Booker, A. R., & Krishnamurthy, M. (2013). Further refinements of the GL(2) converse theorem. *Bulletin of the London Mathematical Society*, *45*(5), 987-1003. https://doi.org/10.1112/blms/bdt031

FURTHER REFINEMENTS OF THE GL(2) CONVERSE THEOREM

ANDREW R. BOOKER AND M. KRISHNAMURTHY

Abstract. We improve the results of [BK11] by allowing restricted sets of poles among the unramified twists. This allows for a clean statement of the GL(2) converse theorem which includes all cases of Eisenstein series.

1. Introduction and statement of results

In this paper, we further weaken the hypotheses of the GL(2) converse theorem, following the methods of [BK11], by allowing some poles among the twists $\Lambda(s,\pi\otimes\omega)$ by unramified characters ω. Precisely, we prove the following:

Theorem 1.1. *Let F be a number field, $\mathbb{A}_F$ its ring of adèles and $\pi=\bigotimes_v \pi_v$ an irreducible, admissible, generic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central idèle class character ω_π. For every (unitary) idèle class character ω, suppose that the complete L-functions*

$$\Lambda(s,\pi\otimes\omega)=\prod_v L(s,\pi_v\otimes\omega_v) \quad \textit{and} \quad \Lambda(s,\widetilde{\pi}\otimes\omega^{-1})=\prod_v L(s,\widetilde{\pi}_v\otimes\omega_v^{-1})$$

(1) *converge absolutely and define analytic functions in some right half-plane $\Re(s)>\sigma$;*
(2) *continue to meromorphic functions on $\mathbb{C}$;*
(3) *satisfy the functional equation*

$$\Lambda(s,\pi\otimes\omega)=\epsilon(s,\pi\otimes\omega)\Lambda(1-s,\widetilde{\pi}\otimes\omega^{-1}), \tag{1.1}$$

where $\epsilon(s,\pi\otimes\omega)$ is as in [JL70, Theorem 11.3].

Further, let $\mathfrak{m}$ be a non-zero integral ideal of F, and let $A(\mathfrak{c})\in\mathbb{C}$ be given for each integral ideal $\mathfrak{c}$ containing $\mathfrak{m}$, with $A(\mathfrak{c})\neq 0$ for at least one such $\mathfrak{c}$. For ω as above, define the twisted Dirichlet polynomial

$$D(s,\omega)=\sum_{\mathfrak{c}\supset\mathfrak{m}} A(\mathfrak{c})\chi_\omega(\mathfrak{c})N(\mathfrak{c})^{1/2-s},$$

where χ_ω is the Größencharakter associated to ω (so that $L(s,\omega)=\sum_{\mathfrak{c}}\chi_\omega(\mathfrak{c})N(\mathfrak{c})^{-s}$). Suppose that $D(s,\omega)\Lambda(s,\pi\otimes\omega)$ continues to an entire function of finite order whenever ω is unramified at every non-archimedean place. Then π is an automorphic representation.

Note that we have removed the requirement from [BK11] that π_v be unitary for all archimedean v, answering a question raised in loc. cit.; thus Theorem 1.1 now directly generalizes the Jacquet–Langlands converse theorem [JL70, Theorem 11.3]. However, the principal motivation for this improvement is that the main theorem of [BK11] applies to some cases of Eisenstein series (those for which $\Lambda(s,\pi\otimes\omega)$ is entire for all unramified ω), but not all. A version of the GL(2) converse theorem which includes all cases of Eisenstein series was first obtained by Li [Li81], but the statement is complicated by the need to specify

A. R. B. was supported by EPSRC Fellowship `EP/H005188/1`.
M. K. was supported by NSA Grant `H98230-12-1-0218`.

the precise locations and residues of all poles. Note that Theorem 1.1 can accommodate any prescribed finite collection of poles for finitely many ω; thus it achieves this goal with a comparatively simple statement.

Another feature is that the result applies to partial L-functions. Precisely, if π satisfies hypotheses (1), (2) and (3) of Theorem 1.1, then for the unramified characters ω it is enough to know that the partial L-function

$$\Lambda^S(s, \pi \otimes \omega) = \prod_{v \notin S} L(s, \pi_v \otimes \omega_v)$$

is entire of finite order for a fixed finite set S of non-archimedean places.

1.1. **Notation and outline of the paper.** First we recall the main items of notation from [BK11]. Let F be a number field and $\mathfrak{o}_F$ its ring of integers. For each place v of F we denote by F_v the completion of F at v. To avoid confusion at archimedean places, we will use the symbol $\|\cdot\|_v$ to denote the normalized absolute value on F_v, and reserve $|\cdot|$ for the usual absolute value of real or complex numbers. Let S_∞ denote the set of archimedean places of F, and define

$$S_{\mathbb{C}} = \{v \in S_\infty : F_v = \mathbb{C}\}, \quad S_{\mathbb{R}} = \{v \in S_\infty : F_v = \mathbb{R}\}.$$

We write $v|\infty$ and $v < \infty$ to mean $v \in S_\infty$ and $v \notin S_\infty$, respectively. For $v < \infty$, let $\mathfrak{o}_v$ denote the ring of integers of F_v, $\mathfrak{p}_v$ the unique prime ideal of $\mathfrak{o}_v$, $\mathfrak{o}_v^\times$ the group of local units, and q_v the cardinality of $\mathfrak{o}_v/\mathfrak{p}_v$. We write $\mathbb{A}_F$ for the ring of adèles of F and $\mathbb{A}_F^\times$ for its group of idèles. The symbol $\mathbb{A}_{F,f}$ will denote the ring of finite adèles and F_∞ will denote $\prod_{v|\infty} F_v$, so that $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$; we write x_∞ and x_f for the corresponding components of $x \in \mathbb{A}_F$. Further, we fix an additive character $\psi = \bigotimes_v \psi_v$ of $F\backslash\mathbb{A}_F$ as in loc. cit. whose conductor is the inverse different $\mathfrak{d}^{-1}$ of F. Let d be a finite idèle such that $(d) = \mathfrak{d}$, where for any $t \in \mathbb{A}_F^\times$ we write (t) to denote the fractional ideal $(t) = \prod_{v<\infty} (\mathfrak{p}_v \cap F)^{\mathrm{ord}_v(t_v)}$.

We fix our choice of Haar measure on the idèle class group as follows: For each finite place v of F, let $d^\times y_v$ be the Haar measure on $F_v^\times$ such that the volume of $\mathfrak{o}_v^\times$ is 1. For $v|\infty$, let dy_v be the ordinary Lebesgue measure; we then set $d^\times y_v = \frac{dy_v}{2\|y_v\|_v}$ for $v \in S_{\mathbb{R}}$ and $d^\times y_v = \frac{dy_v}{\pi\|y_v\|_v}$ for $v \in S_{\mathbb{C}}$. Then this choice of local Haar measures determines a unique Haar measure $d^\times y$ on $\mathbb{A}_F^\times$ such that the volume of $\prod_{v<\infty} \mathfrak{o}_v^\times$ is 1.

We now recall the form of a continuous quasi-character $\chi_v : F_v^\times \to \mathbb{C}^\times$ at an archimedean place v. To be precise, for $v \in S_{\mathbb{R}}$, such a χ_v may be written uniquely in the form $\chi_v(y) = \|y\|_v^{\nu(\chi_v)} \operatorname{sgn}_v(y)^{\epsilon(\chi_v)}$, where $\operatorname{sgn}_v : F_v^\times \to \{\pm 1\}$ is the local sign character, $\nu(\chi_v) \in \mathbb{C}$ and $\epsilon(\chi_v) \in \{0, 1\}$; similarly, for $v \in S_{\mathbb{C}}$ we have $\chi_v(y) = \|y\|_v^{\nu(\chi_v)} \theta_v(y)^{k(\chi_v)}$, where $\theta_v(y) = y\|y\|_v^{-1/2}$, $\nu(\chi_v) \in \mathbb{C}$ and $k(\chi_v) \in \mathbb{Z}$.

Finally, by a *Größencharakter* of conductor $\mathfrak{q}$ we mean a multiplicative function χ of non-zero integral ideals satisfying $\chi(a\mathfrak{o}_F) = \chi_f(a)\chi_\infty(a)$ for associated characters $\chi_f : (\mathfrak{o}_F/\mathfrak{q})^\times \to S^1$ and $\chi_\infty : F_\infty^\times \to S^1$, with χ_f primitive and χ_∞ continuous, and all $a \in \mathfrak{o}_F$ relatively prime to $\mathfrak{q}$. By convention we set $\chi(\mathfrak{a}) = 0$ for any ideal $\mathfrak{a}$ with $(\mathfrak{a}, \mathfrak{q}) \neq 1$. The Größencharakters are in one-to-one correspondence with idèle class characters $\omega : F^\times\backslash\mathbb{A}_F^\times \to S^1$, and the correspondence is such that $\chi_\infty = \omega_\infty^{-1}$. (By a *character* we always mean a unitary character, and use the word *quasi-character* for the more general notion.)

With the required notation in place, we conclude this section with a brief outline of the contents of the paper. We use the same basic setup as [BK11], and assume some familiarity with that paper, particularly in Section 3, where we generalize certain results of [BK11]. In Section 2 we sketch the proof of Theorem 1.1 in broad strokes, stating the main lemmas but deferring their proofs until Sections 3 and 4.

2. Outline of the proof

The primary technical tool that we rely on is a generalization of the method of [BK11], which we summarize in the following proposition:

Proposition 2.1. *Let notation and hypotheses be as in Theorem 1.1. Suppose that ω is an idèle class character such that, for every non-archimedean place v for which π_v is ramified, ω_v is either unramified or sufficiently highly ramified (in a precise sense depending on π_v). Further, let*

$$L(s, \pi \otimes \omega) = \prod_{v<\infty} L(s, \pi_v \otimes \omega_v)$$

denote the finite twisted L-function, and let $\chi_{\omega^{-1}}$ be the Größencharakter associated to ω^{-1}. Then,

$$\sum_{\mathfrak{c}|\mathfrak{m}} A(\mathfrak{c})\chi_{\omega^{-1}}(\mathfrak{m}\mathfrak{c}^{-1})N(\mathfrak{c})^{\frac{1}{2}-s}L(s, \pi \otimes \omega)$$

continues to an entire function of finite order.

