50^\circ$. The final case ($30^\circ$ to $55^\circ$) reaches the commanded position, but the response is erratic and does not resemble the usual short-period mode. Examining the phase plot shows that in the third case, the trajectory is attracted to the stable isola before carrying on to its final
Fig. III-19  Step responses and phase plots of the feedforward HHIRM

Simulations with initial conditions outside the black region will converge to the isola destination at 55 deg. To reach the commanded angle of attack value and avoid the situation seen in the second case, the aircraft must acquire both high angle-of-attack and high pitch rate to keep the trajectory away from the isola’s region of attraction. This can be visualised in Fig. III-20(a), which was created by
running a large number of simulations with different initial conditions in all three states \([\alpha, q, \eta]\). It can be seen that the aircraft is less susceptible to the isola when both the angle-of-attack and pitch rate are high. In addition, the basins of attraction show little dependency on the elevator position \(\eta\). We will see later that this is no longer the case when actuator rate saturation is accounted for.

Finally, there exists a few cases in which the isola is present in the frequency response. One such case is shown in Fig. III-21, where the reference signal is \(\alpha_d = 41 + 15 \sin(2\pi ft)\). This input corresponds to an \(\alpha_d\) sweep between 36° and 56°, which passes the isola that exists in the region \(49.7° \leq \alpha_d \leq 51.2°\) in the unforced bifurcation diagram (Fig. III-18) and leads to the formation of a corresponding second attractor in the frequency domain. The isola in Fig. III-21 is only present at high frequencies, which further indicates that large and aggressive inputs are more likely to be affected.

\[\text{Fig. III-21} \quad \text{Closed-loop frequency response with } \alpha_d = 41 + 15 \sin(2\pi ft), \text{ showing both the main branch and the isola}\]

It has been shown that the presence of an isola can influence the aircraft response if its trajectory approaches the affected region. Although there is no systematic way to detect an isola in practice, its existence can be inferred from the poor performance in time-domain response. The next section will show that nonlinear frequency response analysis can also be exploited to reflect these underlying nonlinearities that are observed in the time-domain. We can then use this information to identify instances where degraded flying qualities could be expected without relying on an excessively large number of time-stepping simulations like in Fig. III-20, which comes with considerable computation expense.
III-4.2. Nonlinear frequency analysis

Consider the aircraft trimmed at 30° angle-of-attack. Fig. III-22 shows the $\alpha_d$-to-$\alpha$ frequency responses at three different forcing amplitudes $A$, i.e., the reference signal is $\alpha_d = 30 + A \sin(2\pi f t)$. Although all three instances look very similar at first glance, it was found that the $A = 25^\circ$ case has a significantly lower roll-off frequency comparing to the other two cases, which have smaller forcing amplitudes. As $A = 25^\circ$ corresponds to an $\alpha_d$ sweep from $5^\circ$ to $55^\circ$ whereas $A = 5^\circ$ covers only $\alpha_d$ between $25^\circ$ and $35^\circ$, the fact that their frequency responses are different suggest that the step response from $30^\circ$ to $55^\circ$ will be substantially different from the $30^\circ$ to $35^\circ$ one. This fact has been verified above in Fig. III-19a. On the other hand, the frequency responses at $A = 5^\circ$ and $A = 1^\circ$ have similar roll-off frequencies, which indicates that their time-domain responses are similar.

As before, the next step is to compare the nonlinear bandwidths at different operating points and for a range of forcing amplitudes $A$, which is shown in Fig. III-23a. Although no bifurcation was detected in Fig. III-23a, it is clear that there is a noticeable bandwidth reduction in the region $30^\circ \leq \alpha_{trim} \leq 50^\circ$. This is shown as a red area and can be attributed to the presence of the isola. Although the unforced bifurcation diagram in Fig. III-18 shows that the isola only exists over a small region of $49.7^\circ \leq \alpha_d \leq 51.2^\circ$, the red region in Fig. III-23a covers a much larger region that what is suggested by the unforced bifurcation diagram. The proposed method can therefore indicate the existence of the isola or any other underlying nonlinearities, which is potentially very useful when there is no systematic way to detect them.

It is also inferred from Fig. III-23a that regions with similar bandwidths will have similar step responses. Of particular interest is those where the step response is similar to the ideal one, such as point A, which corresponds to a step from $0^\circ$ to $1^\circ$. This is expected as the step amplitude is small, and the operating point is at a low angle-of-attack. Points B and C have similar bandwidths, and their step responses in Fig. III-23b are indeed similar to point A’s despite them being far apart in the operating envelope. Additionally, the step
responses of points E, F, and G are the three panels shown in Fig. III-19a, and the frequency response of point H is shown in Fig. III-21.

Apart from the red and brown regions, the envelope in Fig. III-23a also shows an additional yellow area that suggests a different dynamics comparing to those found in the other two regions. The step response at point D is shown as the dotted line in Fig. III-23b, which has noticeably higher overshoot and does not resemble the other three cases. Examining the frequency response at point D (Fig. III-24) shows that the increased bandwidth is caused by the higher resonance peak comparing to the ideal-like response at point A. A higher peak is linked to higher overshoot in the step responses, as seen in the time simulation in Fig. III-23b.

It has been shown that aerodynamic nonlinearities can have significant influence on the aircraft responses in both the time and the frequency domains. In these instances, nonlinear frequency response is a viable method to quantify these effects. The information gained from the triangular envelope can be used to verify whether the controller delivers the intended consistent handling qualities across different operating regions. It is important to point out that the purpose of the coloured envelope is not to evaluate the closed-loop bandwidth but to indicate that bandwidth could be used as an effective metric to quantify the differences in the frequency response at different operating points.
III-5. EFFECT OF ACTUATOR RATE LIMITING

III-5.1. Performance assessment

All results presented so far have assumed that the first-order actuator is ideal. In reality, physical limits can cause the actuator to saturate. This can seriously degrade the aircraft handling qualities [8, 12, 14, 57, 91, 97-101], especially when the demanded manoeuvre rate or the controller gains are too high. Some of the examples presented above already demanded unrealistic elevator movements. For example, although the step response at point B from 22º to 44º looks close to ideal in Fig. III-23, this manoeuvre requires the elevator to travel 46º within the first 0.07 second – an unrealistic rate of over 550 deg/s (see the second panel in in Fig. III-25a) with a high chance of exceeding position limit. It is therefore necessary to expand the analysis to account for actuator saturation. A 45 deg/s rate limit is now imposed on the elevator by replacing the linear first-order actuator with a piecewise function described by equation (c.3). A 45 deg/s rate limit is more representative of actuators found on large airliners as opposed to those on fighter aircraft, which can reach 100 deg/s. This underperforming actuator was chosen to exemplify the problem, which makes nonlinear frequency analysis applicable to cases where the gains are too high or when the aircraft is impaired.

\[
\dot{\eta} = \begin{cases} 
45 \times \text{sgn}(-30\eta + 30\eta_d) & \text{(deg/s)} \quad \text{if } | -30\eta + 30\eta_d | > 45 \\
-30\eta + 30\eta_d & \text{(deg/s)} \quad \text{otherwise} 
\end{cases} 
\] (c.3)

The step response from 22º to 44º with 45 deg/s rate limiting is shown in Fig. III-25a. It is evident that the elevator is heavily rate-saturated, which results in more overshoot and reduced damping – an overall degradation in handling qualities. A less demanding step response from 25º to 37º (point D in Fig. III-23) is shown in Fig. III-25b. In this case, the rate-limited aircraft enters a limit cycle. This dangerous phenomenon has been reported in analysis and test flights [8, 97-99] where rate limiting was not accounted for during the controller design process, although no comprehensive analysis using bifurcation theory was done in those studies.

![Fig. III-25 Time simulations of the ideal and rate-limited aircraft to the step inputs at point B (a) and point D (b). Position limit is not modelled](image)

With the limit cycle detected through time simulation, we can now use numerical continuation to trace out the limit-cycle branch. Fig. III-26 shows the closed-loop unforced bifurcation diagram of the rate-limited
aircraft. It was found that the limit cycle is not connected to the main branch via a Hopf bifurcation. Instead, it forms an isola that exists for $\alpha_d$ between 34.0° and 46.4°, which covers the range of unstable angles-of-attack in the open-loop aircraft (see Fig. III-2). Although it is possible to analytically predict the hidden limit-cycle due to rate saturation using the describing function method \([91, 99, 100]\), the technique can only handle one nonlinear element, meaning that the plant and controller have to be linearised. A numerical approach like continuation, on the other hand, can account for additional nonlinear elements including the aerodynamic terms and gain-scheduling as shown.

As before, the basins of attraction are constructed in Fig. III-27 using time simulations with different initial conditions to further highlight the nonlinear nature of the dynamics when rate limiting is present. In this instance, the coloured region in Fig. III-27 shows large dependency on the actuator state, suggesting that there are many possible ways for the aircraft to enter the limit-cycle, thus making it very hard to define a safe flight envelope. This also contrasts with what we have seen in the equilibrium isola with an ideal actuator (Fig. III-20a), which is almost unaffected by the actuator state. For completeness, Fig. III-20b shows the basins of attraction at $\alpha_d = 50^\circ$ (where the equilibrium isola exists) but with rate limiting active. The elevator state now also has a major influence on whether the aircraft’s trajectory is attracted to the isola, similar to that shown in Fig. III-27 with the limit-cycle isola. To sum up, much of the undesirable and nonlinear behaviours discussed so far are directly caused by actuator rate limiting, which further underlines its deleterious effects on handling qualities.

The basins of attractions in Fig. III-20 and Fig. III-27 are extremely computationally expensive to generate and do not provide much practical information on whether a pilot command is safe from the secondary attractors. On the other hand, there is no systematic way to detect isolas using numerical continuation. The limit cycle in this example was discovered and subsequently traced from the step response in Fig. III-25b. Even without this knowledge, it is still possible to use the nonlinear frequency method to quantify the degradation in handling qualities and infer the existence of additional attractors. This was done in Fig. III-28 by generating the nonlinear frequency responses at every operating point and compare their bandwidth again. This is equivalent to Fig. III-23a above but with rate limiting active. In this case, various bifurcations are
detected in the nonlinear frequency responses when the forcing amplitude exceeds around 5°. Points B and D, which encountered serious performance issues as shown in Fig. III-25, are very deep in the bifurcation zone and underlines the nonlinear nature of their dynamics. In practical terms, Fig. III-28 suggests that the pilot should fly the aircraft gently and not demanding sudden steps above 5° to avoid encountering undesirable and nonlinear responses.

**Fig. III-27** Basins of attraction of the equilibrium (normal flight) solutions at $\alpha_d = 40°$. Simulations with initial conditions outside the coloured region will converge to the limit-cycle. The equilibrium attractor (not visible) has coordinate $[\alpha, q, \eta] = [40, 12.5, -13.3]$

**Fig. III-28** Variation in the closed-loop bandwidth of the rate-limited aircraft. Bandwidths of cases in which bifurcations are detected in the nonlinear frequency responses are not shown

A wide range of interesting dynamics can be observed in the nonlinear frequency responses. For simplicity (without loss of generality), let us investigate the dynamics at points I and J, which are close to the no-bifurcation boundary. Firstly, Fig. III-29b compares the step responses at point I with and without rate limiting. The rate-limited response is slightly slower, as expected, but the performance reduction is minimal.
One might therefore conclude that point I is a safe operating point, especially when it has been reported that aircraft with good handling qualities can still occasionally encounter rate limiting [14, 101]. However, the nonlinear frequency response in Fig. III-29a tells a different story. At higher forcing frequencies $f$, there is a reduction in both gain and phase in the rate-limited response as expected. Moreover, as $f$ approaches the roll-off frequency (around 0.55 Hz) where the effect of rate limiting starts to become dominant, a pair of fold bifurcations is detected that covers the region $f = [0.56 \text{ 0.60}]$ Hz. This region contains two stable solutions with a phase difference of roughly $65^\circ$ for each value of the forcing frequency. Fig. III-29c verifies the existence of multiple solutions, which was generated by running the simulation with the same inputs but different initial conditions. The high phase lag response suffers from heavy saturation in the elevator (not shown), unlike the low phase lag response. In this instance, if the pilot is operating in the low-lag region, then any increase in input amplitude and/or forcing frequency can abruptly add more lag to the response – unlike the smooth transition as seen in the frequency response of the ideal actuator without rate limiting. This behaviour can explain the flying qualities cliff phenomenon, which is described in [101] as a “sudden and dramatic incremental shift in the phase lag, equivalent to the sudden insertion of a significant incremental time delay into the loop, initiated by only a slight change in pilot input command”. Due to its highly nonlinear and elusive nature, the flying qualities cliff has not been systematically investigated. Records of its existence are in the form of anecdotes, mostly after encountering a pilot-induced oscillation [14, 101]. In practice, time delay from the digital flight control computer and sensors can add even more lags to the system. This would further degrade the handling qualities [12], potentially to the point of instabilities [57].

![Fig. III-29](image)

**Fig. III-29** Frequency responses (a), step responses (b), and simulated forced responses (c) at point I (16, 9). The insets in (a) show magnified view near the fold bifurcations. The two $\alpha$ responses in (c) have same inputs but different initial conditions.

Point J (30, 6) in Fig. III-28 is another example of the complex dynamics induced by rate limiting. Its frequency response in Fig. III-30a reveals that apart from the fold bifurcations similar to the case at point I above, there exists a pair of period-doubling bifurcations also near the vicinity of the roll-off frequency. This
leads to a period-2 branch with stable solutions, which is verified in time simulation as shown in Fig. III-30b. The period-2 branch then undergoes further period-doubling cascades (not shown for clarity) and eventually leads to chaotic responses (see Fig. III-30c). These complex dynamics can be explained by the limit cycle isola that exists for \( \alpha_d \) between 34.0° and 46.4° (see Fig. III-26). At point J (30, 6), the forcing parameters involves an \( \alpha_d \) sweep from 24° to 36°, so its trajectory is likely to have been attracted to the limit cycle isola at some frequencies.

We have seen that the inclusion of rate limiting can rapidly deteriorate the aircraft’s handling qualities. Apart from some explosive responses, such as the entry to a limit cycle in Fig. III-25(b), the degradation in flying qualities can be so subtle that even nonlinear time simulation might suggest no potential issues (Fig. III-29(b)). Using the nonlinear frequency response analysis, it is possible to identify the frequency-domain nonlinearities that negatively influence the aircraft’s dynamics during its transient motion, potentially leading to the dangerous flying qualities cliff phenomena and chaotic responses. The bifurcation-free envelope was also determined, which is useful for determining the maximum permissible step inputs that guarantee a linear-like and bifurcation-free response even in the presence of rate limiting. Furthermore, the presence of rate saturation can severely affect the transient dynamics, potentially leading to jump resonance and chaotic motions that significantly degrades the flying qualities.

### III-5.2. Continuation in all controller gains: mitigating the effect of rate limiting

The previous section has shown that rate limiting can shrink the safe operating regions significantly and lead to unacceptable flying qualities. In this final study, unforced bifurcation analysis (without the harmonic oscillator) is used to further reveal the severity of the situation if the bifurcation-free envelope is violated. Although the limit cycle is clearly caused by rate limiting, this is only an issue for \( \alpha_d \) between 34° and 46.4° due to the high controller gains required in this region (see Fig. III-16(c)). Therefore, the link between controller gains and the limit cycles should be investigated. This is done by replacing the values of both feedback gains \([K_\alpha, K_q] \) with \([\Lambda K_\alpha, \Lambda K_q] \), where \( \Lambda \) is a scalar. \( \Lambda \) can be interpreted as the ‘global gain
parameter’, where $\Lambda = 1$ corresponding to setting the gains at their original values and $\Lambda = 0$ means no feedback gains (i.e., feedforward with no stability augmentation). Setting $\Lambda$ as the continuation parameter allows us to directly determine the link between high gains and limit cycle formation. This method of global gain continuation has been applied in [4, 6].

Fig. III-31 $\alpha$ bifurcation diagram (a) and magnified view (b) with $\Lambda$ as the continuation parameter at $\alpha_d = 37^\circ$. Only the maxima of the limit cycle are shown for clarity.

Fig. III-31a shows the bifurcation diagram of the limit-cycle and equilibrium solutions with $\Lambda$ as the continuation parameter when the reference signal is $\alpha_d = 37^\circ$. The limit cycle branch can be calculated by either starting the continuation run with the aircraft already in a limit-cycle or by continuing the periodic solutions at $\alpha_d = 37^\circ$ from Fig. III-26. We can see from Fig. III-31a that the periodic solution branch encounters a fold bifurcation at $\Lambda$ just above 0.38, indicating that for $\alpha_d = 37^\circ$, the limit cycle ceases to exist when the feedback gains are reduced to below 38% of its original values. This also verifies that high gain contributes to the formation of the limit-cycles. It might be concluded that $\Lambda$ should therefore be lowered to 0.37 when the pilot demands an angle-of-attack of $37^\circ$. However, such low gains are inadequate to stabilise the aircraft at the current angle of attack, which is open-loop unstable. Examining the equilibrium solution branch in Fig. III-31a and Fig. III-31b shows that below $\Lambda = 0.48$, the 1:1 mapping between $\alpha_d$ and $\alpha$ is lost. It has been discussed in [6] that this is caused by the equilibrium isola at $\Lambda = 1$ moving closer and merging with the main branch as $\Lambda$ is reduced. This suggests that in the presence of rate limiting, the open-loop unstable operating point $\alpha = 37^\circ$ cannot be globally asymptotically stable. A similar conjecture was made in [102] for a general $n$-th order system, which explains why the limit cycle isola observed here only exists where the open-loop aircraft is unstable. Another implication by this conjecture is that an unforced bifurcation diagram like Fig. III-31 does not provide useful information on the stability conditions in the presence of rate limiting. Traditionally, this information is obtained using computationally expensive time simulations to construct the basins of attraction as demonstrated in Fig. III-20 and Fig. III-27. The proposed harmonic forcing method, on the other hand, provides a time-efficient alternative to define a safe manoeuvring envelope. For our current example, it is paramount that the pilot uses small stick movements whilst operating.
around 37° angle-of-attack to avoid landing on the limit cycle. The amount of maximum permissible stick movement can be inferred from the bifurcation-free envelope in Fig. III-28.

Furthermore, we can still use information from Fig. III-31 to guide the control law design process and reduce the aircraft’s susceptibility to oscillation. A simple limit cycle avoidance system is now implemented for demonstration: if \( \alpha \) exceeds the commanded value \( \alpha_d \) by 25\%, the controller gains will be immediately reduced to \( \Lambda = 0.49 \), which is just above the minimum value required to fly at 37° angle-of-attack. Fig. III-32a compares the step response from 25° to 37° with and without the limit cycle protection. As soon as the aircraft exceeds 46.25° angle-of-attack (equals to 1.25\( \alpha_d \)), \( \Lambda \) is reduced to 0.49, which prevents the transition to a full limit cycle compared to when the gains are kept constant at \( \Lambda = 1 \). Further inspection of the elevator responses reveals that rate saturation disappears as soon as the controller gains are reduced, which achieves the objective of preventing an oscillation. It should be noted that the purpose here is not redesigning the controller (e.g., by varying the two gains independently to maximise the limit-cycle-free operating region), but simply to illustrate the advantages offered by bifurcation analysis in identifying regions where behaviour degrades and to help guide controller design improvements.

\[ \Lambda \]

\[ \frac{\partial \Lambda(\alpha)}{\partial \alpha} \]

The risk of limit cycles has been reduced but still exists. Fig. III-32b shows the time simulation of a more aggressive step response, where the protection system fails. In fact, the gain parameter \( \Lambda \) itself is now in oscillation between the two boundary values. This is due to the scheduling of \( \Lambda \) against a fast variable \( \alpha \), which leads to the hidden coupling term in the form of \( \partial \Lambda(\alpha)/\partial \alpha \) \[5, 46, 95, 96\]. This oscillation of the controller gain due to rate limiting resembles the fatal X-15 accident back in 1967, which has reattracted attention from the research community as recently as 2015 \[8\]. To summarise, the MH-96 flight control system used on the X-15 in the accident had an ‘adaptive gain’, which was automatically increased to the point of limit cycle onset in order to maintain the maximum possible gain. This design did not account for

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**Fig. III-32** Step responses from 25° to 37° (a) and from 0° to 37° (b) with and without limit cycle prevention system. The gains are reduced when \( \alpha \) exceeds 1.25\( \alpha_d \) = 46.25°
actuator rate limiting and was therefore unable to prevent the fatal oscillation induced by the oversaturated actuator with no more than 15 deg/s maximum rate [91]. Referring back to Fig. III-31a, we know that reducing controller gain does not create an asymptotically stable system, and that the limit cycle solutions are directly caused by rate limiting. It is therefore crucial that the impact of rate limiting is fully understood beyond the current limits of linear analysis. The method proposed in this section has demonstrated its capability to provide the control designers with crucial information in highly nonlinear environments. Full bifurcation analysis of this HHIRM model with a discontinuous controller is beyond the scope of this thesis, but may be possible using the method presented in [92].
III-6. PRELIMINARY ANALYSIS OF FLEXIBLE AIRCRAFT DYNAMICS

III-6.1. Overview

For the final section of reduced-order analysis, let us consider the impact of flexible structural dynamics on the aircraft’s flying qualities. The work presented here is a further development of [47], which was the first application of unforced bifurcation analysis on the structural coupling problem in flight dynamics. Nonlinear frequency analysis will be used to highlight how the flexible dynamics can degrade the closed-loop performance and potentially lead to instabilities if its impact is not accounted for in the controller’s design.

The aircraft model considered here is the B-1 supersonic bomber. Its mathematical model, aerodynamic and structural data are outlined in [103-105]. Similar to the HHIRM analysis, the rigid model used here is reduced to second-order to capture the short period mode via two states $\alpha$ and $q$. A controller designed for the rigid aircraft from [47] is included. Its schematic is identical to the scheme shown Fig. III-3a: command augmentation with input gain scheduling to fix the pole positions across the entire operating envelope. For the flexible model, eight additional states $[\varepsilon_1, \dot{\varepsilon}_1, \ldots, \varepsilon_4, \dot{\varepsilon}_4]$ are added to the aircraft’s equations of motion to represent the additional coupling of the four structural modes. Their frequencies can also be artificially adjusted to increase or reduce the rigid-body/flexible frequency separation. For our analysis, use the global parameter $\Lambda$ is used to modify the structural frequencies $[\Lambda \Omega_1, \Lambda \Omega_2, \Lambda \Omega_3, \Lambda \Omega_4]$, with $\Lambda = 1$ corresponds to the original baseline model. The specific values are listed in Table 3. Lowering $\Lambda$ results in a more flexible aircraft with the structural frequencies closer to the rigid-body (short period) frequency, with potentially degraded flying qualities.

