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RATIONAL SOLUTIONS OF PAIRS OF DIAGONAL EQUATIONS, ONE CUBIC AND ONE QUADRATIC

TREVOR D. WOOLEY

ABSTRACT. We obtain an essentially optimal estimate for the moment of order $32/3$ of the exponential sum having argument $\alpha x^3 + \beta x^2$. Subject to modest local solubility hypotheses, we thereby establish that pairs of diagonal Diophantine equations, one cubic and one quadratic, possess non-trivial integral solutions whenever the number of variables exceeds 10.

1. INTRODUCTION

Investigations concerning the integral solubility of simultaneous equations by means of the circle method are in general limited to variable regimes beyond the convexity barrier, so that the underlying number of variables must exceed twice the total degree of the system. This technical limitation has been attained or surmounted in very few cases, almost all quadratic in nature. Thus, one has conclusions for a single quadratic form in 3 or 4 variables (see [10, 11, 12]), and for systems of r diagonal quadratic forms in $4r + 1$ variables (see [3, 8, 9]). Most recently, work of the author joint with Brüdern [5, 6, 7] analyses systems of r diagonal cubic equations in $6r + 1$ variables in general position, and pairs of such equations in 11 or 12 variables possessing a block structure. In this memoir we attain this technical limit in the previously inaccessible case of two diagonal equations, one cubic and one quadratic, in 11 variables. It transpires that progress is possible here owing to the author's recent proof [21] of the cubic case of the main conjecture in Vinogradov's mean value theorem, though wielding the latter to achieve our present purpose entails further innovations beyond the conventional repertoire of Hardy-Littlewood artisans.

By relabelling variables as necessary, there is no loss of generality in supposing that the simultaneous equations central to this paper take the form

$$\Theta(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{z}) = 0, \quad (1.1)$$

where we write

$$\left. \begin{aligned} \Theta(\mathbf{x}, \mathbf{y}) &= a_1 x_1^3 + \dots + a_l x_l^3 + c_1 y_1^3 + \dots + c_m y_m^3 \\ \Phi(\mathbf{x}, \mathbf{z}) &= b_1 x_1^2 + \dots + b_l x_l^2 \qquad \qquad \qquad + d_1 z_1^2 + \dots + d_n z_n^2 \end{aligned} \right\},$$

with the coefficients a_i, b_i, c_j, d_k all non-zero integers. We write $s = l + m + n$ for the total number of variables in the system, and we consider the solubility, in rational integers x_i, y_j, z_k , of the simultaneous diagonal equations (1.1).

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Theorem 1.1. *The simultaneous equations $\Theta(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{z}) = 0$ possess a non-zero integral solution provided only that the following conditions hold:*

- (a) *the system $\Theta(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{z}) = 0$ has a non-trivial real solution, and*
- (b) *the polynomial $\Phi(\mathbf{x}, \mathbf{z})$ is indefinite, and*
- (c) *one has $l + m \geq 7$ and $l + n \geq 5$, and*
- (d) *one has $s = l + m + n \geq 11$.*

The discussion of [17, §5] provides examples that demonstrate conditions (a), (b) and (c) of Theorem 1.1 to be necessary. Thus, by virtue of condition (d), we see that this theorem establishes a conclusion tantamount to the Hasse Principle for pairs of diagonal equations, one cubic and one quadratic, whenever the system has 11 or more variables. We note in this context that (1.1) possesses non-trivial solutions in every p -adic field \mathbb{Q}_p provided only that $s \geq 11$ (see the main theorem of [16]). When p is a prime number with $p \equiv 1 \pmod{3}$, moreover, there are examples of the shape (1.1) with $s = 10$ that possess only the trivial p -adic solution $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{0}$ (see [16, Lemma 7.2]).

It seems that the system (1.1) was first discussed in [17, Theorem 1], where a conclusion analogous to that of Theorem 1.1 was obtained with the condition $s \geq 11$ replaced by the more stringent constraint $s \geq 14$. In subsequent work [19, Theorem 1], the latter was replaced by the condition $s \geq 13$. Not only is our new conclusion superior to these earlier results, but it also attains the technical limit imposed by the convexity barrier for the problem of analysing the integral solubility of the system (1.1).

In most circumstances, one can say much more concerning the density of solutions of the system (1.1) than is apparent from Theorem 1.1. When B is a large positive number, let $N(B)$ denote the number of integral solutions of the system (1.1) with $|\mathbf{x}| \leq B$.

Theorem 1.2. *Suppose that the system $\Theta(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{z}) = 0$ has a non-singular solution in the real field \mathbb{R} , and also in each p -adic field \mathbb{Q}_p . Then provided that $s \geq 11$, $m \leq 5$ and $n \leq 3$, one has $N(B) \gg B^{s-5}$.*

A formal application of the circle method suggests the conjectural asymptotic formula $N(B) \sim CB^{s-5}$, in which C is given by a product of real and p -adic densities. Thus, the conclusion of Theorem 1.2 shows that $N(B)$ grows asymptotically at the expected rate. We remark that our methods would permit the proof of this conjectural formula for $N(B)$ whenever $s \geq 11$, and $m = n = 0$ or $n \in \{1, 2\}$ (see (7.4) below for a slightly more restrictive asymptotic formula). Hitherto, such an asymptotic formula was available only for $s \geq 15$, though this conclusion was apparently absent from the literature.