Our strategy is to choose an ω in the above so that $\chi_{\omega^{-1}}(\mathfrak{m}\mathfrak{c}^{-1})$ vanishes for some $\mathfrak{c}$ but not all, replace π by $\pi \otimes \omega$ and induct on the set of prime factors of $\mathfrak{m}$. One difficulty is that Prop. 2.1 only yields information about the finite L-function $L(s, \pi \otimes \omega)$, and multiplying by the archimedean L-factors $L(s, \pi_\infty \otimes \omega_\infty)$ can introduce undesired poles. However, we can swap the roles of π and $\widetilde{\pi}$, and thereby conclude the same for $L(s, \widetilde{\pi} \otimes \omega^{-1})$; thanks to the functional equation, this places constraints on the possible poles of the complete L-function $\Lambda(s, \pi \otimes \omega)$. Choosing ω with sufficient care, we find that we may avoid poles altogether. Our proof ultimately relies on some algebraic number theory and the following stronger version of the converse theorem, which is a special case of that proved by Piatetski-Shapiro [PŠ75][1]:

Lemma 2.2. *Let $\pi = \bigotimes \pi_v$ be an irreducible, admissible, generic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central idèle class character ω_π and conductor $\mathfrak{N}$. Let S and S' be finite sets of places of F with the following properties:*

1. *$S' \supset S \supset S_\infty$;*
2. *π_v is unramified for every $v \in S' \setminus S_\infty$;*
3. *the ring $\mathfrak{o}_S$ of S-integers is a PID;*
4. *the group $\Gamma_1(\mathfrak{N}\mathfrak{o}_S) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathfrak{o}_S) : c \in \mathfrak{N}\mathfrak{o}_S, d \in 1 + \mathfrak{N}\mathfrak{o}_S \right\}$ admits a set of generators $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $\|d\|_v \geq 1$ for all $v \notin S'$.*

For every idèle class character ω such that ω_v is unramified for all $v \notin S'$, suppose that $\Lambda(s, \pi \otimes \omega)$ continues to an entire function which is bounded in vertical strips and satisfies (1.1). Then there is an automorphic representation π' such that $\pi'_v \cong \pi_v$ whenever v is archimedean or π_v is unramified.

The precise results that we need in order to choose the twisting character ω appropriately and apply the above converse theorem are recorded in the following lemmas.

[1] Piatetski-Shapiro's result assumed, for convenience, that π_v is unitary for all v. However, a careful reading of [PŠ75] reveals that assumption to be unnecessary.

Lemma 2.3. *Let S be a finite set of places of F, containing all archimedean places, and let $\mathfrak{o}_S$ be the ring of S-integers. Let $\mathfrak{N} \subset \mathfrak{o}_S$ be a non-zero ideal, and suppose $p \in \mathbb{Z}$ is a prime number such that $p\mathfrak{o}_S$ is co-prime to $\mathfrak{N}$ and F contains no primitive pth root of unity. Then there is a finite set P of places of F such that*

1. *P is disjoint from S;*
2. *for each $v \in P$, $\mathfrak{p}_v \cap \mathfrak{o}_S$ is co-prime to $\mathfrak{N}$ and $p \nmid q_v(q_v - 1)$;*
3. *$\Gamma_1(\mathfrak{N}) \subset \mathrm{GL}_2(\mathfrak{o}_S)$ has a finite set of generators of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where d is either an S-unit or satisfies $d\mathfrak{o}_S = \mathfrak{p}_v \cap \mathfrak{o}_S$ for some $v \in P$.*

Lemma 2.4. *Let S be a finite set of places of F, containing all archimedean places, and $\pi = \bigotimes \pi_v$ an irreducible, admissible, generic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central idèle class character ω_π. Then for any sufficiently large prime number p (with the meaning of "sufficiently large" depending on F, π and S), there is an idèle class character ω such that*

1. *ω is unramified at the places in $S \setminus S_\infty$ and those dividing p;*
2. *if ω' is an idèle class character such that ω'_v is unramified for every place $v \notin S$ satisfying $p | q_v(q_v - 1)$, then $L(s, \pi_\infty \otimes \omega_\infty \otimes \omega'_\infty)$ and $L(1 - s, \widetilde{\pi}_\infty \otimes \omega_\infty^{-1} \otimes \omega'^{-1}_\infty)$ have no poles in common.*

2.1. **Proof of Theorem 1.1.** With the main lemmas in place, we may now complete the proof of Theorem 1.1.

Let us first treat the base case, $\mathfrak{m} = \mathfrak{o}_F$. Let S_π be the set of non-archimedean places v for which π_v is ramified, and choose a finite set of places $T \supset S_\infty$ which is disjoint from S_π and such that $\mathfrak{o}_T$ is a PID. We apply Lemma 2.4 with $S = T \cup S_\pi$ and a fixed, sufficiently large, prime number p; in particular, we assume that F contains no primitive pth root of unity and that π_v is unramified at all places v dividing p. Let ω be the character given by the lemma, and set $\pi' = \pi \otimes \omega$. Then for any ω' as in the statement of the lemma, $L(s, \pi'_\infty \otimes \omega'_\infty)$ and $L(1 - s, \widetilde{\pi}'_\infty \otimes \omega'^{-1}_\infty)$ have no poles in common. If ω' is also unramified at the places in S_π, then applying Prop. 2.1 to both π and $\widetilde{\pi}$, we have that $L(s, \pi' \otimes \omega')$ and $L(s, \widetilde{\pi}' \otimes \omega'^{-1})$ are entire of finite order. In view of the functional equation, the same conclusion thus applies to $\Lambda(s, \pi' \otimes \omega')$.

Next, let $\mathfrak{N}_{\pi'} \subset \mathfrak{o}_F$ be the conductor of π', and $S_{\pi'}$ the set of finite places of ramification of π'. By construction, $\mathfrak{N}_{\pi'}$ is relatively prime to $p\mathfrak{o}_F$ and the finite places in T. We apply Lemma 2.3 with $S = T$ and $\mathfrak{N} = \mathfrak{N}_{\pi'}\mathfrak{o}_S$ to obtain a suitable set of finite places P. By the previous paragraph, we have that $\Lambda(s, \pi' \otimes \omega')$ is entire of finite order whenever ω' is ramified only at places in $P \cup T$. (Recall also that finite order in this context is the same as bounded in vertical strips, by the Phragmén–Lindelöf convexity principle.)

Finally, we apply Lemma 2.2 with $S = T$, $S' = P \cup T$ and π' in place of π. Thus we see that π' agrees with some automorphic representation at almost all places, including all archimedean places. Applying the stability of γ-factors argument as in [BK11, §5.1], we conclude that π' is automorphic, and it follows that π is as well.

We turn now to the general case. Set $D(s) = \sum_{\mathfrak{c}|\mathfrak{m}} A(\mathfrak{c}) N(\mathfrak{c})^{1/2-s}$ where $\mathfrak{m}$ and $A(\mathfrak{c})$ are as given in the hypotheses, and define $\widetilde{D}(s) = \sum_{\mathfrak{c}|\mathfrak{m}} A(\mathfrak{m}\mathfrak{c}^{-1}) N(\mathfrak{c})^{1/2-s} = N(\mathfrak{m})^{1/2-s} D(1-s)$. Let $\mathfrak{p} \subset \mathfrak{o}_F$ be a prime ideal dividing $\mathfrak{m}$, and set $k = \mathrm{ord}_\mathfrak{p}(\mathfrak{m})$, $\mathfrak{m}_1 = \mathfrak{m}\mathfrak{p}^{-k}$. We may assume without loss of generality that $A(\mathfrak{p}^k\mathfrak{c}) \neq 0$ for at least one choice of $\mathfrak{c}$ dividing $\mathfrak{m}_1$, since otherwise we could replace $\mathfrak{m}$ by $\mathfrak{m}\mathfrak{p}^{-1}$.

Now let us choose an idèle class character ω_1 which is highly ramified at $\mathfrak{p}$ and no other finite places, and set $\pi_1 = \pi \otimes \omega_1$. Let $\mathfrak{N}_{\pi_1}$ be the conductor of π_1, and define

$$D_1(s) = \sum_{\mathfrak{c}|\mathfrak{m}_1} A(\mathfrak{p}^k\mathfrak{c})\chi_{\omega_1}(\mathfrak{c})N(\mathfrak{c})^{1/2-s}.$$

By Prop. 2.1, we see that if ω is an idèle class character which is unramified at all primes dividing $\mathfrak{m}_1\mathfrak{N}_{\pi_1}$ then $D_1(s,\omega)L(s,\pi_1\otimes\omega)$ is entire.

Next, we swap the roles of (π, D) and $(\widetilde{\pi}, \widetilde{D})$. Thus we conclude that there is a Dirichlet polynomial $\widetilde{D}_1(s)$, again of modulus $\mathfrak{m}_1$, such that $\widetilde{D}_1(s,\omega^{-1})L(s,\widetilde{\pi}_1\otimes\omega^{-1})$ is entire for all ω as above. Now set $D_2(s) = D_1(s)\widetilde{D}_1(s)$ and $\widetilde{D}_2(s) = N(\mathfrak{m}_1)^{1-2s}D_2(1-s)$, which are Dirichlet polynomials of modulus $\mathfrak{m}_1^2$. Then for any ω as above we have the functional equation

$$D_2(s,\omega) = \chi_\omega(\mathfrak{m}_1)^2 N(\mathfrak{m}_1)^{1-2s}\widetilde{D}_2(1-s,\omega^{-1}),$$

from which it follows that both $D_2(s,\omega)L(s,\pi_1\otimes\omega)$ and $D_2(s,\omega)L(1-s,\widetilde{\pi}_1\otimes\omega^{-1})$ are entire.