<table>
<thead>
<tr>
<th>Table 3. Baseline and reduced structural frequencies</th>
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<tr>
<td>Rigid</td>
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<tr>
<td>$\Lambda \Omega_1$ (rad/s)</td>
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<td>$\Lambda \Omega_2$ (rad/s)</td>
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<td>$\Lambda \Omega_3$ (rad/s)</td>
</tr>
<tr>
<td>$\Lambda \Omega_4$ (rad/s)</td>
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III-6.2. Limitations of unforced bifurcation analysis

Fig. III-33 shows the closed-loop bifurcation diagrams of the two flexible configurations in the presence of a $S = 40$ deg/s rate limiting with the pilot input $\alpha_d$ as the continuation parameter. Due to the existence of rate saturation, the unstable limit cycle ‘hovers’ above the main equilibrium branch of trimmed flight – indicating that a large enough perturbation could send the aircraft into divergence.

Increasing the maximum actuator rate or the frequency separation could reduce the chance of limit cycle oscillation, but unforced bifurcation analysis cannot reflect these changes. This can be shown by setting $\Lambda$ or $S$ as the continuation parameter. Consider the limit cycle solution marked J in Fig. III-33b. Fig. III-34 shows the unforced bifurcation diagram of $\Lambda$ and $S$ continuation – obtained with point J as the starting solution. We
can see that increasing the frequency separation or the maximum rate takes the limit cycle further away from the equilibrium branch (moving further up the y-axis). However, this rate-limited system cannot be asymptotically stable. Unforced bifurcation analysis therefore does not provide an indication of the aircraft’s proximity to instability.

Fig. III-33  Closed-loop unforced bifurcation diagrams at $\Lambda = 1$ (a) and $\Lambda = 0.5$ (b). Dotted blue lines show limit cycle maxima.

Fig. III-34  Continuation of frequency separation parameter (a) and maximum actuator rate (b) from point J. The actuator is rate limited at $S = 40$ deg/s.

Fig. III-34 may also suggest that frequency separation plays a minor role in the closed-loop response. This is not the case, however. Fig. III-35 compares a small $3^\circ$ step response of the rigid and flexible models. The baseline flexible case ($\Lambda = 1$) has comparable performance to the rigid aircraft, whereas reducing $\Lambda$ to 0.5 leads to much more overshoot and significantly less damping in the closed-loop response – an overall degradation in handling qualities. Therefore, a controller designed on the rigid model will have reduced performance on a flexible aircraft. This feature is not reflected in the unforced bifurcation diagram.

Structural flexibility can also trigger rate limiting and potentially lead to instability. In Fig. III-36a, the aircraft is disturbed from its trim point of $\alpha = 6$ deg and $q = 2$ deg/s by a large perturbation of magnitude $\Delta \alpha = +7$ deg and $\Delta q = +5$ deg/s. Although rate limiting of 40 deg/s is encountered in both instances, the highly-flexible
III-6. Preliminary analysis of flexible aircraft dynamics

Aircraft becomes unstable and diverges whereas the baseline model successfully stabilises itself back to the initial trim point.

![Diagram](image1)

**Fig. III-35** Step responses in $\alpha_d$ from 6° to 9°

![Diagram](image2)

**Fig. III-36** Responses to a large perturbation at two different rate limiting levels

### III-6.3. Nonlinear frequency analysis

We will now examine the nonlinear frequency response to determine the impact of frequency separation and rate limiting on instabilities. Still with the aircraft trimmed at 6° angle-of-attack, Fig. III-37 compares the closed-loop frequency responses under a 3° forcing amplitude. This operating point was chosen because the open-loop bifurcation diagram crosses a fold bifurcation at this angle-of-attack (not shown). As discussed in the HHIRM analysis above, the fold bifurcation reverses the $\eta$-$\alpha$ relationship and can easily saturate the actuator. In the rigid and baseline cases (Fig. III-37a and Fig. III-37b), all solutions are stable. However, the short period resonance becomes unstable in Fig. III-37c and has a much higher gain, indicating a significant degradation in flying qualities and increased risk of instabilities. This high resonance in the frequency response also reflects the increased overshoot of the step response in Fig. III-35.
III-6. Preliminary analysis of flexible aircraft dynamics

Fig. III-37 Closed loop \( \alpha \)-to-\( \alpha_d \) Bode plots. \( \alpha_d = 6 + 3 \sin \omega t \) (deg)

The unstable solutions at the short period frequency of 2 rad/s are now further investigated using time simulation in Fig. III-38. In the rigid and baseline cases, the actuator is far from being saturated. However, heavy rate saturation to the point of instability is encountered in the \( \Lambda = 0.5 \). The intuitive explanation is that the flexible aircraft oscillates more and therefore requires the elevator to work harder. This is exacerbated by the fold bifurcation in the open-loop bifurcation diagram, which causes a non-simple-harmonic \( \eta \) response and places more demand on the actuator.

![Fig. III-38 Forced responses at 2 rad/s](image)

We can assess the role of frequency separation and rate limiting on instabilities by comparing the frequency resonance peaks at different combinations of \( \Lambda \) and \( S \) using the coloured-envelope approach – shown in Fig. III-39. Firstly, Fig. III-39 shows a black region with significantly higher resonance than the rest of the diagram. The sudden drop in gain when \( S \) or \( \Lambda \) is increased beyond the boundary is caused by the unstable solutions detaching from the main branch and forming an isola – one such example is shown in Fig. III-40. The main branch now contains only stable solutions with a significantly reduced gain. Although 6.34 dB is still high, it can be concluded that the risk of instability has been significantly reduced. Fig. III-36b confirms this: the same large perturbation that previously destabilised the aircraft is now controlled, despite the presence of rate limiting. The coloured envelope in Fig. III-39b also indicates that once rate limiting increases
beyond the black region’s boundary, then any further increases does not improve the aircraft’s closed-loop response. Instead, the frequency separation factor $\Lambda$ needs to be increased to make further match the closed-loop response to the ideal case of a rigid aircraft. A $\Lambda$ beyond 1 can be considered adequate for a rigid-like closed-loop response.

![Diagram](image)

**Fig. III-39** Comparison of the resonance peak increase, relative to the rigid aircraft’s value of 2.04 dB

Although the study presented here is still at an early stage and uses a linear structural model, it does highlight the impact of flexible dynamics on the aircraft’s flying qualities. Rate limiting also adds another variable to the problem, which must be fully understood and controlled to ensure satisfactory performance. An extended study on a full-order aircraft model may reveal some modal coupling phenomena in the form of subharmonic and superharmonic resonances. Further development in this topic will be useful for future aircraft designs that make use of slender and high aspect-ratio wings for improved fuel efficiencies at a cost of reduced frequency separations between the flexible and rigid-body modes.
IV. EFFECT OF ACTUATOR SATURATION ON PILOT-INDUCED OSCILLATIONS

Sections III-5 and III-6 have shown that the flying qualities can rapidly deteriorate due to actuator rate saturation. In fact, rate limiting has been identified as the leading cause of many high-profile pilot-induced oscillations (PIO) and limit-cycle oscillations. The problem stretches back to the early days of fly-by-wire technology with the X-15, which encountered both PIO [91, 106] and a fatal limit-cycle oscillation [8] caused by rate saturation in the horizontal stabilisers. Due to its highly nonlinear nature, rate saturation can still be overlooked by many existing linear-based handling qualities criteria [85], which consider the X-15 to meet level-1 requirements and not PIO-prone [91]. The more recent crashes of the highly unstable Gripen fighter jet in 1993 and 1999 were also attributed to rate saturation in its elevon, which was overloaded by intensive pitch and roll control demands during the accident [14]. In all these cases, the impact of rate limiting was not predicted before the accidents, and subsequent investigations only employed quasi-nonlinear methods (e.g., describing functions [8, 14, 91]) to supplement the original linear-based analysis. Nonlinear-based analysis seems like a logical step to improve the capability to predict and prevent PIO. However, this represents a major evolution from the existing procedures that rely on classical linear-based methods, on which many PIO criteria are based [85, 101]. These roadblocks were summarised by Harris in 1996: ‘There are currently no existing metrics that account for these non-linear effects’ [16]. A powerful toolbox that can not only handle nonlinearities efficiently, but also is presented in a familiar manner to facilitate its adoption, can further facilitate the eventual move to nonlinear-based analysis.

This chapter illustrates how both equilibrium and harmonically-forced bifurcation methods can be employed to evaluate the contribution of actuator rate and position saturations on pilot-induced oscillations. Unlike the rate limiting analyses presented in Sections III-5 and III-6, only linear airframes are considered in this chapter. This approach ensures that the only nonlinear element is the actuator in order to highlight the impact of actuator saturation on flying qualities. By analysing historical PIO incidents involving the X-15 and the Space Shuttle Enterprise, it will be shown that bifurcation analysis is not only compatible with existing PIO criteria, but can also predict the elusive and dangerous flying qualities cliff phenomenon.
IV-1. CONTINUATION IN PILOT GAIN: LIMIT CYCLE ANALYSIS

For the first study, unforced bifurcation analysis is employed to repeat the analysis in [91], which investigates the formation of limit cycles in the X-15 configured for landing due to high pilot gain. Consider the pilot-vehicle system shown in Fig. IV-1 consisting of a proportional gain representing the pilot, a first-order actuator, and the pitch-angle-to-elevator transfer function (equation (d.2)). It should be noted that in [91], the gain in equation (d.1) has already been adjusted for 0 dB crossing at –110° phase in the open-loop frequency response, so a pilot gain of $K_p = 1$ in this example refers to the baseline condition.

$$\frac{\theta(s)}{\eta(s)} = \frac{3.476 (s + 0.883) (s + 0.0292)}{(s^2 + 0.038s + 0.01) (s^2 + 1.684s + 5.29)} \quad (d.1)$$

The linear pilot-vehicle system under consideration becomes unstable when $K_p$ exceeds the 17 dB gain margin obtained from the open-loop frequency response, corresponding to $K_p = 7.1$. We can make the same prediction using bifurcation analysis by trimming the aircraft at the origin using zero reference input ($\theta_{dem} = 0^\circ$), then setting the pilot gain as the continuation parameter. Fig. IV-2a shows the resulting bifurcation diagram of the pitch angle $\theta$ as a function of $K_p$, which is a map of the system’s equilibrium solutions as $K_p$ increases. Due to the zero reference input, the equilibrium solution is at the origin regardless of the $K_p$ value, as expected. However, there is a change of stability from stable to unstable at $K_p = 7.1$ – the same value predicted by the linear gain margin. This change of stability is caused by a Hopf bifurcation that gives rise to a limit cycle. As all components are linear, the limit cycle is divergent whenever $K_p$ exceeds 7.1.

The analysis is now repeated but with the actuator rate limited at 15 deg/s. Based on [91, 99], rate saturation is modelled by limiting the magnitude of the signal just before the integrator block as shown in Fig. IV-3a. This results in the nonlinear rate-limited bifurcation diagram shown in Fig. IV-2b. Although the Hopf
Continuation in pilot gain: limit cycle analysis

bifurcation still occurs at $K_p = 7.1$, the limit cycle is now bounded and coexists with the equilibrium solutions in the region $2.4 \leq K_p \leq 7.1$. In other words, the Hopf bifurcation is now subcritical. This is a very undesirable behaviour because a limit cycle now exists for $K_p$ as low as 2.4 and that in the region $2.4 \leq K_p \leq 7.1$, the response can jump from the equilibrium branch to the limit cycle without warning if the perturbation is sufficiently large. We verify this in Fig. IV-4, which shows the time simulation of the aircraft subjected to a small perturbation at the 5s mark and a larger one at the 20s mark. Unlike the first case, the second disturbance is large enough to send the aircraft into the stable limit cycle. The coexistence of the limit cycle and the equilibrium solutions also leads to jump phenomena. Fig. IV-5 shows the rate-limited bifurcation diagram with data from another time simulation superimposed in green. The simulation started in a limit cycle at a high pilot gain, which was then slowly reduced to 0. It can be seen that as the pilot gain reduces past the Hopf bifurcation at $K_p = 7.1$, the response remains in the limit cycle until $K_p$ drops below 2.4, where the oscillation ceases and the response converges to the equilibrium solution again. For further reference, mathematical discussions of limit cycles in a similar setting can be found in [9, 10]; the latter presents a case in which the limit cycle can been measured experimentally.

![Diagram](image)

Fig. IV-3 X-15 nonlinear actuator block diagram: rate limit only (a) and both rate and position limits (b)

![Diagram](image)

Fig. IV-4 X-15 response to two elevator impulse inputs – $K_p = 2.41$

![Diagram](image)

Fig. IV-5 X-15 response to jump phenomenon due to slowly decreasing $K_p$
One might assume that a better actuator with higher travel rate will improve the situation by raising the stability margin. This is not necessarily the case, however. Fig. IV-6 shows the rate-limited bifurcation diagram of the pitch angle $\theta$ and the elevon angle $\eta$ for a range of rate limits. It can be seen that despite the faster actuator, the limit cycle onset is still at $K_p = 2.4$. The oscillation amplitude, on the other hand, increases with the travel rate, i.e. making the situation worse. However, simulations show that a larger perturbation is needed to enter the limit cycle when the actuator travel rate increases. In practice, this means the input perturbation required can be so large that it might exceed the deflection limit of the elevator and therefore has no chance of occurring. To demonstrate, let us impose a deflection limit of 20 deg and a rate limit of 30 deg/s on the actuator using the scheme shown in Fig. IV-3b. The ensuing bifurcation diagram is also shown in Fig. IV-6 (marked as ‘30 deg/s + travel limit’). It is clear from Fig. IV-6b that the deflection limit has confined the elevator state $\eta$ within the 20 deg boundary, which results in a reduced limit cycle amplitude of $\theta$ in Fig. IV-6a compared to when deflection limit is not implemented. The value of Fig. IV-6b is that it demonstrates the susceptibility of an aircraft to position saturation. While time series simulations with perturbation inputs can also indicate this, such a technique is reliant on giving the system the appropriate level of perturbation at the appropriate time, which can be very difficult to predict. Additionally, it does not yield information on the parameter dependence of the dynamics governing the behaviour observed.

![Fig. IV-6 X-15 bifurcation diagrams of the pitch angle (a) and elevator states (b) with different actuator rate limits](image)

The same analysis using the inverse describing function technique was also presented in [107] and it was noted that while the method can predict existence of limit cycles, it does not provide an indication of the aircraft’s susceptibility to entering the oscillation, and hence the PIO. This limitation is also reflected in Fig. IV-6 by the fact that both the limit cycle onset at $K_p = 2.4$ and the Hopf bifurcation at $K_p = 7.1$ are unaffected by the maximum actuator rate. A frequency-domain analysis can overcome this issue, as discussed in the next section.
IV-2. PIO PREDICTION USING EXISTING FREQUENCY-DOMAIN CRITERIA

We now employ frequency-domain bifurcation analysis to generate the open-loop frequency response under the original 15 deg/s rate saturation and no deflection constraint. The demanded pitch angle will now take the form \( \theta_{dem} = A \sin \omega t \), where \( A \) and \( \omega \) are the forcing amplitude and frequency in deg and rad/s, respectively.

Fig. IV-7 compares the \( \theta \)-to-\( \eta_{dem} \) frequency responses of the ideal and the rate-limited open-loop aircraft with no pilot. The 17 dB gain margin mentioned previously can be seen in the linear response. For the rate-limited case, the frequency response is dependent on both the forcing frequency and amplitude – a characteristic of nonlinear systems. Increasing the forcing amplitude will significantly reduce gain margin and lead to a lower roll-off frequency because the actuator cannot keep up with the reference signal even at low frequencies. It was shown in [91] that using the linear frequency-domain criteria, the aircraft meets level 1 handling qualities and is not PIO-susceptible. However, due to the large influence of the actuator rate limiting, the assumptions involved in applying those linear criteria are no longer valid due to the strong mismatch between the linear and nonlinear frequency responses. If we ignore the assumption of linear dynamics and employ nonlinear frequency response analysis, then Fig. IV-7b shows that the rate-limited responses violate the Gibson PIO criterion.

Using the rate-limited frequency response, we can now employ other frequency-domain Category II PIO criteria such as the open-loop onset point (OLOP) boundary [107, 108] in our analysis. The applications of OLOP criterion to various aircraft including the X-15, has previously been discussed in [85] using other techniques such as describing functions. In this section, the procedure using numerical continuation is explained. A pilot gain of \( K_p = 1.5 \) is chosen, which is well below the limit-cycle onset value in Fig. IV-6 and may therefore be listed as not PIO-prone using limit-cycle analysis in Section I-1. The first step is to determine the onset frequency; this is the lowest frequency at which rate limiting first becomes active in the closed-loop system. Fig. IV-8a compares the closed-loop Bode plots of the linear and nonlinear frequency
responses. The onset points are marked as black and white circles for two different levels of rate limiting. Then, plot the onset frequencies on the open-loop Nichols chart (Fig. IV-8b). The following observations can be made:

- At 22.5 deg/s rate limiting, the onset point A matches the prediction made based on the procedure described in [107]. Point A lies well outside the blue boundary in Fig. IV-8b, indicating that the aircraft is safe from PIO.

- At 15 deg/s rate limiting, there are two possible onset points: point B is the one obtained using the method in [107], which lies very close to the PIO boundary in Fig. IV-8b. The OLOP criterion therefore suggests that the aircraft may be on the verge of being PIO prone and requires further examination. Nonlinear frequency response analysis reveals that poor handling qualities is indeed a problem, as a small increase in the forcing frequency beyond point B results in a cliff-like jump in phase lag. Additionally, a second onset point labelled C is also detected at a lower frequency, which falls within the OLOP boundary and provides further evidence of a PIO-prone platform. Time simulation verifies that rate limiting can occur above 2.71 rad/s (and below 2.90 rad/s) if the aircraft is given a large enough perturbation.

![Fig. IV-8 X-15 closed-loop (a) and open-loop (b) θ-to-θ_dem frequency responses at [K_p, A] = [1.5, 2]](image)

The behaviour observed in the 15 deg/s case is caused by a pair of fold bifurcations and explains the jump resonance mentioned in [107, 108]. This highly nonlinear phenomenon and its implications are studied in the next section. For our current discussion, nonlinear frequency analysis has revealed that rate limiting onset can occur in more than one way (at more than one frequency), depending on which side of the 'cliff' the pilot is on.
It has been shown that rate limiting can lead to highly undesirable dynamics even in a conceptually-simple linear system. In both the limit cycle and the frequency response analyses, rate saturation causes the aircraft’s responses to differ significantly from the linear predictions. Bifurcation analysis, on the other hand, not only accurately predicts the dynamics, but can also be directly employed on existing PIO criteria. The main advantages of bifurcation analysis over existing semi-analytical methods are its ability to handle multiple nonlinearities (both rate and position saturations) and to detect the sudden phase jump associated with the flying qualities cliff. In a more complex closed loop system, the effect of the flying qualities cliff is magnified. This is explored in more detail in the next section.
IV-3. DETECTING THE FLYING QUALITIES CLIFF: ANALYSIS OF THE SPACE SHUTTLE PIO INCIDENT

IV-3.1. The Space Shuttle’s pitch control system

We now investigate the PIO incident encountered on the Space Shuttle Enterprise, which also occurred during landing flare on the final flight (FF-5) of the approach and landing test programme (ALT). Unlike the simplified X-15 example above, the full flight control system of the Enterprise with account for time delay is considered in this section. It will be shown that these additional elements magnify the rate limiting effect and thus increase the likelihood of a pilot-induced oscillation.

The Enterprise was built for pilot training and comes without an engine and heat shield [13]. Its typical mission started on the back of a Boeing 747, where the Shuttle was released at altitude to glide back to the landing site. Regarding the PIO, investigations by NASA concluded that a combination of rate limiting and excessive time delay led to the degraded handling quality of the Space Shuttle at landing [12], which escalated into a PIO by the high pilot gain during landing. Subsequent efforts to prevent further PIO included extensive pilot training [13] and the introduction of a new PIO-suppression filter onboard the operational Space Shuttles, which reduces the input gain at certain known problematic frequencies [11]. In this section, the closed-loop frequency response of the unmodified Enterprise at landing configuration is examined. This is the system that suffered from PIO on flight FF-5. The model used for analysis is based on the NASA TM 81366 report [12].

Fig. IV-9 shows the block diagram of the Shuttle’s linearised pitch rate control system with the airframe transfer function described by equation (d.1). Input shaping is not considered in this study as only the actuator nonlinearities are of interest. For the time delay block $e^{-Ts}$, a Padé 3rd-order approximation is used to ensure that the only nonlinear elements in the analysis are actuator rate and deflection saturations.

\[
\frac{q(s)}{\eta(s)} = \frac{-1.3s (s + 0.648) (s + 0.0349)}{(s + 0.887) (s - 0.1) (s^2 + 0.189s + 0.238143)} \tag{d.1}
\]
In the original NASA report, rate and travel limits were implemented using the scheme described by Fig. IV-10a, which involves taking the derivative of the elevon demand signal $\eta_{dem}$. Bifurcation analysis requires the components to be written in autonomous ordinary differential equations form, so it is impractical to use the derivative signal of $\eta_{dem}$. To address this, the scheme shown in Fig. IV-10b is used to model rate and travel saturations (similar to the one used for the X-15 above). In essence, the difference between these two setups is that Fig. IV-10a represents software rate limiting whereas Fig. IV-10b represents hardware rate limiting. Their difference in the simulated responses is small (see Fig. IV-11), although it can be seen that hardware rate limiting results in a system with slightly more phase lag when rate saturation is active. For the rest of this section, hardware rate limiting is used in bifurcation analysis for numerical convenience while all the time simulations presented use software rate limiting to remain consistent with the original NASA report and ensure the validity of the results.

Fig. IV-10 Actuator rate and travel limit implementations for time simulation (a) and bifurcation analysis (b)

Fig. IV-11 Comparison of the two rate limiting implementations

**IV-3.2. Frequency and amplitude responses**

The original NASA report [12] considers time delays $T$ between 0.1 and 0.4 s. For the first analysis, the worst-case scenario of $T = 0.4$ s is considered. Fig. IV-12a compares the linear and nonlinear frequency responses. Due to the nonlinearity introduced by rate limiting, the nonlinear responses are functions of both forcing frequency $\omega$ and forcing amplitude $A$. More specifically, larger $A$ reduces the roll-off frequency due
to the actuator not travelling fast enough to follow the input signal. In extreme cases like at $A = 15^\circ$, a pair of fold bifurcations at 1.19 and 1.58 rad/s is detected. Each fold bifurcation reverses the direction of the solution branch in the gain plot, causing a change of stability from stable to unstable (and vice versa). As a consequence, the resonance curve leans to the left and leads to a region of two stable solutions between these two fold bifurcations. Moreover, the stable solutions in this range have a phase difference of almost 180°. One might attribute this highly nonlinear behaviour to the large forcing amplitude, which requires the elevon to travel at a higher speed at lower frequency. However, it is shown that reducing the forcing amplitude to 5° does little to improve the situation as the fold bifurcation and the large phase jump are still observed.