The proof of Theorems 1.1 and 1.2 depend on new mean value estimates for Weyl sums, only now available as a consequence of the proof [21] of the cubic case of the main conjecture in Vinogradov's mean value theorem. In order to describe these estimates, we define $\mathfrak{F}(\alpha, \beta) = \mathfrak{F}(\alpha, \beta; X)$ by putting

$$\mathfrak{F}(\alpha, \beta; X) = \sum_{1 \leq x \leq X} e(\alpha x^3 + \beta x^2), \quad (1.2)$$

where $e(z)$ denotes $e^{2\pi iz}$. Then, when s is a positive real number, we put

$$T_s(X) = \int_0^1 \int_0^1 |\mathfrak{F}(\alpha, \beta)|^{2s} d\alpha d\beta. \quad (1.3)$$

Theorem 1.3. *When $1 \leq s \leq 4$, one has $T_s(X) \ll X^{s+\varepsilon}$. In addition,*

$$T_5(X) \ll X^{31/6+\varepsilon} \quad \text{and} \quad T_s(X) \ll X^{2s-5} \quad (s > 16/3). \quad (1.4)$$

The bound $T_4(X) \ll X^{4+\varepsilon}$ establishes strongly diagonal behaviour for the exponential sum $\mathfrak{F}(\alpha, \beta)$ for the first time for moments exceeding the sixth. We direct the reader to [17, Theorem 4.1] for the bound $T_3(X) \ll X^{3+\varepsilon}$, and [18, Theorem 1] for the sharper conclusion $T_3(X) = 6X^3 + O(X^{7/3+\varepsilon})$. Meanwhile, the estimate $T_s(X) \ll X^{2s-5}$ follows for $s > 7$ by applying an argument based on that underlying the proof of Hua's lemma, in combination with [17, Theorem 4.1]. A classical approach to bounding $T_5(X)$, using Hölder's inequality to interpolate between the estimates $T_3(X) \ll X^{3+\varepsilon}$ and $T_7(X) \ll X^{9+\varepsilon}$ just cited, yields $T_5(X) \ll X^{6+\varepsilon}$ in place of the first estimate of (1.4).

We remark that Theorem 1.3 also improves on the sharpest estimates previously available for moments incorporating smooth Weyl sums. Denote the set of R -smooth integers not exceeding X by

$$\mathcal{A}(X, R) = \{n \in [1, X] \cap \mathbb{Z} : p \text{ prime and } p|n \Rightarrow p \leq R\}, \quad (1.5)$$

and put

$$\mathfrak{f}(\alpha, \beta) = \sum_{x \in \mathcal{A}(X, R)} e(\alpha x^3 + \beta x^2).$$

Then [19, Theorem 2] shows that whenever $\eta > 0$ is sufficiently small in terms of ε , and $R \leq X^\eta$, then

$$\int_0^1 \int_0^1 |\mathfrak{F}(\alpha, \beta)^4 \mathfrak{f}(\alpha, \beta)^6| d\alpha d\beta \ll X^{17/3+\varepsilon}.$$

As is clear, however, the exponent $17/3$ here may be replaced by $31/6$, by virtue of the estimate $T_5(X) \ll X^{31/6+\varepsilon}$ made available in Theorem 1.3.

The proof of Theorem 1.3 depends on two auxiliary mean value estimates established in §2, these being exploited in §3 by means of an argument motivated by the translation invariance of a related Vinogradov system. We then turn to the proof of Theorems 1.1 and 1.2, attending to some preliminary simplifications in §4, and establishing the former theorem as a consequence of the latter. The proof of Theorem 1.2, using the Hardy-Littlewood method, is accomplished in §§5–13 in three parts according to a classification of systems of type (1.1) depending on the values of m and n . Each such part proceeds in three phases, the first discussing such auxiliary estimates as are required in the argument, the second addressing the minor arcs of the Hardy-Littlewood dissection, and the third disposing of the major arc contribution.

Throughout, the letter s will denote a positive integer, and ε and η will denote sufficiently small positive numbers. We take X and P to be large positive real numbers depending at most on s , ε and η . The implicit constants

in Vinogradov's notation \ll and \gg will depend at most on s , ε and η , unless otherwise indicated. We adopt the following convention concerning the numbers ε and R . Whenever ε or R appear in a statement, we assert that for each $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon, s)$ such that the statement holds whenever $R \leq P^\eta$. Finally, we employ the convention that whenever $G : [0, 1]^k \rightarrow \mathbb{C}$ is integrable, then

$$\oint G(\boldsymbol{\alpha}) \, d\boldsymbol{\alpha} = \int_{[0,1]^k} G(\boldsymbol{\alpha}) \, d\boldsymbol{\alpha}.$$

Here and elsewhere, we use vector notation in the natural way.

