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Undecidability and the problem of outcomes in quantum measurements

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We argue that it is fundamentally impossible to recover information about quantum superpositions when a quantum system has interacted with a sufficiently large number of degrees of freedom of the environment. This is due to the fact that gravity imposes fundamental limitations on how accurate measurements can be. This leads to the notion of undecidability: there is no way to tell, due to fundamental limitations, if a quantum system evolved unitarily or suffered wavefunction collapse. This in turn provides a solution to the problem of outcomes in quantum measurement by providing a sharp criterion for defining when an event has taken place. We analyze in detail in examples two situations in which in principle one could recover information about quantum coherence: a) “revivals” of coherence in the interaction of a system with the measurement apparatus and the environment and b) the measurement of global observables of the system plus apparatus plus environment. We show in the examples that the fundamental limitations due to gravity and quantum mechanics in measurement prevent both revivals from occurring and the measurement of global observables. It can therefore be argued that the emerging picture provides a complete resolution to the measurement problem in quantum mechanics.

I. INTRODUCTION

The Copenhagen interpretation of quantum mechanics is formulated with the aid of a classical macroscopic realm in order to explain the measurement process. Given the growing number [1] of experiments showing the existence of superpositions of macroscopically distinct quantum states, it is becoming increasingly desirable to understand quantum mechanics entirely as a standalone quantum paradigm without having to refer to an external classical world (see for instance [2, 3] for previous attempts). Notice that the problem transcends the actual physical measurement of quantities in that it is present every time one wishes to assign a probability to an event. We use the word event to denote that a physical observable of coupled system plus apparatus plus environment takes a definite value. (For more details on the notion of event in light of the ideas of this paper see appendix 2.)

Most physicists claim that environmentally induced decoherence may play an important role in the solution of the measurement problem of quantum mechanics (see [3] for a recent review). Decoherence allows us to understand how the interaction with the environment induces a local suppression of interference between a set of preferred states, associated with the “pointer basis”. It leads to a fast suppression of the interference terms of the reduced matrix describing the system S coupled to the measurement apparatus A , and it selects a preferred set of states that are robust in spite of their interaction with the environment. All this is achieved with the standard unitary time evolution for the composite system plus apparatus plus environment $S + A + E$ and therefore the global phase coherence is not destroyed but simply transferred from the system S to the environment.

There are two main criticisms to the solution of the problem of measurement by invoking the environment (decoherence). The first criticism is that the “system plus apparatus plus environment” evolves unitarily and therefore all information about the original superposition is still present at the end and could in principle be retrieved (see for instance [4]). The second type of criticism is related to the fact that the system plus the measurement apparatus is left in a superposition of states through the interaction with the environment and therefore it would not generate a definite event (or measurement) but a superposition of them. Bell [5] calls this “and/or problem”, since the states corresponding to the diagonal of the density matrix coexist with each other and therefore one has “A and B” as possibilities. In the classical world, however, one has “A or B” as the resulting outcomes.

In previous work [6] we have argued that fundamental limitations to the measurement of time and distances due to gravitational effects imply that there are no mechanisms to retrieve the information about a quantum superposition from the system plus apparatus plus environment. Essentially, gravity puts fundamental limitations on the accuracy of clocks and this adds a new source of loss of quantum coherence. This in turn implies that experiments that “wait a long time” for quantum coherence to be restored (“revivals”) [4] actually worsen the situation due to the increasing inaccuracy of clocks as one measures longer times (see [6] for details). A second proposal to recover quantum coherence, by measuring global observables of the “system plus measuring apparatus plus environment” is also limited by fundamental arguments based on the laws of quantum mechanics and gravity, as we shall see.

If one accepts that quantum coherence cannot be retrieved, this leads to another key point we have recently emphasized, undecidability [7]: one cannot decide if the quantum state has suffered reduction or if it evolved unitarily. In fact, it could even be conceivable that sometimes there might be reduction, sometimes not, and we do not have

reasons to expect one or the other in a given instance. Let us get back to the second objection to the decoherence solution to the measurement problem: that at the end of the interaction with the environment, the measuring apparatus is generically left in a superposition of (eigen)-states corresponding to different “positions of the needle of the gauge”. That would not correspond to what one usually calls a “measurement” (production of an event) in which the apparatus is in a given (eigen-)state, corresponding to the “needle of the gauge” taking a definite position. This is the and/or problem of Bell that we mentioned above. We would like to argue that this problem can be solved by defining the appearance of events without necessarily implying a change in the quantum state. We will claim that an event occurs when the distinction between the “system plus apparatus plus environment” being in a superposition or in a given state becomes *undecidable*. At this point a definite choice of outcome, be it A or B, is compatible with the laws of physics. All possible experiments would give the same results if the complete system plus apparatus plus environment suffered a reduction of its state or evolved unitarily.

The organization of this paper is as follows. In section II we briefly review the fundamental limitations that gravity imposes on the measurement process and how it leads to a fundamental loss of coherence in quantum states. In section III we review how the limitations on measurement lead to a notion of undecidability in which one cannot tell if a quantum collapse has occurred or not and how this provides a sharp criterion for when a quantum event took place. In section IV we analyze the proposal in the context of models of environmentally-induced decoherence related to the Zurek model of spins. In particular we show how undecidability occurs and how the problem of “revivals” and the “problem of outcomes” are solved. In section V we analyze if extreme physics in quantum field theory could violate the results of section IV. We end with a discussion.

II. FUNDAMENTAL LOSS OF COHERENCE DUE TO THE USE OF REAL CLOCKS AND MEASURING RODS

Quantum mechanics as ordinarily formulated has a unitary evolution. In the usual formulation one assumes that time t is given by a classical parameter. This parameter can be measured with arbitrary precision. (A similar assumption is made about space if measurements if one is dealing with quantum field theory). But one knows that no observable quantity in nature can be measured with absolute precision. At the very least one will have quantum fluctuations in the observable. But theoretically one could choose observables with arbitrarily small quantum fluctuations and use those as the “clock”. This hits a fundamental limit if one includes gravity in measurements. To make accurate measurements one need to expend energy. When one considers gravity one is limited in the amount of energy one can invest in a measurement. Too large an energy density creates a black hole and spoils the measurement. This argument has been put forward by many authors. It is heuristic and without a full theory of quantum gravity cannot be rigorously worked out. The heuristic limits all imply that the error in the measurement of a time T goes as $\delta T \sim T_{\text{Planck}}^{1-a} T^a$ with $T_{\text{Planck}} \sim 10^{-44} \text{ s}$ is Planck’s time and a is a positive power. Some variations of the argument yield $a = 1/3$, some yield $a = 1/2$ [8]. The particular value of a is not very relevant, as we shall discuss later. We will choose $a = 1/3$ for the rest of the paper since the value is suggested by several arguments ([9, 10, 11]). As long as the error in measuring the time goes as a positive power of T then one loses quantum coherence. The reason for this is simple: although quantum evolution is unitary in terms of the fiduciary parameter t that appears in the Schrödinger equation, our clocks are not good enough to keep track of unitarity in evolution. After some time has evolved, even if we started with a pure state, our inaccurate clocks will force us to consider a superposition of states at different values of t as corresponding to the value of “time” T . Therefore pure states evolve into mixed states. A similar argument can be worked out for the measurements of distance in the context of quantum field theory but we will not expand it here, readers can refer to [12] for more details.

The loss of coherence due to imperfect clocks makes the off diagonal elements of the density matrix of a quantum system in the energy eigenbasis to decrease exponentially. The exponent for the mn -th matrix element is given by $\omega_{mn}^2 T_{\text{Planck}}^{4/3} T^{2/3}$, where $\omega_{mn} = E_{mn}/\hbar$ is the difference of energy between levels m and n divided by \hbar (the Bohr frequency between n and m). One could see this effect in the lab in reasonable times (hours) only if one were handling “macroscopic” quantum states corresponding to about 10^{13} atoms in coherence. The direct observation of this effect is therefore beyond our current experimental capabilities. However, as we shall see, it has important implications for the measurement problem in quantum mechanics.