Fig. IV-12 Space Shuttle closed-loop $q$-to-$u$ frequency (a) and amplitude (b) responses – $T = 0.4$ s

An advantage of numerical continuation methods is that the dependency on the forcing amplitude can be directly assessed. By fixing $\omega$ while allowing $A$ to vary, the system’s gain and phase responses as a function of only the forcing amplitude can be generated as shown Fig. IV-12b, in which the forcing frequency is fixed at 1.55 rad/s. As before, a region of multiple solutions (two stable and one unstable) arising from a pair of fold bifurcations is detected for $3.9^\circ \leq A \leq 15.7^\circ$, which has dangerous implications. Fig. IV-13 shows the simulated response with fixed amplitude at $15^\circ$ and a frequency of 1.55 rad/s, which lies inside the region where two stable responses exist. Before the perturbation at the 159 s mark, the motion corresponds to the in-phase solution with a phase lag of roughly 22°. The perturbation causes a violent transition to the out-of-phase solution and eventually converges to the stable response with a 177° phase lag. This transition resembles the ‘flying qualities cliff’ phenomenon, which is described in [91] as a “sudden and dramatic incremental shift in the phase lag, equivalent to the sudden insertion of a significant incremental time delay into the loop, initiated by only a slight change in pilot input command.” Studies have identified actuator rate limiting as the main cause of the flying qualities cliff, making modern high-performance fly-by-wire aircraft especially vulnerable [101]. Despite its potential consequence, the issue has not been thoroughly analysed or documented due to its highly nonlinear and elusive nature. As shown in the time simulation in Fig. IV-13, the transition to the out-of-phase motion can be sudden and violent, which catches the pilot off guard and contributes to PIO. This also makes it extremely hard to predict when the pilot might encounter the cliff and...
to replicate the phenomenon using time simulation unless the exact parameters are known beforehand. On the other hand, it has been shown that nonlinear frequency response analysis can circumvent this challenge, and Fig. IV-12 is the first successful attempt at characterising the flying qualities cliff using bifurcation analysis and numerical continuation.

Fig. IV-13  Space Shuttle simulated response to an input perturbation for \([T, A, \omega] = [0.4, 15, 1.55]\)

With the PIO fully developed, the pilot may attempt to stop the out-of-phase motion by reducing the forcing amplitude. However, the fold bifurcation at \(A = 3.9^\circ\) in Fig. IV-12b will impede the recovery process. As the response will always follow the nearest stable solution, once the motion lands on the out-of-phase region at \(A = 15^\circ\), reducing \(A\) sees no change in the pitch rate amplitude along with a very small reduction in phase lag all the way down to \(A = 3.9^\circ\), at which point, a sudden jump back to the low-amplitude and in-phase response is observed. We verify this by forcing the aircraft at 1.55 rad/s but with the forcing amplitude reducing linearly at a rate of 0.1 deg/s, shown in Fig. IV-14. It can be seen that before \(A\) passes 3.9\(^\circ\), there is no reduction in the pitch rate response amplitude despite a constantly reducing \(A\). In practical terms, this behaviour suggests that the only way to recover the aircraft in this situation is to release the stick, which is hardly an option when the PIO is fully developed with the aircraft already close to the ground at landing, potentially while also at a dangerous attitude.

Fig. IV-14  Space Shuttle simulated response with a linearly reducing forcing amplitude

**IV-3.3. Effects of time delay**

Fig. IV-15 shows the frequency responses at \(A = 15^\circ\) for a range of time delay constant \(T\). Although reducing \(T\) from 0.4 s to 0.1 s improves the situation by narrowing the frequency separation between the fold
bifurcations and reducing the magnitude of the phase jump, the fold bifurcations still exist. In fact, they only disappear by reducing the time delay further. This suggests that both rate limiting and time delay contribute to the formation of the fold bifurcations and consequently the phase jump that caused the PIO, but not time delay alone since the Padé approximation is linear and therefore cannot create the fold bifurcations. Although the original NASA report also drew similar conclusions through experience, frequency-domain bifurcation analysis can systematically identify the precise mechanism of the flying qualities cliff that contributed to the PIO experienced on the Enterprise. The existence of multiple stable solutions at $T = 0.1$ s, $A = 15^\circ$, and $\omega = 1.96$ rad/s is validated using time simulation with different initial conditions. Fig. IV-16 confirms that there are indeed two possible responses for the same forcing input: one with $45^\circ$ and one with $95^\circ$ phase lag, and that the elevon is rate-saturated in both instances.

![Fig. IV-15](image)

**Fig. IV-15** Space Shuttle rate-limited frequency responses for a range of time delay levels at $A = 15^\circ$

![Fig. IV-16](image)

**Fig. IV-16** Space Shuttle pitch rate responses from two different simulations with different initial conditions. $[T, A, \omega] = [0.1, 15, 1.96]$

### IV-3.4. Detecting position saturation

Finally, let us explore another advantage of continuation methods: the frequency response of an element inside a closed-loop can be examined. Of particular interest in this study is that of the elevon state $\eta$. Its
frequency responses at \( T = 0.4 \text{ s} \) for three different forcing amplitudes are shown in Fig. IV-17. It can be seen that for \( A = 6.5^\circ \) and above, the resonance peak is flattened at \( \eta = 21.5^\circ \), which is the elevon travel limit. We verify this by forcing the Space Shuttle at \( A = 6.5^\circ \) with a chirp signal that linearly decreases the forcing frequency at a rate of \(-0.005 \text{ rad/s}^2\). Its pitch rate response in Fig. IV-18a shows the jump phenomenon as discussed previously. The elevon response between 1.35 and 1.25 rad/s is shown in Fig. IV-18b, which is clearly position-saturated as predicted (in addition to being rate-saturated).

![Fig. IV-17 Frequency responses of the Space Shuttle’s elevon state \( \eta \) at \( T = 0.4 \text{ s} \)](image)

![Fig. IV-18 Space Shuttle pitch rate (a) and elevon (b) responses to a chirp signal – \([T, A] = [0.4, 6.5]\)](image)

The actuator’s susceptibility to position saturation can be analysed by examining the movement of the upper fold bifurcation as the forcing amplitude and frequency varies. This locus of the fold bifurcation was calculated using the two-parameter continuation method and is shown in Fig. IV-17, along with its projection onto the \( A-\eta \) plane in Fig. IV-19. It can be seen in Fig. IV-19 that for \( T = 0.4 \text{ s} \), the line asymptotes to \( \eta = 21.5^\circ \) from around \( A = 5^\circ \), indicating that pumping the stick beyond \( 5^\circ \) amplitude will lead to position saturation. Reducing the time delay to 0.3 s sees a significant improvement, and now a forcing amplitude of \( 12^\circ \) is required to induce travel saturation. In practice, a margin can be defined to ensure the
The limit is not reached. The benefit of the two-parameter continuation technique is that it can find the locus of a bifurcation point in the two-parameter space in a single run, which is much more computationally efficient than generating the individual frequency responses as shown in Fig. IV-17. Additionally, the only way to extract information on position saturation without bifurcation analysis is through time simulation, which is very time-consuming and is essentially ‘hit or miss’ if the range of susceptible parameters is not known in advance.

*Fig. IV-19 Two-parameter continuation of the upper fold bifurcation in the A-\( \eta \) plane*
IV-4. CONCLUDING REMARKS

The use of bifurcation analysis can provide valuable information on the link between actuator saturation and PIO. Re-examinations of the X-15 illustrate how bifurcation analysis can be directly implemented on existing Category I and II PIO prediction techniques, and the Space Shuttle studies further highlight the method’s advantages by explaining the dangerous flying qualities cliff phenomenon, which is very hard to predict and is not reflected in many existing PIO criteria. It is worth reiterating that all the nonlinear phenomena observed in this section were caused by actuator saturation alone. This further highlights the importance of understanding the impact of rate limiting in flying qualities assessment, and the method proposed here can be a valuable addition to the existing analysis techniques.
V. DEEP STALL RECOVERY

All analyses presented so far were on reduced-order or linear aircraft models, which are generally considered adequate for initial controller design. However, there are instances where the nonlinear dynamics of the slow mode plays a major role and cannot be neglected. This is demonstrated via a study on the nonlinear flight dynamics at deep stall conditions. At such high angles-of-attack, the slow (phugoid) mode moves closer to the short period mode in the Bode diagram, which results in highly unconventional dynamics that only be studied sufficiently using nonlinear frequency analysis.
V-1. BACKGROUND

Deep stall (also known as super stall) is a dangerous phenomenon in which the aircraft is locked into a high angle-of-attack attitude that results in a steep descending trajectory. In serious cases, this descending trajectory is maintained even with the nose horizontal or pointing upward (i.e., the aircraft falls belly-first – see Fig. V-1b). A deep stall is deemed unrecoverable when there is insufficient pitch control authority to bring the nose down and reduce the angle-of-attack. This problem has resulted in many accidents of early T-tail airliners – a design that is especially susceptible to deep stall [109]. Although there are many successful safety measures in use to prevent excursion into the deep stall region, most commonly via stick shaker & stick pusher [109] and digital angle-of-attack limiter in full-authority fly-by-wire systems [17, 110], research into deep stall recovery methods are few and of limited scope. These studies either involve simplified flight dynamics models [18-21] or empirical methods [17, 111], making it hard to determine a safe and consistent procedure to guarantee recovery. Moreover, recent research into advanced landing techniques for small unmanned aerial vehicles involves deliberately bringing the aircraft into a deep stall to minimise the landing distance [112-114]. These developments further emphasise the need to improve our understanding of the flight characteristics in the deep stall regime. Future research into this topic can also benefit from recent data published by NASA [115-117], which provides high-fidelity flight dynamics modelling of a hypothetical T-tail passenger aircraft as part of the global effort to reduce airliner loss-of-control incidents [58-60].

![Fig. V-1 A T-tail aircraft in normal flight (a) and deep stall (b)*](en.wikipedia.org/wiki/File:Deep_stall.svg (accessed 26 May 2021))

Previous works on deep stall recovery have indicated that it is possible to rock aircraft in the pitch sense to gain momentum, regain some pitch control, and eventually push the nose down below the critical angle-of-attack [17-19, 111]. This chapter demonstrates that nonlinear frequency analysis can provide further insights into this problem and help derive a successful recovery manoeuvre.

*Adapted from en.wikipedia.org/wiki/File:Deep_stall.svg (accessed 26 May 2021)
### V-2. PROBLEM DESCRIPTION

The conventional way to predict a deep stall is to examine the relationship between pitching moment coefficient $C_M$ and angle-of-attack $\alpha$. Referring to the pitch rate equation (d.2), a trimmed aircraft will have zero pitch rate ($q = 0$), thereby requiring $C_M = 0$ (for definitions of the terms in equation (d.2), see appendix C). Because $C_M$ is a function of angle of attack, a statically stable trim point has $\frac{\partial C_M}{\partial \alpha} < 0$. Fig. V-2 shows an example $C_M(\alpha)$ plot of a T-tail design. The third trim point at $34^\circ$ angle-of-attack is stable. If there is not enough pitch control authority to bring the nose down, the aircraft will be stuck in this high $\alpha$ condition, leading to an unrecoverable deep stall.

\[
\dot{q} = \frac{1}{2} \rho V^2 Sc \frac{C_M}{I_y}
\]

![Diagram](image.png)

**Fig. V-2** Typical pitching moment coefficient plot for the T-tail configuration. Negative slope indicates positive static stability in pitch

Unrecoverable (locked-in) deep stall due to insufficient pitch control power is usually attributed to the following features:

- A T-tail configuration, for which the airflow at the elevator or stabilator (all-moving tailplane) is blocked by the wing/fuselage wake at high angles-of-attack, rendering them ineffective (see Fig. V-1b).

- An aft centre of gravity, which reduces the elevator/stabilator moment arm. This design is usually found in statically unstable fighter aircraft for improved manoeuvrability, controllability in the presence of shock waves, and reduced trim drag. Consideration of centre of gravity movement is also relevant in terms of ensuring safe flight of an airliner over the full range of possible loading conditions.

For our discussion, the deep stall dynamics of the statistically F-16 fighter jet is investigated. This model is suitable for the analysis because it is unstable at low and stable at high angles-of-attack, meaning that the aircraft has a natural tendency to pitch up into a deep stall when flown manually. Its open-loop dynamics is presented in the rest of this section to provide a basic overview of the airframe under consideration. Using aerodynamic data from [17], a fourth-order version of the F-16 is constructed with four states $\alpha$ (angle-of-
attack), $V$ (velocity), $q$ (pitch rate), and $\theta$ (pitch angle), thereby restricting the motion to the longitudinal plane. The equations of motion and aerodynamic data tables for this reduced-order model are provided in appendix C. The use of secondary control surfaces such as leading edge device and speed brake is not considered for simplicity without loss of generality. A fixed thrust of 8,785 N and centre of gravity at 37.5% mean aerodynamic chord are used. At 30,000 feet altitude, this combination results in a trimmed level flight at Mach 0.6, which matches the most severe test cases done by NASA [17]. Pitch control is achieved using a pair of all-moving horizontal stabilator that can deflect up to 25° in both directions.

![Bifurcation diagram: stabilator continuation](image)

Fig. V-3 Bifurcation diagram: stabilator continuation. The pitch rate bifurcation diagram is not shown as all equilibrium solutions have $q = 0$ deg/s. Solutions beyond $\delta_s = 25°$ were found using spline extrapolation.

Fig. V-3 shows the bifurcation diagram with the horizontal stabilator $\delta_s$ as the continuation parameter. From the angle-of-attack bifurcation diagram in Fig. V-3a, it can be seen that the aircraft is statically unstable below 20° angle-of-attack. Normal operation in this regime therefore requires the use of a full-authority feedback controller (which usually limits the maximum $\alpha$ to 25° [17, 22]). There is also a branch of stable solutions at very high angles-of-attack (around 60°) that exists throughout the whole stabilator deflection range. Because this stable high $\alpha$ solution exists at full nose-down elevator (25°), a locked-in deep stall is possible.

In this instance, there are actually two possible deep stall trajectories at $\delta_s = 25°$. Fig. V-4 illustrates the two possible flight trajectories at deep stall. It can be seen that the aircraft converges to one of the two stable solutions that results in a steep descending trajectory at low speed (refer to the bifurcation diagram in Fig. V-3b for the speed at equilibrium) – despite the combination of a slight nose-up attitude, full nose-down
stabilator, and a constant thrust adequate for cruise at Mach 0.6 (182 m/s). The existence of the second branch at a slightly lower angle-of-attack of 47° makes recovery more challenging. The reason is that in normal operation with the full-authority flight control system active, the \( \alpha \) limiter will detect that the maximum 25° limit has been exceeded and will continually try to push the nose down, thereby keeping the stabilator fixed at 25°. As seen from Fig. V-3, two stable deep stall trim points exist at \( \delta_s = 25° \), so the aircraft has a high probability of being stuck in one of these two high-\( \alpha \) descending trajectories. The pilot loses all command of the aircraft via the fly-by-wire control law in this situation, so direct open-loop control is required to recover from deep stall [17]. Past studies have also not revealed the second deep stall branch connected to point B [17, 22], potentially due to their inclusion of the leading edge device.

![Diagram](image)

Fig. V-4 Trajectories of two possible trimmed conditions at deep stall with max nose-down stabilator (\( \delta_s = 25° \)). The vertical axis \( z \) is positive down due to sign convention. Aircraft not drawn to scale and are shown 5 seconds apart, indicating a rapid drop in altitude

The existence of these equilibrium trim points can be verified by examining the \( C_M(\alpha) \) curves in Fig. V-5. Noting that the pitching moment coefficient must be zero at equilibrium, there are two stable trim points at full nose-down stabilator (\( \delta_s = 25° \)): \( \alpha = 47° \) and \( \alpha = 57° \), which confirms the prediction of deep stall by bifurcation analysis. Furthermore, the \( C_M \) for \( \delta_s = 25° \) is very close to zero for high angles-of-attack (above 40°), indicating that there is very limited pitch-down capability in this region.

![Diagram](image)

Fig. V-5 Pitching moment coefficients for three stabilator deflection values. Trim points exist where \( C_M = 0 \)
Finally, the link between recoverability and centre of gravity placement is examined. The current deep stall is unrecoverable because the fold bifurcation labelled A in Fig. V-3a lies beyond the stabilator deflection physical limit of $25^\circ$. Moving the centre of gravity forward will bring point A back within the stabilator deflection range, making it possible to push the nose down below the very high $\alpha$ region. Using two-parameter continuation, we can verify this by computing the locus of the fold bifurcations labelled A and B in Fig. V-3a as functions of stabilator deflection and centre of gravity position. Fig. V-6 shows that as the centre of gravity moves forward, the branch associated with point A drops below $\delta_s = 25^\circ$ whereas the one linked to point B increases beyond the deflection limit, indicating that there is no equilibrium solution for this branch (either stable or unstable) at $\delta_s = 25^\circ$ or lower.

![Fig. V-6 Two-parameter continuation of points A and B in Fig. V-3a](image)
V-3. NONLINEAR FREQUENCY ANALYSIS

As discussed above, previous studies have indicated that pumping the stick at the aircraft’s natural rigid-body frequency is a possible strategy to recover from a deep stall. However, it was not known that this resonance frequency can vary considerably depending on how the aircraft enters a deep stall. Fig. V-7 illustrates this point by showing the nonlinear simulations of the open-loop F-16 trimmed at $\delta_e = 0^\circ$ ($\alpha = 59^\circ$) in response to different levels of angle-of-attack perturbation. For a small perturbation of 4°, the response resembles the linear simulation (not shown), and its fast Fourier transform in Fig. V-8b indicates a single resonant frequency of 1.35 rad/s. For larger perturbations, Fig. V-7 shows that their responses are now highly nonlinear as indicated by the non-symmetric oscillation about the trim point and the varying oscillation frequencies. In fact, Fig. V-7b shows that after the transient motions have damped out, the ensuing small-amplitude oscillations are not lined up with each other unlike in a linear system. The existence of multiple frequency components can also be highlighted by the fast Fourier transform of time history data. Fig. V-8c and Fig. V-8d indicate not only the additional harmonics, but also a reduction in the natural frequency as the perturbation amplitude increases, which is indicative of a softening forced system from a frequency analysis perspective. All of this highlights the nonlinear nature of the dynamics at such a high angle-of-attack. As the oscillation frequency is not constant, the way the aircraft enters a deep stall will affect the ensuing response and consequently the forcing input required from the pilot. If a non-optimal frequency is chosen by the pilot, the recovery process will be more challenging or even unsuccessful [17, 18].

![Simulated responses to angle-of-attack perturbation. $\Delta\alpha_0$ is the perturbation amplitude. The blue line corresponds to the trimmed (unperturbed) aircraft. Both panels show the same sets of simulations but at different time intervals](image)

Fig. V-7 Simulated responses to angle-of-attack perturbation. $\Delta\alpha_0$ is the perturbation amplitude. The blue line corresponds to the trimmed (unperturbed) aircraft. Both panels show the same sets of simulations but at different time intervals.
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The stabilator deflection now takes the form \( \delta_s = 0 + A \sin \omega t \) in order to generate the nonlinear frequency response, where \( \delta_{s_0} = 0^\circ \) is stabilator deflection at neutral position, \( A \) is the forcing amplitude in deg, and \( \omega \) is the forcing frequency in rad/s. This replicates the open-loop stick pumping action in [22]. A linear model was also obtained from trimmed flight in deep stall at neutral stabilator for comparison with the nonlinear analysis. Its angle-of-attack-to-stabilator transfer function is:

\[
\frac{\alpha(s)}{\delta_s(s)} = -0.0044843 \left( s + 114.9 \right) \frac{s^2 + 0.3006s + 0.03046}{(s^2 + 0.3017s + 0.03056)(s^2 + 0.04681s + 1.731)} \quad \text{(deg/deg)}
\] (d.3)

For a very small forcing amplitude like \( A = 0.1^\circ \), Fig. V.9a shows that the linear and nonlinear frequency responses are identical, which is expected. What is more notable is that there is only one prominent peak from the short-period (fast) mode at 1.32 rad/s, whereas the contribution from the slow (phugoid) mode in the \( \alpha \) response is almost negligible. This is very different from the typical frequency response at lower angles-of-attack, where both the fast and slow modes are distinctively visible (refer to Fig. VII-14a on page 163 for an example). The non-standard shape of the frequency response also agrees with the time simulations observed in Fig. V-7, where only one lightly damped mode is visible (even though there are technically two).

When \( A \) is increased slightly to \( 1^\circ \) in Fig. V.9b (which is still very small), the resonant peak now leans to the left, indicating a nonlinear softening system with an amplitude dependency. At \( A = 8^\circ \) in Fig. V.9c, the leaning increases considerably, and the subharmonic components at low frequencies are now visible. This explains the time-domain responses in Fig. V-7 and their Fourier analysis in Fig. V-8, where increasing the amplitude led to a gradual reduction in the oscillation frequency (as well as making the additional harmonics more prominent). From the pilot’s perspective, these behaviours bring the following implications:
Different ways of entering the deep stall will result in oscillations with different natural frequencies. This requires active monitoring from the pilot if the current recovery procedures are followed, resulting in very high workload with possibilities of unsuccessful recovery.

Max-amplitude (stop-to-stop) stabilator movement will be intuitively used by the pilot to achieve the highest oscillation amplitude. Due to the softening behaviour, the best forcing frequency for an optimal recovery may be lower than the frequency observed at deep stall entry.

Fig. V-9 Linear and nonlinear frequency responses at four different forcing amplitudes. The two ‘x’ marks in (d) are the points at which the numerical solver failed due to the large-amplitude solutions that exceeded the viable aerodynamic data region

We verify the hypothesis made in the second point by examining the nonlinear frequency response at \( A = 25^\circ \), which is the value for stop-to-stop stabilator movement in Fig. V-9. At such a large amplitude, there are a few intervals of \( \omega \) where no stable solution exists due to various period-doubling and torus bifurcations. Time simulations at some randomly chosen frequencies below 1.32 rad/s (the linear resonant frequency) verify that the oscillation amplitude diverges to infinity in most cases, which is exactly what we need to escape the deep stall. In practice, once the angle-of-attack drops to a reasonably low value (e.g., below the 25° limit imposed by the control system, where Fig. V-5 indicates the return of substantial pitch control power), the pilot can stop the harmonic forcing input, turn the flight control computer back on, and resume normal flight. Furthermore, it is not recommended to rely on linear analysis to devise an escape manoeuvre. If we force the aircraft at the linear resonant frequency of 1.32 rad/s, then the magnified view of the \( A = 25^\circ \) frequency response in Fig. V-10a shows that the nonlinear solution at this frequency is stable, suggesting it may not be an optimal frequency for escaping this flight regime. This is verified by time simulation in Fig. V-10b, where the forced responses at 1.32 rad/s (linear resonant frequency) and 1.0 rad/s (a nearby frequency...
with no stable solution) are compared. The latter clearly has a much larger amplitude, which successfully reduces the angle-of-attack to below 25° in less than 15 seconds and exhibits exponentially growing oscillation amplitude. In practice, the pilot would stop pumping the stick and resume normal flight at low \( \alpha \) by turning the flight control computer back on as soon as the angle-of-attack drops to a reasonable value. On the other hand, the 1.32 rad/s response is stable and never dropped below 37° angle-of-attack. Its steady-state oscillation troughs are still above \( \alpha = 50° \) – well within the deep stall regime.