2. AUXILIARY MEAN VALUE ESTIMATES

In this section we establish estimates for certain auxiliary mean values required in our proof of Theorem 1.3. We begin by introducing some notation with which to describe these mean values. Let Y and H be positive numbers, and consider the exponential sum $\mathbf{g}(\boldsymbol{\alpha}) = \mathbf{g}(\boldsymbol{\alpha}; Y, H)$ defined by

$$\mathbf{g}(\boldsymbol{\alpha}; Y, H) = \sum_{0 < |h| \leq H} \sum_{1 \leq y \leq Y} e(h\alpha_1 + hy\alpha_2 + hy^2\alpha_3). \quad (2.1)$$

We seek to obtain estimates for mean values of the shape

$$I_s(Y, H) = \oint |\mathbf{g}(\boldsymbol{\alpha})|^{2s} \, d\boldsymbol{\alpha}. \quad (2.2)$$

Lemma 2.1. *For each $\varepsilon > 0$, one has $I_2(Y, H) \ll H^3 Y + (HY)^{2+\varepsilon}$.*

Proof. By orthogonality, the mean value $I_2(Y, H)$ counts the number of integral solutions of the simultaneous equations

$$\sum_{i=1}^4 h_i y_i^j = 0 \quad (0 \leq j \leq 2), \quad (2.3)$$

with $0 < |h_i| \leq H$ and $1 \leq y_i \leq Y$ ($1 \leq i \leq 4$). We divide the solutions counted by $I_2(Y, H)$ into three classes. Denote by T_0 the number of solutions of the system (2.3) counted by $I_2(Y, H)$ in which $y_1 = y_2 = y_3 = y_4$, by T_1 the corresponding number with $h_3 y_3^2 + h_4 y_4^2 = 0$, and by T_2 the number with $y_3 \neq y_4$ and $h_3 y_3^2 + h_4 y_4^2 \neq 0$. Then by symmetry, it follows that

$$I_2(Y, H) \ll T_0 + T_1 + T_2. \quad (2.4)$$

Observe first that, by considering the linear equation in \mathbf{h} in (2.3), one finds that the number of possible choices for \mathbf{h} is $O(H^3)$, and consequently

$$T_0 \ll H^3 Y. \quad (2.5)$$

Given a solution \mathbf{h}, \mathbf{y} of (2.3) counted by T_1 , meanwhile, it follows from the equation with $j = 2$ that

$$h_1 y_1^2 + h_2 y_2^2 = 0 = h_3 y_3^2 + h_4 y_4^2.$$

Given fixed choices of h_2, y_2, h_4, y_4 , it follows that h_1 and y_1 are divisors of the fixed non-zero integer $h_2y_2^2$, and that h_3 and y_3 are divisors of the fixed non-zero integer $h_4y_4^2$. An elementary estimate for the divisor function therefore shows that there are $O((HY)^\varepsilon)$ possible choices for h_1, y_1, h_3, y_3 , and hence

$$T_1 \ll (HY)^{2+\varepsilon}. \quad (2.6)$$

In order to estimate T_2 , we begin by considering the polynomial identity

$$(a+b)(ax^2+by^2) - (ax+by)^2 = ab(x-y)^2.$$

Given any solution \mathbf{y}, \mathbf{h} of the system (2.3), one therefore has

$$h_1h_2(y_1-y_2)^2 = h_3h_4(y_3-y_4)^2. \quad (2.7)$$

Consider a fixed choice of h_3, h_4, y_3, y_4 corresponding to a solution \mathbf{h}, \mathbf{y} counted by T_2 . Since $y_3 \neq y_4$, it follows from (2.7) that h_1, h_2 and $z = y_1 - y_2$ are each divisors of the fixed non-zero integer $h_3h_4(y_3 - y_4)^2$. A standard estimate for the divisor function shows that there are $O((HY)^\varepsilon)$ possible such choices. Fix any one choice, and consider the equation with $j = 2$ in (2.3). One has

$$h_1(z+y_2)^2 + h_2y_2^2 = -h_3y_3^2 - h_4y_4^2 \neq 0. \quad (2.8)$$

Since the integer $h_1z^2 + 2h_1zy_2 + (h_1 + h_2)y_2^2$ is non-zero, it follows that the equation (2.8) is non-trivial in terms of y_2 . For if one were to have $h_1 + h_2 = 2h_1z = 0$, then one would have also $h_1z^2 = 0$, yielding a contradiction. There are consequently at most 2 solutions for the remaining undetermined variable y_2 , and then $y_1 = z + y_2$ is also determined. We therefore conclude that

$$T_2 \ll (HY)^{2+\varepsilon}. \quad (2.9)$$

By substituting the estimates (2.5), (2.6) and (2.9) into (2.4), we conclude that $I_2(Y, H) \ll H^3Y + (HY)^{2+\varepsilon}$, thereby completing the proof of the lemma. \square

We employ Lemma 2.1 to establish a bound for $I_3(Y, H)$ via the Hardy-Littlewood method.