III. UNDECIDABILITY AS THE CRITERION FOR THE PRODUCTION OF EVENTS

The “problem of outcomes” associated with the measurement problem in quantum mechanics has to do with the fact that even if one assumes the quantum system interacts with an environment and decoherence takes place, and the density matrix of the system (plus the measurement apparatus if there is one present) is diagonal, the density matrix

represents an improper mixture and therefore the quantum system (plus the measurement apparatus) is generically left in a superposition of states. Yet, when a measurement occurs one usually assumes the system and the measurement apparatus are left in an individual, definite state, not in a superposition. Therefore it appears that decoherence alone cannot account for the measurement process in quantum mechanics.

There have been several proposals in the literature to address this problem. One type of proposals consists in modifying the dynamics of the quantum theory to actually drive the system to an individual definite state. Examples of this point of view would be the modified dynamics of Ghirardi, Rimini and Weber [13] or Penrose [14]. Another point of view is to consider that the evolution is given by the traditional, unitary, Schrödinger equation, and to associate a criterion to decide which are the definite-valued observables, and therefore which are the events that may occur. This type of proposals may be generically called as “modal” interpretations. The many worlds interpretation can also be considered among this group. Particular proposals differ in the criteria chosen to determine which event occurs. A discussion of various modal interpretations and some of their issues can be seen in [15].

We here propose a new point of view. We will give a criterion for the production of events that is compatible both with a unitary evolution and with a reduction postulate. We will claim that an event has occurred when the evolution of the composite system plus apparatus plus environment reaches a point such that we cannot determine if a collapse or a unitary evolution has taken place. Usually this will happen when a quantum system interacts with an environment i.e. it does not occur in quantum systems in isolation. In such case a unitary evolution or an abrupt change as the one given by a collapse would be obviously distinguishable. When any system reaches this state, we will say that the system has become “undecidable”. The appearance of undecidability is therefore the criterion for the production of an event. An event occurs when the system plus apparatus plus environment becomes undecidable. We shall see that the loss of coherence due to the use of realistic clocks and measuring rods we mentioned in the previous section, together with the usual uncertainties of quantum mechanics, is what is at the root of the undecidability, and is therefore what enables the new point of view presented in this paper.

To illustrate how this would work in practice in what follows we would like to consider a concrete example.

IV. MODELS OF ENVIRONMENTALLY INDUCED DECOHERENCE

A. Zurek’s model

As is customary in studies of environmental decoherence in quantum mechanics these types of ideas cannot be proved in general, but have to be explored in examples. Therefore in spite of the universality of the idea of undecidability, there is of course no way to reach conclusions in generality, it must be studied in specific examples of increasing level of realism. The example we wish to consider is a more realistic version of a model introduced by Zurek [16] to probe ideas of environmentally-induced decoherence. Let us start by recalling that model. Although it does not include all the effects of a realistic situation, it exhibits how the information is transferred from the measuring apparatus to the environment. It consists of taking a spin one-half system that encodes the information about the microscopic system plus the measuring device. A basis in its two dimensional Hilbert space will be denoted by $\{|+\rangle, |-\rangle\}$. The environment is modeled by a bath of many similar two-state systems called atoms. There are N of them, each denoted by an index k and with associated two dimensional Hilbert space $\{|+\rangle_k, |-\rangle_k\}$. The dynamics is very simple, when there is no coupling with the environment the two spin states have the same energy, which is taken to be 0. All the atoms have zero energy as well in the absence of coupling. The whole dynamics is contained in the coupling, given by the following interaction Hamiltonian

$$H_{\text{int}} = \hbar \sum_k (g_k \sigma_z \otimes \sigma_z^k \otimes_{j \neq k} I_j). \quad (1)$$

σ_z is a Pauli spin matrix acting on the state of the system. It has eigenvalues $+1$ for the spin eigenvector $|+\rangle$ and -1 for $|-\rangle$. The operators σ_z^k are similar, acting on the state of the k -th atom. I_j denotes the identity matrix acting on atom j and g_k is the coupling constant that has dimensions of frequency and characterizes the coupling energy of one of the spins k with the system. In spite of the abstract character of the model, it can be thought of as providing a sketchy model of a photon propagating in a polarization analyzer.

Starting from a normalized initial state

$$|\Psi(0)\rangle = (a|+\rangle + b|-\rangle) \prod_{k=1}^N \otimes [\alpha_k |+\rangle_k + \beta_k |-\rangle_k], \quad (2)$$

it is easy to solve the Schroedinger equation and one gets for the state at the time t ,

$$|\Psi(t)\rangle = a|+\rangle \prod_{k=1}^N \otimes [\alpha_k \exp(ig_k t) |+\rangle_k + \beta_k \exp(-ig_k t) |-\rangle_k] \\ + b|-\rangle \prod_{k=1}^N \otimes [\alpha_k \exp(-ig_k t) |+\rangle_k + \beta_k \exp(ig_k t) |-\rangle_k]. \quad (3)$$

Writing the complete density operator $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$, one can take its trace over the environment degrees of freedom to get the reduced density operator,

$$\rho_c(t) = |a|^2 |+\rangle\langle +| + |b|^2 |-\rangle\langle -| + z(t) ab^* |+\rangle\langle -| + z^*(t) a^* b |-\rangle\langle +|, \quad (4)$$

where

$$z(t) = \prod_{k=1}^N [\cos(2g_k t) + i(|\alpha_k|^2 - |\beta_k|^2) \sin(2g_k t)]. \quad (5)$$

The complex number $z(t)$ controls the value of the non-diagonal elements. If this quantity vanishes the reduced density matrix ρ_c would correspond to a totally mixed state. Ignoring the “problem of outcomes” for a minute, that form of the matrix is the desired result, one would have several classical outcomes with their assigned probabilities. However, although the expression we obtained vanishes quickly assuming the α ’s and β ’s take random values, it behaves like a multiperiodic function, i.e. it is a superposition of a large number of periodic functions with different frequencies. Therefore this function will retake values arbitrarily close to the initial value for sufficiently large times (in closed systems). This implies that the apparent loss of information about the non-diagonal terms reappears if one waits a long enough time. This problem is usually called “recurrence of coherence” or “revival of coherence”. The characteristic time for these phenomena is proportional to the factorial of the number of involved frequencies. This time is usually large, perhaps exceeding the age of the universe—at least for sufficiently large systems—, making the problem unobservable at a fundamental rather than just a practical level. We will address in section V A how the fundamental limits on the measurements of time and distance completely eliminates the presence of revivals for sufficiently large systems, therefore confirming one cannot recover quantum coherence even in principle.

Another way of distinguishing if a reduction has taken place has been proposed by d’Espagnat. His proposal consists in analyzing a global observable for which the predictions of decoherence differ from those of a reduction postulate. The observable \hat{M} consists in measuring the x component of the spin of the combined system plus environment, $\hat{M} = \hat{\sigma}_x^S \otimes \prod_{k=1}^N \hat{\sigma}_x^k$. This observable commutes with the model’s Hamiltonian and is therefore a conserved quantity that generically will have a non-zero expectation value. If we think in terms of collapse, the spin has been measured and is therefore in the state $|+\rangle$ or $|-\rangle$. The expectation value of the observable can be shown to vanish. If we measure the observable several times we can obtain its expectation value and check if collapse has taken place or not. Although in practice, for this example and for other systems, it is unrealistic to expect that one will be able to measure observables like this one, one can ask if at least in principle they could not be observed.