Finally, applying Lemma 2.4, we may choose an ω which is unramified at the primes dividing $\mathfrak{m}_1\mathfrak{N}_{\pi_1}$ such that $L(s,\pi_{1,\infty}\otimes\omega_\infty\otimes\omega'_\infty)$ and $L(1-s,\widetilde{\pi}_{1,\infty}\otimes\omega_\infty^{-1}\otimes\omega'^{-1}_\infty)$ have no poles in common for any ω' which is unramified at all finite places. Thus, if $\pi' = \pi_1\otimes\omega$ and $D'(s) = D_2(s,\omega)$ then $D'(s,\omega')\Lambda(s,\pi'\otimes\omega')$ is entire for all unramified characters ω'. Moreover, D' has modulus $\mathfrak{m}_1^2$, which has fewer prime factors than $\mathfrak{m}$. The result follows by induction. □

3. Modifications to the method of [BK11]

The object of this section is to prove Prop. 2.1. Rather than copying the arguments nearly verbatim from [BK11], we will describe the changes necessary to generalize the proof, referring heavily to [BK11, §4–5].

3.1. **Piatetski-Shapiro's lemma.** Our first step is to prove a generalization of [PŠ75, Lemma 4], which is the starting point for the method of [BK11]. For every place v, let V_{π_v} be the space of π_v and let $W(\pi_v,\psi_v)$ be the ψ_v-Whittaker model of π_v. By definition, for each archimedean place v, V_{π_v} is a Harish-Chandra module, i.e., an irreducible admissible $(\mathfrak{gl}_2(F_v), K_v)$-module. For such a place v, we write $V_{\pi_v}^\infty$ to denote the canonical completion of V_{π_v} [Cas89]. In particular, $V_{\pi_v}^\infty$ is a smooth representation of $\mathrm{GL}_2(F_v)$ of moderate growth.

If we identify the space V_π of π with $\otimes_v V_{\pi_v}$, then $W(\pi,\psi) = \otimes_v W(\pi_v,\psi_v)$ is the global Whittaker model of π with respect to ψ. For each place v, we fix a choice of test vector as in [BK11, §4.1–2]. In particular, for every v the vector $\xi_v \in V_{\pi_v}$ is chosen so that it satisfies

$$\int_{F_v^\times} W_{\xi_v}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\|y\|_v^{s-\frac{1}{2}}d^\times y = \|d\|_v^{\frac{1}{2}-s}L(s,\pi_v).$$

Let us set $\xi = \otimes\xi_v$. For any Ξ in $\bigotimes_{v|\infty} V_{\pi_v}^\infty \otimes \bigotimes_{v<\infty} V_{\pi_v}$, we may form the corresponding Whittaker function W_Ξ and we write ϕ_Ξ to denote the function

$$\phi_\Xi(g) = \sum_{\gamma\in F^\times} W_\Xi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\right), \quad g\in\mathrm{GL}_2(\mathbb{A}_F).$$

By construction we have

$$\Lambda(s,\pi) = N(\mathfrak{d})^{\frac{1}{2}-s} \int_{F^\times\backslash\mathbb{A}_F^\times} \phi_\xi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) \|y\|^{s-\frac{1}{2}}\, d^\times y \tag{3.1}$$

provided $\Re(s) \gg 1$. Since the unramified twists $\Lambda(s,\pi\otimes\omega)$ may have poles, [BK11, (5.1)] is not valid anymore but we will soon derive a variant of this equation.

To this end, let $(t,0)\in F_\infty^\times \times \mathbb{A}_{F,f}$ and let $u(t)$ denote the unipotent matrix $\left(\begin{smallmatrix} 1 & t \\ & 1\end{smallmatrix}\right)$. For an unramified character ω of $F^\times\backslash\mathbb{A}_F^\times$, consider the integral

$$\mathcal{M}_t(\xi,\omega)(s) = \omega(d) N(\mathfrak{d})^{\frac{1}{2}-s} \int_{F^\times\backslash\mathbb{A}_F^\times} (R_{u(t)}\phi_\xi)\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) \omega(y)\|y\|^{s-\frac{1}{2}}\, d^\times y \tag{3.2}$$

which converges for $\Re(s)\gg 1$. A calculation similar to the one in [BK11, §4.4] reveals that

$$\begin{aligned}\mathcal{M}_t(\xi,\omega)(s) &= L(s,\pi\otimes\omega)\int_{F_\infty^\times} \psi_\infty(ty_\infty) W_{\xi_\infty}\left(\begin{pmatrix} y_\infty & \\ & 1\end{pmatrix}\right)\omega_\infty(y_\infty)\|y_\infty\|_\infty^{s-\frac{1}{2}}\, d^\times y_\infty \\ &= \Lambda(s,\pi\otimes\omega) F_t(s,\xi_\infty,\omega_\infty),\end{aligned}$$

where we define

$$F_t(s,\xi_v,\omega_v) = \frac{\int_{F_v^\times} \psi_v(ty) W_{\xi_v}(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right))\,\omega_v(y)\|y\|_v^{s-\frac{1}{2}}\, d^\times y}{L(s,\pi_v\otimes\omega_v)} \tag{3.3}$$

for $t\in F_v$, and

$$F_t(s,\xi_\infty,\omega_\infty) = \prod_{v|\infty} F_{t_v}(s,\xi_v,\omega_v)$$

for $t\in F_\infty$.

Now on the dual side consider the integral

$$\widetilde{\mathcal{M}}_t(\xi,\omega)(s) = \omega(d) N(\mathfrak{d})^{\frac{1}{2}-s} \int_{F^\times\backslash\mathbb{A}_F^\times} (R_{u(t)}\phi_\xi)\left(w\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) \omega(y)\|y\|^{s-\frac{1}{2}}\, d^\times y. \tag{3.4}$$

This integral converges for $-\Re(s)\gg 1$. Then a similar calculation as above combined with the functional equation yields

$$\begin{aligned}\widetilde{\mathcal{M}}_t(\xi,\omega)(s) &= \frac{\Lambda(s,\pi\otimes\omega)}{\omega_\infty(-1)\epsilon(s,\pi_\infty\otimes\omega_\infty,\psi_\infty)L(1-s,\tilde{\pi}_\infty\otimes\omega_\infty^{-1})} \\ &\cdot \int_{F_\infty^\times} W_{\xi_\infty}\left(w\begin{pmatrix} y_\infty & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & t \\ & 1\end{pmatrix}\right)\omega_\infty(y_\infty)\|y_\infty\|_\infty^{s-\frac{1}{2}}\, d^\times y_\infty.\end{aligned} \tag{3.5}$$

For each $v|\infty$, keeping the above notation, set

$$\Psi_v(s;\xi_v,t_v,\omega_v) = \int_{F_v^\times} W_{\xi_v}\left(\begin{pmatrix} y_v & y_v t_v \\ & 1\end{pmatrix}\right)\omega_v(y_v)\|y_v\|_v^{s-\frac{1}{2}}\, d^\times y_v,$$

and

$$\widetilde{\Psi}_v(s;\xi_v,t_v,\omega_v) = \int_{F_v^\times} W_{\xi_v}\left(w\begin{pmatrix} y_v & y_v t_v \\ & 1\end{pmatrix}\right)\omega_v(y_v)\|y_v\|_v^{s-\frac{1}{2}}\, d^\times y_v.$$

Lemma 3.1. *For each $v|\infty$, the local integral $\Psi_v(s;\xi_v,t_v,\omega_v)$ converges absolutely for $\Re(s) \gg 1$ and extends to a meromorphic function of s such that the the ratio*

$$\frac{\Psi_v(s;\xi_v,t_v,\omega_v)}{L(s,\pi_v\otimes\omega_v)}$$

is entire. If π_v is unitary (resp. tempered), the integral converges absolutely in the half-plane $\Re(s)\geq 1$ (resp. $\Re(s)>0$).

Proof. Observe that $\Psi_v(s;\xi_v,t_v,\omega_v)$ may be expressed as the Mellin transform of a Whittaker function, namely,

$$\Psi_v(s;\xi_v,t_v,\omega_v) = \int_{F_v^\times} W_{u(t_v)\cdot\xi_v}\left(\begin{pmatrix} y_v & \\ & 1\end{pmatrix}\right)\omega_v(y_v)\|y_v\|_v^{s-\frac{1}{2}}\,d^\times y_v,$$

where $u(t_v)\cdot\xi_v \in V_{\pi_v}^\infty$ and $W_{u(t_v)\cdot\xi_v}$ is the corresponding Whittaker function. Then the lemma follows from [JS90, Theorem 5.1], and the last part is a consequence of the well known bounds on the parameters of π_v (see [BK11, §4.3]). □

Since, for $W_{\xi_v}\in W(\pi_v,\psi_v)$, the function $g\mapsto W_{\xi_v}(w\,{}^tg^{-1})$ belongs to $W(\tilde{\pi}_v,\psi_v^{-1})$, the local integral $\widetilde{\Psi}_v(s;\xi_v,t_v,\omega_v)$ converges in the half-plane $-\Re(s)\gg 1$ and extends to a meromorphic function of s, so that the ratio

$$\frac{\widetilde{\Psi}_v(s;\xi_v,t_v,\omega_v)}{L(1-s,\tilde{\pi}_v\otimes\omega_v^{-1})}$$

is entire.

Lemma 3.2. *For $v|\infty$, $\omega_v^\times$ any character of $F_v^\times$ and $t_v\in F_v^\times$, the following local functional equation is satisfied:*

$$\widetilde{\Psi}_v(s;\xi_v,t_v,\omega_v) = \omega_v(-1)\gamma(s,\pi_v\otimes\omega_v,\psi_v)\Psi_v(s;\xi_v,t_v,\omega_v), \tag{3.6}$$

where $\gamma(s,\pi_v\otimes\omega_v,\psi_v) = \dfrac{\epsilon(s,\pi_v\otimes\omega_v,\psi_v)L(1-s,\tilde{\pi}_v\otimes\omega_v^{-1})}{L(s,\pi_v\otimes\omega_v)}$.