Lemma 2.2. *For each $\varepsilon > 0$, one has $I_3(Y, H) \ll (HY)^\varepsilon(H^5Y^2 + H^4Y^3)$.*

Proof. We begin by obtaining an estimate of Weyl-type for the exponential sum $\mathfrak{g}(\boldsymbol{\alpha})$. By applying Cauchy's inequality to (2.1), one obtains the bound

$$\begin{aligned} |\mathfrak{g}(\boldsymbol{\alpha})|^2 &\ll H \sum_{0 < |h| \leq H} \left| \sum_{1 \leq y \leq Y} e(h\alpha_1 + hy\alpha_2 + hy^2\alpha_3) \right|^2 \\ &\leq 2H^2Y + 2H \sum_{0 < |h| \leq H} \left| \sum_{1 \leq y_2 < y_1 \leq Y} e(h(y_1 - y_2)\alpha_2 + h(y_1^2 - y_2^2)\alpha_3) \right|. \end{aligned}$$

Thus, by substituting $z = y_1 - y_2$ and $w = y_1 + y_2$, we find that

$$|\mathfrak{g}(\boldsymbol{\alpha})|^2 \ll H^2Y + H \sum_{1 \leq h \leq H} \sum_{1 \leq z \leq Y} \left| \sum_{\substack{w \in I(z) \\ 2|(w+z)}} e(hz\alpha_2 + hzw\alpha_3) \right|,$$

where $I(z)$ is an interval of integers lying in $[1, 2Y]$. Consequently, one finds that

$$\begin{aligned} |\mathbf{g}(\boldsymbol{\alpha})|^2 &\ll H^2Y + H \sum_{1 \leq h \leq H} \sum_{1 \leq z \leq Y} \min\{Y, \|2hz\alpha_3\|^{-1}\} \\ &\ll H^2Y + (HY)^\varepsilon H \sum_{1 \leq m \leq 2HY} \min\{HY^2/m, \|m\alpha_3\|^{-1}\}. \end{aligned}$$

An application of [15, Lemma 2.2] therefore shows that whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha_3 - a/q| \leq q^{-2}$, then

$$|\mathbf{g}(\boldsymbol{\alpha})|^2 \ll (HY)^{2+\varepsilon}(q^{-1} + Y^{-1} + q(HY^2)^{-1}). \quad (2.10)$$

Thus, a standard transference principle (see Lemma 14.1 below) reveals that whenever $\alpha_3 \in [0, 1)$, $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ satisfy $(b, r) = 1$, then

$$|\mathbf{g}(\boldsymbol{\alpha})| \ll (HY)^{1+\varepsilon}(\lambda^{-1} + Y^{-1} + \lambda(HY^2)^{-1})^{1/2}, \quad (2.11)$$

where $\lambda = r + HY^2|r\alpha_3 - b|$.

We now apply the Hardy-Littlewood method. Denote by \mathfrak{M} the union of the intervals

$$\mathfrak{M}(q, a) = \{\theta \in [0, 1) : |q\theta - a| \leq (HY)^{-1}\},$$

with $0 \leq a \leq q \leq Y$ and $(a, q) = 1$. In addition, put $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. Given $\alpha_3 \in \mathfrak{m}$, an application of Dirichlet's approximation theorem shows that there exist $q \in \mathbb{N}$ and $a \in \mathbb{N}$ with $0 \leq a \leq q \leq HY$, $(a, q) = 1$ and $|q\alpha_3 - a| \leq (HY)^{-1}$. Since $\alpha_3 \in \mathfrak{m}$, one therefore has $q > Y$, and so it follows from (2.10) that

$$|\mathbf{g}(\boldsymbol{\alpha})| \ll H^{1+\varepsilon}Y^{1/2+\varepsilon}. \quad (2.12)$$

Thus, we infer from (2.2) that

$$\begin{aligned} \int_{\mathfrak{m}} \int_0^1 \int_0^1 |\mathbf{g}(\boldsymbol{\alpha})|^6 d\alpha_1 d\alpha_2 d\alpha_3 &\ll \left(\sup_{\alpha_3 \in \mathfrak{m}} |\mathbf{g}(\boldsymbol{\alpha})| \right)^2 \oint |\mathbf{g}(\boldsymbol{\alpha})|^4 d\boldsymbol{\alpha} \\ &\ll (H^2Y)^{1+\varepsilon} I_2(Y, H). \end{aligned}$$

An application of Lemma 2.1 consequently delivers the bound

$$\begin{aligned} \int_{\mathfrak{m}} \int_0^1 \int_0^1 |\mathbf{g}(\boldsymbol{\alpha})|^6 d\alpha_1 d\alpha_2 d\alpha_3 &\ll (H^2Y)^{1+\varepsilon} (H^3Y + (HY)^{2+\varepsilon}) \\ &\ll (HY)^{3\varepsilon} (H^5Y^2 + H^4Y^3). \end{aligned} \quad (2.13)$$