B. A modified Zurek model with a more realistic interaction

Since we have argued that Zurek’s model is based on an idealization of the interaction with the environment, we would like to propose a model with a more realistic interaction between the spins. This could allow to study the behavior of d’Espagnat’s observable in a more realistic setting. The model we propose is a cavity with a uniform magnetic field in the z direction and in which there is a static spin S that represents the “needle” of the measuring apparatus. We assume a flux of spins (“the environment”) is pumped into the cavity and the spins interact with S during a finite time τ (the time will depend on the speed at which the spins are pumped into the cavity and the length of it. We can avoid having interactions among the spins of the “environment” by making the stream sufficiently “dilute” (i.e. essentially the spins go through the cavity one at a time). We will propose a more realistic interaction Hamiltonian than Zurek’s. Assume γ_1 is the magnetic moment of the spin S and γ_2 the magnetic moment of the spins of the environment and f_k the coupling between the k -th spin of the environment and S . The interaction Hamiltonian we choose is,

$$\hat{H}_k^{\text{int}} = f_k \left(\hat{S}_x \hat{S}_x^k + \hat{S}_y \hat{S}_y^k + \hat{S}_z \hat{S}_z^k \right). \quad (6)$$

The Hamiltonian due to the presence of the magnetic field when the k -th particle is in the cavity is,

$$\hat{H}_k^B = \gamma_1 B \hat{S}_z \otimes \hat{I}_k + \gamma_2 B \hat{I} \otimes S_z^k, \quad (7)$$

where \hat{I} is the identity matrix acting on the Hilbert space of the needle and I_k is the identity in the Hilbert space of the k -th particle. The introduction of a constant magnetic field pointing in a given direction, which we choose to be the z -direction, is in order to have a definite pointer basis. Recall that pointer states are distinguished by their ability to persist in spite of environmental monitoring. The system will therefore be quasi-diagonal in the basis associated with the direction z . Paz and Zurek have studied in detail the which pointer basis arises depending on which term in the Hamiltonian is the dominant one [17]. We here assume the dominant term is the one giving the coupling of the magnetic field with the needle. We would like to see if we can either measure global observables or observe revivals for this system (the needle).

The complete interaction Hamiltonian (acting on the Hilbert space of all particles) when the particle k is in the cavity is,

$$\begin{aligned} \hat{H}_k &= \left(\hat{H}_k^{\text{int}} + \hat{H}_k^B \right) \otimes \prod_{j \neq k} \hat{I}_j \\ &= \left[f_k \left(\hat{S}_x \hat{S}_x^k + \hat{S}_y \hat{S}_y^k + \hat{S}_z \hat{S}_z^k \right) + \gamma_1 B \hat{S}_z \otimes \hat{I}_k + \gamma_2 B \hat{I} \otimes \hat{S}_z^k \right] \otimes \prod_{j \neq k} \hat{I}_j. \end{aligned} \quad (8)$$

We are not explicitly writing the Hamiltonian describing the spatial evolution of the particles of the environment, which we assume to be a free-particle Hamiltonian.

We will first analyze qualitatively the possibility of measuring the global observable. We make the particles traverse the cavity where they interact with the magnetic field and the spin representing the “needle”. One could try to measure the spins after they exit the cavity using a Stern–Gerlach apparatus, and use the results to compute the observable \hat{M} or a similar observable involving all the spins. Repeating the experiment for an ensemble one could compute the expectation value $\langle \hat{M} \rangle$ and therefore determine if collapse has taken place or not. Let us analyze the spin coupling. The coupling constant is given by $f = \mu\gamma_1\gamma_2/\hbar r^3$ where μ is the vacuum permeability, $\gamma_{1,2}$ are the magnetic moments of the spins, r is the separation of the spins. In order for the system to experiment environmental decoherence the couplings between the spins cannot be arbitrarily small. For instance, if we reworked the state (3) for this model, one gets,

$$\begin{aligned} |\Psi(t)\rangle &= a|+\rangle \prod_{k=1}^N \otimes \left[\alpha_k \exp\left(i \int dt f_k\right) |+\rangle_k + \beta_k \exp\left(-i \int dt f_k\right) |-\rangle_k \right] \\ &\quad + b|-\rangle \prod_{k=1}^N \otimes \left[\alpha_k \exp\left(-i \int dt f_k\right) |+\rangle_k + \beta_k \exp\left(i \int dt f_k\right) |-\rangle_k \right]. \end{aligned} \quad (9)$$

and if the interaction is too weak we would get products of quantities with modulus close to one and the off diagonal terms of the density matrix (5) would not cancel. Therefore in order to have environmental decoherence the condition is $\int dt f_k > 1$. To compute the integral we refer the reader to figure 1,

$$\int f_k dt = \frac{\mu\gamma_1\gamma_2}{\hbar} \int_0^\tau \left(d^2 + (L - vt)^2 \right)^{-\frac{3}{2}} dt = \frac{2\mu\gamma_1\gamma_2}{\hbar v d^2} \frac{1}{\sqrt{1 + \frac{d^2}{L^2}}}, \quad (10)$$

where d is the impact parameter, v is the velocity of the spins of the environment and $2L$ is the length of the cavity. So we can conclude that for the existence of environmental decoherence due to the interaction with the spins one must satisfy the condition,

$$\frac{\mu\gamma_1\gamma_2}{\hbar d^2 v} > 1, \quad (11)$$

and we therefore see that one either needs large values for the magnetic moments or a small separation or velocity. However, there is an interaction acting on the spins, its energy of interaction is $E \sim -\mu\gamma_1\gamma_2/r^3$, and this leads to a force with a component in the y direction,

$$F_y = -\frac{2\mu\gamma_1\gamma_2}{r^5} y, \quad (12)$$

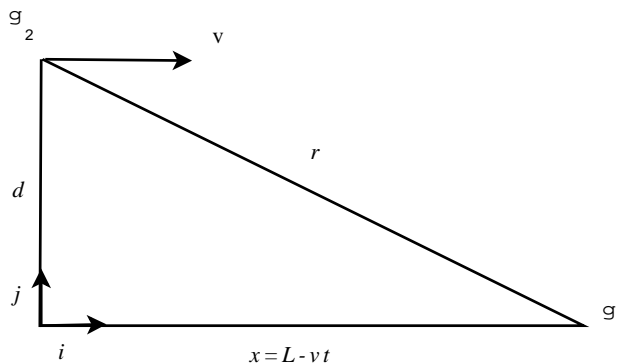


FIG. 1: The geometry of the interaction.

which causes to an acceleration in the point of closest approach between the spins $a_y = -\mu\gamma_1\gamma_2/(md^4)$. If the relative speed with respect to the environment particles is v then we have a period of time of at least d/v in which they will remain at a distance of the order of d from the spin S , being influenced by the repulsive force. This gives us a lower bound to the speed that the particles will acquire in the direction perpendicular to the motion of $v_y > a_y d/v \sim \mu\gamma_1\gamma_2(md^3v)$. Coupling this with the bound (11) one gets

$$v_y > \hbar/(md), \quad (13)$$

which for instance, would yield $v_y > 10^{-7}/d$ in MKS units, if the spins of the environment are neutrons. We will see that the force responsible for this variation in the velocity, together with the uncertainty in the position of the particles in the incident package will make impossible to determine the final position with enough precision to measure the spins of the particles of the environment after they finish passing through the cavity. To show this let us figure out the minimum time it will take to perform the experiment. We see that equation (11) gives us a bound on the time of flight since $\tau > d/v$. We then have $\tau > \hbar d^3/(\mu\gamma_1\gamma_2)$. If we take as the spin S a proton and the environment spins to be neutrons $m \sim 10^{-27}kg$, $\gamma_{1,2} \sim 10^{-26}$ Joule/Tesla, we have that the total time of the experiment $T > N\tau \sim Nd^3 10^{25}$. The shortest time for the experiment is characterized by the separation of the particles of the environment. We will take such separation to be much larger than the smallest possible d , let us say $d \sim 10^{-13}m$. If, for instance, the number of particles in the environment is $N \sim 10^{10}$, then at least $T \sim 1s$. This gives a lower bound for the length in time of the experiment (in practice it will be much longer). This leads to a dispersion of the particles. In order to estimate the dispersion in the length of time considered, we note that for a Gaussian packet, for a minimum dispersion packet one has that the dispersion is [18],