There is also a region of unstable solutions between 1.81 and 4.24 rad/s as seen in Fig. V-9d, which is bounded by a pair of torus bifurcations. It may be tempting to use them since they are higher than the resonant frequency and therefore may promise a faster recovery. However, time simulation at 3.0 rad/s forcing in (Fig. V-11) shows that the motion is not divergent but are probably bounded quasi-periodic oscillations. The amplitude also grows rather slowly comparing to the response at 1.0 rad/s, making this unstable region unsuitable for our purpose. The lesson learned here is similar to any previous work using bifurcation methods: that time simulation must supplement the analysis to ensure that the transient dynamics is satisfactory.

![Fig. V-10 Magnified view of the max-amplitude frequency response in Fig. V-9d (a) and simulated responses at two different forcing frequencies (b)](image)

![Fig. V-11 Forced \( \alpha \) response at \( \omega = 3.0 \) rad/s](image)
The method presented here relies on there being an unstable region in the nonlinear frequency response which can then be exploited to initiate recovery. Deep stall requires pitch instability, as reflected in the pitching moment variations with $\alpha$ in Fig. V-5 for the F-16 aircraft (positive slope unstable, negative slope stable). This type of pitching moment trend is a characteristic of deep stall and always involves ranges of alpha in which the aircraft is statically unstable. Hence it can be anticipated that for any aircraft in deep stall, harmonic forcing at a large enough amplitude is likely to result in the aircraft experiencing excursions sufficiently far into the unstable region as to provoke divergence from deep stall. The nonlinear frequency response approach, derived from numerical bifurcation analysis, provides the necessary information on the range(s) of forcing frequency at which the aircraft is indeed unstable.

Based on the results presented in this section, it is likely that an appropriate value to select for $\omega$ will be below the linear resonant frequency due to the softening characteristic. The proposed procedure for deep stall recovery is then to use nonlinear frequency analysis at stop-to-stop stabilator movements (max amplitude) to determine a range of harmonic forcing frequencies where only unstable solutions exist. This list can be further narrowed down using time simulation to determine the frequencies that diverge the aircraft quickly. When a deep stall is encountered, either the pilot or the pre-programmed flight control computer will force the pitch control device at one of these frequencies to gain momentum and bring the nose down. Unlike in [17], this method allows the manoeuvre to be done in an open-loop manner with reduced workload and results in a higher chance of recovery. The proposed low-frequency manoeuvre also minimises the chance of encountering actuator rate limiting and does not rely on quick, square wave stabilator movements as proposed in [18], which cannot be achieved physically.
V-4. ANALYSIS OF THE NASA GTT

The previous section has discussed the deep stall behaviour caused by aft centre-of-gravity placement. Since the same phenomenon can also be experienced on a T-tail aircraft, a separate study on this configuration is necessary. To do so, the symmetric 4th-order version of the NASA Generic T-Tail Transport (GTT) model is used based on geometric, inertial, and aerodynamic data provided by NASA Langley. A schematic of this aircraft is shown in Fig. V-12. Because the GTT is statically stable and can be flown manually (unlike the unstable F-16), the result in this section provides a more realistic example of a deep stall encounter. Our discussion therefore focuses on the recovery procedure and the flying qualities at high angles-of-attack.

Fig. V-12 The NASA GTT. Source: [115]

V-4.1. Aircraft model

As the name suggests, the GTT represents a generic mid-size regional jet airliner with a T-tail configuration. Its aerodynamic data was collected from a series of low-speed sub-scale wind tunnel and water tunnel tests, and some preliminary studies have been reported in recent conferences [115-117]. Computational fluid dynamics was also deployed by NASA to estimate the influence of Reynolds number on the measured aerodynamics data, allowing corrections to the pitching moment and pitch damping data to be implemented to represent the equivalent full-scale aircraft. These Reynolds-corrected data was used here to construct a 4th-order (longitudinal) model and an 8th-order (6 degrees of freedom) model without flaps and spoilers, which are deemed adequate for our purpose. In the 4th-order implementation, the model contains 19 aerodynamics tables, which are 1D and 3D functions of angle-of-attack and $\alpha$/stabilator/elevator deflections. The valid angle-of-attack range is from $-8^\circ$ to $60^\circ$.

It has been reported in [115] that a locked-in deep stall is possible when the centre of gravity is at 40% mean aerodynamic chord (MAC) and the horizontal tailplane is in full nose-down position ($-10^\circ$). This is confirmed by what we refer to as an unforced bifurcation diagram in Fig. V-13, which shows the stable and unstable equilibrium solutions of the aircraft over the full $\pm 20^\circ$ travel range of the elevator (achieved by solving for steady states of the autonomous longitudinal equations of motion). The pitch rate bifurcation diagram is not shown in Fig. V-13 since all equilibrium solutions have zero pitch rate. Fig. V-13 indicates that the stable deep stall branch at high angles-of-attack (above $30^\circ$) extends all the way to $\delta_e = 20^\circ$ – the maximum nose-down elevator position – which is indicative of a locked-in deep stall that agrees with sub-scale experimental
results [115]. We assume zero thrust for our analysis due to the lack of an engine model for the GTT. This is a reasonable assumption because at the high angles-of-attack involved in a deep stall, any civil engines will be susceptible to significant performance degradation, so the zero thrust assumption corresponds to the worst-case scenario. This also ensures that no nose-down moment is generated by the high-mounted engines on the GTT, which can affect the outcomes concerning deep stall recovery.

The existence of locked-in deep stall can be explained by examining the values of pitching moment coefficient $C_M$ against the angle-of-attack at different elevator positions (Fig. V-14). For a deep stall around $\alpha = 45^\circ$ (elevator close to neutral), there is nose-down elevator power available; but as the pilot or autopilot pushes the stick forward, the locked-in deep stall will move to a lower angle-of-attack ($37^\circ$ at $\delta_e = +20^\circ$). In this state, no nose-down moment can be produced.

**V-4.2. Frequency analysis – longitudinal**

The existence of a locked-in deep stall has been verified by unforced bifurcation analysis as well as the $C_M$-$\alpha$ curves. To devise a recovery manoeuvre, the 4th-order longitudinal aircraft’s response to a harmonic forcing in the form of $\delta_e = 20 \sin \omega t$ is now examined using nonlinear frequency analysis. Here, $\omega$ is the forcing frequency in rad/s and 20 is the maximum possible amplitude in degree for stop-to-stop elevator movement on the GTT. This input replicates the stick pumping action in [22], albeit done in an open-loop manner. Fig.
V-15 shows the resulting linear and nonlinear Bode plots. The former was obtained by linearising the aircraft in deep stall at neutral elevator ($\delta_e = 0^o$), resulting in the state-space matrices and transfer functions shown in appendix D. In the nonlinear Bode plot, stability information was obtained by examining the Floquet multipliers of the periodically-forced system, with unstable solutions indicating divergent oscillatory responses.

![Fig. V-15 GTT linear vs nonlinear $\alpha$-to-$\delta_e$ frequency responses, $A = 20^o$](image)

At such a large forcing amplitude of $20^o$, it is expected that the linear and nonlinear frequency responses will differ significantly. One of the first notable features is that the nonlinear frequency response leans to the left in the region of maximum gain and has lower resonance frequency compared to its linear counterpart, indicating a softening system (i.e., the restoring moment becomes weaker as the aircraft moves further away from the deep stall trim point). More importantly, unstable solutions are detected near resonance in the nonlinear response, which can coexist with the stable ones at some frequencies. For $\omega$ between 0.29 and 0.51 rad/s, there are only unstable solutions in the nonlinear frequency response, all of which result in oscillations of growing amplitude. Fig. V-16 confirms that the forced responses at 0.40 rad/s (one of the unstable solutions) is divergent, which crashed the simulation after the 57s mark due to the ensuing large-amplitude oscillation. On the other hand, the response at the linear resonant frequency of 0.68 rad/s is verified to be stable and does not give the highest oscillation amplitude as indicated by the linear analysis. Large oscillations with growing amplitude like in the 0.40 rad/s case are beneficial for deep stall recovery because the aircraft can gain considerable momentum, making it easier to push the nose down and bring the angle-of-attack back to the low-$\alpha$ regime. The forced responses at resonance also cause large variation in the total velocity $V$ and the pitch angle $\theta$ (not shown). Therefore, it can be inferred that past studies that only considered the short-period mode and ignored variations in $V$ and $\theta$ [18, 20, 21] cannot accurately reflect the aircraft dynamics in this high-$\alpha$ regime, making them unsuitable for analysing dynamic deep stall recovery.
The flying quality in the deep stall region is also different from that observed at lower angles of attack. For example, it is known that the rigid-body dynamics of an open-loop conventional aircraft is usually separated into the short-period (fast) mode and the phugoid (slow) mode, which should be distinctly visible in the Bode plot and separated by an order of magnitude in frequency. This is not the case here, as seen in Fig. V-15 as well as Fig. V-17, where only one apparent peak is visible in the linear frequency response of the fast states $[\alpha, q]$. In fact, the second peak at 0.23 rad/s is barely able to be discerned in the frequency responses of the slow states $[V, \theta]$ and is much closer to the peak evident in the fast modes at 0.68 rad/s. This indicates a significant frequency reduction between the two oscillatory modes, and the aircraft response no longer resembles the conventional short-period and phugoid dynamics seen at lower angles-of-attack. Examining the time-domain responses can further highlight this feature. Fig. V-18a is an example of a typical step response in normal flight, where the fast and slow dynamics are clearly visible. At a high angle-of-attack in the deep stall region like in Fig. V-18b, the dynamics is notably different with only one apparent mode (even though there are technically two).

![Fig. V-17 Linear frequency responses at deep stall ($\delta_e = 0'$) for all four longitudinal states. Natural frequencies: 0.23 and 0.73 rad/s](image)

The unconventional high-$\alpha$ dynamics can be further demonstrated by examining the movement of the poles of the linearised system from low to high angles-of-attack. Fig. V-19 verifies that as the angle-of-attack increases, both the short-period and phugoid roots move toward each other, leading to the reduced frequency separation observed in both the time- and frequency-domain responses. This suggests that the local dynamics...
in the deep stall region can be well-approximated by the short-period model, which was the approach adopted in [18, 20, 21]. However, such an approximation is not adequate for the large-scale motion induced by harmonic forcing (or more generally by the dynamic recovery method). It has been shown above that the total velocity $V$ and pitch angle $\theta$ can vary considerably at resonance. Nevertheless, these two states are ignored in [18, 20, 21] due to their traditional association with the slow mode at low angles-of-attack. Accurate analysis of dynamic deep stall recovery therefore requires that all four longitudinal states be considered. It can also be said that any controller designed for normal flight will also be much less effective in this deep stall region due to the unconventional dynamics that does not follow the traditional short-period and phugoid model, upon which the design of most controllers is based.

**Fig. V-18** Elevator step responses: $0^\circ$ to $-1^\circ$ (a) and $17^\circ$ to $16^\circ$ (b)

**Fig. V-19** Pole positions: normal flight vs deep stall

**Fig. V-20** Magnified view of the $\alpha$-to-$\delta_e$ frequency response – $A = 18.48^\circ$

It is known that most nonlinear systems exhibit an amplitude dependency in their frequency responses. In the case of the GTT, Fig. V-20 shows that by reducing the forcing amplitude slightly to $18.48^\circ$, the nonlinear resonance near 0.4 rad/s now contains only stable responses. This is because the unstable solutions observed at $A = 20^\circ$ have detached from the main branch and formed an isola (isolated family of solutions that is mainly unstable in this case). The isola continues to move away from the main branch as $A$ is further reduced, indicating that a smaller forcing amplitude is less likely to induce an unstable oscillation with large amplitude.
that promotes deep stall recovery. Chapter VII will discuss the method for detecting the isola and track its movement. Lastly, it is also worth noting that softening behaviour is still strong in Fig. V-20, leading to a lower resonance frequency compared to the linear prediction.

For future work, the transition to the ‘single-mode’ dynamics observed in the deep stall region can be further studied using the eigenvector similarity metric outlined in reference [118], which is the first application of the method in a flight dynamics context. The insight gained from this methodology can improve our understanding of the flight dynamics at high angles-of-attack and potentially be used to aid controller design. The use of a fourth-order longitudinal model will also provide a simple example for demonstrating the eigenvector analysis framework because the example provided in reference [118] involves lateral-directional coupling, which made the stall and post-stall upset behaviours more difficult to interpret.

**Fig. V-21 Example recovery attempts.**

---

**V-4.3. Recovery procedure**

It has been established using nonlinear frequency analysis that the actual resonance frequency of the nonlinear aircraft is lower than the predicted value using linear analysis, and that the oscillation near resonance can be divergent. To leverage this phenomenon in a deep stall recovery, the pilot should pump the stick at a frequency in the unstable range to build up momentum, then push nose-down once the oscillation amplitude is large enough. Two example recovery attempts are presented in Fig. V-21 using the two forcing frequencies discussed previously. In both cases, the pilot forces the aircraft for 1.25 cycles (initiated by a nose-up input), then applies a full nose-down elevator step at the top of the second nose-up pull. It can be seen that the 0.40 rad/s forcing results in a much larger oscillation. When the nose-down step was applied, the angle-of-attack of the 0.40 rad/s response quickly ‘overtakes’ (having lower $\alpha$) the 0.68 rad/s one at the 25s mark thanks to the large built-up momentum from the preceding harmonic forcing, despite the large 8s gap between their nose-down inputs. Both examples converge to the same low-$\alpha$ attitude at around the same time, but the time
from nose-down input to recovery is far shorter in the 0.4 rad/s case. Therefore, the divergence observed in Fig. V-16 is beneficial for deep stall recovery. It can also be inferred that an aircraft with more serious deep stall characteristics may not recovery if the pilot relies on the linear resonance frequency (that turns out to be stable on the nonlinear Bode plot). To prevent this, nonlinear frequency analysis should be employed to identify the unstable (diverging) frequencies that guarantee large-amplitude oscillations that contribute to a successful recovery.

The proposed method of destabilising the deep stall trim point via harmonic forcing is effective because there is always a region of unstable trimmed (equilibrium) conditions across the angles-of-attack range between the deep stall and normal flight regions. This can be verified in the $C_M$ plot (Fig. V-14) as well as the unforced bifurcation diagram (Fig. V-13), both of which show that the aircraft is unstable between $9^\circ$ and $30^\circ$ angles-of-attack. This type of pitching moment trend is a characteristic of deep stall in general and T-tailed aircraft in particular. When the elevator travels across these unstable regions under the harmonic oscillatory input, it becomes possible to induce a large-amplitude oscillation due to the existence of the unstable trim points mentioned. The nonlinear frequency response approach, derived from numerical bifurcation analysis, allows us to identify the forcing parameters that can excite this resonance and increase the chance of a successful recovery.

We now compare the pitch rocking method with the other alternative of invoking a lateral control input (aileron in this case) to roll the aircraft about the body axis. In principle, a body-axis roll at high angles-of-attack will covert $\alpha$ into sideslip $\beta$ due to kinematic coupling. This may help reduce $\alpha$ to regain control of the aircraft, but at a cost of potentially invoking large lateral-directional motions that can take the aircraft into an upset/loss-of-control situation. To demonstrate, a recovery is now attempted by rolling the deep-stalled 8th-order aircraft model (containing lateral-directional dynamics) using maximum aileron ($\delta_a = 25^\circ$). Fig. V-22 compares the responses using two different elevator positions: neutral ($\delta_e = 0^\circ$) and full nose-down ($\delta_e = 20^\circ$). The former enters a stable limit-cycle at high $\alpha$, thereby showing no signs of recovery. On the other hand, combining the nose-down and rolling inputs will bring the angle-of-attack down to the normal range after around 30 seconds. Although this manoeuvre successfully reduces $\alpha$ in a similar time frame to the pitch rocking example in Fig. V-21, the aircraft is now in an upset condition involving large bank and sideslip angles ($\phi$ approaching $-90^\circ$ and $\beta$ around $-50^\circ$), thereby making it more susceptible to entering a spin, autorotation, or other loss-of-control situations. This lateral-directional approach also incurs significant height loss, making it more dangerous compared to the proposed pitch rocking method.
Fig. V-22 8th-order time simulation at full left roll aileron ($\delta_a = 25^\circ$)

V-4.4. Influence of aerodynamic asymmetries

All real aircraft contain aerodynamic asymmetries, especially at high angles-of-attack. Therefore, it is important to verify the dynamic recovery technique in the presence of these features. The analysis now considers the 8th-order (6 degrees of freedom) asymmetric GTT model. The asymmetric data was compiled by NASA from wind tunnel tests in the NASA Langley Research Center 12-Foot Low-Speed Tunnel (12-Foot LST) and the Boeing North American Aviation Research Tunnel (NAART), yielding two separate sets of data with the latter having smaller magnitudes of asymmetries. Data from the NAART is considered more indicative and is used in this section, mainly because no detailed investigation into aerodynamic asymmetry of this type of aircraft has been conducted.

As a result of asymmetries, the rolling and yawing coefficients $C_L$ and $C_N$ are non-zero at $\alpha$ beyond $10^\circ$ (see Fig. V-23). It is evident that without aileron and rudder input beyond $10^\circ$ angle-of-attack, the aircraft would no longer maintain wing-level flight with zero sideslip. Accordingly, the trimmed flight condition in deep stall at $\delta_e = \delta_a = 0^\circ$ is $\alpha = 44^\circ$, $V = 66$ m/s, $q = 0.58^\circ$/s, $\theta = -2.7^\circ$, $\beta = 2.8^\circ$, $p = -0.14^\circ$/s, $r = -2.86^\circ$/s, and $\phi = -11.4^\circ$. 
For our analysis, the following three different implementations of the 8th-order model are considered:

- Symmetric: contains no aerodynamic asymmetries. The responses are equivalent to that of the 4th-order responses as long as no lateral-directional input or perturbation is given. This model is considered here for reference only.

- Asymmetric: contains aerodynamic asymmetries using the NAART dataset.

- Asymmetric – CL: same as above but with the addition of roll rate feedback to the aileron. The addition of a roll damping controller allows us to ascertain that it is possible to overcome aerodynamic asymmetries using conventional control methods, which is important for a safe recovery.

The recovery manoeuvre using elevator movement as shown in Fig. V-21 is now tested on these three 8th-order models. Their responses are shown in Fig. V-24, noting that all longitudinal variables are placed in the first column. It can be seen that there is minimal variation in the longitudinal responses in all three cases, indicating that our proposed deep stall recovery manoeuvre is effective even in the presence of aerodynamic asymmetries. The main differences are in the lateral-directional responses – shown in the second column. Specifically:

- The symmetric model contains no lateral-directional motions (as expected).

- The open-loop asymmetric model invokes large bank angles (ϕ) that exceeded 40° during the manoeuvre. Notably, ϕ did not vary much during the initial stick pumping phase, but then increased rapidly when the nose-down push was initiated at the 19.6s mark.

- The roll-damping controller is very effective at reducing bank angle, and accordingly other lateral-directional variables as well. In this example, the proportional stability-augmentation gain was set to 2.0. This resulted in elevator movements that can be considered reasonable as seen in the δa time history, showing a –7° peak in travel and 10°/s peak in rate. We can therefore conclude that the control input
required to keep the wing close to level throughout the manoeuvre can be achieved, whether through the use of a controller or potentially via pilot input.

Fig. V-24. Comparing deep stall recovery on the 8th-order models.

To sum up, the pitch rocking technique remains effective for deep stall recovery even in the presence of aerodynamic asymmetries. These effects, however, can incur large lateral-directional motions and potentially lead to an upset/loss-of-control incident. This can be prevented by minimising the rolling (and potentially also yawing) motions throughout the manoeuvre. In the GTT example considered, the stability augmentation controller successfully stabilised the aircraft while also showed that the aileron movement required to keep the wing close to level is reasonable. Such a control input can therefore be expected from a pilot doing manual flying, at least after the stick-pumping phase.
V-5. CONCLUDING REMARKS

In this paper, we have shown that bifurcation methods implemented in the form of nonlinear frequency analysis can facilitate a systematic study to identify possible deep stall recovery manoeuvres in a T-tailed aircraft. Despite the existence of the locked-in deep stall at full nose-down elevator, it is still possible to initiate recovery by forcing the pitch control device at one of the nonlinear resonant frequencies. This manoeuvre destabilises the statically stable trim point by via a divergent pitch oscillation, which enables the pilot to rock the aircraft out of a potentially unrecoverable deep stall without invoking large lateral-directional motions that can lead to an upset and loss-of-control situations. The divergent responses are reflected as asymptotically unstable solutions on the nonlinear Bode plot, and the forcing frequencies that trigger this resonance can be identified using nonlinear frequency analysis. On the other hand, the resonance frequency predicted by linear analysis is incorrect. This can prevent recovery due to the linear natural frequency being insufficiently close to the true resonance peak, thereby presenting a rare example of unstable (divergent) solutions being beneficial in the context of flight dynamics and control. Furthermore, the pitch rocking method is safer than other recovery methods that induce lateral-directional motions, which come with high risk of triggering an upset condition. The presence of aerodynamic asymmetries does not preclude recovery, since the aircraft considered here has enough control power to counteract the lateral-directional motions generated due to asymmetries.

It was also found that at high angles-of-attack, the frequency separation between the conventional short-period and phugoid modes is significantly reduced, leading to non-conventional dynamics that resemble only one single mode. This further highlights the nonlinear nature of high angles-of-attack behaviours that may have hindered previous studies of dynamic deep stall recovery, most of which were also limited to empirical methods. The proposed nonlinear frequency approach provides not only a systematic study of dynamic deep stall recovery, but also expands the nonlinear analysis toolbox to account for both stationary and non-stationary nonlinearities.
VI. EVALUATING UNSTEADY AERODYNAMIC EFFECTS

A range of applications for nonlinear frequency analysis as a standalone tool has been discussed. By extension, it is of interest to explore how this technique fits in with other tools in a more comprehensive study. Accordingly, this chapter outlines the steps to combine the nonlinear frequency approach with unforced bifurcation analysis. The example problem addresses the choice of aerodynamic modelling methods, specifically quasi-steady and unsteady. It will be shown that combining unforced and forced bifurcation analysis provides further insights that cannot be obtained when either method is used separately.

Although quasi-steady aerodynamic modelling is widely used, its shortcomings in representing the stall and post-stall dynamics have been noted in the literature. Various methods to model the unsteady aerodynamics effects have been proposed as a result, but their direct implications on the aircraft’s flight dynamics characteristics have not been widely studied. As mathematical models that incorporate unsteady aerodynamics become more widespread, so does the need to systematically assess the transient behaviour in a nonlinear environment. In the context of unsteady aerodynamic analysis, nonlinear frequency response is an especially suitable tool because a harmonically-forced aircraft is non-stationary by nature, meaning that the time-dependent aerodynamic effects observed during the aircraft’s transient motion can be directly reflected in the analysis. The insights gained by nonlinear frequency analysis cannot be obtained from linear-based methods and conventional (unforced) bifurcation analysis.