Proof. This follows from [JS90, Theorem 5.1] as well. □

By assumption, we know that for an unramified idèle class character ω of F, $\Lambda(s,\pi\otimes\omega)$ continues to a meromorphic function of s. From Lemma 3.1, we have that $F_t(s,\xi_v,\omega_v)$ is entire and consequently $\mathcal{M}_t(\xi,\omega)(s)$ extends to a meromorphic function of s. Likewise, $\widetilde{\mathcal{M}}_t(\xi,\omega)(s)$ also extends to a meromorphic function of s. Further, by Lemma 3.2, it follows that these meromorphically continued functions satisfy the relation

$$\widetilde{\mathcal{M}}_t(\xi,\omega)(s) = \mathcal{M}_t(\xi,\omega)(s) = \Lambda(s,\pi\otimes\omega)F_t(s,\xi_\infty,\omega_\infty),$$

away from the poles. Now, in this setup, we have the following generalization of Piatetski-Shapiro's lemma:

Lemma 3.3. *We have*

$$(R_{u(t)}\phi_\xi)\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) - (R_{u(t)}\phi_\xi)\left(w\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)$$
$$= c_F\sum_\omega \omega^{-1}(dy)\frac{1}{2\pi i}\int_{\downarrow\uparrow}\Lambda(s,\pi\otimes\omega)F_t(s,\xi_\infty,\omega_\infty)N(\mathfrak{d})^{s-\frac{1}{2}}\|y\|^{\frac{1}{2}-s}\,ds,$$

where $\int_{\downarrow\uparrow}$ is short-hand for $\int_{\Re(s)=\sigma+1} - \int_{\Re(s)=-\sigma}$, and the sum ranges over all unramified idèle class characters ω, normalized so that $\prod_{v|\infty} \omega_v(y) = 1$ for all $y \in \mathbb{R}_{>0}$.

Proof. The proof is a straightforward application of the Mellin inversion formula for the idèle class group $\mathcal{C}_F$. Note that

$$\widehat{\mathcal{C}_F} \cong \widehat{\mathcal{C}_F^1} \times \mathbb{R},$$

where $\mathcal{C}_F^1 \subset \mathcal{C}_F$ denotes the norm 1 subgroup, and $\chi \leftrightarrow (\omega, t)$ if and only if $\chi(x) = \omega(x)\|x\|^{it}$. Now, suppose g is any continuous function such that $x \mapsto g(x)\|x\|^{\sigma'}$ belongs to $L^1(\mathcal{C}_F)$ for some positive real number σ'. Then its Fourier transform $\hat{g}(\chi, \sigma') = \int_{\mathcal{C}_F} g(x)\chi(x)\|x\|^{\sigma'} d^\times x$ is well-defined. Further, if $\hat{g}(\cdot, \sigma') \in L^1(\widehat{\mathcal{C}_F})$, then the Mellin inversion formula reads

$$g(x) = \int_{\widehat{\mathcal{C}_F}} \hat{g}(\chi, \sigma')\chi^{-1}(x)\|x\|^{-\sigma'}\, d\chi$$

for a unique choice of Haar measure $d\chi$ on $\widehat{\mathcal{C}_F}$. Since $\mathcal{C}_F^1$ is compact, its dual $\widehat{\mathcal{C}_F^1}$ is discrete. Now, suppose the function g is also right $\prod_v \mathfrak{o}_v^\times$-invariant. Then $\hat{g}(\omega, \sigma') = 0$ for any idèle class character ω that is ramified at a finite place. Therefore, in this situation, the above formula takes the form

$$g(x) = c_F \sum_\omega \omega^{-1}(x) \frac{1}{2\pi i} \int_{\Re(s)=\sigma'} \hat{g}(\omega, s)\|x\|^{-s}\, ds,$$

where ω runs through all unramified idèle class characters satisfying $\prod_{v|\infty} \omega_v(y) = 1$ for all $y \in \mathbb{R}_{>0}$. Here c_F is some constant related to the choice of the dual measure, which only depends on F. (Note also that the line of integration may be moved further to the right.) Now we apply this formula to $g(y) = (R_{u(t)}\phi_\xi)(\begin{pmatrix} y & \\ & 1 \end{pmatrix})$ and $g(y) = (R_{u(t)}\phi_\xi)(w\begin{pmatrix} y & \\ & 1 \end{pmatrix})$ to obtain the desired result. ☐

Our goal for the next couple of sections is to modify our main results from [BK11, §5.3–4] by changing the test vector. To be precise, suppose $\mathfrak{c}$ is any non-zero integral ideal, write $\mathfrak{c} = \prod_{i=1}^r \mathfrak{p}_i^{k_i}$ and let $v_i \leftrightarrow \mathfrak{p}_i$, $1 \le i \le r$, be the corresponding finite places of F. Let $t_\mathfrak{c}$ denote the finite idèle (t_v), where

$$t_v = \begin{cases} 1 & \text{if } v \notin \{v_1, \ldots, v_r\}, \\ \varpi_{v_i}^{k_i} & \text{if } v = v_i \text{ for some } i \in \{1, \ldots, r\}. \end{cases}$$

Note that the ideal generated by $t_\mathfrak{c}$ is precisely $\mathfrak{c}$. In what follows we will consider test vectors of the form $\xi^\mathfrak{c} = \bigotimes_{v\notin\{v_1,\ldots,v_r\}} \xi_v \otimes \bigotimes_{i=1}^r \pi_{v_i}\begin{pmatrix} \varpi_{v_i}^{-k_i} & \\ & 1 \end{pmatrix}\xi_{v_i}$ and possible linear combinations of such $\xi^\mathfrak{c}$. Namely, given a non-zero integral ideal $\mathfrak{m}$ and $A(\mathfrak{c}) \in \mathbb{C}$ for each $\mathfrak{c} \supset \mathfrak{m}$ as in Theorem 1.1, we consider $\xi^D = \sum_{\mathfrak{c}|\mathfrak{m}} A(\mathfrak{c})\xi^\mathfrak{c}$.

3.2. **Dirichlet series.** Recall from [BK11, §4.4] that for $y \in \mathbb{A}_F^\times$ and $\gamma \in F^\times$, we define $a_\xi(y, \gamma) = \prod_{v<\infty} W_{\xi_v}\begin{pmatrix} \gamma y_v & \\ & 1 \end{pmatrix}$, so that $a_\xi(y, \gamma) = 0$ unless $(\gamma y) \subset \mathfrak{d}^{-1}$. Set $\mathfrak{a}_j' = (t_j)\mathfrak{d}$ and $\mathfrak{a}_j = (t_j^{-1})\mathfrak{d}\mathfrak{N}$, $1 \le j \le h$. For any non-zero integral ideal $\mathfrak{a}$ of F, the normalized Dirichlet coefficient $\lambda_\pi(\mathfrak{a})$ is defined as follows:

$$\lambda_\pi(\mathfrak{a}) = a_\xi(t_j, \gamma)\sqrt{N(\mathfrak{a})},$$

where j is the unique index satisfying $\mathfrak{a} = (\gamma)\mathfrak{a}'_j$. Likewise, by definition we have

$$\tilde{a}_\xi(y^{-1}, \gamma) = \prod_{v<\infty} W_{\xi_v}\left(\begin{pmatrix} \gamma y_v^{-1} & \\ & 1 \end{pmatrix} w\right) \omega_{\pi_v}(\gamma y_v^{-1})^{-1}$$

which vanishes unless $\gamma \in (y)\mathfrak{N}^{-1}\mathfrak{d}^{-1}$, where $\mathfrak{N}$ is the conductor of π. For a non-zero integral ideal $\mathfrak{a}$ of F, let $\tilde{a}_\pi(\mathfrak{a}) = \tilde{a}_\xi(t_j^{-1}, \gamma)\sqrt{N(\mathfrak{a})}$, where j is such that $\mathfrak{a} = (\gamma)\mathfrak{a}_j$. These are defined so that when one unfolds the integrals defining $\Lambda(s, \pi)$ and $\Lambda(1-s, \widetilde{\pi})$ (as explained in [BK11, §4]), the finite part is given as a Dirichlet series involving the coefficients $\lambda_\pi(\mathfrak{a})$ and $\tilde{a}_\pi(\mathfrak{a})$, respectively.

Now, given a non-zero integral ideal $\mathfrak{m}$ of F and an integral ideal $\mathfrak{c}$ that contains $\mathfrak{m}$, let $\xi^{\mathfrak{c}}$ be as above. Let $a_{\xi^{\mathfrak{c}}}(y, \gamma)$ be the unnormalized coefficient obtained from replacing ξ by $\xi^{\mathfrak{c}}$; in other words, $a_{\xi^{\mathfrak{c}}}(y, \gamma) = a_\xi(yt_{\mathfrak{c}}^{-1}, \gamma)$. In particular, $a_{\xi^{\mathfrak{c}}}(y, \gamma) = 0$ unless $\gamma \in \mathfrak{d}^{-1}(y)^{-1}\mathfrak{c}$. For a non-zero integral ideal $\mathfrak{b}$ we define the normalized coefficient $\lambda_\pi^{\mathfrak{c}}(\mathfrak{b})$ as

$$\lambda_\pi^{\mathfrak{c}}(\mathfrak{b}) = a_{\xi^{\mathfrak{c}}}(t_j, \gamma)\sqrt{N(\mathfrak{b})}, \quad \text{where } \mathfrak{b} = (\gamma)\mathfrak{a}_j.$$

Then it follows that

$$\lambda_\pi^{\mathfrak{c}}(\mathfrak{b}) = \begin{cases} \lambda_\pi(\mathfrak{b}\mathfrak{c}^{-1})\sqrt{N(\mathfrak{c})} & \text{if } \mathfrak{b} \subset \mathfrak{c}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

We now turn to the dual side. For reasons that will soon be apparent, we consider the effect of replacing ξ by $\xi^{\mathfrak{m}\mathfrak{c}^{-1}}$. Let $\tilde{a}_{\xi^{\mathfrak{m}\mathfrak{c}^{-1}}}(t_j^{-1}, \gamma) = \tilde{a}_\xi(t_j^{-1}t_{\mathfrak{m}\mathfrak{c}^{-1}}, \gamma)$ be the corresponding unnormalized coefficient, which vanishes unless $\gamma \in (\mathfrak{a}_j\mathfrak{m})^{-1}\mathfrak{c}$. Let us define $\tilde{a}_\pi^{\mathfrak{c}}(\mathfrak{b})$ as

$$\tilde{a}_\pi^{\mathfrak{c}}(\mathfrak{b}) = \tilde{a}_{\xi^{\mathfrak{m}\mathfrak{c}^{-1}}}(t_j^{-1}, \gamma)\sqrt{N(\mathfrak{b})}, \quad \text{where } \mathfrak{b} = (\gamma)\mathfrak{a}_j\mathfrak{m}. \tag{3.8}$$