$$\Delta x(t) = \frac{\delta}{2} \sqrt{1 + \frac{4\hbar^2 t^2}{m^2 \delta^4}}, \quad (14)$$

where δ is a parameter that gives the width of the Gaussian. If we minimize δ for a given time T one gets $\delta^2 = 2\hbar T/m$ and therefore the minimum dispersion is given by,

$$\Delta x \sim \sqrt{\frac{\hbar T}{m}} \sim 10^{-5}m, \quad (15)$$

for the example in question. Due to this dispersion we will have very different velocities for the different values of the impact parameter d ranging from $v_y \sim 10^6$ m/s to 10^2 m/s for $d \sim 10^{-13}m$ and $10^{-5}m$ respectively. These velocity differences give rise to deviations from the original trajectory one sent the spins into the cavity without any control of the experimenter. To avoid these problems and have more or less predictable trajectories for the spins of the environment one should have to make the interaction distance much larger than before, let us say $d \sim 10^{-4}m$. This would lead to too weak an interaction with the spins of the environment and therefore no decoherence takes place.

One could ask what would happen if one did not use a package which minimizes uncertainty growth, as we did before. One can see that in that case, if one chooses a sufficiently small packet initially to avoid large deviations in the y direction the size of the packet grows rapidly with time. That would force to send the spins of the environment with large separations in time in order to avoid them from interacting among each other. Allowing out of control interactions among the environment spins will prevent us from computing the observable. Using packets that are very

spaced in time avoids that problem initially, but the large size of the packages at the end of the experiment does not allow us to have particles of the environment that do not interact with each other, as we discuss in the appendix.

The conclusion is that, at least with nuclear particles, it is impossible to compute the observable because if one requires enough interaction among the spins for decoherence to take place, one ends up losing control of the experiment. That is, one cannot measure the observable.

Could it be that the experiment is still feasible with larger masses and magnetic momenta? We will analyze this situation in the next section.

V. AN EXPERIMENT WITH LARGE MASSES AND MAGNETIC MOMENTA

A. The experiment and the appearance of decoherence

As we mentioned in the previous section, the problems in measuring the observable could in principle be overcome if one used a larger separation between the spins of the environment and the spin of the “needle”, and using larger magnetic moments. The larger magnetic moment will enable decoherence in spite of the larger separation. One will still have large deviations from the initial direction of the trajectory due to the interactions, but since the separation is large compared to the size of the packet, one will have little dispersion in the direction of exit. This will require a full quantum treatment of the problem. Let us go back to the interaction Hamiltonian (8), and rewrite it in terms of the vectors of the spin basis along the z direction.

Using that

$$\hat{S}_x = |+\rangle\langle-| + |- \rangle\langle+| \quad (16)$$

$$\hat{S}_y = -i|+\rangle\langle-| + i|- \rangle\langle+| \quad (17)$$

$$\hat{S}_z = |+\rangle\langle+| - |- \rangle\langle-| \quad (18)$$

we can represent the Hamiltonian as,

$$\hat{H}_k = \begin{pmatrix} f_k + B\Gamma_+ & 0 & 0 & 0 \\ 0 & -f_k + B\Gamma_- & 2f_k & 0 \\ 0 & 2f_k & -f_k - B\Gamma_- & 0 \\ 0 & 0 & 0 & f_k - B\Gamma_+ \end{pmatrix}, \quad (19)$$

where the elements of the basis are ordered as $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ respectively and $\Gamma_{\pm} \equiv \gamma_1 \pm \gamma_2$

With this Hamiltonian we can find the evolution of the system when the spin $k = 1$ is interacting with S (we initially focus in the passage of the first spin in the stream through the cavity),

$$i\frac{d|\psi\rangle}{dt} = \hat{H}_1|\psi\rangle, \quad (20)$$

$$|\psi(t=0)\rangle = (a|+\rangle + b|-\rangle) \otimes (\alpha_1|+\rangle_1 + \beta_1|-\rangle_1), \quad (21)$$

$$|\psi(t)\rangle = R(t)|+\rangle|+\rangle_1 + T(t)|+\rangle|-\rangle_1 + U(t)|-\rangle|+\rangle_1 + V(t)|-\rangle|-\rangle_1, \quad (22)$$

and using (19-21) in (22) we get a set of differential equations for the coefficients,

$$i\dot{R} = (f_1 + B\Gamma_+)R, \quad (23)$$

$$i\dot{U} = 2f_1T + (-f_1 - B\Gamma_-)U, \quad (24)$$

$$i\dot{T} = 2f_1U + (-f_1 + B\Gamma_-)T, \quad (25)$$

$$i\dot{V} = (f_1 - B\Gamma_+)V, \quad (26)$$

where we have assumed that f_k is time independent. Notice that we are neglecting a proper treatment of the spatial dependence of the quantum states. We are carrying out a semiclassical analysis in which each spin is treated like a classical point particle that flies through the cavity in a well defined trajectory and such that the distance between spin and the “needle” spin changes little during its flight in the cavity. The only quantum aspect we are treating is the spin-spin and spin-field interactions. The treatment is quantum-statistical mechanics in nature, in the sense that the couplings f_k must be taken to be a random variable whose statistical properties will be determined by the spread of the wavefunction associated with the particle k . This is a good approximation if the distance from spin k to spin S is approximately constant during the time of flight of k within the cavity, as would be the case in a thin cavity if

k impacts far away from S . The initial conditions for the differential equations are $R(t=0) = a\alpha_1$, $T(t=0) = a\beta_1$, $U(t=0) = b\alpha_1$, $V(t=0) = b\beta_1$. We can solve for R and V and get,

$$R(t) = a\alpha_1 \exp(-i(f_1 + B\Gamma_+)t), \quad (27)$$

$$V(t) = b\beta_1 \exp(-i(f_1 - B\Gamma_+)t), \quad (28)$$

and for T and U ,

$$T(t) = a\beta_1 e^{if_1 t} \left(\cos(\Omega_1 t) - i \frac{B\Gamma_-}{\Omega_1} \sin(\Omega_1 t) \right) - 2ib\alpha_1 \frac{f_1}{\Omega_1} e^{if_1 t} \sin(\Omega_1 t), \quad (29)$$

$$U(t) = b\alpha_1 e^{if_1 t} \left(\cos(\Omega_1 t) + i \frac{B\Gamma_-}{\Omega_1} \sin(\Omega_1 t) \right) - 2ia\beta_1 \frac{f_1}{\Omega_1} e^{if_1 t} \sin(\Omega_1 t), \quad (30)$$

$$(31)$$

where we have introduced the frequency $\Omega_1 = \sqrt{4f_1^2 + B^2\Gamma_-^2}$.