Next, define the function $\Upsilon(\beta)$ for $\beta \in [0, 1)$ by taking

$$\Upsilon(\beta) = (q + HY^2|q\beta - a|)^{-1},$$

when $\beta \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}$, and by taking $\Upsilon(\beta) = 0$ otherwise. Then it follows from (2.11) that when $\alpha_3 \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}$, one has

$$|\mathbf{g}(\boldsymbol{\alpha})| \ll H^{1+\varepsilon}Y^{1/2+\varepsilon} + (HY)^{1+\varepsilon}\Upsilon(\alpha_3)^{1/2}. \quad (2.14)$$

A comparison of (2.12) and (2.14) therefore leads via the argument underlying (2.13) to the estimate

$$\int_{\mathfrak{M}} \int_0^1 \int_0^1 |\mathbf{g}(\boldsymbol{\alpha})|^6 d\alpha_1 d\alpha_2 d\alpha_3 \ll (HY)^{3\varepsilon} (H^5 Y^2 + H^4 Y^3) + (HY)^{2+\varepsilon} J_0, \quad (2.15)$$

where

$$J_0 = \sum_{1 \leq q \leq Y} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\mathfrak{M}(q,a)} \int_0^1 \int_0^1 \Upsilon(\alpha_3) |\mathbf{g}(\boldsymbol{\alpha})|^4 d\alpha_1 d\alpha_2 d\alpha_3.$$

Write

$$\Psi(\beta) = \int_0^1 \int_0^1 |\mathbf{g}(\alpha_1, \alpha_2, \beta)|^4 d\alpha_1 d\alpha_2,$$

so that $\Psi(\beta)$ has the Fourier expansion

$$\Psi(\beta) = \sum_{|n| \leq 2HY^2} \psi_n e(\beta n),$$

with

$$\psi_n = \oint |\mathbf{g}(\boldsymbol{\alpha})|^4 e(-n\alpha_3) d\boldsymbol{\alpha}.$$

Then it follows from [2, Lemma 2] that

$$J_0 \ll (HY^2)^{\varepsilon-1} \left(Y\psi_0 + \sum_{n \neq 0} |\psi_n| \right). \quad (2.16)$$

On the one hand, we find from (2.2) that

$$\psi_0 = \oint |\mathbf{g}(\boldsymbol{\alpha})|^4 d\boldsymbol{\alpha} = I_2(Y, H).$$

Meanwhile, one has

$$\sum_{n \neq 0} |\psi_n| \leq \sum_{n \in \mathbb{Z}} \oint |\mathbf{g}(\boldsymbol{\alpha})|^4 e(-n\alpha_3) d\boldsymbol{\alpha} = \int_0^1 \int_0^1 |\mathbf{g}(\alpha_1, \alpha_2, 0)|^4 d\alpha_1 d\alpha_2.$$

We therefore deduce from (2.16), Lemma 2.1 and orthogonality that

$$J_0 \ll (HY^2)^{\varepsilon-1} (H^3 Y^2 + H^2 Y^3 + J_1), \quad (2.17)$$

where J_1 denotes the number of integral solutions of the simultaneous equations

$$\left. \begin{aligned} h_1 y_1 + h_2 y_2 &= h_3 y_3 + h_4 y_4 \\ h_1 + h_2 &= h_3 + h_4 \end{aligned} \right\}, \quad (2.18)$$

with $0 < |h_i| \leq H$ and $1 \leq y_i \leq Y$ ($1 \leq i \leq 4$). A crude estimate for J_1 is obtained by simply observing that for each fixed choice of h_i and y_i ($1 \leq i \leq 3$), the equations (2.18) uniquely determine h_4 , and hence also y_4 . Thus we find that $J_1 \ll (HY)^3$, so that (2.17) yields the bound $J_0 \ll H^{2+\varepsilon} Y^{1+\varepsilon}$. On substituting this estimate into (2.15), we therefore deduce that

$$\int_{\mathfrak{M}} \int_0^1 \int_0^1 |\mathbf{g}(\boldsymbol{\alpha})|^6 d\alpha_1 d\alpha_2 d\alpha_3 \ll (HY)^\varepsilon (H^5 Y^2 + H^4 Y^3).$$

Since $\mathfrak{M} \cup \mathfrak{m} = [0, 1)$, it therefore follows from (2.13) that

$$\oint |\mathfrak{g}(\boldsymbol{\alpha})|^6 d\boldsymbol{\alpha} \ll (HY)^\varepsilon (H^5 Y^2 + H^4 Y^3),$$

and the conclusion of the lemma follows at once on recalling (2.2). \square

3. NEW MEAN VALUE ESTIMATES FOR EXPONENTIAL SUMS

The mean value estimates for $\mathfrak{g}(\boldsymbol{\alpha})$ obtained in the previous section can be applied to convert estimates associated with the cubic case of Vinogradov's mean value theorem into estimates for mean values of the exponential sum $\mathfrak{F}(\alpha, \beta)$ defined in (1.2). In this section we discuss this conversion, and hence establish the bounds recorded in Theorem 1.3. We recall the exponential sum $\mathfrak{g}(\boldsymbol{\alpha}; Y, H)$ defined in (2.1), and write

$$\mathfrak{g}^*(\boldsymbol{\alpha}; Y, H) = \mathfrak{g}(\alpha_1, 2\alpha_2, 3\alpha_3; Y, H).$$