Observe that the dependency on $\mathfrak{m}$ on the right-hand side cancels out, so we are justified in omitting $\mathfrak{m}$ from the notation on the left-hand side. One can verify that

$$\tilde{a}_\pi^{\mathfrak{c}}(\mathfrak{b}) = \begin{cases} \tilde{a}_\pi(\mathfrak{b}\mathfrak{c}^{-1})\sqrt{N(\mathfrak{c})} & \text{if } \mathfrak{b} \subset \mathfrak{c}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

We also define

$$\lambda_{\widetilde{\pi}}^{\mathfrak{c}}(\mathfrak{b}) = \begin{cases} \lambda_{\widetilde{\pi}}(\mathfrak{b}\mathfrak{c}^{-1})\sqrt{N(\mathfrak{c})} & \text{if } \mathfrak{b} \subset \mathfrak{c}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.10}$$

where $\lambda_{\widetilde{\pi}}(\mathfrak{a})$ denotes the Dirichlet coefficient of the contragredient representation $\widetilde{\pi}$. At this point it is worth recalling from [BK11, §4.4] that $\tilde{a}_\pi(\mathfrak{a}) = \omega_\pi(d)\epsilon_f(\pi, \psi)\lambda_{\widetilde{\pi}}(\mathfrak{a})$, from which it follows that

$$\tilde{a}_\pi^{\mathfrak{c}}(\mathfrak{b}) = \omega_\pi(d)\epsilon_f(\pi, \psi)\lambda_{\widetilde{\pi}}^{\mathfrak{c}}(\mathfrak{b}) \tag{3.11}$$

for all integral ideals $\mathfrak{b}$.

It is trivial to check that

$$\begin{aligned} N(\mathfrak{d})^{\frac{1}{2}-s}\int_{\mathcal{C}_F} \phi_{\xi^{\mathfrak{c}}}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \|y\|^{s-\frac{1}{2}}\, d^\times y &= N(\mathfrak{c})^{\frac{1}{2}-s}N(\mathfrak{d})^{\frac{1}{2}-s}\int_{\mathcal{C}_F} \phi_\xi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \|y\|^{s-\frac{1}{2}}\, d^\times y \\ &= N(\mathfrak{c})^{\frac{1}{2}-s}\Lambda(s, \pi) \end{aligned}$$

and (see [BK11, §4.4])

$$N(\mathfrak{d})^{\frac{1}{2}-s}\int_{\mathcal{C}_F}\phi_{\xi^{\mathfrak{mc}^{-1}}}\left(w\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)\|y\|^{s-\frac{1}{2}}\,d^\times y = N(\mathfrak{mc}^{-1})^{\frac{1}{2}-s}\int_{\mathcal{C}_F}\phi_{\xi}\left(w\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)\|y\|^{s-\frac{1}{2}}\,d^\times y$$
$$= N(\mathfrak{mc}^{-1})^{\frac{1}{2}-s}\epsilon(s,\pi)L(1-s,\widetilde{\pi}).$$

Consequently, for ξ^D as above, we obtain

$$N(\mathfrak{d})^{\frac{1}{2}-s}\int_{\mathcal{C}_F}\phi_{\xi^D}\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)\|y\|^{s-\frac{1}{2}}\,d^\times y = D(s)\Lambda(s,\pi),$$

where $D(s)=D(s,\mathbf{1})=\sum_{\mathfrak{c}|\mathfrak{m}}A(\mathfrak{c})N(\mathfrak{c})^{1/2-s}$. In fact, borrowing notation from Section 3.1, for any unramified idèle class character ω and for $\Re(s)\gg 1$, we may conclude that

$$\mathcal{M}_t(\xi^D,\omega)(s)=D(s,\omega)\Lambda(s,\pi\otimes\omega)F_t(s,\xi_\infty,\omega_\infty)$$

and similarly

$$\widetilde{\mathcal{M}_t}(\xi^D,\omega)(s)=D(s,\omega)\epsilon(s,\pi\otimes\omega)\Lambda(1-s,\tilde{\pi}\otimes\omega^{-1})F_t(s,\xi_\infty,\omega_\infty)$$

for $-\Re(s)\gg 1$. Here, as expected, the functional equation satisfied by $\Lambda(s,\pi\otimes\omega)$ implies that the analytic continuation of these two expressions are equal.

Bearing in mind that π satisfies the hypotheses of Theorem 1.1, we take the appropriate linear combination of the result of Lemma 3.3 applied to $\xi^{\mathfrak{c}}$ to see that

$$(R_{u(t)}\phi_{\xi^D})\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)=(R_{u(t)}\phi_{\xi^D})\left(w\begin{pmatrix}y & \\ & 1\end{pmatrix}\right),$$

for $t\in F_\infty^\times$, $y\in\mathbb{A}_F^\times$. From this, as argued in [BK11, §5.2], it follows that

$$\phi_{\xi^D}\left(\begin{pmatrix}(\beta_\infty a_\infty,a_f) & (\beta_\infty,0)\\ & 1\end{pmatrix}\right)=\phi_{\xi^D}\left(w\begin{pmatrix}(\beta_\infty a_\infty,a_f) & (\beta_\infty,0)\\ & 1\end{pmatrix}\right) \tag{3.12}$$

for all $a\in\mathbb{A}_F^\times$ and $\beta\in F^\times$.

3.3. **Additive twists.** We continue with the notation of the previous section. In particular, $\mathfrak{m}$ is a fixed non-zero integral ideal of F. We also fix a j with $1\le j\le h$, a non-zero integral ideal $\mathfrak{q}$ and a continuous character $\chi_\infty:\Gamma_{\mathfrak{q}}\backslash F_\infty^\times\longrightarrow S^1$. For $\alpha\in\mathfrak{a}_j\mathfrak{m}\mathfrak{d}\mathfrak{q}^{-1}\cap F^\times$, $y\in F_\infty^\times$, and any $\Xi\in V_\pi$, let

$$\Phi_\Xi(y,\alpha)=\chi'_\infty(y)^{-1}\phi_\Xi\left(\begin{pmatrix}(\beta_\infty y,t_j) & (\beta_\infty,0)\\ & 1\end{pmatrix}\right),$$

where $\chi'_\infty=\omega_{\pi_\infty}\chi_\infty$ and $\beta=-\alpha^{-1}$. Let $\mathfrak{q}'\subset\mathfrak{q}$ be the integral ideal defined in [BK11, §5.3] and consider the integral

$$\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]}\int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times}\Phi_{\xi^{\mathfrak{c}}}(y,\alpha)\|y\|_\infty^{s-\frac{1}{2}}\,d^\times y. \tag{3.13}$$

Then a calculation identical to that in [BK11, §5.3] shows that this integral equals
(3.14)

$$\|\beta\|_\infty^{\frac{1}{2}-s}\sum_{\gamma\in\mathfrak{o}_F^\times\backslash(\mathfrak{a}_j'^{-1}\cap F^\times)}\frac{a_{\xi^{\mathfrak{c}}}(t_j,\gamma)e_{\mathfrak{q}'}((\gamma\beta),\chi'_\infty)}{\|\gamma\|_\infty^{s-\frac{1}{2}}}\int_{F_\infty^\times}W_{\xi_\infty}\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)\chi'_\infty(y)^{-1}\|y\|_\infty^{s-\frac{1}{2}}\,d^\times y,$$

where

$$e_{\mathfrak{q}'}((\gamma\beta),\chi'_\infty)=\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]}\sum_{\eta\in\Gamma_{\mathfrak{q}'}\backslash\mathfrak{o}_F^\times}\psi_\infty(\eta\gamma\beta)\chi'_\infty(\eta\gamma\beta).$$

Then using the definition of $\lambda_\pi^{\mathfrak{c}}(\mathfrak{a})$ from Section 3.2 we see that

$$\sum_{\gamma\in\mathfrak{o}_F^\times\backslash(\mathfrak{a}_j'^{-1}\cap F^\times)}\frac{a_{\xi^{\mathfrak{c}}}(t_j,\gamma)e_{\mathfrak{q}'}((\gamma\beta),\chi'_\infty)}{\|\gamma\|_\infty^{s-\frac{1}{2}}}=N(\mathfrak{a}'_j)^{s-\frac{1}{2}}\sum_{\{\mathfrak{a}\sim\mathfrak{a}'_j\}}\frac{\lambda_\pi^{\mathfrak{c}}(\mathfrak{a})e_{\mathfrak{q}'}(\mathfrak{a}\mathfrak{a}_j'^{-1}(\beta),\chi'_\infty)}{N(\mathfrak{a})^s}.$$

This is precisely the additive twist $N(\mathfrak{a}'_j)^{s-1/2}L_{\mathfrak{a}'_j}(s,\lambda_\pi^{\mathfrak{c}},\beta,\chi'_\infty)$, and therefore

$$\begin{aligned}\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]}\int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times}\Phi_{\xi^{\mathfrak{c}}}(y,\alpha)\|y\|_\infty^{s-\frac{1}{2}}\,d^\times y&=N((\beta)^{-1}\mathfrak{a}'_j)^{s-1/2}L_{\mathfrak{a}'_j}(s,\lambda_\pi^{\mathfrak{c}},\beta,\chi'_\infty)\prod_{v|\infty}L(s,\pi_v\otimes{\chi'_v}^{-1})\\&=\Lambda_{\mathfrak{a}'_j}(s,\lambda_\pi^{\mathfrak{c}},\beta,\chi'_\infty).\end{aligned}$$

We now change $\xi^{\mathfrak{c}}$ to $\xi^D=\sum_{\mathfrak{c}|\mathfrak{m}}A(\mathfrak{c})\xi^{\mathfrak{c}}$ and obtain, by linearity,

$$\begin{aligned}&\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]}\int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times}\Phi_{\xi^D}(y,\alpha)\|y\|_\infty^{s-\frac{1}{2}}\,d^\times y\\&\quad=N((\beta)^{-1}\mathfrak{a}'_j)^{s-1/2}L_{\mathfrak{a}'_j}(s,\lambda_\pi^D,\beta,\chi'_\infty)\prod_{v|\infty}L(s,\pi_v\otimes{\chi'_v}^{-1})\\&\quad=\Lambda_{\mathfrak{a}'_j}(s,\lambda_\pi^D,\beta,\chi'_\infty),\end{aligned}\tag{3.15}$$

where $\lambda_\pi^D=\sum_{\mathfrak{c}|\mathfrak{m}}A(\mathfrak{c})\lambda_\pi^{\mathfrak{c}}$.