Replacing the solutions (27-31) in (22) we have the state of the system after the passage of spin $k=1$ is given by,

$$|\psi(t)\rangle_1 = a|+\rangle \left[\alpha_1 e^{-i(f_1 + B\Gamma_+)t} |+\rangle_1 + \beta_1 e^{if_1 t} \left(\cos(\Omega_1 t) - i \frac{B\Gamma_-}{\Omega_1} \sin(\Omega_1 t) \right) |-\rangle_1 - 2i \frac{b\alpha_1}{a} \frac{f_1}{\Omega_1} e^{if_1 t} \sin(\Omega_1 t) |-\rangle_1 \right] \quad (32)$$

$$+ b|-\rangle \left[\beta_1 e^{-i(f_1 - B\Gamma_+)t} |-\rangle_1 + \alpha_1 e^{if_1 t} \left(\cos(\Omega_1 t) + i \frac{B\Gamma_-}{\Omega_1} \sin(\Omega_1 t) \right) |+\rangle_1 - 2i \frac{a\beta_1}{b} \frac{f_1}{\Omega_1} e^{if_1 t} \sin(\Omega_1 t) |+\rangle_1 \right], \quad (33)$$

and given τ the time of flight of the spins in the cavity we can find the state of the system after N spins have passed,

$$|\psi\rangle = a|+\rangle|A\rangle + b|-\rangle|B\rangle \quad (34)$$

with,

$$|A\rangle = \prod_k^N \left[\alpha_k e^{-i(f_k + B\Gamma_+)\tau} |+\rangle_k + \beta_k e^{if_k \tau} \left(\cos(\Omega_k \tau) - i \frac{B\Gamma_-}{\Omega_k} \sin(\Omega_k \tau) \right) |-\rangle_k - 2i \frac{b\alpha_k}{a} \frac{f_k}{\Omega_k} e^{if_k \tau} \sin(\Omega_k \tau) |-\rangle_k \right] \quad (35)$$

$$|B\rangle = \prod_k^N \left[\beta_k e^{-i(f_k - B\Gamma_-)\tau} |-\rangle_k + \alpha_k e^{if_k \tau} \left(\cos(\Omega_k \tau) + i \frac{B\Gamma_-}{\Omega_k} \sin(\Omega_k \tau) \right) |+\rangle_k - 2i \frac{a\beta_k}{b} \frac{f_k}{\Omega_k} e^{if_k \tau} \sin(\Omega_k \tau) |+\rangle_k \right]. \quad (36)$$

The density matrix operator of the system is given by,

$$\hat{\rho} = |\psi\rangle\langle\psi| = |a|^2 |+\rangle\langle+| |A\rangle\langle A| + ab^* |+\rangle\langle-| |A\rangle\langle B| + a^* b |-\rangle\langle+| |B\rangle\langle A| + |b|^2 |-\rangle\langle-| |B\rangle\langle B|, \quad (37)$$

and the reduced density matrix of the ‘‘needle gauge’’ system S is

$$\hat{\rho}_S = |a|^2 |+\rangle\langle+| \langle A|A\rangle + ab^* |+\rangle\langle-| \langle B|A\rangle + a^* b |-\rangle\langle+| \langle A|B\rangle + |b|^2 |-\rangle\langle-| \langle B|B\rangle. \quad (38)$$

We now need to show that the crossed terms in the needle basis go to zero in the large N limit. We have that,

$$\begin{aligned} \langle A|A\rangle &= \prod_k^N \left[|\alpha_k|^2 + |\beta_k|^2 \left(\cos^2(\Omega_k \tau) + \left(\frac{B\Gamma_-}{\Omega_k} \right)^2 \sin^2(\Omega_k \tau) \right) \right. \\ &\quad + 4 \frac{|b|^2 |\alpha_k|^2}{|a|^2} \frac{f_k^2}{\Omega_k^2} \sin^2(\Omega_k \tau) + 2i\beta_k \left(\frac{b\alpha_k}{a} \right)^* \frac{f_k}{\Omega_k} \left(\cos(\Omega_k \tau) - i \frac{B\Gamma_-}{\Omega_k} \sin(\Omega_k \tau) \right) \sin(\Omega_k \tau) \\ &\quad \left. - 2i \frac{b\alpha_k}{a} \beta_k^* \frac{f_k}{\Omega_k} \left(\cos(\Omega_k \tau) + i \frac{B\Gamma_-}{\Omega_k} \sin(\Omega_k \tau) \right) \sin(\Omega_k \tau) \right] \end{aligned} \quad (39)$$

with a similar expression for $\langle B|B\rangle$. For the last term we have,

$$\begin{aligned} \langle A|B\rangle = & \prod_k^N \left[|\alpha_k|^2 e^{2if_k\tau + iB\Gamma_+\tau} \left(\cos(\Omega_k\tau) + i\frac{B\Gamma_-}{\Omega_k} \sin(\Omega_k\tau) \right) - \alpha_k^* e^{i(2f_k + B\Gamma_+)\tau} 2i\frac{a\beta_k}{b} \frac{f_k}{\Omega_k} \sin(\Omega_k\tau) \right. \\ & \left. + |\beta_k|^2 e^{-i(2f_k - B\Gamma_+)\tau} \left(\cos(\Omega_k\tau) + i\frac{B\Gamma_-}{\Omega_k} \sin(\Omega_k\tau) \right) + 2i\left(\frac{b\alpha_k}{a}\right)^* \beta_k \frac{f_k}{\Omega_k} e^{-i(2f_k - B\Gamma_+)\tau} \sin(\Omega_k\tau) \right]. \end{aligned} \quad (40)$$

If we now consider the case in which the coupling between the spins is much weaker than the coupling with the magnetic field, and that $\gamma_1 \neq \gamma_2$ i.e., $f_k \ll |B\Gamma_-|$, one has that,

$$\Omega_k = \sqrt{4f_k^2 + B^2\Gamma_-^2} \sim B\Gamma_- \left(1 + \frac{1}{2} \frac{4f_k^2}{B^2(\gamma_1 - \gamma_2)^2} \right), \quad (41)$$

so the leading term is $\Omega_k \sim B\Gamma_-$ and therefore $f_k/\Omega_k \ll 1$.

Using these approximations in the above expressions for the inner products,

$$\langle A|A\rangle = \prod_k^N [|\alpha_k|^2 + |\beta_k|^2 (\cos^2(\Omega_k\tau) + \sin^2(\Omega_k\tau))] = 1, \quad (42)$$

with a similar expression for $\langle B|B\rangle$. and

$$\langle A|B\rangle = \prod_k^N [|\alpha_k|^2 e^{2if_k\tau + i\Omega_k\tau} (\cos(\Omega_k\tau) + i\sin(\Omega_k\tau)) + |\beta_k|^2 e^{-2if_k\tau + i\Omega_k\tau} (\cos(\Omega_k\tau) + i\sin(\Omega_k\tau))] \quad (43)$$

$$= \prod_k^N e^{2i\Omega_k\tau} [\cos(2f_k\tau) + i(|\alpha_k|^2 - |\beta_k|^2) \sin(2f_k\tau)]. \quad (44)$$

This last expression goes to zero for large N since it is the product of complex numbers of unit modulus but with random phases. We can therefore construct the reduced density matrix,

$$\begin{aligned} \hat{\rho}_S = & |a|^2 |+\rangle\langle +| + ab^* |+\rangle\langle -| \left(\prod_k^N e^{2i\Omega_k\tau} (\cos(2f_k\tau) + i(|\alpha_k|^2 - |\beta_k|^2) \sin(2f_k\tau)) \right)^* \\ & + a^* b |-\rangle\langle +| \prod_k^N e^{2i\Omega_k\tau} (\cos(2f_k\tau) + i(|\alpha_k|^2 - |\beta_k|^2) \sin(2f_k\tau)) + |b|^2 |-\rangle\langle -|, \end{aligned} \quad (45)$$

so we have decoherence,

$$\hat{\rho}_S = \begin{pmatrix} |a|^2 \langle A|A\rangle & ab^* \langle B|A\rangle \\ a^* b \langle A|B\rangle & |b|^2 \langle B|B\rangle \end{pmatrix} \xrightarrow{N \gg 1} \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \quad (46)$$

There is a caveat in this proof of decoherence. For a closed system, the off diagonal terms are given really by multiperiodic functions, i.e. they are given by a superposition of a large number of periodic functions with different frequencies. Therefore this function will retake values arbitrarily close to the initial value for sufficiently large times. This implies that the apparent loss of information about the non-diagonal terms reappears if one waits a long enough time. This problem is usually called ‘‘recurrence of coherence’’ (‘‘revivals’’). The characteristic time for these phenomena is proportional to the factorial of the number of involved frequencies.