In addition, we define the exponential sum $\mathfrak{h}(\boldsymbol{\alpha}) = \mathfrak{h}(\boldsymbol{\alpha}; X)$ by putting

$$\mathfrak{h}(\boldsymbol{\alpha}; X) = \sum_{1 \leq x \leq X} e(\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3).$$

Then, with the standard notation associated with Vinogradov's mean value theorem in mind, we put

$$J_{s,3}(X) = \oint |\mathfrak{h}(\boldsymbol{\alpha}; X)|^{2s} d\boldsymbol{\alpha}. \quad (3.1)$$

Finally, we recall the definition of the mean value $T_s(X)$ from (1.3).

Lemma 3.1. *When s is a natural number, one has*

$$T_s(X) \ll J_{s,3}(X) + X^{-1} \oint |\mathfrak{h}(\boldsymbol{\alpha}; 2X)|^{2s} \mathfrak{g}^*(-\boldsymbol{\alpha}; X, sX) d\boldsymbol{\alpha}.$$

Proof. Define δ_j to be 1 when $j = 1$, and to be 0 otherwise. Our starting point for the proof of this lemma is the observation that, by orthogonality, the mean value $T_s(X)$ counts the number of integral solutions of the system of equations

$$\sum_{i=1}^s (x_i^j - y_i^j) = \delta_j h \quad (1 \leq j \leq 3), \quad (3.2)$$

with $1 \leq x_i, y_i \leq X$ ($1 \leq i \leq s$) and $|h| \leq sX$. Here, the constraint on

$$\sum_{i=1}^s (x_i - y_i) \quad (3.3)$$

imposed by the linear equation in (3.2) is redundant, since the range for h automatically accommodates all possible values of (3.3) within (3.2).

Let $T_s^*(X)$ denote the number of integral solutions of the system (3.2) counted by $T_s(X)$ in which $h \neq 0$. Then on considering the underlying Diophantine systems, we see that

$$T_s(X) = T_s^*(X) + J_{s,3}(X). \quad (3.4)$$

Next, we consider the effect of shifting all of the variables by an integer z . By applying the binomial theorem, one finds that \mathbf{x}, \mathbf{y} is a solution of (3.2) if and only if \mathbf{x}, \mathbf{y} is a solution of the system

$$\sum_{i=1}^s ((x_i + z)^j - (y_i + z)^j) = jz^{j-1}h \quad (1 \leq j \leq 3).$$

We therefore infer that for each fixed integer z with $1 \leq z \leq X$, the mean value $T_s^*(X)$ is bounded above by the number of integral solutions of the system

$$\sum_{i=1}^s (u_i^j - v_i^j) = jz^{j-1}h \quad (1 \leq j \leq 3),$$

with $1 \leq u_i, v_i \leq 2X$ ($1 \leq i \leq s$) and $0 < |h| \leq sX$. Thus, on recalling the definition (2.1) and applying orthogonality, one finds that

$$\begin{aligned} \sum_{1 \leq z \leq X} T_s^*(X) &\leq \sum_{1 \leq z \leq X} \oint |\mathfrak{h}(\boldsymbol{\alpha}; 2X)|^{2s} \sum_{0 < |h| \leq sX} e(-h\alpha_1 - 2hz\alpha_2 - 3hz^2\alpha_3) d\boldsymbol{\alpha} \\ &= \oint |\mathfrak{h}(\boldsymbol{\alpha}; 2X)|^{2s} \mathfrak{g}^*(-\boldsymbol{\alpha}; X, sX) d\boldsymbol{\alpha}. \end{aligned}$$

We therefore arrive at the relation

$$T_s^*(X) \ll X^{-1} \oint |\mathfrak{h}(\boldsymbol{\alpha}; 2X)|^{2s} \mathfrak{g}^*(-\boldsymbol{\alpha}; X, sX) d\boldsymbol{\alpha}.$$

The conclusion of the lemma follows by combining this estimate with (3.4). \square

We are now equipped to establish estimates for the moments of $\mathfrak{F}(\alpha, \beta)$.