Now the strategy in [BK11, §5.2] is to recompute (3.15) using (3.12). Since we are not altering the test vector at the infinite places it is evident that the archimedean analysis of loc. cit. remains valid in the current setup as well.[2] Here, we mainly want to point out the precise form of the additive twist that arises while implementing the aforementioned strategy. Fix a finite subset $T\subset(1+\mathfrak{q})\cap F^\times$ as in [BK11, Lemma 5.3]. Let $M\in\mathbb{Z}_{\geq 0}$ and fix $m_0\in 4\mathbb{Z}$ with $0\leq m_0<M$. Let us first consider the sum over γ on the right-hand side of [BK11, (5.12)] with $\xi^{\mathfrak{m}\mathfrak{c}^{-1}}$ in place of ξ. Then, in the notation of [BK11, §5.4], after performing the integration over $\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times$, we obtain

$$\begin{aligned}&\chi'_\infty(-1)\sum_{\gamma\in\mathfrak{o}_F^\times\backslash(\mathfrak{a}_j^{-1}\mathfrak{m}^{-1}\mathfrak{c}\cap F^\times)}\tilde{a}_{\xi^{\mathfrak{m}\mathfrak{c}^{-1}}}(t_j^{-1},\gamma)e_{\mathfrak{q}'}((\gamma\alpha),\chi_\infty)\|\gamma\alpha\|_\infty^{\frac{1}{2}-\frac{m_0}{2}-s}\\&\qquad\cdot P\Big(s+\frac{m_0}{2};m_0\Big)\,L\Big(s+\frac{m_0}{2},\widetilde{\pi}_\infty\otimes\chi_\infty^{-1}\Big),\end{aligned}\tag{3.16}$$

where

$$P(s;m_0)=\left(\frac{(2\pi i)^{m_0/2}}{(m_0/2)!}\right)^{[F:\mathbb{Q}]}\frac{L(1-s+\frac{m_0}{2},\pi_\infty\otimes\chi_\infty)}{L(1-s,\pi_\infty\otimes\chi_\infty)}.$$

[2] One minor alteration is required since we no longer assume that π_v is unitary for archimedean v: in [BK11, Lemma 5.2], one must change the line of integration from $\Re(s)=\frac{1}{2}$ to a vertical line sufficiently far to the right (depending on the Langlands parameters of π_∞). This requires only cosmetic changes to the proof.

Hence using (3.8) we see that (3.16) is precisely

$$\chi'_\infty(-1)P\Big(s+\frac{m_0}{2};m_0\Big)\,\Lambda_{\mathfrak{a}_j\mathfrak{m}}\Big(s+\frac{m_0}{2},\tilde{a}^{\mathfrak{c}}_{\pi},\alpha,\chi_\infty\Big)\,,$$

which in turn, using (3.11), equals

$$\kappa P\Big(s+\frac{m_0}{2};m_0\Big)\,\Lambda_{\mathfrak{a}_j\mathfrak{m}}\Big(s+\frac{m_0}{2},\lambda^{\mathfrak{c}}_{\widetilde{\pi}},\alpha,\chi_\infty\Big)\,,$$

where $\kappa=\chi'_\infty(-1)\omega_\pi(d)\epsilon_f(\pi,\psi)$. As before, changing $\xi^{\mathfrak{m}\mathfrak{c}^{-1}}$ to $\xi^D=\sum_{\mathfrak{c}|\mathfrak{m}}A(\mathfrak{m}\mathfrak{c}^{-1})\xi^{\mathfrak{m}\mathfrak{c}^{-1}}$ in the above calculation and using linearity, we see that (3.16) equals

$$\kappa P\Big(s+\frac{m_0}{2};m_0\Big)\,\Lambda_{\mathfrak{a}_j\mathfrak{m}}\Big(s+\frac{m_0}{2},\lambda^{\widetilde{D}}_{\widetilde{\pi}},\alpha,\chi_\infty\Big)\,,$$

where $\lambda^{\widetilde{D}}_{\widetilde{\pi}}=\sum_{\mathfrak{c}|\mathfrak{m}}A(\mathfrak{m}\mathfrak{c}^{-1})\lambda^{\mathfrak{c}}_{\widetilde{\pi}}$.

Gathering the above information and incorporating it into the argument in [BK11, §5.4] we obtain the following identity:

$$\kappa P\Big(s+\frac{m_0}{2};m_0\Big)\Lambda_{\mathfrak{a}_j\mathfrak{m}}\Big(s+\frac{m_0}{2},\lambda^{\widetilde{D}}_{\widetilde{\pi}},\alpha,\chi_\infty\Big)$$
$$=\sum_{\lambda\in T}c_\lambda\chi_\infty(\lambda)^{-1}N((\lambda))^{s-\frac{1}{2}}\Lambda_{\mathfrak{a}'_j}(s,\lambda^D_\pi,\lambda^{-1}\beta,\chi'_\infty)-H_{M,m_0,\alpha}(s),$$

where $H_{M,m_0,\alpha}(s)$ is holomorphic for $\Re(s)>\sigma+\frac{3}{2}-\frac{M}{[F:\mathbb{Q}]}$. Furthermore, although it was not pointed out in [BK11, §5.4], since $H_{M,m_0,\alpha}(s)$ arises as a Mellin transform (of the function $E_{M,m_0,\alpha}(y)$ defined in [BK11, (5.12)]), it must be bounded in vertical strips.

From this, choosing $m_0>4\sigma-2$ and $M>\max(m_0,[F:\mathbb{Q}](m_0+2\sigma+2K-1)/2)$, we see that

$$\frac{\Lambda_{\mathfrak{a}_j\mathfrak{m}}(s,\lambda^{\widetilde{D}}_{\widetilde{\pi}},\alpha,\chi_\infty)}{L(1-s,\pi_\infty\otimes\chi_\infty)}$$

is entire of finite order.

Remark. Let $\widetilde{D}(s)=\sum_{\mathfrak{c}|\mathfrak{m}}A(\mathfrak{m}\mathfrak{c}^{-1})N(\mathfrak{c})^{1/2-s}$. Then the coefficients λ^D_π and $\lambda^{\widetilde{D}}_{\widetilde{\pi}}$ are precisely the Dirichlet coefficients of $D(s)L(s,\pi)$ and $\widetilde{D}(s)L(s,\widetilde{\pi})$, respectively.

Suppose ω is an idèle class character and $\chi_{\omega^{-1}}$ is the Größencharakter associated to ω^{-1}. Define

$$\Lambda\big(s,\lambda^{\widetilde{D}}_{\widetilde{\pi}}\times\chi_{\omega^{-1}}\big)=L\big(s,\lambda^{\widetilde{D}}_{\widetilde{\pi}}\times\chi_{\omega^{-1}}\big)\prod_{v|\infty}L(s,\widetilde{\pi}_v\otimes\omega_v^{-1}),$$

where $L\big(s,\lambda^{\widetilde{D}}_{\widetilde{\pi}}\times\chi_{\omega^{-1}}\big)$ is the multiplicative twist introduced in [BK11, §3]. Then from [BK11, Prop. 3.1][3] we know that $\Lambda(s,\lambda^{\widetilde{D}}_{\widetilde{\pi}}\times\chi_{\omega^{-1}})$ is a $\mathbb{C}$-linear combination $\sum_{i=1}^m c_i\Lambda_{\mathfrak{a}_{j_i}\mathfrak{m}}\big(s,\lambda^{\widetilde{D}}_{\widetilde{\pi}},\alpha_i,\omega_\infty\big)$ of additive twists. Therefore

$$\frac{\Lambda(s,\lambda^{\widetilde{D}}_{\widetilde{\pi}}\times\chi_{\omega^{-1}})}{L(1-s,\pi_\infty\otimes\omega_\infty)}$$

is entire of finite order. Next, we need the following simple lemma which follows from the stability of local L-functions under highly ramified twists.

[3]The proposition includes the hypothesis that the L-series is given by an Euler product, which might not be true in this case; however, that hypothesis is not used in this direction of the proof.

Lemma 3.4. *Let $\pi = \bigotimes \pi_v$ be an irreducible, admissible, generic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$*, and suppose ω is an idèle class character such that, for every non-archimedean place v for which π_v is ramified, ω_v is either unramified or sufficiently highly ramified (depending in a precise way on π_v). Let $L(s, \pi \otimes \omega) = \prod_{v<\infty} L(s, \pi_v \otimes \omega_v)$. Then $L(s, \pi \otimes \omega) = L(s, \lambda_\pi \times \chi_\omega)$.*

Now for ω satisfying the conditions of Lemma 3.4 it follows from the above remark that

$$\frac{\Lambda(s, \widetilde{\pi} \otimes \omega^{-1})\widetilde{D}(s, \omega^{-1})}{L(1-s, \pi_\infty \otimes \omega_\infty)}$$

is entire of finite order, where $\widetilde{D}(s, \omega^{-1}) = \sum_{\mathfrak{c}|\mathfrak{m}} A(\mathfrak{m}\mathfrak{c}^{-1})\chi_{\omega^{-1}}(\mathfrak{c})N(\mathfrak{c})^{\frac{1}{2}-s}$. Thus, using the functional equation we obtain that $\widetilde{D}(1-s, \omega^{-1})L(s, \pi \otimes \omega)$ is entire of finite order. This concludes the proof of Prop. 2.1.

4. Proofs of Lemmas 2.3 and 2.4

Proof of Lemma 2.3. It is well-known that $\Gamma_1(\mathfrak{N})$ is finitely-generated. Let Δ be any finite set of generators. We will show that one can modify the elements of Δ, possibly enlarging the set in the process, so that the conclusion is satisfied. First note that if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(\mathfrak{N})$ is triangular (i.e., $bc = 0$) then d must be an S-unit. Thus, we may add triangular matrices to Δ, so we are free to multiply a given generator by triangular matrices on either side.