The above derivation was done using ordinary quantum mechanics in which one assumes an ideal clock is used to measure time. As we have argued in our previous paper, If one redoes the derivation using the effective equation we derived for quantum mechanics with real clocks one gets the same expression for the off-diagonal terms except that it is multiplied by $\prod_k \exp\left(-2(g_k)^2 T_{\text{Planck}}^{4/3} t^{2/3}\right)$. That means that asymptotically the off diagonal terms indeed vanish, the off-diagonal terms are not periodic anymore. Although the exponential term decreases slowly with time, the fact that there is a product of them makes the effect quite relevant, especially for the long periods of time involved in ‘‘revivals’’. Therefore we see that including real clocks in quantum mechanics offers a mechanism to turn pure states into mixed states in a way that is desirable to explain the problem of measurement in quantum mechanics.

How many particles does one need to consider for the exponential decrease to kill the possibility of revivals? A criterion would be that the magnitude of the off diagonal term in the revivals be smaller than the magnitude of the off diagonal terms in the intermediate region between revivals. If that were the case the revival would be less than the “background noise” in regions where there is no revival. The magnitude of the interference terms in the density matrix were studied by Zurek [16] and go as $\rho_{+-} \sim 1/2^{N/2}$. The time for revivals to occur goes as $T \sim N!/\Omega$ where Ω is the mean value of the Ω_k 's. This implies that if one has more than hundreds of particles the loss of coherence will make impossible the observation of revivals.

At this point it is worthwhile emphasizing the robustness of this result in practical terms. One could, for instance, question how reliable the fundamental limit for the inaccuracy of clocks of [11] is. Some authors have characterized the fundamental limit as too optimistically large, arguing that the real fundamental limit is of the order of Planck time itself. In view of this it is interesting to notice that if one posits a much more conservative estimate of the error of a clock, for instance $\delta T \sim T^\epsilon T_{\text{Planck}}^{1-\epsilon}$, for any small value of ϵ the only modification would be to change the number of particles $N_0 \sim 100$ to at least $N \sim N_0/(3\epsilon)$.

B. Non-observability of d'Espagnat's global observable

As we mentioned above, d'Espagnat proposed a global observable for Zurek's model that could be used to test the presence of decoherence versus state reduction in the model we are considering. Let us analyze how the observable behaves in the latter. It is defined as,

$$\hat{M} \equiv \hat{S}_x \otimes \prod_k^N \hat{S}_x^k. \quad (47)$$

Let us assume that at some point in the experiment the wavefunctions collapses. When the collapse occurs we would have to project the system onto the final state. For instance, if we assume the measured spin is up we would have $a = 1, b = 0$. From there on the system would continue its evolution, as discussed above. The important point is that the evolution considered will not move the system away from its collapsed state, that is a would still be unity after evolution. That is, from the moment the collapse happens we would have $\langle \hat{M} \rangle = 0$.

Let us go back and consider that no collapse occurs and compute the evolution of the expectation value of the observable.

From the approximations (41) one has that the state is,

$$|\psi\rangle = a|+\rangle \prod_k^N \left[\alpha_k e^{-i(f_k + \Omega_k)\tau} |+\rangle_k + \beta_k e^{i(f_k - \Omega_k)\tau} |-\rangle_k \right] + b|-\rangle \prod_k^N \left[\alpha_k e^{-i(f_k + \Omega_k)\tau} |+\rangle_k + \beta_k e^{-i(f_k - \Omega_k)\tau} |-\rangle_k \right] \quad (48)$$

so the expectation value of the observable¹ is,

$$\langle \psi | M | \psi \rangle = ab^* \prod_k^N [\alpha_k \beta_k^* + \alpha_k^* \beta_k] e^{-2i\Omega_k \tau} + a^* b \prod_k^N [\alpha_k \beta_k^* + \alpha_k^* \beta_k] e^{2i\Omega_k \tau}, \quad (49)$$

and the density matrix for the pair S, S_k , is given by,

$$\rho = \begin{pmatrix} |a|^2 |\alpha_k|^2 & |a|^2 \alpha_k \beta_k^* & ab^* |\alpha_k|^2 & ab^* \alpha_k \beta_k^* \\ |a|^2 \alpha_k^* \beta_k & |a|^2 |\beta_k|^2 & ab^* \alpha_k^* \beta_k & ab^* |\beta_k|^2 \\ a^* b |\alpha_k|^2 & a^* b \alpha_k \beta_k^* & |b|^2 |\alpha_k|^2 & |b|^2 \alpha_k \beta_k^* \\ a^* b \alpha_k^* \beta_k & a^* b |\beta_k|^2 & |b|^2 \alpha_k^* \beta_k & |b|^2 |\beta_k|^2 \end{pmatrix}. \quad (50)$$

The expression for the state clearly differs from the one obtained in the case with collapse. For instance, taking into account that we are working in the limit $\Omega_k \sim B\Gamma_-$ and therefore Ω is k -independent, and choosing the time of flight τ such that $e^{\pm 2i\Omega N \tau} = 1$ one has that $\langle M \rangle$ is obviously different from zero.

¹ The reader may ponder if this observable is measurable even in principle, since it is generically very small since it involves large products of small factors. However, one could in principle prepare the initial state in such a way that the products are not small, for instance choosing $\alpha_k = \beta_k = 1/\sqrt{2}$.

As we discussed above, an experiment measuring this type of global observables is not possible using nuclear particles. However, for quantum systems with larger masses and magnetic momenta it is clearly possible to measure them. One could therefore distinguish collapse from reduction in the wavefunction. We will see that the fundamental loss of coherence due to the use of realistic clocks and rods eliminates the possibility of measuring the global observable.

Let us consider the evolution of the observable when we consider the modified evolution due to real rods and clocks. In that case we have shown [19] that the off-diagonal terms of the density matrix in the energy pointer basis decrease exponentially as,

$$\rho_{mn}(T) = \rho_{mn}(0) \exp(i\omega_{mn}T) \exp\left(-\omega_{mn}T_{\text{Planck}}^{4/3}T^{1/3}\right) \quad (51)$$

with $a > 0$ and ω_{mn} are the Bohr frequencies between the levels n and m . The relevant Bohr frequencies for this case can be obtained by considering consider the interaction Hamiltonian (19) in the approximation where $f_k < |B\gamma_{1,2}|$,

$$\hat{H}_k = \text{diag}(B(\gamma_1 + \gamma_2), B(\gamma_1 - \gamma_2), -B(\gamma_1 - \gamma_2), -B(\gamma_1 + \gamma_2)). \quad (52)$$

The end result is that the evolution of the density matrix is given by,

$$\rho = \begin{pmatrix} |a|^2|\alpha_k|^2 & |a|^2\alpha_k\beta_k^*e^{-(2B\gamma_2)^2\theta} & ab^*|\alpha_k|^2e^{-(2B\gamma_1)^2\theta} & ab^*\alpha_k\beta_k^*e^{-(2B(\gamma_1+\gamma_2))^2\theta} \\ |a|^2\alpha_k^*\beta_k e^{-(2B\gamma_2)^2\theta} & |a|^2|\beta_k|^2 & ab^*\alpha_k^*\beta_k e^{-(2B(\gamma_1-\gamma_2))^2\theta} & ab^*|\beta_k|^2e^{-(2B(\gamma_1))^2\theta} \\ a^*b|\alpha_k|^2e^{-(2B\gamma_1)^2\theta} & a^*b\alpha_k\beta_k^*e^{-(2B(\gamma_1-\gamma_2))^2\theta} & |b|^2|\alpha_k|^2 & |b|^2\alpha_k\beta_k^*e^{-(2B(\gamma_2))^2\theta} \\ a^*b\alpha_k^*\beta_k e^{-(2B(\gamma_1+\gamma_2))^2\theta} & a^*b|\beta_k|^2e^{-(2B(\gamma_1))^2\theta} & |b|^2\alpha_k^*\beta_k e^{-(2B(\gamma_2))^2\theta} & |b|^2|\beta_k|^2 \end{pmatrix} \quad (53)$$

where $\theta = 3T_{\text{Planck}}^{4/3}\tau^{2/3}/2$.