To initiate work on the topic, this chapter utilised bifurcation analysis to examine the effect of unsteady aerodynamics on the longitudinal dynamics of the NASA GTT (Generic T-tail Transport) airliner model. Specifically, we use both conventional bifurcation analysis and the nonlinear frequency response to assess the sensitivity of stall and post-stall behaviours to the choice of aerodynamic modelling method: quasi-steady or unsteady. The GTT uses wind tunnel test data by NASA and Boeing [115-117] from both static and forced oscillation tests, resulting in a nonlinear but conventional (quasi-steady) flight dynamics model. We then augment the quasi-steady GTT model with unsteady aerodynamics effects using the state-space method (sometimes referred to as the Goman-Khrabrov model in the literature) [119], so as to compare the two aerodynamic modelling techniques and validate the capability of the state-space approach in matching the forced oscillation test results. Finally, the study is expanded to consider a hypothetical highly-unsteady version of the GTT by artificially increasing the magnitude of delay-relaxation parameters to the values typical of a delta wing configuration [120]. This is done to demonstrate the advantages of our proposed approach to the study of transient dynamics while also providing a speculative study on the impacts of strong unsteady effects in terms of flight dynamics and control.

The NASA GTT was chosen based on the following assumption associated with the T-tail configuration: that the effect of wing-tail aerodynamic coupling on the pitching moment in the stall zone can be neglected, i.e., the nonlinear unsteady aerodynamics is totally described by the flow separation processes over the wing. To illustrate, Fig. VI-1 compares the stalled wake of the wing sector in the stall zone of the T-tailed GTT and
the conventional NASA GTM (General Transport Model). For the low-tail GTM, the wake will impinge on the horizontal tailplane due to downwash, therefore affecting its aerodynamics. The wing-tail coupling is significant in this instance and therefore must be included in the analysis [121, 122]. For the GTT, on the other hand, the horizontal stabilizer is above the wake region with negligible downwash (up to the post-stall angle-of-attack at around 18°). This assumption of no wing-tail coupling for a T-tail aircraft is further justified by the fact that the identified time scales for the NASA GTT match very well with the time scales identified for the NASA Common Research Model (CRM) wing-body no tail configuration [123]. This is discussed in more detail in section VI-1.3.

Fig. VI-1 The NASA GTT (a) and GTM (b). Image sources: [27, 115]

The work presented in this chapter was done in collaboration with Professor Mikhail G. Goman at De Montfort University. The author was responsible for idea formulation, model validation, bifurcation analysis, and project management. Professor Goman created the unsteady state-space version of the GTT aerodynamic model using his experience in applying state-space method to measured or estimated aerodynamic data. He also provided guidance on unsteady aerodynamic analysis.
VI. MODELLING AND VALIDATION

The first mathematical formulation of the aeroplane’s longitudinal dynamics was introduced in 1911 by Bryan, where the aerodynamic derivatives in the equations of motion were treated as constant coefficients [23]. This was later expanded to accommodate flights at different conditions by turning these coefficients into functions of the angles of attack (and often other states such as speed or Mach number, and of control surface deflections), thereby creating the so-called quasi-steady modelling method. Despite its role as the foundation for modern flight dynamics and control analyses [25], it is known that the quasi-steady method cannot accurately model the unsteady aerodynamic effects in the stall region, such as delayed flow separation and attachment. In these instances, the state-space aerodynamic modelling method has been shown to be a better way to reflect these phenomena as it is capable of reflecting amplitude and frequency/rate effects on the aerodynamic loads [119]. The formulation and implication of both approaches are discussed in this section.

VI-1.1. Quasi-steady modelling

As mentioned, the aerodynamic coefficients in the quasi-steady approach are functions of the angle of attack (as well as other states such as sideslip angle or Mach number). Their nonlinear relationships are usually measured in wind tunnel static tests or predicted computationally, in either case keeping the angle of attack and sideslip angle constant. The contributions to aerodynamic loads due to the pitch rate $q$, rate of change in the angle of attack $\dot{\alpha}$, control surface deflections (such as elevator $\delta_e$), are usually considered as linear increments with an aerodynamic derivative for a given angle of attack. If only the longitudinal motion is considered, the normal force and pitching moment coefficients $C_z$ and $C_m$ can be represented in the following form:

$$C_z = C_{z,ST}(\alpha) + C_{zq}(\alpha) \frac{qc}{2V} + C_{z\dot{\alpha}}(\alpha) \frac{\dot{\alpha}c}{2V} + C_{z\delta_e}(\alpha) \delta_e$$  \hspace{0.5cm} (e.1)

$$C_m = C_{m,ST}(\alpha) + C_{mq}(\alpha) \frac{qc}{2V} + C_{m\dot{\alpha}}(\alpha) \frac{\dot{\alpha}c}{2V} + C_{m\delta_e}(\alpha) \delta_e$$  \hspace{0.5cm} (e.2)

where the first terms in equations (e.1) and (e.2) are the static nonlinear dependencies on the angle-of-attack obtained from static wind tunnel tests, the linear terms representing rotary and unsteady aerodynamic derivatives are normally obtained in wind tunnel forced oscillations tests, while control derivatives are calculated based on linearisation of the experimental nonlinear dependences from static tests. The experimental forced oscillation rigs normally involve pure angular oscillations so that pitch rate $q$ and the rate of change in angle of attack $\dot{\alpha}$ are identical. This leads to the measurement of rotary and unsteady aerodynamic derivatives as mixed combinations, for example, $C_{zq}(\alpha) + C_{z\dot{\alpha}}(\alpha)$ and $C_{mq}(\alpha) + C_{m\dot{\alpha}}(\alpha)$. Experimental rigs capable of generating heave motions for separate measurement of these two aerodynamic derivatives exist but are not widely used. As a compromise, representations (e.1) and (e.2) are often modified by inclusion of the measured cumulative pairs $C_{zq}^* = C_{zq}(\alpha) + C_{z\dot{\alpha}}(\alpha)$, $C_{mq}^* = C_{mq}(\alpha) + C_{m\dot{\alpha}}(\alpha)$, which is
partly justified by noting that the changes in stability characteristics due to translational ($\dot{\alpha}$) effects are slight (at least for conventional aircraft geometries). This results in the following simplified representation:

\[
C_z = C_{z, st}^* (\alpha) + C_{zq} (\alpha) \frac{q_c}{2V} + C_{z\delta_e} (\alpha) \delta_e
\]  
\[
C_m = C_{m, st}^* (\alpha) + C_{mq} (\alpha) \frac{q_c}{2V} + C_{m\delta_e} (\alpha) \delta_e
\]

Issues arise when the rotary/unsteady aerodynamic derivatives are measured at stall conditions. This is caused by the flow separation creating a dependency of the measured aerodynamic derivatives on the frequency $\omega$ and amplitude of forced oscillations:

\[
C_z = C_{z, st}^* (\alpha, \omega) \frac{q_c}{2V} + C_{z\delta_e} (\alpha) \delta_e
\]  
\[
C_m = C_{m, st}^* (\alpha, \omega) \frac{q_c}{2V} + C_{m\delta_e} (\alpha) \delta_e
\]

It is not possible to directly translate these frequency and amplitude dependencies into time simulation of free flights involving arbitrary motions. This can be partially resolved by replacing the dependence on the frequency $\omega$ with the dependence on the angular pitching velocity $q$, as was done in [116, 117].

VI-1.2. State-space method for unsteady aerodynamic modelling

The state-space method provides a more accurate representation of the unsteady aerodynamic effect in the normal force coefficient $C_z$ [119]. Recent development has shown that this approach is successful in modelling the effects of blowing-type plasma actuators for active control of flow separation as well as vertical wind gusts [124, 125]. At its core, the state-space method uses two separate envelope functions $C_{z, att} (\alpha)$ and $C_{z, sep} (\alpha)$ to describe the dependencies of attached and fully separated flow on the angle of attack, plus an internal state variable $x_z$ characterising the delayed transition between $C_{z, att} (\alpha)$ and $C_{z, sep} (\alpha)$. This delay and relaxation process is reflected by a first-order lag in $x_z$, which includes two time constants $T_1$ characterising flow relaxation process and $T_2$ characterising the delay in flow separation due to the rate of change in angle of attack ($\dot{\alpha}$). Accordingly, the unsteady model for normal force coefficient $C_z$ can be represented in the following form:

\[
C_z = C_{z, att} (\alpha) x_z + C_{z, sep} (\alpha) (1 - x_z) + C_{zq} (\alpha) \frac{q_c}{2V} + C_{z\delta_e} (\alpha) \delta_e
\]  
\[
T_1 \frac{dx_z}{dt} + x_z = x_{z0} (\alpha_{eff})
\]

where:

- $C_{z, att} (\alpha)$ is the dependence of the normal force coefficient assuming that flow is attached
- $C_{z, sep} (\alpha)$ is the dependence of the normal force coefficient assuming that flow is fully separated
- $x_z \in [0,1]$ is a normalised internal state variable characterising transition from attached to separated flow
- $x_{z0}(\alpha)$ is a smooth function describing transition between the attached and separated flow so that $x_{z0} = 1$ at angles of attack below the stall zone and $x_{z0} = 0$ at angles of attack above the stall zone
- $T_1$ is physical time in seconds characterising the relaxation process
- $\tau_1 = T_1 V/c$ is the non-dimensional relaxation time
- $\alpha_{eff}$ is the effective angle of attack describing the delay in separation due to the rate of change in angle of attack $\dot{\alpha}$; it can also reflect influence of the Reynolds number $\Delta Re$ [119] and intensity of jet blowing [124] or vertical wind gust $w$ [125] when present, i.e. $\alpha_{eff} = \alpha - T_2 \dot{\alpha} - \Delta \alpha Re$, where $\tau_2 = T_2 V/c$ is the non-dimensional time characterising the delay in onset of flow separation
- $C_{zq0}$ is the rotary aerodynamic derivative reflecting contribution from an airframe without account for flow separation on the wing
- $C_{z\delta_e}$ is the aerodynamic control derivative with respect to elevator deflection.

Unsteady aerodynamic modelling in pitching moment is more complicated because the process depends on both the magnitude of the aerodynamic force $C_z$ and the centre of its application $x_p$ [121]. Nevertheless, a structure similar to equations (e.7) and (e.8) can still be used on a T-tail aircraft because the wing-tail interaction is minimal and hence will not influence the aerodynamic pitching moment in the stall region ($\alpha$ between 10° and 25°). This is contrary to the conventional (low tail) configuration, where both the development of flow separation on the wing and the delayed action of wing downwash on the horizontal tail must be considered [121]. Accordingly, the contribution to the moment coefficient from flow separation on the wing can be described by a smooth function $C_{m,ws}(\alpha)$, which equals zero outside of the stall region, and results in the following unsteady representation of the pitching moment coefficient:

\[
C_m = C_{m0}(\alpha) + x_m + C_{mq0}(\alpha) \frac{qc}{2V} + C_{m\delta_e}(\alpha) \delta_e
\]

\[
T_1 \frac{dx_m}{dt} + x_m = C_{m,ws}(\alpha_{eff})
\]  

where:

- $C_{m0}(\alpha), C_{m,ws}(\alpha)$ are the functions describing static dependence of the pitching moment coefficient without account of flow separation over the wing and the contribution from the wing separation to the pitch break, respectively. The sum of these two functions should be equal to the static dependence $C_{m,ST}(\alpha) = C_{m0}(\alpha) + C_{m,ws}(\alpha)$
- $x_m$ is a normalised internal state variable characterising transition from attached to separated flow
- $C_{mq0}$ is the rotary aerodynamic derivative reflecting contribution from an airframe without account of flow separation on the wing
- $\alpha_{eff}$ is the same effective angle of attack as in equation (e.8)
VI-1. Modelling and validation

Fig. VI-2 illustrates the contributions of the static terms in equations (e.7-e.8) and (e.9-e.10) to the normal force and pitching moment coefficients of the GTT airplane. Using our proposed modelling method, $C_{m,ws}(\alpha)$ captures the nonlinear change in pitching moment in the stall region, which is characterised by the onset of an unstable positive slope and subsequent restoration of a negative slope. A similar process can be used for the initial choice of functions $C_{z,q0}(\alpha)$ and $C_{m,q0}(\alpha)$. This methodology could be extended to low-tail geometries by considering aerodynamic coefficients for different wing-body-tail, wing-body, and body-tail combinations, obtained from experiments or CFD, supplemented by visualisation of distributed flow parameters.

![Fig. VI-2 Static unsteady aerodynamic components: normal force (a) and pitching moment (b)](image)

VI-1.3. GTT implementation and validation

The state-space approach outlined above is now applied to the NASA’s Generic T-tail Transport (GTT) model. For this study, the 4th-order model is based on the original set of data from sub-scale tests and ignores the horizontal tailplane, flaps, and spoilers, which is deemed adequate for this study. All lateral-directional states and inputs are therefore zero. This 4th-order implementation contains 19 aerodynamics tables, which are 1D and 3D functions of angle-of-attack and of angle-of-attack/horizontal tailplane/elevator deflections. The valid $\alpha$ range is from $-8^\circ$ to $60^\circ$.

In order to compare quasi-steady and unsteady aerodynamics modelling techniques, the following three models based on the GTT are created for this study:

- The ‘quasi-steady’ model is basically the original GTT but with the tabular data for static normal force and pitching moment replaced by spline functions as shown in Fig. VI-3 (as well as Fig. VI-2). The use of spline functions instead of lookup tables ensures consistency with the two unsteady models explained below while resulting in negligible difference comparing to the unmodified GTT. This also has the secondary effect of making the model smoother, which is beneficial for bifurcation analysis.
- The ‘nominal unsteady model’ is augmented with two additional states $x_z$ and $x_m$ to describe the internal unsteady dynamics of $C_Z$ and $C_M$ using the approach outlined in equations (e.7-e.10).

- A hypothetical highly-unsteady model is also examined to demonstrate the use of bifurcation analysis in studying unsteady aerodynamics phenomena. In this implementation, the time delay constants are both multiplied by a scale factor of 2.5 (i.e., $\tau_1$ and $\tau_2$ become $\Lambda \tau_1$ and $\Lambda \tau_2$, where $\Lambda$ equals 2.5). This version is referred to as the ‘$\Lambda = 2.5$’ or ‘highly-unsteady’ model. Note that $\Lambda = 1$ is the nominal unsteady model.

It can be seen from Fig. VI-3 that the spline functions demonstrate an acceptable level of agreement between experimental and modelling results for static dependencies. The envelope boundaries for the normal force coefficients are quite well approximated as sinusoidal $C_{Z,att}(\alpha) = a \sin \alpha$ and $C_{Z,sep}(\alpha) = b \sin \alpha$, where $a = -5.0$, $b = -2.4$, and $C_{z0} = -0.27$. The upper limit $C_{Z,sep}(\alpha) = -5$ is close to the experimental data at small angles of attack below the stall, the lower limit $C_{Z,sep}(\alpha) = -2.4 \sin \alpha$ is close to the experimental data beyond the stall to $90^\circ$, while the function $x_{z0}(\alpha)$ is responsible for the transition between the two envelope functions inside the stall region.

![Fig. VI-3](image)

**Fig. VI-3  Fitting of the static aerodynamic coefficients**

To verify the dynamic dependencies, the damping derivatives $C_{zq}^*(\alpha)$ and $C_{mq}^*(\alpha)$ of the quasi-steady and nominal unsteady models are now compared in Fig. VI-4. The simulation condition to used obtain these derivatives was selected to match the amplitude and frequency of angle-of-attack oscillations during wind tunnel experiments [115]: $5^\circ$ in amplitude, 0.44 Hz in frequency at $Re = 230,000$ and flow speed $V = 18$ m/s. Correct identification of $\tau_1$ and $\tau_2$ should result in an unsteady model with damping derivatives $C_{zq}^*(\alpha)$ and $C_{mq}^*(\alpha)$ matching experimental data (quasi-steady modelling) in the condition specified above, especially in the stall region. The tuning process was done empirically, giving $\tau_1 = 4.5$ and $\tau_2 = 3.5$ for a reasonably good match. A more comprehensive approach would involve calculating $\tau_1$ and $\tau_2$ using a formal identification method to ensure the best fit between experimental and modelling result [120, 126], but this is beyond the scope of the thesis.
It is worth noting that the values of $\tau_1$ and $\tau_2$ are similar to those obtained in [116] based on CFD simulations of the NASA Common Research Model (CRM) wing-body configuration. The aerodynamic model structure for the normal force coefficient in [116] was identical to (e.7-e.8) and the considered wing-body configuration excludes the interference with the horizontal tail. The dimensionless time constants found in [14] for this case were $\tau_1 = 4.86$ and $\tau_2 = 3.89$. These values are very close to those obtained in this study for the GTT aircraft, based on the NASA wind tunnel test data [115] and utilising the aerodynamic models (e.7-e.8) and (e.9-e.10), namely $\tau_1 = 4.5$ and $\tau_2 = 3.5$. This provides a reasonable independent verification of the obtained time constants in our study. Additionally, the time scale for the model of the normal force coefficient of the wing-body-tail configuration in [116] shows an increase in the relaxation time $\tau_1 = 7.77$ and no change in the delay $\tau_2 = 3.89$. This increase in relaxation time $\tau_1$ can be justified as it now includes additional time for the downwash of the wing to act on the horizontal tail.

Aerodynamic derivatives obtained from forced oscillation tests in a wind tunnel in a stall region usually depend on the frequency and amplitude of the oscillations. To illustrate such dependencies in the normal force and the pitching moment coefficients, Fig. VI-5 shows the estimated traditional rotary aerodynamic derivatives (e.3-e.4) for the GTT aircraft obtained as an out-of-phase aerodynamic derivative from simulated forced oscillation tests using the identified unsteady aerodynamic models (e.7-e.8) and (e.9-e.10). The peaks of the derivatives $C_{zq}^* (\alpha)$ and $C_{mq}^* (\alpha)$ in the stall region decrease with increasing frequency $\omega$, demonstrating a significant dependence on frequency. Such dependence on frequency makes the use of the quasi-steady aerodynamic model (e.3-e.4) problematic for simulations in the time domain in the stall region. The increase of frequency correlates with the increase of pitch rate amplitude $q_{max}$ during forced oscillations executed at the same amplitude. This is why in [116, 117], for example, the damping terms are the equivalent of equations (e.5-e.6), where frequency was replaced with angular rate $q$. Such modification of the quasi-steady aerodynamic model allows only the delay process in flow separation to be represented, while the relaxation process is not accounted for. The relaxation process is important in modelling of a vertical wind gust effect in the stall region. In equations (e.7-e.10), the gust effect can be incorporated in the effective angle of attack...
as follows: \( \alpha_{\text{eff}} = \alpha - (\tau_2 c/V)\dot{\alpha} - \Delta\alpha_{\text{Re}} + \alpha_w - (\tau_2 c/V)\dot{\alpha}_w \), where \( \alpha_w \) is the change in angle of attack generated by the vertical gust [125].

![Image](image1.png)

**Fig. VI-5** Out-of-phase aerodynamic derivatives \( C_{Zq} \) and \( C_{mq} \) of the unsteady models – obtained in simulated forced pitch oscillations with amplitude 5\(^\circ\) and different frequencies

![Image](image2.png)

**Fig. VI-6** Normal force (a) and pitching moment (b) variation in a sinusoidal \( \alpha \) forcing at 0.4446 Hz and 5\(^\circ\) amplitude – quasi-steady vs unsteady. A clockwise loop in \( C_M \) indicates negative damping

Now that the unsteady model has been verified to show a good match to the static and quasi-steady data under equivalent conditions, the effects of increasing the unsteady effects via the scaling factor \( \Lambda \) are examined. Fig. VI-6 compares the quasi-steady and unsteady force and moment coefficients when the angle-of-attack is subjected to a sinusoidal forcing. It can be observed that a good match is achieved between the quasi-steady and nominal-unsteady model. In fact, the unsteady effects only become more prominent by increasing \( \Lambda \) to 2.5, resulting in the differences seen around the stall region of the highly-unsteady model. Specifically, negative damping results in the twisted \( C_M \) loop of the \( \Lambda = 2.5 \) response as well as a thicker \( C_Z \) loop, which are indicative of more unsteady dynamics due to the increased delay in flow separation and reattachment.
The choice of $\Lambda = 2.5$ in this study can be regarded as representative of a configuration in which the unsteady effects play a bigger role. For reference, the equivalent scaling factor in the case of the delta wing study in [120] is approximately $\Lambda = 3.4$.

It has been shown that the state-space method is a feasible alternative to the quasi-steady approach because their responses are more or less the same in regions where the quasi-steady data is known to be valid (i.e., for forced oscillation conditions equivalent to those used in the wind tunnel tests from which the quasi-steady dynamic derivatives were defined). Furthermore, a modification to the time delay constants using the scaling factor $\Lambda$ brings out the unsteady aerodynamics effects of interest for an aircraft configuration in which there are stronger time dependencies; this facilitates a speculative study on the potential ramifications of unsteady effects on the aircraft’s dynamics. These three models are now implemented on the fourth-order equations of motion to create the corresponding longitudinal flight dynamics models with four states $[\alpha, V, q, \theta]$ (plus two additional states $x_z$ and $x_m$ for the unsteady versions), which will be studied using bifurcation analysis in the sections to follow.
VI-2. RESULTS AND DISCUSSIONS

Let us compare the flight dynamics characteristics of the NASA GTT using two different aerodynamic modelling methods: quasi-steady and unsteady. In both cases, the following standard fourth-order equations of motion for longitudinal dynamics are used:

\[
\dot{\alpha} = \frac{1}{m} \left[ \frac{1}{2} \rho V^2 S (C_x \cos \alpha - C_x \sin \alpha) - T \sin \alpha + mg \cos(\theta - \alpha) \right] + q \tag{e.11}
\]

\[
\dot{V} = \frac{1}{m} \left[ \frac{1}{2} \rho V^2 S (C_z \sin \alpha + C_x \cos \alpha) + T \cos \alpha - mg \sin(\theta - \alpha) \right] \tag{e.12}
\]

\[
\dot{q} = \frac{1}{2} \rho V^2 Sc \frac{C_m}{I_y} \tag{e.13}
\]

\[
\dot{\theta} = q \tag{e.14}
\]

The aircraft parameters are shown in Table 4. The three total aerodynamic coefficients \([C_x, C_z, C_m]\) are made up of static and dynamic components. Table 5 lists the data type for each static coefficient, and Table 6 summarises how the total component was calculated in the quasi-steady and unsteady models.