The proof of Theorem 1.3. We first establish the estimate $T_4(X) \ll X^{4+\varepsilon}$. We therefore apply Lemma 3.1, with $s = 4$, in combination with Hölder's inequality. On recalling (2.2) and (3.1), we obtain $T_4(X) \ll J_{4,3}(X) + X^{-1}I_1$, where

$$I_1 = (J_{5,3}(2X))^{1/2} (J_{6,3}(2X))^{1/4} (I_2(X, 4X))^{1/4}.$$

But [21, Theorem 1.1] establishes the main conjecture in the cubic case of Vinogradov's mean value theorem, and thus

$$J_{s,3}(2X) \ll X^{s+\varepsilon} \quad (1 \leq s \leq 6). \quad (3.5)$$

Meanwhile, Lemma 2.1 establishes the bound $I_2(X, 4X) \ll X^{4+\varepsilon}$. Hence

$$T_4(X) \ll X^{4+\varepsilon} + X^{-1}(X^{5+\varepsilon})^{1/2}(X^{6+\varepsilon})^{1/4}(X^{4+\varepsilon})^{1/4} \ll X^{4+\varepsilon}.$$

This confirms the first estimate of Theorem 1.3 in the case $s = 4$. Meanwhile, when $1 \leq s \leq 4$, an application of Hölder's inequality provides the bound $T_s(X) \leq (T_4(X))^{s/4} \ll X^{s+\varepsilon}$, yielding the first estimate of the theorem in full.

We next take $s = 5$, and apply Lemma 3.1 together with Hölder's inequality. Again recalling (2.2) and (3.1), we obtain $T_5(X) \ll J_{5,3}(X) + X^{-1}I_2$, where

$$I_2 = (J_{6,3}(2X))^{5/6} (I_3(X, 5X))^{1/6}.$$

Making use of Lemma 2.2 together with the main conjecture (3.5) once again, we conclude that

$$T_5(X) \ll X^{5+\varepsilon} + X^{-1}(X^{6+\varepsilon})^{5/6}(X^{7+\varepsilon})^{1/6} \ll X^{31/6+\varepsilon}. \quad (3.6)$$

This completes the proof of the second estimate of the theorem.

The final estimate of the theorem is confirmed via the Hardy-Littlewood method. When $q \in \mathbb{N}$ and $a_2, a_3 \in \mathbb{Z}$, denote a typical major arc by

$$\mathfrak{M}(q, a_2, a_3) = \{\alpha \in [0, 1)^2 : |q\alpha_i - a_i| < (18X^{i-1})^{-1} \ (i = 2, 3)\}.$$

We take \mathfrak{M} to be the union of these major arcs with $0 \leq a_i \leq q \leq P$ ($i = 2, 3$) and $(q, a_2, a_3) = 1$, and then put $\mathfrak{m} = [0, 1)^2 \setminus \mathfrak{M}$. Then it follows from [17, Lemma 9.2] that whenever $t > 9$, one has

$$\int_{\mathfrak{M}} |\mathfrak{F}(\alpha, \beta)|^t d\alpha d\beta \ll X^{t-5}. \quad (3.7)$$

Meanwhile, the argument leading to [17, equation (7.9)] shows that

$$\sup_{(\alpha, \beta) \in \mathfrak{m}} |\mathfrak{F}(\alpha, \beta)| \ll X^{3/4+\varepsilon}.$$

We consequently deduce from (3.6) that for $t > 10$, one has

$$\int_{\mathfrak{m}} |\mathfrak{F}(\alpha, \beta)|^t d\alpha d\beta \ll (X^{3/4+\varepsilon})^{t-10} T_5(X) \ll X^{t-5+(32-3t)/12+t\varepsilon}.$$

By combining this estimate with (3.7), we conclude that whenever $t > 32/3$ and ε is sufficiently small, then

$$T_t(X) = \int_{\mathfrak{m}} |\mathfrak{F}(\alpha, \beta)|^t d\alpha d\beta + \int_{\mathfrak{M}} |\mathfrak{F}(\alpha, \beta)|^t d\alpha d\beta \ll X^{t-5}.$$

This completes the proof of the final estimate of the theorem. \square

4. PRELIMINARY SIMPLIFICATION OF THE DIAGONAL EQUATIONS

In the remainder of this memoir we focus on the system of equations (1.1) and seek to prove Theorem 1.1. Our application of the Hardy-Littlewood method to this system is simplified by some preliminary observations, much of the necessary work having been accomplished previously in [17, §6] and [19, §3]. We start by showing that the conditions of Theorem 1.1 permit us to assume that (1.1) has non-singular real solutions, and also non-singular p -adic solutions for each prime p . Thus, in our application of the circle method, we can expect both the singular series and singular integral to be non-zero.

Lemma 4.1. *Suppose that conditions (a), (b) and (c) of Theorem 1.1 hold for the system of equations (1.1). Then, either:*

- (i) *the system has a real solution $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \boldsymbol{\theta}$ with the property that no θ_i is zero, and for which, locally, there is an $(s-2)$ -dimensional subspace \mathcal{S} of positive $(s-2)$ -volume in the neighbourhood of $\boldsymbol{\theta}$ on which $\Theta = \Phi = 0$, or else*
- (ii) *the system has a non-zero rational solution.*

Proof. The conclusion of the lemma follows by combining [17, Lemmata 6.1 and 6.2], just as in [19, Lemma 3.1]. \square

Let $M(q)$ denote the number of solutions of the simultaneous congruences $\Theta(\mathbf{x}, \mathbf{y}) \equiv \Phi(\mathbf{x}, \mathbf{z}) \equiv 0 \pmod{q}$ with $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{Z}/q\mathbb{Z})^s$.