A calculation very similar to the one we did before but with the modified evolution yields,

$$\langle \hat{M} \rangle = ab^*e^{-i2N\Omega T}e^{-4NB^2(\gamma_1-\gamma_2)^2\theta} \prod_k^N \left[\alpha_k\beta_k^*e^{-16B^2\gamma_1\gamma_2\theta} + \alpha_k^*\beta_k \right] \quad (54)$$

$$+ba^*e^{i2N\Omega T}e^{-4NB^2(\gamma_1-\gamma_2)^2\theta} \prod_k^N \left[\alpha_k\beta_k^* + \alpha_k^*\beta_k e^{-16B^2\gamma_1\gamma_2\theta} \right] \quad (55)$$

where $\Omega = B(\gamma_1 - \gamma_2)$. We therefore see that the expectation value of the observable decreases exponentially with time with an exponent proportional to $B(\gamma_1 - \gamma_2)$. This is a large number since as we noted $B(\gamma_1 - \gamma_2) > f_k$ and the f_k 's are large as we argued above.

Let us study if there is a range of parameters where at least in principle, one could measure the observable and therefore conclude that collapse has occurred or not. Let us summarize the set of conditions we have encountered in this work: a) that the couplings be large enough so decoherence can occur, b) that there is no significant dispersion in the particles, c) that the interaction with the magnetic field be larger than the inter-spin interaction so one has decoherence in a predetermined basis, d) the effects implied by the use of real rods and clocks in measurement. These correspond to:

$$a) \quad 1 < f\tau = \frac{\mu\gamma_1\gamma_2}{\hbar} \frac{\tau}{d^3}, \quad (56)$$

$$b) \quad \Delta x \sim \sqrt{\frac{\hbar T}{m}}, \quad (57)$$

$$c) \quad f \ll |B(\gamma_1 - \gamma_2)|, \quad (58)$$

$$d) \quad \langle \hat{M} \rangle \sim \exp\left(-6NB^2(\gamma_1 - \gamma_2)^2T_{\text{Planck}}^{4/3}\tau^{2/3}\right), \quad (59)$$

where T is the total length of the experiment and τ the time of flight within the cavity. Let us consider (57) with the condition $T > N\tau$ since the particles are assumed to enter the cavity one at a time, then,

$$\Delta x > \sqrt{\frac{\hbar N\tau}{m}}. \quad (60)$$

Now, from (56) one has that,

$$d^3 < \frac{\mu\gamma_1\gamma_2}{\hbar}\tau. \quad (61)$$

In addition to that we need that $\Delta x < d$ otherwise, i) if the dispersion were larger than the distance between the environment particles and the needle, we could not avoid collisions among them, ii) the deflection due to the magnetic interaction would be very large if the particles could impact arbitrarily close (as would happen if the dispersion were larger than the impact parameter), iii) condition (58) would be violated if the particles flew by closely, since the coupling would become large. Combining (60) and (61) we have that,

$$\tau^{1/3} < \frac{m(\gamma_1\gamma_2)^{2/3}}{h^{5/3}N}\mu^{2/3}. \quad (62)$$

Considering now the exponent in (59) $K = NB^2(\gamma_1 - \gamma_2)^2 T_{\text{Planck}}^{4/3} \tau^{2/3}$ we have that, using (56) and (57),

$$B^2(\gamma_1 - \gamma_2)^2 \tau^2 \gg f^2 \tau^2 > 1 \quad (63)$$

and therefore

$$K = \frac{NB^2(\gamma_1 - \gamma_2)^2 \tau^2 T_{\text{Planck}}^{4/3} \tau^{2/3}}{\tau^2} \gg \frac{NT_{\text{Planck}}^{4/3}}{\tau^{4/3}}, \quad (64)$$

and replacing in this expression (62) we have that,

$$K \gg \frac{NT_{\text{Planck}}^{4/3}}{\tau^{4/3}} > \frac{NT_{\text{Planck}}^{4/3} \hbar^{20/3} N^4}{m^4(\gamma_1\gamma_2)^{8/3} \mu^{8/3}}. \quad (65)$$

Let us recall that if K is large we will not be able to decide if the system underwent collapse or not. We therefore have to impose that $K < 1$ in order to distinguish if there was collapse or not. From the previous expression this implies that,

$$m(\gamma_1\gamma_2)^{2/3} \gg \frac{T_{\text{Planck}}^{1/3} \hbar^{5/3} N^{5/4}}{\mu^{2/3}}. \quad (66)$$

Let us see if we can meet that condition. Taking the “needle” spin as a proton and environment spins to be neutrons (the particles have to be different, see ²) one gets that $10^6 \gg N^{5/4}$, which is violated already for an “environment” with 10^5 spins. We know this case is already not feasible due to the interactions and the dispersions, but it is good to verify that even if one did not have those difficulties, the fundamental loss of coherence would prevent us from deciding that collapse has occurred.

It is worthwhile asking what would happen if one considered particles with significantly more mass and magnetic moment, to avoid the issues of the close interaction and associated dispersion. Let us assume that we consider an environment of 10^{23} spins. one has that,

$$m(\gamma_1\gamma_2)^{2/3} > 10^{-38} \quad (67)$$

which requires masses and magnetic moments much larger than those of elementary particles. For instance if we consider particles with $m = M_{\text{Planck}}$ the gyromagnetic ratios needed are of the order of 10^{20} and that have to live at least 10^{-4} s. Such objects are highly unlikely to exist due to quantum field theory effects.

VI. DISCUSSION

We have argued that fundamental limitations in the process of measurement, both due to gravitational and quantum mechanical effects lead to a loss of coherence in quantum evolution. Coupled to environmentally-induced decoherence, we conjecture that this effect provides a solution to the problem of measurement in quantum mechanics. Basically, we propose that the collapse or not of a state becomes undecidable and when that happens an event has taken place. In this paper we have examined these ideas in the context of models that are usually considered when discussing decoherence. We have shown that indeed the collapse or not of the states becomes undecidable, and that using

² The particles need to have different magnetic moments in order for an external magnetic field to determine the pointer basis. This is reflected in the $\gamma_1 - \gamma_2$ factors that appear in various expressions. Decoherence still takes place due to the interaction of the needle and the environment, but there is not a preferred basis in which the events will occur.

undecidability to characterize when an event has taken place solves the “problem of outcomes”. We have also studied the possibility of using global observables, as proposed by d’Espagnat, to characterize if collapse has taken place and we have shown that the limitations in measurement we point out make impossible the measurement of the relevant global observables.

The reader may question how distinguishable are the fundamental limitations we are discussing in this paper from the practical limitations that have often been invoked when discussing the solution of the problem of measurement through environmentally-induced decoherence. From our point of view the key difference is the exponential decay of observables due to fundamental limitations in measurement. For instance, in the example we consider, to distinguish an expectation value that is exponentially small and one that is identically zero would require a measurement of an ensemble with a number of elements that quickly becomes prohibitive, say, comparing to the total degrees of freedom of the universe inside our horizon. This limitation bears some resemblance to the non-polynomial computations in quantum computing. In that case, an NP problem cannot be worked out no matter what the details of the computer in question. In our case, the effect we discuss will occur irrespective of the details of the experiment in question and the particulars of the mechanism for fundamental loss of coherence, and is essentially limited by the size of the universe.