### Table 4. Aircraft parameters

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(S)</td>
<td>wing area</td>
</tr>
<tr>
<td>(c)</td>
<td>mean aerodynamic chord</td>
</tr>
<tr>
<td>(m)</td>
<td>mass</td>
</tr>
<tr>
<td>(\rho)</td>
<td>air density (at 10,000 ft)</td>
</tr>
<tr>
<td>(I_y)</td>
<td>pitch moment of inertia</td>
</tr>
<tr>
<td>(T)</td>
<td>thrust</td>
</tr>
<tr>
<td>(g)</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td></td>
<td>70.08 m²</td>
</tr>
<tr>
<td></td>
<td>3.37 m</td>
</tr>
<tr>
<td></td>
<td>25,332 kg</td>
</tr>
<tr>
<td></td>
<td>0.90463 kg/m³</td>
</tr>
<tr>
<td></td>
<td>1,510,624 kg m²</td>
</tr>
<tr>
<td></td>
<td>22,000 N</td>
</tr>
<tr>
<td></td>
<td>9.81 m/s²</td>
</tr>
</tbody>
</table>

### Table 5. Static aerodynamic data types

<table>
<thead>
<tr>
<th></th>
<th>Quasi-steady</th>
<th>Unsteady ((\Lambda = 1) and 2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_{x, st})</td>
<td>Tabular (not shown)</td>
<td>Tabular (not shown)</td>
</tr>
<tr>
<td>(C_{z, st})</td>
<td>Splines (Fig. VI-3a)</td>
<td>Splines (Fig. VI-3a)</td>
</tr>
<tr>
<td>(C_{m, st})</td>
<td>Splines (Fig. VI-3b)</td>
<td>Splines (Fig. VI-3b)</td>
</tr>
</tbody>
</table>

### Table 6. Modelling methods of the three total aerodynamic coefficients

<table>
<thead>
<tr>
<th></th>
<th>Quasi-steady</th>
<th>Unsteady ((\Lambda = 1) and 2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_x)</td>
<td>Similar to eq. (e.3)</td>
<td>Similar to eq. (e.3)</td>
</tr>
<tr>
<td>(C_z)</td>
<td>eq. (e.3)</td>
<td>eq. (e.7-e.8)</td>
</tr>
<tr>
<td>(C_m)</td>
<td>eq. (e.4)</td>
<td>eq. (e.9-e.10)</td>
</tr>
</tbody>
</table>
VI-2.1. Unforced bifurcation analysis

Fig. VI-7 shows the unforced bifurcation diagrams of the quasi-steady and unsteady GTT with the elevator deflection $\delta_e$ as the continuation parameter. The insets are magnified views of regions where limit cycles exist. Firstly, it is noted that Fig. VI-7a and Fig. VI-7b are very similar, which suggests that dynamics of the quasi-steady and nominal-unsteady models are comparable. This observation is further verified in Fig. VI-8, which compares the pole positions of all equilibrium solutions shown in the first two bifurcation diagrams. The rigid-body roots of the quasi-steady and nominal-unsteady models are comparable, although the latter model contains two additional real roots on the far-left due to the two internal aerodynamic states $x_z$ and $x_m$. These two roots overlap each other and travel as a real pair (as opposed to a common complex-conjugate pair). Apart from this minor difference, the similar rigid-body roots verify that the state-space method is a valid alternative to quasi-steady modelling while also highlighting that unsteady aerodynamics in the current application (a T-tail transport aircraft that does not undergo rapid manoeuvres) is not sufficiently influential to require a time-dependent model. One can also make an opposite conclusion: that quasi-steady modelling is adequate for the GTT and possibly for any general T-tail transport aircraft applications.

![Fig. VI-7 Unforced bifurcation diagrams – elevator continuation](image1)

![Fig. VI-8 Pole positions of all equilibrium solutions in Fig. VI-7a and Fig. VI-7b](image2)

Fig. VI-7a and Fig. VI-7b also feature a pair of Hopf bifurcations between 8.2° and 9.4° angle-of-attack. They give rise to a branch of stable limit cycles, meaning that the aircraft may encounter pitch oscillation in the
region. This pair of Hopf bifurcations disappears in Fig. VI-7c when $\Lambda$ is increased to 2.5. To verify the pitch oscillation, we trim the aircraft at $\delta_e = -6.3^\circ$, then step down to $-6.47^\circ$. The resulting responses in Fig. VI-9 verify the limit cycle’s existence in the quasi-steady and nominal-unsteady cases. For the $\Lambda = 2.5$ case, the aircraft is technically stable, although marginally damped.

![Fig. VI-9 Response to an elevator step from $-6.3^\circ$ to $-6.47^\circ$](image)

Another Hopf bifurcation is detected at a higher angle-of-attack. This occurs at around $\alpha = 17^\circ$ for the first two cases, resulting in a branch of unstable limit cycles that collides with the unstable equilibrium branch (i.e., a global homoclinic bifurcation). These unstable limit cycles are not directly observable in time simulation. However, the similarity between figures 7a and b further emphasises that quasi-steady modelling is adequate to characterise the dynamics of the aircraft for the current application. When $\Lambda$ is increased to 2.5, this Hopf bifurcation moves further to the left to reside at a higher angle-of-attack and create a branch of stable limit cycles that can be observed in time simulation – one of which is shown in Fig. VI-10 alongside a plot of how $C_M$ varies with $\alpha$ throughout the oscillation. In the latter, it was found that the oscillation is linked to the damping loop in $C_M$ being partly undamped, which in turn was a result of the highly unsteady aerodynamics. In other words, increasing $\Lambda$ reduces damping at high angles-of-attack, which can lead to stability loss and pitch oscillation in extreme cases.

![Fig. VI-10 A high-$\alpha$ stable limit cycle at $[\Lambda, \delta_e] = [2.5, -7]$ and its associated pitching moment hysteresis](image)

Finally, the reduced damping at high $\alpha$ is further highlighted by a large elevator step from $-2^\circ$ to $-8^\circ$ (Fig. VI-11). Based on the bifurcation diagrams, this manoeuvre equates to moving between the two stable trim points at $3.6^\circ$ and $20.6^\circ$ angle-of-attack. It can be seen from Fig. VI-11 that the quasi-steady and nominal-unsteady responses are very similar. On the other hand, the $\Lambda = 2.5$ case is very different in addition to being significantly less damped as inferred above.
Results and discussions

In conclusion, unforced bifurcation analysis and time simulations verify that the state-space method provides a feasible alternative to the quasi-steady modelling approach. Conversely, it can also be said that, based on the cases studied here, a quasi-steady model can be considered adequate for transport aircraft applications that do not undergo rapid manoeuvres. On the other hand, the responses become very different in the hypothetical highly-unsteady case, which underline the shortcomings of the quasi-steady method in instances where the unsteady effects are significant. The analyses also demonstrated the potential of combining the state-space modelling method with bifurcation analysis for studying the aircraft’s flight dynamics in these highly nonlinear instances.

VI-2.2. Forced bifurcation analysis

It has been shown that unforced bifurcation analysis provides valuable insights on the effects of unsteady aerodynamics on the aircraft dynamics especially at high angles-of-attack. However, this approach becomes less effective in closed-loop applications in which the controller provides stability. To illustrate this, consider a manoeuvre-demand system as shown Fig. VI-12a, where the input is demanded angle-of-attack $\alpha_d$. Fig. VI-12b is the resulting unforced bifurcation diagram of the closed-loop system, which is identical for all three cases (quasi-steady, nominal-unsteady, and highly unsteady). This is because unforced bifurcation analysis only provides information on the equilibrium solutions. Since the controller provides stability and allows us to trim the aircraft at any angle-of-attack within the elevator deflection range, the influence of unsteady aerodynamics is not noticable in Fig. VI-12b.

![Fig. VI-12 Closed-loop block diagram (a) and closed-loop unforced bifurcation diagram (b) (same for all cases)](image-url)
In order to assess the controlled aircraft using bifurcation analysis, let us now employ the nonlinear frequency analysis method to observe the impact of the unsteady effects when the aircraft is non-stationary. The pilot input now takes the form \( \alpha_d = 20 + 2 \sin 2\pi f t \) (deg). This is equivalent to trimming the aircraft at 20° angle-of-attack, then applying a forcing input with amplitude 2° and frequency \( f \) (Hz). The resulting Bode plots are shown in Fig. VI-13. As before, the quasi-steady and nominal-unsteady cases are very similar, and both become unstable for a range of frequencies near resonance below 1 Hz. The resonance peak also includes a region of observable hysteresis; this feature is not discussed further as it is not caused by the unsteady effects. When \( \Lambda \) is increased to 2.5, these unstable solutions increase beyond 1 Hz and only become stable again at 1.17 Hz. In all instances, stability is lost via a torus bifurcation, which gives rise to a large-amplitude quasi-periodic oscillation. This is verified by comparing the forced responses at 1 Hz in time simulation (Fig. VI-14). As predicted by nonlinear frequency analysis, this high-frequency stick pumping results in very small-amplitude oscillations at exactly 1 Hz for the quasi-steady and nominal-unsteady responses. However, the \( \Lambda = 2.5 \) simulation is remarkably different. In addition to having an extremely large amplitude, the oscillation is quasi-periodic at around 0.14 Hz, which is significantly lower than the 1 Hz input. This behaviour marks a degraded controller performance. Since the quasi-periodic (unstable) region expands as \( \Lambda \) increases, it can be said that unsteady aerodynamics can negatively affect the controller’s performance in a manner that cannot be detected using the quasi-steady modelling approach and even unforced bifurcation analysis with full unsteady aerodynamics modelling. These shortcomings can be addressed by using forced bifurcation analysis.

**Fig. VI-13** \( \alpha \)-to-\( \alpha_d \) frequency responses. \( \alpha_d = 20 + 2 \sin 2\pi f t \) (deg)

**Fig. VI-14** Forced response at 1 Hz
The dynamics at resonance is also affected by the presence of unsteady aerodynamics. To illustrate, consider the motions at 0.35 Hz forcing, at which there are no stable solutions in all three cases according to the nonlinear Bode plots. The resulting phase plots in Fig. VI-15 show quasi-periodic oscillations for the first two cases and chaotic motion for the highly unsteady one. As before, the quasi-steady and $\Lambda = 1$ phase plots are very similar despite the large-amplitude oscillations ($\alpha$ varies between 12° and 28°), which further validates the accuracy of the state-space modelling method.

![Phase plots at 0.35 Hz forcing, showing trajectories between 350s and 500s](image_url)

*Fig. VI-15 Phase plots at 0.35 Hz forcing, showing trajectories between 350s and 500s*
VI-3. CONCLUDING REMARKS

This chapter presents one of the first attempts at combining the state-space aerodynamic modelling method with bifurcation analysis for a fixed-wing aircraft, and at extending this to investigate the effects of the aerodynamic modelling on transient behaviour via nonlinear frequency responses generated using numerical continuation. Whilst previous studies were limited to assessments of only aerodynamic characteristics, it has been shown here that the influence of unsteady aerodynamics in the context of dynamics and control can be directly evaluated using bifurcation analysis. Combining the state-space aerodynamic modelling method and bifurcation analysis allows us to verify the following observations that have not been thoroughly assessed previously:

- Quasi-steady aerodynamic modelling is adequate for the NASA GTT model, and potentially more generally for other T-tail transport aircraft that do not undergo rapid manoeuvring.

- Likewise, the state-space modelling approach is a feasible alternative that provides comparable responses in regions where the quasi-steady results are known to be valid.

- When the unsteady effects are significant, the aircraft’s open- and closed-loop performance can be severely degraded, especially at high angles-of-attack and while transitioning between the stall and post-stall regimes.

In the hypothetical highly-unsteady study, unforced bifurcation analysis detected the formation of stable limit cycles in the post-stall regime, and harmonically forced bifurcation analysis confirmed the significant reduction in pitch damping via the widened resonance region. Both of these indicate a severely degraded flying qualities caused by the aerodynamic phenomena that cannot be reflected using the traditional quasi-steady modelling technique. Furthermore, the time delay parameters have been chosen to resemble a delta wing configuration found in high-performance fighter aircraft. This suggests that an accurate mathematical model of these highly manoeuvrable platforms must account for unsteady aerodynamic effects in order to accurately represent the stall and post-stall behaviours. To this end, the combination of state-space aerodynamic modelling and bifurcation analysis presents a possible alternative to the traditional quasi-steady method and conventional analysis methods.
VII. SECONDARY RESONANCES AND MODAL COUPLINGS

As seen in the F-16’s frequency response, the inclusion of the phugoid mode can give rise to complex subharmonic or superharmonic resonances that are not present in reduced-order and linear analysis. These highly nonlinear phenomena and the methods to suppress them are investigated here. Following this, the impact of aerodynamic asymmetries is examined.
VII-1. SYMMETRIC AIRCRAFT ANALYSIS

VII-1.1. Description of the aircraft model

The dynamics of the NASA Generic Transport Model (GTM) is considered in this section. The GTM is a nonlinear simulation of a 5.5% scale, remotely piloted, generic twin-under-wing engine civil transport aircraft, which was developed to study airliner upsets and loss-of-control. For our discussion, the fourth-order (longitudinal) polynomial version of the GTM is used [127]. This polynomial implementation is symmetric about the body x-z plane, unlike the full version of the GTM, which simplifies the study and making it ideal to demonstrate our method. In addition, the use of polynomials instead of lookup tables to represent the aerodynamics coefficients significantly reduces computation time. Table 7 lists the two operating points to be studied, which represent medium and high angles of attack, and Fig. VII-1 shows the positions of their roots in the complex plane. The aircraft is trimmed for level and horizontal flight in both cases by adjusting the elevator deflection and throttle. This is appropriate as linear gain-scheduled controllers are typically designed around these trim points with zero flight path angles.

Table 7. Operating points to be studied

<table>
<thead>
<tr>
<th>Operating point</th>
<th>( \alpha = \theta ) (deg)</th>
<th>( V ) (m/s)</th>
<th>( q ) (deg/s)</th>
<th>( \delta_e ) (deg)</th>
<th>Throttle</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>29.6</td>
<td>0</td>
<td>0.68</td>
<td>12.7%</td>
<td>Phugoid mode is marginally damped</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>25.0</td>
<td>0</td>
<td>-7.2</td>
<td>59%</td>
<td>Short period mode is unstable</td>
</tr>
</tbody>
</table>

Fig. VII-1  Longitudinal root loci of the polynomial GTM for elevator deflections between -30° and 6°

VII-1.2. Nonlinear frequency response

Consider the open-loop frequency response at operating point 1. The pitch-angle-to-elevator Bode plots are shown in Fig. VII-2a for the linear model and Fig. VII-2b for the nonlinear model. For clarity, bifurcation symbols are not shown in Fig. VII-2b. However, they are shown in the magnified view given in Fig. VII-4a using the notation listed in Table 8. Apart from the two peaks at the phugoid and short-period frequencies (0.07 Hz and 0.60 Hz), the nonlinear model contains additional peaks at low frequencies (between 0.017 and 0.035 Hz), known as subharmonic resonance, as well as a peak at 0.14 Hz (to the right side of the phugoid resonance) due to a pair of period-doubling bifurcations; all of which are not captured in the linear model. The peak at 0.14 Hz can be referred to as superharmonic resonance. Additionally, the resonance curves in the
nonlinear response lean to the right, indicating that the aircraft resembles a hardening system (i.e., the restoring force increases with higher oscillation amplitude, which is expected for a dynamically stable airliner). Unstable solutions are seen in the nonlinear response, which cause the aircraft to diverge if it is forced at one of those frequencies.

Despite operating point 1 being statically stable, the nonlinear resonances instability observed in this region are caused by the phugoid mode being marginally damped (see Fig. VII-1). A common method to improve phugoid stability is to use pitch angle feedback. For this study, a simple proportional stability augmentation controller as shown in Fig. VII-3 is used, and the elevator demand signal is given a harmonic input in the form of \( \sin \omega t \). The effect of increasing the proportional gain from zero on the pitch-angle-to-elevator-demand frequency response will be studied, which is referred to as the pitch-angle-to-elevator frequency response (with the word ‘demand’ omitted).

Fig. VII-3 Proportional stability augmentation controller

Fig. VII-4 shows how the nonlinear frequency response is modified due to the controller gain increasing from 0 to 0.08, and Table 9 summarises the notable features. The sections to follow will explain how numerical continuation is used to identify the features listed in Table 9.
Table 8. Notation as used in the nonlinear frequency responses and bifurcation diagrams

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>Stable</td>
<td>Frequency responses remain stable beyond the specified range.</td>
</tr>
<tr>
<td>Unstable</td>
<td>Frequency responses become unstable beyond the specified range.</td>
</tr>
<tr>
<td>Torus bifurcation</td>
<td>Period-1 response loses stability beyond the specified range.</td>
</tr>
<tr>
<td>Fold bifurcation</td>
<td>Period-doubling response repeats itself after twice the forcing cycle.</td>
</tr>
<tr>
<td>Period-doubling bifurcation</td>
<td>Period-doubling response repeats itself after twice the forcing cycle.</td>
</tr>
</tbody>
</table>

Fig. VII-4 Frequency responses at different controller gains. The dotted rectangles are used to highlight the notable features; solid rectangles are magnified views.
Table 9. Summary of the frequency responses in Fig. VII-4

<table>
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<th>Gain</th>
<th>Figure</th>
<th>Description</th>
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<tr>
<td>0</td>
<td>Fig. VII-4a</td>
<td>Open-loop frequency response (note the change of y-axis unit from dB to deg).</td>
</tr>
<tr>
<td>0.00046</td>
<td>Fig. VII-4b</td>
<td>The unstable subharmonic resonance detaches from the main branch and forms an isola.</td>
</tr>
<tr>
<td>0.00820</td>
<td>Fig. VII-4c</td>
<td>The unstable superharmonic resonance detaches from the main branch and forms an isola.</td>
</tr>
<tr>
<td>0.014</td>
<td>Fig. VII-4d</td>
<td>The stable superharmonic (period-2) resonance disappears.</td>
</tr>
<tr>
<td>0.024</td>
<td>Fig. VII-4e</td>
<td>The unstable phugoid resonance detaches from the main branch and forms an isola.</td>
</tr>
<tr>
<td>0.080</td>
<td>Fig. VII-4f</td>
<td>Most of the subharmonic resonance is suppressed.</td>
</tr>
<tr>
<td>0.113</td>
<td>not shown</td>
<td>The isola formed by the superharmonic (period-2) resonance disappears.</td>
</tr>
<tr>
<td>0.129</td>
<td>not shown</td>
<td>The isola formed by the subharmonic resonance disappears.</td>
</tr>
<tr>
<td>0.148</td>
<td>not shown</td>
<td>The isola formed by the phugoid resonance disappears.</td>
</tr>
</tbody>
</table>

VII-1.3. Detecting the formation of isolas

The inset panels in Fig. VII-4a and Fig. VII-4b show the process of the unstable solutions in the subharmonic resonance detaching from the main branch as the controller gain increases from 0 to 0.00046. This process is shown in more detail in Fig. VII-5 with an intermediate step at gain 0.00040. The isola is formed when the torus bifurcation and the lower fold bifurcation collide, which ‘connects’ the stable solutions while ‘cutting off’ the unstable solutions from the main branch. Detecting isolas using numerical continuation is difficult as the method can only map out the solutions connected to the initial solution supplied by the user. However, this can be overcome by tracing the movement of the bifurcation points (fold and torus bifurcations in this case) using two-parameter continuation, shown as the blue and grey lines in Fig. VII-5. When an isola formation is suspected, such as when the torus and fold bifurcation collide, one of the fold bifurcation points can be then used as the initial solution and the isola can then be traced.

![Formation of the isola in the subharmonic resonance region](image)

To verify that the divergent motion due to unstable solutions no longer exists, the aircraft is forced at the frequency 0.03503 Hz using the three different controller gains shown in Fig. VII-5. Their time histories (Fig. VII-6) show that in the first two cases (gains 0 and 0.00040), the responses still diverge to infinity, leading to the simulation failing, while in the third case (gain 0.00046), the response is stable, confirming that the
unstable solution has been removed from the main branch. Although all solutions in the isola are unstable in this case, which removes the possibility of two possible responses for the same forcing frequency (one in the main branch and one in the isola with much higher amplitude), the presence of the unstable isola can still influence dynamic response in this region, e.g., in the case of a large pitch disturbance.

Using the same technique, the isolas formed in the superharmonic and phugoid resonances can also be traced. The point at which they detach from the main branch are shown in Fig. VII-4c and Fig. VII-4e.

**VII-1.4. Eliminating the superharmonic (period-2) resonance**

In the period-2 superharmonic region at open-loop (see Fig. VII-4a), the aircraft’s response repeats itself after two instead of one forcing cycle – a highly nonlinear behaviour that is not captured in the linear model. Fig. VII-7 shows the simulation results when the aircraft is forced at 0.137 Hz and 0.138 Hz, corresponding to the stable and unstable solutions in the period-2 region, respectively. In both cases, period-2 motions are observed, and the oscillation amplitude is very large compared to the linear model’s prediction at these frequencies. When the forcing frequency lies within the unstable region such as in the second case (0.138 Hz), the aircraft diverges. This large amplitude period-2 motion is undesirable.

Using our proportional pitch-angle-feedback controller, it is not possible to use linear design techniques to determine the controller’s effectiveness in eliminating the period-2 region as this motion is not captured by the linear model. Instead, the two-parameter continuation technique can be used to determine the controller’s effectiveness by tracking the movement of the period-doubling bifurcation in the frequency-proportional gain space – thereby providing information on how the period-2 region is modified as the controller gain increases. The result in Fig. VII-8 shows that as the controller gain increases, the two period-doubling bifurcations approach each other, reducing the size of the period-2 region, and finally merge when the controller gain
reaches 0.01379, meaning that this is the minimum gain required to eliminate the period-2 motion. To verify this, the aircraft is forced again at 0.138 Hz but with the gain set at 0.0140, shown in Fig. VII-9. Comparing to Fig. VII-7b, the oscillation is now stable and has the same period as the forcing term, confirming that the period-doubling bifurcations no longer exist in the main solution branch.

**Fig. VII-7** Open-loop period-2 response at 0.137 Hz (a) and 0.138 Hz (b)

**Fig. VII-8** Locus of the period-doubling bifurcations

**Fig. VII-9** Closed-loop response at 0.138 Hz with gain 0.014
VII-1.5. Eliminating the subharmonic resonance

Fig. VII-4e shows that at gain 0.024, all unstable and divergent solutions have been removed from the main branch, so the forced response is stable at all frequencies. However, the subharmonic resonance is still prominent at frequencies below the phugoid resonance, so a higher controller gain is needed to remove them. Since there is no longer any bifurcation on the main branch, two-parameter continuation cannot be used. Instead, one-parameter continuation of the controller gain is utilised. Referring back to Fig. VII-2a, it is known from the standard elevator-to-pitch-angle transfer function of the linear model that below the phugoid frequency (0.065 Hz), the response amplitude is proportional to the forcing frequency (i.e., higher forcing frequency leads to larger response, up to 0.065 Hz). This is not the case in the nonlinear model in the presence of the subharmonic resonance peaks. It is desirable to know the controller gain required to remove these subharmonic resonances, therefore making the full aircraft behave more like its linear counterpart. To do this, one-parameter continuation of the controller gain is done at two frequencies: 0.0350 Hz (near the peak of the subharmonic resonance in Fig. VII-2a) and a nearby point at higher frequency, chosen as 0.0385 Hz in this case. The result in Fig. VII-10 shows that as the controller gain increases from 0, the response amplitude at the lower frequency (0.0350 Hz) rapidly decreases and eventually becomes lower than the response at the higher frequency (0.0385 Hz) for controller gain beyond 0.07. This shows that in order to remove the subharmonic resonance, the controller gain cannot be less than 0.07. As shown in Fig. VII-4f, at gain 0.08, most of the subharmonic resonance has been removed.