With that in mind, let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta$. Let T be the set of prime ideals of $\mathfrak{o}_S$ dividing $p\mathfrak{o}_S$, and define $T_1 = \{\mathfrak{q} \in T : c \in \mathfrak{q}\}$, $T_2 = \{\mathfrak{q} \in T : c \notin \mathfrak{q}\}$. Note that since $ad - bc$ is an S-unit, we must have $a \notin \mathfrak{q}$ for every $\mathfrak{q} \in T_1$. Further, since $p\mathfrak{o}_S$ is co-prime to $\mathfrak{N}$, by the Chinese remainder theorem there exists $n \in \mathfrak{N}$ such that $n \notin \mathfrak{q}$ for every $\mathfrak{q} \in T_1$ and $n \in \mathfrak{q}$ for every $\mathfrak{q} \in T_2$. Thus, $c' = an + c$ satisfies $c' \notin \mathfrak{q}$ for every $\mathfrak{q} \in T$. Moreover, shifting n by an element of $p\mathfrak{N}$ if necessary, we may assume that c' and $d' = bn + d$ are non-zero. Hence, in view of the equality

$$\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c' & d' \end{pmatrix},$$

we may assume without loss of generality that $cd \neq 0$ and that $c\mathfrak{o}_S$ and $p\mathfrak{o}_S$ are co-prime.

Next, for any $z_1, z_2 \in \mathfrak{o}_S^\times$ with $z_2 \in 1 + \mathfrak{N}$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 z_2^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a z_1 z_2^{-1} & b \\ c z_1 & d z_2 \end{pmatrix}.$$

By choosing z_1 and z_2 appropriately, we may assume without loss of generality that c and d are elements of $\mathfrak{o}_F$ and that c is a unit at every place in S.

Finally, for any $x \in \mathfrak{o}_F$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ax + b \\ c & cx + d \end{pmatrix}.$$

To conclude the proof, it suffices to show that there are infinitely many choices for x such that $cx + d$ generates a prime ideal of $\mathfrak{o}_F$ and $N(cx + d) \not\equiv 1 \pmod{p}$. In fact, since F contains no primitive pth root of unity, it follows from the Chebotarev density theorem that the set of prime ideals generated by such elements has density $\frac{1}{\#\mathrm{Cl}^{(c)}} \frac{p-2}{p-1} > 0$, where $\mathrm{Cl}^{(c)}$ is the ray class group for the modulus $c\mathfrak{o}_F$. □

Proof of Lemma 2.4. First, since π is assumed to be generic, by Lemma 4.1, any common poles between $L(s, \pi_\infty \otimes \omega_\infty)$ and $L(1-s, \widetilde{\pi}_\infty \otimes \omega_\infty^{-1})$ for any character ω_∞ must arise from the local L-factors at distinct archimedean places.

Let ω be an idèle class character. If $L(s, \pi_\infty \otimes \omega_\infty)$ and $L(1-s, \widetilde{\pi}_\infty \otimes \omega_\infty^{-1})$ have a common pole then the same is true with ω replaced by $\omega \| \cdot \|^{it}$ for any $t \in \mathbb{R}$. Hence, we may assume without loss of generality that ω_∞ is trivial on the diagonal embedding of $\mathbb{R}_{>0}$ in $F_\infty^\times$, i.e. $\sum_{v|\infty} \nu(\omega_v) = 0$. Let $v_1, \ldots, v_{n+1}$ denote the archimedean places of F. To any such ω, we associate the vector $x(\omega) \in \mathbb{R}^{n+1}$ defined by

$$x(\omega) = \left(\frac{\nu(\omega_{v_1})}{2\pi i}, \ldots, \frac{\nu(\omega_{v_{n+1}})}{2\pi i} \right).$$

By our normalization above, $x(\omega)$ is an element of the trace 0 hyperplane

$$H = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j = 0 \right\}.$$

Next, let $\epsilon_1, \ldots, \epsilon_n$ be a system of fundamental units for $\mathfrak{o}_F$, and let M be the $n \times (n+1)$ real matrix with entries $\log \|\epsilon_i\|_{v_j}$, $1 \le i \le n$, $1 \le j \le n+1$. By Dirichlet's unit theorem, M defines an isomorphism between H and $\mathbb{R}^n$. Set $y(\omega) = Mx(\omega)$; then, since ω is an idèle class character, we have $y(\omega) \in \mathbb{Q}^n$.

Let v_j and v_k be distinct archimedean places of F. From the definition of the local L-factors we see that if ρ is a pole of $L(s, \pi_{v_j} \otimes \omega_{v_j})$ then

$$-\rho - \frac{\nu(\omega_{\pi_{v_j}})}{2} - \nu(\omega_{v_j}) \in \delta_j \nu(\pi_{v_j}) + \frac{1}{2}\mathbb{Z}_{\ge 0}$$

for some $\delta_j \in \{\pm 1\}$. If ρ is also a pole of $L(1-s, \widetilde{\pi}_{v_k} \otimes \omega_{v_k}^{-1})$ then

$$\rho - 1 + \frac{\nu(\omega_{\pi_{v_k}})}{2} + \nu(\omega_{v_k}) \in \tilde{\delta}_k \nu(\pi_{v_k}) + \frac{1}{2}\mathbb{Z}_{\ge 0}$$

for some $\tilde{\delta}_k \in \{\pm 1\}$, and thus

$$\nu(\omega_{v_k}) - \nu(\omega_{v_j}) \in \frac{\nu(\omega_{\pi_{v_j}}) - \nu(\omega_{\pi_{v_k}})}{2} + \delta_j \nu(\pi_{v_j}) + \tilde{\delta}_k \nu(\pi_{v_k}) + 1 + \frac{1}{2}\mathbb{Z}_{\ge 0}.$$

Considering real parts, we see that this can hold for at most two choices of the pair $(\delta_j, \tilde{\delta}_k) \in \{\pm 1\}^2$. Considering imaginary parts, we thus see that any ω such that $L(s, \pi_{v_j} \otimes \omega_{v_j})$ and $L(1-s, \widetilde{\pi}_{v_k} \otimes \omega_{v_k}^{-1})$ have a common pole must satisfy

$$\frac{\nu(\omega_{v_k}) - \nu(\omega_{v_j})}{2\pi i} = c$$

for one of at most two numbers $c \in \mathbb{R}$ (depending on j and k).

Collecting this result for all distinct pairs (v_j, v_k), we see that there is a set of at most $2n(n+1)$ hyperplanes in H whose union contains $x(\omega)$ for any ω such that $L(s, \pi_\infty \otimes \omega_\infty)$ and $L(1-s, \widetilde{\pi}_\infty \otimes \omega_\infty^{-1})$ have a common pole. The rational points of the image of those hyperplanes under M therefore lie in a finite union of rational hyperplanes, which may be defined by equations of the type $a_1 y_1 + \ldots + a_n y_n = b$, where $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ is non-zero and $b \in \mathbb{Z}$. Thus, there is a non-negative integer $m \le 2n(n+1)$ and non-zero vectors

$w_1, \dots, w_m \in \mathbb{Z}^n$ such that $w_j \cdot y(\omega) \in \mathbb{Z}$ holds for at least one $j \le m$ whenever $L(s, \pi_\infty \otimes \omega_\infty)$ and $L(1-s, \widetilde{\pi}_\infty \otimes \omega_\infty^{-1})$ have a common pole.

Now let $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$ be a vector which is not orthogonal to any of $w_1, \dots, w_m$, and let p be a prime which does not divide

$$\#\mu(F) \cdot \prod_{j=1}^{m} (w_j \cdot z) \cdot \prod_{v \in S \setminus S_\infty} \big(q_v(q_v - 1)\big),$$

where $\mu(F)$ is the group of roots of unity in F. By Lemma 4.2 below, there are infinitely many ideals $\mathfrak{q} \subset \mathfrak{o}_F$ admitting a character $\chi : (\mathfrak{o}_F/\mathfrak{q})^\times \to S^1$ satisfying $\chi(\epsilon_i) = e(z_i/p)$ for $i = 1, \dots, n$. Any of these may be completed to an idèle class character ω with conductor dividing $\mathfrak{q}$ and satisfying $\omega_\infty(\epsilon_i) = e(z_i/p)$ for $i = 1, \dots, n$. In fact, Lemma 4.2 guarantees that there is an infinite, pairwise co-prime sequence of such $\mathfrak{q}$, so clearly we may choose one so that part (1) of the lemma is satisfied.

As for part (2), if ω' is as given in the hypotheses then it is not hard to see that the numbers $\omega'_\infty(\epsilon_i)$, $i = 1, \dots, n$, are kth roots of unity for some k relatively prime to p. Thus, for every $j \le m$, $w_j \cdot (y(\omega) + y(\omega'))$ must have denominator divisible by p. The conclusion follows. □

Lemma 4.1. *Let v be an archimedean place of F and π_v an irreducible, admissible $(\mathfrak{gl}_2(F_v), K_v)$-module. Then π_v is non-generic, i.e. it does not have a Whittaker model, if and only if there is a character $\omega_v : F_v^\times \to \mathbb{C}^\times$ such that $L(s, \pi_v \otimes \omega_v)$ and $L(1-s, \widetilde{\pi}_v \otimes \omega_v^{-1})$ have a common pole.*

Proof. We first note that according to [JL70, Theorem 5.13, Theorem 6.3, and the concluding paragraph of §5], the representation π_v is non-generic if and only if it is finite-dimensional. Now, suppose ω_v is a local character. Since the hypothesis that $L(s, \pi_v \otimes \omega_v)$ and $L(1-s, \widetilde{\pi}_v \otimes \omega_v^{-1})$ have a common pole is invariant under shift, we may assume without loss of generality that $\frac{\nu(\omega_{\pi_v})}{2} + \nu(\omega_v) = 0$. If π_v is a discrete series then it is both infinite-dimensional and unitary, so the conclusion is immediate. Otherwise, we have $\pi_v \cong \pi(\mu_1, \mu_2)$ for some quasi-characters μ_1, μ_2 of $F_v^\times$.