To conclude we point out that our results have been derived in a particular model, and further work in better models is needed to build a stronger case that the resolution of the problem of measurement presented is a robust one.

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Appendix 1

Here we would like to discuss what happens if one attempts to use packets that do not minimize the uncertainty growth. These would be packets that start small and grow over time. We shall see that the growth is too large in the end, leading to packets that cannot be considered non-interactive and therefore for which one cannot compute the global observables.

The speed that the particles acquire due to the magnetic interaction is given by the condition (13) $v_y \geq \hbar/(md)$ and the dispersion as a function of time is (14),

$$\Delta x(t) = \frac{\delta}{2} \sqrt{1 + \frac{4\hbar^2 t^2}{m^2 \delta^4}}. \quad (68)$$

We will choose the initial width of the packet in such a way that they do not suffer very different deviations in their trajectories due to the magnetic interaction. This way, even though they deviate, they will all do it in approximately the same way and we know where to find them at the end of the experiment. If we chose the dispersion of the order of the impact parameter some particles would come very close to the needle and would deviate a lot. We will take $\delta = d/10$, that is one order of magnitude smaller than the impact parameter and we will study if the experiment is feasible. The dispersion at the end of the experiment therefore is,

$$\Delta x(T) = \frac{d}{20} \sqrt{1 + \frac{4\hbar^2 T^2}{m^2 (0.1d)^4}} \geq \frac{\hbar T}{0.1md}. \quad (69)$$

The condition to have loss of coherence (sufficiently strong couplings) was given by $1 < \mu\gamma_1\gamma_2/(\hbar d^2 v)$, so we can conclude for the speed that $v < \mu\gamma_1\gamma_2/(\hbar d^2)$. The total length to be traversed in the experiment can be estimated as

$$l = vT < \mu\gamma_1\gamma_2 T/(\hbar d^2). \quad (70)$$

Let us now recall that the final dispersion, caused by the force (12) and the spread of the wave packet in all directions, will be present not only in the transverse direction to the motion but in the longitudinal one as well. We therefore have to ensure that the particles of the environment do not end up interacting among themselves. If the distance that the particles traverse would be smaller than the final dispersion the particles could interact. It would

even lead to a non-vanishing probability of finding particles of the environment in the cavity after the experiment is over. We will therefore impose $\Delta x(T) < l$. Using (69) and (70) we have,

$$\frac{\hbar T}{0.1md} \leq \Delta X(T) \leq l \leq \frac{\mu\gamma_1\gamma_2 T}{\hbar d^2}, \quad (71)$$

which leads to $d \leq 0.1\mu\gamma_1\gamma_2 m/\hbar^2$. This, for neutrons, implies $d < 10^{-19}\text{m}$, which cannot be satisfied for known elementary particles. What is going on is that the particles disperse faster than the distance they are traveling if we impose that they travel slow enough in order for decoherence with the environment to take place. Clearly for more massive particles or with larger magnetic moments one could avoid this problem, but as we discussed in section VB, the effect of the fundamental decoherence renders the observable unmeasurable in that case.

Appendix 2

A. Events

Let us briefly outline the formal structure that underlies the notion of event in our interpretation of quantum mechanics.

Let $|\Psi\rangle$ be the resulting state of the evolution of a system S that has interacted with a measuring device A and an environment E . It is therefore the state of the system $\mathbf{S} = S \otimes A \otimes E$. Events generically occur in very complex systems that include a lot of elementary subsystems. To simplify the presentation, we will assume that the system \mathbf{S} is composed of three spins. A possible state could be, for instance,

$$|\Psi\rangle = \frac{c_1}{\sqrt{2}} (|+, +, -\rangle + |+, -, +\rangle) + c_2 |-, +, +\rangle \quad (72)$$

with $|c_1|^2 + |c_2|^2 = 1$. In this greatly simplified picture we will assume $S \otimes A$ is the first spin and E is the other two.

The reduced density matrix ρ is the state of the subsystem $S \otimes A$. It is obtained taking the partial trace over the states of the unobserved system E ,

$$\rho = \text{Tr}_{23} (|\Psi\rangle\langle\Psi|) = |c_1|^2 |+\rangle\langle+| + |c_2|^2 |-\rangle\langle-|. \quad (73)$$

This density matrix, as was extensively discussed by d'Espagnat, admits two interpretations. One of them is that ρ is an improper mixture, since it was obtained as a partial trace of a pure state. It represents partial information about a system which we are ignoring a portion of. ρ represents the options “spin up” and “spin down” in coexistence as potentialities with probabilities $|c_1|^2$ and $|c_2|^2$. The same mathematical expression can be used to describe a situation where the particle was left in the state “up” with probability $|c_1|^2$ or with state “down” with probability $|c_2|^2$. Equivalently, it can be taken to describe a system where a number $|c_1|^2$ of particles were introduced in “up” state and $|c_2|^2$ were introduced in “down” state. The “problem of outcomes” in quantum mechanics is how to pass from the first interpretation to the second. Our proposal is that one should shift the interpretation when one cannot distinguish the state $|\Psi\rangle$ of the whole system \mathbf{S} and a statistical superposition of its projections,

$$\rho = |c_1|^2 |\Psi_+\rangle\langle\Psi_+| + |c_2|^2 |\Psi_-\rangle\langle\Psi_-|, \quad (74)$$

with,

$$|\Psi_+\rangle = |+\rangle\langle+| \otimes I_2 \otimes I_3 |\Psi\rangle \quad (75)$$

$$|\Psi_-\rangle = |-\rangle\langle-| \otimes I_2 \otimes I_3 |\Psi\rangle, \quad (76)$$

since in such a situation any empirical distinction between both situations is impossible. We say that *undecidability* has taken place. In simple models like the above this does not happen, those states are always distinguishable. In systems with many more degrees of freedom, the fundamental loss of coherence due to gravity we introduce turns the state indistinguishable from its projections. That is how undecidability develops, in a universal fashion.

The event implies that the system is now represented by exclusive alternatives and one of them is adopted by the system. Which alternative is realized is not predicted by physics, only the probabilities with which they occur. The chosen alternative is represented by the projector P_{123} with eigenvectors $|\Psi_+\rangle$ (or $|\Psi_-\rangle$)

$$P_{123} = \frac{1}{2} (|+, +, -\rangle + |+, -, +\rangle) (|+, +, -\rangle + |+, -, +\rangle) \quad (77)$$

and all the properties compatible with it. Those are the properties associated with subsystems with projectors P_i such that $P_i P_{123} = P_{123}$ (they also commute with P_{123}).

For example if the property associated to the $|\Psi_+\rangle$ state is updated, a compatible property is given by the projector,

$$P_1 = |+\rangle\langle+| \otimes I_2 \otimes I_3, \quad (78)$$

which satisfies $P_1 P_{123} = P_{123}$ and characterizes the property “spin 1 is up”. Another compatible property has projector,

$$P_{23} = I_1 \otimes \frac{1}{2} (|+, -\rangle + |-, +\rangle) (\langle+, -| + \langle-, +|) \quad (79)$$

which satisfies $P_{23} P_{123} = P_{123}$ and represents “spins 2 and 3 are opposite”.

A quantum event is, in this interpretation, a bundle of properties. There is one of them that covers the whole content of the event, in this case P_{123} . This property is in general not accessible experimentally given its complexity. We call this property the *essential property* of the event since it characterizes it completely. All other properties are defined by projectors P_i such that $P_i P_{\text{essential}} = P_{\text{essential}}$.

One last point is that the above discussion could be carried out without a measuring apparatus, just with the quantum system S and the environment E , as we emphasized earlier in the paper.

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