![Fig. VII-10 Continuation of the controller gain at the subharmonic region for two different frequencies](image)

VII-1.6. Eliminating the unstable isolas

Fig. VII-4f indicates that at gain 0.08, the nonlinear frequency response is stable across the entire frequency range considered and resembles a linearised response. However, the three unstable isolas that emerged from the subharmonic, phugoid, and superharmonic resonances still exist and can influence the aircraft’s dynamics, such as in the case of a large disturbance. The edges of these isolas are three pairs of fold bifurcations labelled A to F as shown in Fig. VII-4f. Again, by using two-parameter continuation to track the movement of these fold bifurcations, the gain required to remove each isola can be determined. Fig. VII-11 shows the two-
parameter continuation of the points A to F as the controller gain changes. Despite their complex trajectory, it can be seen that each pair of fold bifurcation will merge when the controller is high enough, indicating the disappearance of the isola. The gains required to remove each isola are listed as the last three items of Table 9. For a gain above 0.148, the final isola that emerges from the phugoid resonance ceases to exist.

![Graph showing parameter continuation of all fold bifurcations](image)

**Fig. VII-11 Two-parameter continuation of all fold bifurcations shown in Fig. VII-4f**

In summary, this section has so far shown the advantages of using nonlinear analysis in identifying the undesirable dynamics in the frequency domain that are not captured by the linear model. Using both one and two-parameter continuation of the controller gain, we can assess the controller’s effectiveness in removing those undesirable resonances.

### VII-1.7. Dynamics at an unstable operating point

Now consider the polynomial GTM trimmed for straight-and-level flight 18° angle-of-attack (operating point 2 in Table 7). As short-period mode is unstable here, the unforced aircraft will enter a limit cycle without any external input. Forcing the open-loop aircraft at this angle-of-attack will result in motions that appear to be quasi-periodic as shown in Fig. VII-12. The linear model cannot capture this behaviour but can only indicate that the aircraft is unstable. Therefore, the linear aircraft diverges to infinity as soon as it is perturbed from the trim condition, whether forced or unforced. Due to these differences, the open-loop frequency response is not examined for this operating point.

![Phase plot of the open-loop nonlinear aircraft](image)

**Fig. VII-12 Phase plot of the open-loop nonlinear aircraft responding to an elevator forcing at 1 Hz**
To stabilise the short-period mode, a pitch-rate-feedback controller with proportional gain 0.05 is used (similar to the scheme in Fig. VII-3 but using pitch rate for the feedback signal). Fig. VII-13 compares the linear and nonlinear closed-loop unforced responses when subjected to an elevator perturbation of 0.1° and duration 0.1s. Whilst the short-period responses are similar, it can be seen that the low-frequency content from the phugoid mode is much less damped in the nonlinear model.

Fig. VII-13 Linear (a) and nonlinear (b) closed-loop aircraft responding to an elevator perturbation

Fig. VII-14 Linear and nonlinear closed-loop pitch-angle-to-elevator Bode plots (a) and forced response at 0.07358 Hz (b)

The reduced damping of the phugoid mode in the nonlinear response is more also reflected in the frequency domain. Fig. VII-14a compares the linear and nonlinear closed-loop frequency responses, where large discrepancies are seen at low frequencies. In particular, a large difference in gain and phase is observed around the phugoid frequency, approximately near 10⁻¹ Hz. As the forcing frequency is reduced further, the gain again becomes similar while the phase difference increases to as high as 180°. This means that at low frequencies, the nonlinear pitch angle response is completely out of phase with the linear response and hence with the pilot input. To verify this, the linear and nonlinear models are simulated with a harmonic elevator input of magnitude 1° and frequency 0.07358 Hz (Fig. VII-14b). This frequency was chosen as it is near the peak of the phugoid resonance, where the largest discrepancies in both gain and phase are observed between the linear and nonlinear models. It can be seen that although the controller has stabilised the aircraft, the
responses are completely out of phase and have different magnitudes. Although this reduction in handling quality is not captured by the linear model, the use of continuation methods can provide insights on the issue, specifically to inform the control designers that further investigation is required. It can also be inferred that the reduced performance is only encountered during an active manoeuvre, as simple time-domain tests like Fig. VII-13b do not reflect the severity of the situation. This further underlines the importance of a nonlinear frequency-based analysis in ensuring that the controller achieves satisfactory performance and robustness.
VII-2. ASYMMETRIC AIRCRAFT ANALYSIS

For the final study, the frequency responses of the full GTM is examined without any reduction in the nonlinear equations of motion and the aerodynamic data. This results in an asymmetric 8th-order asymmetric aircraft with three control surfaces: elevator, aileron, and rudder. Apart from the goal of exploring lateral-directional dynamics, this section also highlights the limitations of nonlinear frequency methods. All analysis was done with the aircraft trimmed at 8.7° angle-of-attack, where the phugoid instability is observed previously. Similarly, a small forcing amplitude of 1° is used in all cases. Because the full GTM is asymmetric, any perturbation to one of its control surfaces will induce both longitudinal and lateral-directional motions.

Background on the full GTM model: this is a Simulink implementation referred to as DesignSim by NASA [128]. Its aerodynamic data is stored in lookup tables and are based on wind-tunnel tests for angles-of-attack between -5° and 85° and sideslip angles up to 45°. Spline interpolation is utilised to ensure smoothness for the continuation algorithm. Unforced bifurcation analysis of this model has been done in [4, 56, 57].

VII-2.1. Elevator forcing

The open-loop elevator-to-pitch-angle Bode plot of the linearised and nonlinear full GTM are shown in Fig. VII-15. Solutions between 0.057 and 0.0606 Hz could not be computed due to numerical difficulties. Similar to the polynomial GTM, a pair of period-doubling bifurcation exist between the phugoid and short-period frequencies (see the right inset in Fig. VII-15b), and simulation verifies the existence of the period-2 motion (Fig. VII-16). Two-parameter continuation of these period-doubling bifurcations (Fig. VII-17) shows that this region is removed if a proportional pitch-angle-feedback controller with gain 0.047 or larger is used.

![Fig. VII-15 Open-loop elevator-to-pitch-angle frequency responses of the linearised (a) and nonlinear (b) NASA GTM. The period-2 solution branch around 0.11 Hz is not solved](image-url)
Subharmonic resonance is observed below the phugoid frequency and a region of two stable solutions is found (see the left inset in Fig. VII-15b). Fig. VII-18 shows two simulated responses with different initial conditions, which confirms the existence of two stable solutions in this region. More notably, elevator forcing in this case has excited the lateral-directional modes and resulted in large-amplitude oscillation that is not restricted to the longitudinal plane. This can be seen in the time histories of the bank angle ($\phi$). It took the aircraft a very long time to converge to the periodic orbit shown in Fig. VII-18, which indicates that the dynamics involves a slow mode.
VII-2. Aileron forcing

The open-loop aileron-to-bank-angle frequency responses of the linearised and nonlinear full GTM are shown in Fig. VII-19. The inset in the nonlinear response highlights a region with three stable solutions at just below 0.062 Hz. As a result, the same forcing frequency will lead to three very different responses with a peak pitch angle of around 9\(^\circ\), 30\(^\circ\), and 50\(^\circ\), depending on the aircraft’s initial condition. Their existence is verified in time simulation (Fig. VII-20). We note that 0.062 Hz is the phugoid frequency, and that the 30\(^\circ\) and 50\(^\circ\) responses both induce large-amplitude oscillations in the longitudinal plane (see the \(\theta\) time history) even though we are forcing the aileron at a very small amplitude of 1\(^\circ\). This cross-coupling between the longitudinal and lateral-directional modes is not reflected in the linear response – only a small ‘blip’ around 0.062 is shown in the linear Bode plot. Nevertheless, these secondary resonances consist of only weak attractors that requires a long duration of forcing to converge to their final values.

![Fig. VII-19 Linear (a) and nonlinear (b) aileron-to-bank-angle frequency response](image1)

![Fig. VII-20 Three different possible responses to aileron forcing at 0.0617 Hz](image2)
The coexistence of multiple solutions is further investigated by running a large number of simulations at 0.0617 Hz forcing with different initial yaw and bank angles (while keeping the initial conditions of the remaining six states at their trim values). Fig. VII-21 shows the result of those simulations with the initial bank and side slip angles on the x and y axes. The colour of each point indicates peak pitch angle in the final oscillation cycle after a 1000s simulation, which would converge to one of the three stable solutions of $\theta$ (9°, 30°, and 50°; the first one corresponds to normal flight). The following observations can be made:

- The normal flight response (9° – blue) dominates the central-right region, which reflects the asymmetric nature of the GTM.

- The highest amplitude response (50° – yellow) is the least likely to occur. However, it can scatter almost randomly in a green-dominant region. One such example is shown in the magnified view, in which for the same sideslip angle of 0°, the three bank angles of -13°, -12°, and -11° converge to three different solutions. This means that changing the initial condition by just one degree can result in a vastly different response, which highlights the nonlinear nature of the GTM. Similar behaviour can also be observed in other sections of Fig. VII-21.

*Fig. VII-21  Basin of attractions – aileron forcing at 0.0617 Hz*
Fig. VII-22 shows the basin of attraction generated from a similar set of simulations but with the initial pitch angle reduced to $-50^\circ$, which now resembles an upset condition. Accordingly, blue region (normal flight) has shrunk considerably, and the effect of asymmetries is more visible.

VII-2.3. Rudder forcing

The linear and nonlinear open-loop rudder-to-yaw-rate frequency responses are shown in Fig. VII-23. In addition to the subharmonic resonances around the phugoid frequency as discussed above, a pair of torus bifurcation between 0.63 and 0.78 Hz (near the middle of Fig. VII-23b) results in a region of stable quasi-periodic response. Time simulation of this quasi-periodic motion is shown in Fig. VII-24a, in which the longitudinal state $\theta$ appears to be oscillating at a completely different frequency to the rudder forcing. At lower frequencies, the inset in the nonlinear frequency response also indicates a pair of period-doubling bifurcation, which gives rise to a very complex period-7 motion shown in Fig. VII-24b. As before, strong and complex cross coupling between the longitudinal and lateral-directional modes are seen in all these attractors.

Fig. VII-23 Linear (a) and nonlinear (b) open-loop rudder-to-yaw-rate frequency responses
Fig. VII-24  Rudder forcing: 0.65 Hz (a) and 0.0604 Hz (b)
VII-3. CONCLUDING REMARKS

We have observed a few examples of how small-amplitude forcing can induce complex modal coupling when aerodynamic asymmetries are included. Bifurcation analysis has correctly predicted the dynamics in all cases, although time simulations indicate that these secondary attractors are rather weak and slow. The dynamics may be fascinating to observe, but it could be argued that including these complex features in nonlinear frequency analysis is redundant. These slow attractors can ‘clog up’ the Bode plot while providing little useful information to the control designers. Therefore, nonlinear frequency analysis as a design tool may function best in reduced-order or full-order symmetric models. As seen in the second-order, PIO, and deep stall analyses, our proposed approach still reveals many valuable insights in reduced-order and symmetric applications that are otherwise not reflected using existing linear-based methods.
Nonlinearities remain a challenging problem in flight dynamics and are an inherent feature of all aircraft designs. With the ongoing drive to build more complex and efficient air vehicles that continuously push the operating envelope, nonlinear dynamics will be an increasingly common feature. Furthermore, the application of future flight-control technologies will introduce new challenges that may not be adequately captured by the existing techniques. This thesis has proposed a new analysis method to bridge this gap. Using bifurcation analysis of harmonically forced systems, it is possible to assess the non-stationary nonlinear elements that play a major role in the aircraft’s flying characteristics. This systematic consideration of the transient dynamics is necessary to uncover many important phenomena that are otherwise undetected. Additionally, presenting the results in the form of a nonlinear Bode plot provides an intuitive platform for the practising engineers to interpret the data. It could be said that the nonlinear frequency method is an extension to bifurcation methods in the flight dynamics and control context – therefore presenting a new contribution to the growing literature on nonlinear analysis.

In this thesis, it has been shown that bifurcation analysis in the form of nonlinear frequency response has the ability to:

- Assess the closed-loop transient dynamics as the aircraft transition across different operating points (chapter III). In the high angles-of-attack regions where elevator effectiveness is reversed, nonlinear frequency analysis provides an indication of degraded controller performance that corresponds to the poor performance observed in the step responses. This is in contrast to unforced bifurcation analysis and linear-based methods, which indicate no potential issues and predict an aircraft with consistent handling qualities. The use of nonlinear frequency analysis to quantify closed-loop performance can also infer the existence of highly undesirable behaviours, including secondary attractors, rate-limited limit cycles, and jump resonance. These examples show that the closed-loop transient dynamics can be inferred from the nonlinear closed-loop Bode plot, therefore extending the notion of ‘frequency response’ beyond the open-loop stability margin idea.

- Inform the control designers of instances where actuator saturation can severely degrade flying qualities and can potentially lead to pilot-induced oscillations (chapter IV). Specifically, it was shown that both unforced and forced bifurcation analysis can be integrated into existing pilot-induced oscillation criteria to account for the effects of both rate and travel saturations. This results in a more accurate and efficient PIO prediction method comparing to existing procedures. Furthermore, nonlinear frequency analysis of the Space Shuttle reveals that when heavy rate saturation is present in a feedback loop, a pair of fold bifurcations can form in the nonlinear closed-loop frequency response. They cause a large phase jump of 180° and lead to the infamous flying qualities cliff phenomenon, which is extremely hard to predict and has been attributed to many high-profile PIO incidents. Both rate saturation and time delay contributed to the flying qualities cliff of the Space Shuttle, which agree with NASA’s conclusion based
on experience. However, nonlinear frequency analysis can accurately predict these phenomena and present the result in a simple and intuitive manner. Jump resonance like this can be found in very simple feedback systems with saturation as shown in section II-5.

- Identify the optimal stick-pumping frequency to assist deep stall recovery (chapter V). The idea to rock the aircraft nose out of a locked-in deep stall had been explored previously in the literature but brought mixed results, mainly due to the high pilot workload required to observe and match the input frequency with the aircraft’s natural modes. The physics behind this manoeuvre was verified using nonlinear frequency analysis, which detected unstable solutions that lead to diverging oscillations and showed that it is possible to destabilise the deep stall trim point via harmonic forcing of the pitch control device. However, the optimal forcing frequency is slightly below the linear resonance frequency due to softening behaviour at high angles-of-attack. In fact, recovery may not be possible if the pilot pumps the stick at the linear resonance frequency. The information obtained from nonlinear frequency response allows recovery to be done in an open-loop manner, which significantly reduces pilot workload. The analysis was also made interesting by the fact that unstable solutions in this instance are desirable, unlike what is observed in the rest of the thesis.

- Reflect on the effects of unsteady aerodynamics on an aircraft’s transient motion (chapter VI). This observation is similar to the discussion on closed-loop transient dynamics in chapter III. It was found that quasi-steady aerodynamic modelling – the most common method in use at the moment – is adequate for T-tailed transport aircraft that do not undergo rapid manoeuvring. Conversely, the hypothetical highly-unsteady study resembling a high-performance aircraft revealed remarkably different dynamics with a general degradation in handling qualities. Both examples showed that the state-space method for aerodynamic modelling is a feasible alternative for quasi-steady modelling in both instances. These results are also one of the first to directly investigate the link between unsteady aerodynamics and flight dynamics and control. Furthermore, chapter VI also demonstrates how unforced bifurcation analysis and nonlinear frequency response can be combined to gain a deeper understanding than what can be achieved by using either method separately.

- Detect sub- and super-harmonic resonances (section VII-1 of chapter VII). These secondary resonances are reflected as torus and period-doubling bifurcations in the nonlinear frequency response, which led to slow divergence at certain frequencies. Two-parameter continuation in the gain-frequency plane revealed the exact mechanism of how the controller gain shrunk and eventually eliminated the unstable regions, thereby providing a method to assess controller effectiveness in scenarios where linear analysis cannot be used. At a higher angle-of-attack where the open-loop aircraft is unstable, the linear and nonlinear closed-loop frequency responses differ significantly at low frequencies, which led to a 180° difference in phase response. Time simulation confirmed this prediction, despite the fact that the linear and nonlinear impulse responses were very similar. This suggested that simple time-domain tests like step and impulse responses would not detect the reduced performance, which would only surface during
a dynamic manoeuvre. A nonlinear frequency-domain analysis is therefore important to ensure the controller achieves the desired level of robustness and performance.

- Reveal the complex modal coupling dynamics due to aerodynamic asymmetries (section VII-2 of chapter VII). Large-amplitude lateral-directional oscillations can be induced by forcing the pitch control surfaces and vice versa, thereby revealing a case of strong nonlinear coupling. Quasi-periodic and period-7 motions were detected, in addition to coexisting stable solutions. Although fascinating from a dynamics point of view, these attractors are weak and may not provide useful information to the control designers. Therefore, it may be necessary to apply appropriate simplifications to the model to make the most use of the method.

Overall, the work has demonstrated the potential for bifurcation analysis in the form of nonlinear frequency responses to give valuable information to assess and respond to a range of rich dynamical features exhibited by aircraft in flight. Two promising topics for further work are in flexible aircraft dynamics and unsteady aerodynamics – both of which have been briefly discussed in the thesis. The former reflects the trends of increasingly slender designs with higher fuel efficiency in the aerospace industry, which comes with even higher potential for geometric nonlinearity in the flexible wing and can lead to coupling with the rigid-body modes. As seen in the sub-harmonic resonance of the GTM, it is expected that similar phenomena could exist in a flexible aircraft model with active control and flutter suppression systems and lead to degraded flying qualities. Regarding unsteady aerodynamics, expanding the analysis to include lateral-directional motions presents a major step up from the results presented in this thesis. Doing so provides the capability to analyse spin and upset/loss-of-control dynamics in the post-stall regimes, which poses a threat to both civil and military applications and involve highly unsteady aerodynamic phenomena. With all the techniques developed here, it is hoped that future flight controllers could be designed in a manner that pre-emptively anticipates these nonlinear non-stationary phenomena and eventually suppresses them.

The next step of development is to facilitate industrial usage of nonlinear frequency analysis, possibly by showcasing the method using an industrial-standard model as discussed in [1]. The numerical analysis software DST used in this thesis is already approved by Airbus [55], thereby removing the challenge of certification for aerospace applications. Finally, the use of nonlinear frequency analysis in other engineering applications should also be explored.
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REFERENCES


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APPENDIX

A. PROOF OF THE HARMONIC OSCILLATOR EQUATIONS

The third and fourth states of equation (b.3) on page 44 are reproduced below:

\[ \begin{align*}
\dot{x}_3 &= x_3 + \omega x_4 - x_3(x_3^2 + x_4^2) \\
\dot{x}_4 &= -\omega x_3 + x_4 - x_4(x_3^2 + x_4^2)
\end{align*} \tag{f.1} \]

Introducing the complex variable \( z = x_4 + ix_3 \). We have:

\[ |z|^2 = x_3^2 + x_4^2 \tag{f.2} \]

and

\[ \begin{align*}
\dot{z} &= \dot{x}_4 + i\dot{x}_3 \\
\dot{z} &= [-\omega x_3 + x_4 - x_4(x_3^2 + x_4^2)] + i[x_3 + \omega x_4 - x_3(x_3^2 + x_4^2)] \\
\dot{z} &= (x_4 + ix_3) - (x_3^2 + x_4^2)(x_4 + ix_3) + i\omega(x_4 + ix_3) \\
\dot{z} &= z - |z|^2 z + i\omega z \\
\dot{z} &= z(1 - |z|^2 + i\omega)
\end{align*} \tag{f.3} \]

Now transform \( z \) to its polar form of \( z = re^{i\theta} \), giving \( |z|^2 = r^2 \). Substitute these in equation (f.3):

\[ \dot{z} = re^{i\theta}(1 - r^2 + i\omega) \tag{f.4} \]

Furthermore, the first derivative of \( z = re^{i\theta} \) is:

\[ \dot{z} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} = e^{i\theta}(\dot{r} + i\dot{\theta}r) \tag{f.5} \]

Equating (f.4) and (f.5) and cancelling \( e^{i\theta} \) gives:

\[ r(1 - r^2) + i\omega r = \dot{r} + i\dot{\theta}r \tag{f.7} \]

Therefore:

\[ \begin{align*}
\dot{r} &= r(1 - r^2) \\
\dot{\theta} &= \omega
\end{align*} \tag{f.8} \]

It can be seen that \( r = 1 \) gives \( \dot{r} = 0 \). In this instance, the system becomes:
\[
\begin{align*}
\dot{r} &= 0 \\
\dot{\theta} &= \omega
\end{align*}
\] (f.9)

Which describes a phasor of constant radius 1 and constant angular velocity \( \omega \) rad/s. Its real and imaginary components are \( x_4 = \cos \omega t \) and \( x_3 = \sin \omega t \), respectively. To ensure that \( r = 1 \) at the start of the simulation, set \( x_3(t = 0) = 0 \) and \( x_4(t = 0) = 1 \).

We can now couple the states \( x_3 \) and/or \( x_4 \) into the system’s equations of motion to generate the harmonic forcing term in autonomous form for bifurcation analysis.
B. MATLAB CODE TO GENERATE THE FREQUENCY RESPONSE OF THE DUFFING EQUATION

For future reference, the code for open-loop forced bifurcation analysis of the Duffing equation is presented in this section. This code has been tested on MATLAB 2014a, 2015b, 2016b, and 2019a. There are seven files listed below. Dynamical Systems Toolbox 3.1 is required for bifurcation analysis. For time simulation, only the first two files are needed.