First suppose that $v \in S_\mathbb{R}$. Let $\nu_j = \nu(\mu_j)$, $\epsilon_j = \epsilon(\mu_j)$ for $j = 1, 2$. As detailed in [God70, §2.8 Thm. 2], $\pi_v \cong \pi(\mu_1, \mu_2)$ is finite-dimensional precisely when $\sigma(\mu_1, \mu_2)$ is defined, viz. when $2\nu(\pi_v) = \nu_1 - \nu_2$ is a non-zero integer satisfying[4] $2\nu(\pi_v) \equiv \epsilon_1 - \epsilon_2 + 1 \pmod 2$. If $k(\pi_v) = 0$, then from the definition of the local L-factor [BK11, p. 683], ρ is a pole of $L(s, \pi_v \otimes \omega_v)$ if and only if

$$\rho \in \pm\nu(\pi_v) - |\epsilon(\omega_v) - \epsilon(\pi_v)| - 2\mathbb{Z}_{\ge 0}.$$

On the other hand, ρ is a pole of $L(1-s, \widetilde{\pi}_v \otimes \omega_v^{-1})$ if and only if

$$1 - \rho \in \pm\nu(\pi_v) - |\epsilon(\omega_v) - \epsilon(\pi_v)| - 2\mathbb{Z}_{\ge 0}.$$

[4]This was stated incorrectly in [BK11, §4.2.1]. Specifically, on the last line of p. 681 it was stated that $\mathcal{B}(\mu_1, \mu_2)$ is irreducible unless $\nu_1 - \nu_2$ is a non-zero integer and $\epsilon_1 \ne \epsilon_2$; the latter condition should be corrected to $\nu_1 - \nu_2 \equiv \epsilon_1 - \epsilon_2 + 1 \pmod 2$. Also, in the following paragraph it was stated that for a discrete series or limit of discrete series representation we may assume that $\nu_1 - \nu_2 \in \mathbb{Z}_{\ge 0}$ and $(\epsilon_1, \epsilon_2) = (0, 1)$; here the latter should be corrected to $\epsilon_1 \le \epsilon_2$. Fortunately these errata cause no harm to any of the subsequent arguments in [BK11].

These conditions have a non-empty intersection if and only if $\nu(\pi_v)$ is real and

$$|2\nu(\pi_v)| \in 1 + 2|\epsilon(\omega_v) - \epsilon(\pi_v)| + 2\mathbb{Z}_{\geq 0},$$

which, in turn, happens for some character ω_v if and only if $2\nu(\pi_v)$ is an odd integer, i.e. π_v is finite-dimensional. Similarly, if $k(\pi_v) = 1$, any common pole ρ satisfies both

$$\rho \in -\frac{1}{2} \pm \left(|\epsilon(\omega_v) - \epsilon(\pi_v)| + \nu(\pi_v) - \frac{1}{2} \right) - 2\mathbb{Z}_{\geq 0}$$

and

$$1 - \rho \in -\frac{1}{2} \pm \left(|\epsilon(\omega_v) - \epsilon(\pi_v)| - \nu(\pi_v) - \frac{1}{2} \right) - 2\mathbb{Z}_{\geq 0},$$

and this is possible if and only if $2\nu(\pi_v)$ is a non-zero even integer, i.e. π_v is finite-dimensional.

Turning to the case $v \in S_{\mathbb{C}}$, let $l = -k(\omega_{\pi_v}) - 2k(\omega_v)$. Then from the definition of the local L-factors [BK11, p. 685], any common pole ρ of $L(s, \pi_v \otimes \omega_v)$ and $L(1 - s, \widetilde{\pi}_v \otimes \omega_v^{-1})$ satisfies

$$\rho \in -\frac{\max(|l|, k(\pi_v))}{4} \pm \left(\nu(\pi_v) - \operatorname{sgn}(l) \frac{\min(|l|, k(\pi_v))}{4} \right) - \mathbb{Z}_{\geq 0}$$

and

$$1 - \rho \in -\frac{\max(|l|, k(\pi_v))}{4} \pm \left(\nu(\pi_v) + \operatorname{sgn}(l) \frac{\min(|l|, k(\pi_v))}{4} \right) - \mathbb{Z}_{\geq 0}$$

so that $\nu(\pi_v)$ is real and

$$|2\nu(\pi_v)| \in 1 + \frac{\max(|l|, k(\pi_v))}{2} + \mathbb{Z}_{\geq 0}.$$

This happens for some ω_v if and only if $|2\nu(\pi_v)| - \frac{k(\pi_v)}{2}$ is a positive integer, which in turn holds if and only if $\mu_1(t)\mu_2(t)^{-1}$ is of the form $t^p \bar{t}^q$, where $p, q \in \mathbb{Z}$ and $pq > 0$. By [God70, §2.9 Thm. 3], this is precisely the condition under which π_v is finite-dimensional. □

Lemma 4.2. *Let p be a prime number such that F contains no primitive pth root of unity, and let $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ be pth roots of unity. Let $\epsilon_1, \ldots, \epsilon_n$ be a system of fundamental units for $\mathfrak{o}_F$. Then there is an infinite sequence $\mathfrak{q}_1, \mathfrak{q}_2, \ldots$ of pairwise co-prime ideals of $\mathfrak{o}_F$ and characters $\chi_i : (\mathfrak{o}_F/\mathfrak{q}_i) \to S^1$ satisfying $\chi_i(\epsilon_j) = \zeta_j$ for all i, j.*

Proof. Fix $x \in \mathbb{Z}_{\geq 0}$, and let $T(x, p)$ be the set of prime ideals $\mathfrak{p} \subset \mathfrak{o}_F$ with norm $N(\mathfrak{p})$ satisfying $N(\mathfrak{p}) > x$ and $N(\mathfrak{p}) \equiv 1 \pmod{p}$. For any $\mathfrak{p} \in T(x, p)$, fix a generator $g_{\mathfrak{p}}$ for the cyclic group $(\mathfrak{o}_F/\mathfrak{p})^\times \cong \mathbb{F}_{N(\mathfrak{p})}^\times$, and consider the composite map

$$\varphi_{\mathfrak{p}} : \mathfrak{o}_F^\times \xrightarrow{\text{mod } \mathfrak{p}} (\mathfrak{o}_F/\mathfrak{p})^\times \xrightarrow{\log_{g_{\mathfrak{p}}}} \mathbb{Z}/(N(\mathfrak{p}) - 1)\mathbb{Z} \xrightarrow{\text{mod } p} \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p.$$

We define $w(\mathfrak{p}) = (\varphi_{\mathfrak{p}}(\epsilon_1), \ldots, \varphi_{\mathfrak{p}}(\epsilon_n)) \in \mathbb{F}_p^n$.

We claim that there exist $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in T(x, p)$ such that $\{w(\mathfrak{p}_1), \ldots, w(\mathfrak{p}_n)\}$ is a basis for $\mathbb{F}_p^n$. If not then there must be a non-zero linear functional which vanishes at each $w(\mathfrak{p})$, i.e. there are integers $a_1, \ldots, a_n \in [0, p)$, not all zero, such that

$$\varphi_{\mathfrak{p}}(\epsilon_1^{a_1} \cdots \epsilon_n^{a_n}) = a_1\varphi_{\mathfrak{p}}(\epsilon_1) + \ldots + a_n\varphi_{\mathfrak{p}}(\epsilon_n) = 0$$

for all $\mathfrak{p} \in T(x, p)$. Thus, the unit $\epsilon = \epsilon_1^{a_1} \cdots \epsilon_n^{a_n}$ is a pth power modulo $\mathfrak{p}$ for every $\mathfrak{p} \in T(x, p)$, but is not a pth power in $\mathfrak{o}_F^\times$. To see that this is not possible, consider the field extensions

$F' = F(\zeta)$ and $F'' = F'(\sqrt[p]{\epsilon})$, where ζ is a primitive pth root of unity. If $\mathfrak{p} \subset \mathfrak{o}_F$ is a prime ideal not dividing the relative discriminant of F''/F, then:

(1) $\mathfrak{p}$ splits completely in F' if and only if $N(\mathfrak{p}) \equiv 1 \pmod{p}$;
(2) $\mathfrak{p}$ splits completely in F'' if and only if $N(\mathfrak{p}) \equiv 1 \pmod{p}$ and ϵ is a pth power modulo $\mathfrak{p}$.

Therefore, by the Kronecker–Frobenius density theorem, the set of primes $\mathfrak{p} \subset \mathfrak{o}_F$ such that $N(\mathfrak{p}) \equiv 1 \pmod{p}$ and ϵ is *not* a pth power modulo $\mathfrak{p}$ has density

$$\frac{1}{[F' : F]} - \frac{1}{[F'' : F]} = \frac{1}{p-1} - \frac{1}{p(p-1)} = \frac{1}{p} > 0.$$

Thus, there is such a prime $\mathfrak{p} \in T(x, p)$, proving the claim.

Now take any $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in T(x, p)$ such that $\{w(\mathfrak{p}_1), \ldots, w(\mathfrak{p}_n)\}$ is a basis for $\mathbb{F}_p^n$, and set $\mathfrak{q} = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. For each $i = 1, \ldots, n$, let $b_i \in \mathbb{F}_p$ be such that $e(b_i/p) = \zeta_i$. Choose $c_1, \ldots, c_n \in \mathbb{F}_p$ so that

$$c_1 w(\mathfrak{p}_1) + \ldots + c_n w(\mathfrak{p}_n) = (b_1, \ldots, b_n),$$

and define characters $\chi_j : (\mathfrak{o}_F/\mathfrak{p}_j)^\times \to S^1$, $j = 1, \ldots, n$, by $\chi_j(g_{\mathfrak{p}_j}^a) = e(c_j a/p)$. Thus, the character $\chi : (\mathfrak{o}_F/\mathfrak{q})^\times \to S^1$ defined by $\chi = \chi_1 \cdots \chi_n$ satisfies the desired conclusion. To see that there is a sequence of such characters with relatively prime moduli, it suffices to repeat the construction with larger and larger values of x. ☐

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School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom

E-mail address: `andrew.booker@bristol.ac.uk`

Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, USA

E-mail address: `muthu-krishnamurthy@uiowa.edu`