<table>
<thead>
<tr>
<th>File</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>run.m</td>
<td>Run the time simulation and bifurcation analysis</td>
</tr>
<tr>
<td>eom.m</td>
<td>Equations of motion</td>
</tr>
<tr>
<td>c_1.m</td>
<td>AUTO constant file 1 for one-parameter continuation</td>
</tr>
<tr>
<td>c_2para.m</td>
<td>AUTO constant file 2 for two-parameter continuation</td>
</tr>
<tr>
<td>f_AUTO_duffing.m</td>
<td>AUTO function file – used to call the eom.m file and feed the state derivatives to AUTO</td>
</tr>
<tr>
<td>steady_state_f.m</td>
<td>Collect time simulation data for generate_data_f_MHL.m (written by Dr Djamel Rezgui)</td>
</tr>
<tr>
<td>generate_data_f_MHL.m</td>
<td>Generate the starting periodic solution for AUTO (written by Dr Djamel Rezgui)</td>
</tr>
</tbody>
</table>

File 1: run.m

```matlab
% Frequency response of the Duffing equation
% See DOI 10.2514/1.G005197 for the published version of these results
% DST version: 3.1.0

% Control parameters and initial conditions
c = 0.2; k = 1; alpha = 0.05; A = 2.5; omega=3;
u0 = [c; k; alpha; A; omega];
x0 = [0; 0; 0; 1]; % x, xdot, sin(0), cos(0)

% Simulate
options = odeset('reltol',1e-6,'abstol',1e-6);
numCyclesOfSimulation = 50;
tmax = numCyclesOfSimulation*(2*pi/omega); % number of forcing cycles to be simulated
tspan = [0 tmax];
[t,y] = ode45(@eom,tspan,x0,options,u0);
figure
subplot(4,1,1); plot(t,y(:,1)); ylabel('x');
subplot(4,1,2); plot(t,y(:,2)); ylabel('V');
subplot(4,1,3); plot(t,y(:,3)); ylabel('sin(\omegat)');
subplot(4,1,4); plot(t,y(:,4)); ylabel('cos(\omegat)'); xlabel('Time (s)')

% Generate the initial solution of one cycle (written by M Lowenberg and D Rezgui)
period=2*pi/omega;
[time_steady,states_steady] = steady_state_f(t,y,period);
datfilename='data_duffing_one_cycle'
generate_data_f_MHL(time_steady,states_steady,datfilename)

% Plot a cycle if desired - e.g. x vs time:
% fni=char(datfilename); fn2=char('.dat');
% fn=strcat(fni,fn2);
% load(fn);
% figure
% plot(data_duffing_one_cycle(:,1),data_duffing_one_cycle(:,2))
% ylabel('x')
% xlabel('Time (s)')

% Generate the frequency response
a{1}=auto;
a{1}.s.FileNames.DatFileName='data_duffing_one_cycle.dat';
a{1}.s.FileNames.FunctionFileName='f_AUTO_duffing';
a{1}.s.InitialConditions.Par0=u0;
a{1}.s.InitialConditions.U0=x0;
```

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% Phase calculation - based on time history data in the a{1}.f8.Ups object
% see DOI 10.2514/1.G005197 for the full definition of phase
% find the normalised time at the peak of states 1 and 2
peak_x=[];
[temp1, temp2] = max(a{1,1}.f8.Ups(:,1,:)); peak_x = [peak_x squeeze(temp2(1,1,:))];
[temp1, temp2] = max(a{1,1}.f8.Ups(:,2,:)); peak_x = [peak_x squeeze(temp2(1,1,:))];

% find the normalised time at the peak of states 4 (cos(\omega t)), which is linked to the forcing term
peak_u=[];
[temp1, temp2] = max(a{1,1}.f8.Ups(:,4,:)); peak_u = [peak_u squeeze(temp2(1,1,:))];

phase=[]; temp=[];
for i = 1:2
    for j = 1:size(peak_x(:,i),1);
        temp=[temp; (a{1,1}.f8.Tm(peak_u(j,1),j)-a{1,1}.f8.Tm(peak_x(j,i),j))];
        if temp(j,1)>90/360; temp(j,1)=temp(j,1)-1;
        end
        phase(j,i)=temp(j,1);
    end
    temp=[];
end
phase=phase*360;

% Save phase data to a{1,1}.f7.U(:,5/6)
a{1,1}.f7.U(:,5) =phase(:,1);
a{1,1}.f7.U(:,6) =phase(:,2);

% Plot the frequency response
p=plautobj; set(p,'title','Frequency response','lcStyle','o','labelPts','on');
figure;
subplot(211); set(p,'title','Frequency response','xEqStr','P(5)','xLab','\omega (rad/s)','yEqStr','U(1)','yLab','x_{max}','newPlot','off');
ploteq(p,a(1)); set(p,'newPlot','on');
subplot(212); set(p,'title','Two-parameter continuation trace the locus of the fold bifurcations as a function of amplitude and frequency
% Generate data for the two-parameter continuation from point 45
a[2]=a[1];
a[2].c=copy(a[2].c);
a[2].c=c_2para(a[2].c);
a[2]=runauto(a[2]);

% Start the two-parameter continuation
a[3]=a[2];
a[3].c=copy(a[3].c);
a[3].c=c_2para(a[3].c);
a[3].c.Irs=135;
a[3]=runauto(a[3]);

% Plot the locus of the fold bifurcation
p=plautobj;
set(p,'title','Two-parameter continuation','lcStyle','o','labelPts','on');
set(p,'xEqStr','P(5)','xLab','\omega (rad/s)','yEqStr','P(4)','yLab','A');
ploteq(p,a(1));
File 2: eom.m

```matlab
function xd = eom(t,x,u)
c=u(1); k=u(2); alpha=u(3); A=u(4); omega=u(5);
x1=x(1); x2=x(2); x3=x(3); x4=x(4);

% Duffing equation: xdotdot + c*xdot + k*x + alpha*x^3  = A*cos(omega*t)
x1dot = x2;
x2dot = -c*x2-k*x1-alpha*x1^3+A*x4;

% Harmonic oscillator equations
x3dot = x3+omega*x4
   - x3*(x3^2+x4^2);

x4dot = -omega*x3+x4 - x4*(x3^2+x4^2);

xd = [x1dot; x2dot; x3dot; x4dot];
```

File 3: c_1.m

```matlab
function c=c_1(c)
set(c,'Ndim',4,'Ips',2,'Irs',1,'Ilp',1);
set(c,'Icp',[5,11]);
set(c,'Ntst',50,'Ncol',4,'Iad',3,'Isp',2,'Isw',1,'Ilplt',0,'Nbc',0,'Nint',0);
set(c,'Nmx',200,'Rl0',0.1,'Rl1',1,'A0','-2500','A1',2500);
set(c,'Npr',1,'Mxbf',10,'Iid',2,'Itnx',8,'Itnw',5,'Nwtn',3,'Jac',0);
set(c,'Epsl',1e-007,'Epsu',1e-007,'Epss',0.00001);
set(c,'Ds','-0.2','Dsmin',0.01,'Dsmax',0.5,'Iads',1);
set(c,'Thl',[]);
set(c,'Thu',[]);
set(c,'Uzr',[]);
```

File 4: c_2para.m

```matlab
function c=c_2para(c)
set(c,'Ndim',4,'Ips',2,'Irs',45,'Ilp',1);
set(c,'Icp',[4,5,11]);
set(c,'Ntst',50,'Ncol',4,'Iad',3,'Isp',2,'Isw',1,'Ilplt',0,'Nbc',0,'Nint',0);
set(c,'Nmx',500,'Rl0',0.1,'Rl1',2.501,'A0','-2500','A1',2500);
set(c,'Npr',1,'Mxbf',10,'Iid',2,'Itnx',8,'Itnw',5,'Nwtn',3,'Jac',0);
set(c,'Epsl',1e-007,'Epsu',1e-007,'Epss',0.00001);
set(c,'Ds','-0.1','Dsmin',0.01,'Dsmax',0.5,'Iads',1);
set(c,'Thl',[]);
set(c,'Thu',[]);
set(c,'Uzr',[]);
```

File 5: f_AUTO_duffing.m

```matlab
function [f,o,dfdu,dfdp]=f_AUTO_duffing(u,x,ijac)
f=[]; % derivative values, same size as Ndim
o=[]; % additional outputs, size automatically detected
dfdu=[]; % user-defined derivatives for states, this parameter empty when Jac=0
ddfp=[]; % user-defined derivatives for parameters, this parameter empty when Jac=0
xd = eom(0,x,u);
f = xd;
```
File 6: steady_state_f.m

```matlab
function [time_steady,states_steady] = steady_state_f(time,states,period)
% [time_steady,states_steady] = steady_state_f(time,states,period)
% This function calculates the states variation for one period of time
% at steady state region.
% time: is the time spent for simulation.
% states: is the array that holds the state values
% period: is the period for one cycle.
% states_steady: is the steady state array.
% time_steady: is the time at steady state for one period.
% Written by Djamel Rezgui in June 2006.
% University of Bristol.
% See also generate_data_f.
%*************************************************************************

n = length(time);
j = 1;

for i = n:-1:2
    rev_n = mod(time(i),period);
    rev_n1 = mod(time(i-1),period);
    if ((rev_n1 - rev_n) > period/2)
        limit(j) = i;
        j = j+1;
    end
end
l1 = limit(2);
l2 = limit(1)-1;
states_steady = states(l1:l2,:);
%states_steady(:,1) = mod(states_steady(:,1),2*pi);
time_steady = time(l1:l2) - time(l1);
```

---

File 6: steady_state_f.m
function generate_data_f(time_steady,states_steady,datfilename)
% This function generates steady state data for one cycle. The data is
% written as .dat file and is needed for Auto97 to start periodic
% continuation runs.
% This file needs two input data. (They can be generated by steady_state.m)
% time_steady: is the column vector of the time spent in one cycle (sec).
% states_steady: is an array of the states history in one cycle. The state
% values have to be saved as column vectors of the states_steady array.
% Written by Djamel Rezgui in June 2006.
% University of Bristol.
% See also steady_state_f.
%***************************************************************
clc;
% [fn,pn]=uiputfile('*.dat', 'Specify a File Name');
fn1=datfilename; fn2='.dat';
fn=strcat(fn1,fn2);
pn='';
file_id = fopen([pn,fn], 'wt');
steady = [time_steady,states_steady];
nn = size(steady);
n = nn(1,1);
m = nn(1,2);
i_count=0;
for i = 1:1:n
    %for i = 1:5:n
    i_count=i_count+1;
    m_auto=0;
    for j = 1:m
    % Only print out the columns that correspond to states in the AUTO system
    % (the "+1" is a reminder that there is an extra column relative to AUTO,
    % namely time in column 1):
    % if (j==1+1) & (j==7+1) & (j==8+1) & (j==10+1) & (j==11+1) & ...
    % (j==12+1) & (j==13+1) & (j==14+1) & (j==17+1) & (j==18+1) & ...
    % (j==21+1) & (j==22+1) & (j==27+1) & (j==28+1) & (j==43+1)
    if (j==1+9)
        m_auto=m_auto+1;
        fprintf(file_id,' % 12.8f ',steady(i,j));
    end;
    fprintf(file_id,'
');
    %fprintf('** Steady state data for
    Steady state data for one cycle has been generated in file: %s
',fn);
end;
fclose(file_id);
C. LONGITUDINAL F-16 MODEL

This section presents the steps to construct the 4th-order longitudinal F-16 simulation. The model contains only longitudinal aerodynamic data from reference [17] and is valid for angles-of-attack between –20° and 90°. The use of leading edge devices, flaps, and speed brake is not considered.

Flow chart of the F-16 simulation

Aircraft parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$ wing area</td>
<td>28.87 m$^2$</td>
</tr>
<tr>
<td>$c$ mean aerodynamic chord</td>
<td>3.4503 m</td>
</tr>
<tr>
<td>$m$ mass</td>
<td>9,294 kg</td>
</tr>
<tr>
<td>$\rho$ air density (at 30,000 ft)</td>
<td>0.45831 kg/m$^3$</td>
</tr>
<tr>
<td>$I_y$ pitch moment of inertia</td>
<td>75,643 kg m$^2$</td>
</tr>
<tr>
<td>$g$ gravitational acceleration</td>
<td>9.81 m/s$^2$</td>
</tr>
</tbody>
</table>

Input parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_s$ stabiliser deflection</td>
<td>$-25^\circ \leq \delta_s \leq 25^\circ$</td>
</tr>
<tr>
<td>$T$ thrust</td>
<td>8,785 N</td>
</tr>
<tr>
<td>$cg$ C.G. position</td>
<td>37.5 (% MAC)</td>
</tr>
</tbody>
</table>

Equations of motion:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$ angle-of-attack</td>
<td>(rad)</td>
</tr>
<tr>
<td>$V$ velocity</td>
<td>(m/s)</td>
</tr>
<tr>
<td>$q$ pitch rate</td>
<td>(rad/s)</td>
</tr>
<tr>
<td>$\theta$ pitch angle</td>
<td>(rad)</td>
</tr>
</tbody>
</table>

\[
\dot{\alpha} = \frac{1}{mV} \left[ \frac{1}{2} \rho V^2 S (C_Z \cos \alpha - C_X \sin \alpha) - T \sin \alpha + mg \cos(\theta - \alpha) \right] + q
\]

\[
\dot{V} = \frac{1}{m} \left[ \frac{1}{2} \rho V^2 S (C_Z \sin \alpha + C_X \cos \alpha) + T \cos \alpha - mg \sin(\theta - \alpha) \right]
\]

\[
\dot{q} = \frac{1}{2} \rho V^2 S c \frac{C_M}{I_y}
\]

\[
\dot{\theta} = q
\]

Note that $\theta - \alpha$ equals the flight path angle $\gamma$. 
To plot the flight trajectory as in Fig. V-4 on page 119, integrate the following two equations:

\[
\begin{align*}
\dot{x} &= V \cos(\theta - \alpha) \\
\dot{z} &= -V \sin(\theta - \alpha)
\end{align*}
\]

Lookup tables:

\[
\begin{align*}
C_X &= C_{X_{\alpha,\delta_s}}(\alpha, \delta_s) + \frac{c_q}{2V} C_{X_q}(\alpha) \\
C_Z &= C_{Z_{\alpha,\delta_s}}(\alpha, \delta_s) + \frac{c_q}{2V} C_{Z_q}(\alpha) \\
C_M &= C_{M_{\alpha,\delta_s}}(\alpha, \delta_s) + \frac{c_q}{2V} C_{M_q}(\alpha) + \Delta C_{M_\alpha}(\alpha) + \left(0.35 - \frac{c_g}{100}\right) C_Z
\end{align*}
\]

The result presented in this thesis used MATLAB’s spline interpolation/extrapolation for 2D tables and ‘pchip’ interpolation for 1D tables.

<table>
<thead>
<tr>
<th>(\alpha) (deg)</th>
<th>(-25)</th>
<th>(-10)</th>
<th>(0)</th>
<th>(10)</th>
<th>(25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-20)</td>
<td>-0.1868</td>
<td>-0.1223</td>
<td>-0.0933</td>
<td>-0.0884</td>
<td>-0.1141</td>
</tr>
<tr>
<td>(-15)</td>
<td>-0.1875</td>
<td>-0.1188</td>
<td>-0.0978</td>
<td>-0.101</td>
<td>-0.145</td>
</tr>
<tr>
<td>(-10)</td>
<td>-0.1787</td>
<td>-0.1147</td>
<td>-0.0982</td>
<td>-0.1092</td>
<td>-0.1633</td>
</tr>
<tr>
<td>(-5)</td>
<td>-0.1432</td>
<td>-0.0893</td>
<td>-0.0784</td>
<td>-0.0902</td>
<td>-0.1337</td>
</tr>
<tr>
<td>(0)</td>
<td>-0.1132</td>
<td>-0.0617</td>
<td>-0.0489</td>
<td>-0.0606</td>
<td>-0.1075</td>
</tr>
<tr>
<td>(5)</td>
<td>-0.0693</td>
<td>-0.0172</td>
<td>-0.0066</td>
<td>-0.02</td>
<td>-0.0785</td>
</tr>
<tr>
<td>(10)</td>
<td>-0.012</td>
<td>0.0399</td>
<td>0.049</td>
<td>0.0313</td>
<td>-0.0336</td>
</tr>
<tr>
<td>(15)</td>
<td>0.0537</td>
<td>0.1027</td>
<td>0.1072</td>
<td>0.0829</td>
<td>0.0212</td>
</tr>
<tr>
<td>(20)</td>
<td>0.0951</td>
<td>0.1322</td>
<td>0.1283</td>
<td>0.0971</td>
<td>0.0187</td>
</tr>
<tr>
<td>(25)</td>
<td>0.1111</td>
<td>0.1407</td>
<td>0.13</td>
<td>0.0949</td>
<td>0.0198</td>
</tr>
<tr>
<td>(30)</td>
<td>0.1435</td>
<td>0.1651</td>
<td>0.1536</td>
<td>0.1104</td>
<td>0.0381</td>
</tr>
<tr>
<td>(35)</td>
<td>0.1663</td>
<td>0.1795</td>
<td>0.1605</td>
<td>0.1201</td>
<td>0.0479</td>
</tr>
<tr>
<td>(40)</td>
<td>0.1739</td>
<td>0.1798</td>
<td>0.1552</td>
<td>0.1127</td>
<td>0.0418</td>
</tr>
<tr>
<td>(45)</td>
<td>0.1659</td>
<td>0.1671</td>
<td>0.1382</td>
<td>0.0996</td>
<td>0.0363</td>
</tr>
<tr>
<td>(50)</td>
<td>0.1693</td>
<td>0.1544</td>
<td>0.1281</td>
<td>0.1071</td>
<td>0.0472</td>
</tr>
<tr>
<td>(55)</td>
<td>0.1804</td>
<td>0.1488</td>
<td>0.12</td>
<td>0.103</td>
<td>0.0484</td>
</tr>
<tr>
<td>(60)</td>
<td>0.1718</td>
<td>0.1383</td>
<td>0.1147</td>
<td>0.0914</td>
<td>0.046</td>
</tr>
<tr>
<td>(70)</td>
<td>0.1695</td>
<td>0.1328</td>
<td>0.1025</td>
<td>0.119</td>
<td>0.0641</td>
</tr>
<tr>
<td>(80)</td>
<td>0.1598</td>
<td>0.1211</td>
<td>0.0821</td>
<td>0.0519</td>
<td>-0.0173</td>
</tr>
<tr>
<td>(90)</td>
<td>0.166</td>
<td>0.1247</td>
<td>0.0864</td>
<td>0.0504</td>
<td>-0.0173</td>
</tr>
</tbody>
</table>
### APPENDIX C

#### $C_{z_{\alpha,\delta_s}}(\alpha, \delta_s)$

<table>
<thead>
<tr>
<th>$\alpha$ (deg)</th>
<th>$\delta_s$ (deg)</th>
<th>-25</th>
<th>-10</th>
<th>0</th>
<th>10</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td></td>
<td>1.315</td>
<td>1.228</td>
<td>1.116</td>
<td>1.039</td>
<td>0.71</td>
</tr>
<tr>
<td>-15</td>
<td></td>
<td>1.171</td>
<td>1.059</td>
<td>0.959</td>
<td>0.849</td>
<td>0.476</td>
</tr>
<tr>
<td>-10</td>
<td></td>
<td>0.925</td>
<td>0.815</td>
<td>0.692</td>
<td>0.596</td>
<td>0.205</td>
</tr>
<tr>
<td>-5</td>
<td></td>
<td>0.469</td>
<td>0.356</td>
<td>0.287</td>
<td>0.205</td>
<td>0.125</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0.155</td>
<td>0.064</td>
<td>-0.025</td>
<td>-0.114</td>
<td>-0.228</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>-0.189</td>
<td>-0.287</td>
<td>-0.367</td>
<td>-0.49</td>
<td>-0.578</td>
</tr>
<tr>
<td>10</td>
<td></td>
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<td>-0.65</td>
<td>-0.75</td>
<td>-0.849</td>
<td>-0.946</td>
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<tr>
<td>15</td>
<td></td>
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<td>-0.98</td>
<td>-1.112</td>
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</tr>
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<td>-1.789</td>
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<td>-2.2</td>
<td>-2.308</td>
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<tr>
<td>40</td>
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<td>-2.216</td>
<td>-2.328</td>
<td>-2.411</td>
<td>-2.337</td>
</tr>
<tr>
<td>45</td>
<td></td>
<td>-2.185</td>
<td>-2.263</td>
<td>-2.311</td>
<td>-2.306</td>
<td>-2.327</td>
</tr>
<tr>
<td>55</td>
<td></td>
<td>-2.01</td>
<td>-2.088</td>
<td>-2.252</td>
<td>-2.231</td>
<td>-2.231</td>
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<tr>
<td>60</td>
<td></td>
<td>-1.916</td>
<td>-2.051</td>
<td>-2.208</td>
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<td>-2.174</td>
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<tr>
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<td></td>
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<td>-2.134</td>
<td>-2.268</td>
<td>-2.259</td>
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<tr>
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<td>-1.998</td>
<td>-2.14</td>
<td>-2.057</td>
<td>-2.069</td>
</tr>
</tbody>
</table>

#### $C_{M_{\alpha,\delta_s}}(\alpha, \delta_s)$

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<th>$\delta_s$ (deg)</th>
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<th>-10</th>
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Linear analysis:
The following state-space model was obtained from the trimmed aircraft in deep stall at $\delta_s = 0^\circ$, $T = 8785$ N, $c_g = 37.5$, $q = 0$ deg/s, $\alpha = 58.5$ deg, $V = 79.8$ m/s, and $\theta = 8.5$ deg. Note that $q$, $\alpha$, and $\theta$ have units rad/s or rad in the state-space matrices.

$x = [q, \alpha, V, \theta]^T$, \quad $u = \delta_s$

\[ A = \begin{bmatrix} -0.16093 & -1.77576 & 4.40366e-07 & 0 \\ 0.98316 & 0.01311 & -0.00173 & 0.09415 \\ -2.00203 & 1.25473 & -0.20067 & -6.30705 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ B = \begin{bmatrix} -0.00920 \\ -7.82661e-05 \\ -0.03019 \\ 0 \end{bmatrix} \]

\[ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
D. LINEAR GTT MODEL AT DEEP STALL

The following state-space model was obtained from the trimmed aircraft at \( \delta_e = 0^\circ, T = 0 \text{ N}, c_g = 40\% \text{ MAC}, \alpha = 44.2 \text{ deg}, V = 64.5 \text{ m/s}, q = 0 \text{ deg/s}, \) and \( \theta = 0.87 \text{ deg}. \) Note that \( \alpha, q, \) and \( \theta \) have units rad/s or rad in the state-space matrices.

\[ x = [\alpha, V, q, \theta]^T \quad u = \delta_e \]

\[
A = \begin{bmatrix}
-0.13858 & -0.00343 & 0.92943 & 0.10426 \\
-7.14144 & -0.20869 & -4.27044 & -7.13799 \\
-0.62887 & 2.74314 \times 10^{-6} & -0.34515 & -6.30705 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-0.00024411 \\
-0.011471 \\
-0.0035998 \\
0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Corresponding linear transfer functions:

\[
\frac{\alpha(s)}{\delta_e(s)} = \frac{-0.013986 (s + 13.77) (s^2 + 0.3328s + 0.04953)}{(s^2 + 0.3345s + 0.05439) (s^2 + 0.3579s + 0.5347)} \quad \text{(deg/deg)}
\]

\[
\frac{V(s)}{\delta_e(s)} = \frac{-0.011471 (s + 2.572) (s + 1.461) (s + 0.1019)}{(s^2 + 0.3345s + 0.05439) (s^2 + 0.3579s + 0.5347)} \quad \text{(m/s/deg)}
\]

\[
\frac{q(s)}{\delta_e(s)} = \frac{-0.20625s (s + 0.2965) (s + 0.008109)}{(s^2 + 0.3345s + 0.05439) (s^2 + 0.3579s + 0.5347)} \quad \text{(deg/s/deg)}
\]

\[
\frac{\theta(s)}{\delta_e(s)} = \frac{-0.20625s (s + 0.2965) (s + 0.008109)}{(s^2 + 0.3345s + 0.05439) (s^2 + 0.3579s + 0.5347)} \quad \text{(deg/deg)}
\]