Lemma 4.2. *Suppose that $s \geq 11$ and the conditions (a), (b) and (c) of Theorem 1.1 hold for the system of equations (1.1). Then, either:*

(i) *for each rational prime p , there is a natural number $w = w(p)$ with the property that for all $t \geq w$, one has $M(p^t) \geq p^{(t-w)(s-2)}$, or else*

(ii) *the system has a non-zero rational solution.*

Proof. This is [17, Lemma 6.7]. We note that the conclusion (i) follows from the existence of a non-singular p -adic solution of the system (1.1). \square

Solubility is easily established when there are many vanishing coefficients.

Lemma 4.3. *Suppose that $s \geq 11$ and the conditions (a), (b) and (c) of Theorem 1.1 hold for the system of equations (1.1). Then the system has a non-zero solution in rational integers when either $m \geq 6$ or $n \geq 4$.*

Proof. The respective conclusions follow from [17, Lemmata 6.3 and 6.5]. \square

In view of Lemma 4.3, we may assume henceforth that $s \geq 11$, $0 \leq m \leq 5$ and $0 \leq n \leq 3$, whence $l = s - m - n \geq s - 8$. Also, by Lemma 4.1 together with the homogeneity of the system (1.1), we may suppose that the latter equations have a non-singular real solution $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = \boldsymbol{\theta}$ with the property that $0 < |\theta_i| < \frac{1}{2}$ ($1 \leq i \leq s$). Since whenever necessary the a_i can be replaced by $-a_i$ by interchanging x_i and $-x_i$, and similarly c_j may be replaced by $-c_j$ by interchanging y_j and $-y_j$, we may suppose without loss that in fact $\theta_i > 0$ ($1 \leq i \leq s$). Finally, as a consequence of Lemma 4.2, we may suppose that for every rational prime p , there is a natural number $w = w(p)$ with the property that for all $t \geq w$, one has $M(p^t) \geq p^{(t-w)(s-2)}$. The latter bound also holds whenever (1.1) has a non-singular p -adic solution for each prime p .

Our initial simplifications complete, we now record some notation to assist in our later deliberations. Let P be a positive number sufficiently large in terms of ε , \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} and $\boldsymbol{\theta}$, and let α_i ($i = 2, 3$) be real variables. Also, write

$$t = \max_{i,j,k} \{|a_i|, |b_i|, |c_j|, |d_k|\}.$$

We take η to be a positive number sufficiently small in terms of ε , put $R = P^\eta$, and then define $\mathcal{A}(P, R)$ via (1.5). We define the exponential sums

$$\begin{aligned} f_i(\alpha_2, \alpha_3) &= \sum_{\frac{1}{2}\varepsilon_i P < x \leq 2\varepsilon_i P} e(a_i \alpha_3 x^3 + b_i \alpha_2 x^2) \quad (1 \leq i \leq l), \\ g_j(\alpha_3) &= \sum_{\frac{1}{2}\eta_j P < y \leq 2\eta_j P} e(c_j \alpha_3 y^3) \quad (1 \leq j \leq m), \\ h_k(\alpha_2) &= \sum_{\frac{1}{2}\zeta_k P < z \leq 2\zeta_k P} e(d_k \alpha_2 z^2) \quad (1 \leq k \leq n), \end{aligned}$$

and, when we wish to make use of analogous exponential sums in which the variable of summation is restricted to lie in the set of smooth numbers $\mathcal{A}(P, R)$, we decorate this notation with a tilde. Thus, the exponential sum $\tilde{f}_i(\alpha_2, \alpha_3)$ denotes such a smooth Weyl sum. For the sake of concision, we abbreviate

$$|f_i(\boldsymbol{\alpha})| \text{ to } f_i, \quad |g_j(\alpha_3)| \text{ to } g_j, \quad \text{and} \quad |h_k(\alpha_2)| \text{ to } h_k,$$

with similar conventions for other generating functions. We also write

$$f(\boldsymbol{\alpha}; X) = \sum_{1 \leq x \leq X} e(\alpha_3 x^3 + \alpha_2 x^2) \quad \text{and} \quad g(\alpha; X) = \sum_{1 \leq x \leq X} e(\alpha x^3).$$

Our aim is to estimate the number $R(P)$ of solutions of the Diophantine system (1.1) in rational integers x_i, y_j, z_k satisfying the conditions

$$\frac{1}{2}\boldsymbol{\xi}P < \mathbf{x} \leq 2\boldsymbol{\xi}P, \quad \frac{1}{2}\boldsymbol{\eta}P < \mathbf{y} \leq 2\boldsymbol{\eta}P, \quad \frac{1}{2}\boldsymbol{\zeta}P < \mathbf{z} \leq 2\boldsymbol{\zeta}P. \quad (4.1)$$

Recall that there is no loss of generality in assuming in our proof of Theorem 1.1 that, in general, we have $0 \leq m \leq 5$ and $0 \leq n \leq 3$. The technical difficulties associated with our application of the Hardy-Littlewood method force us to divide systems of the shape (1.1) into three classes:

- (A) $m = n = 0$ or $n \in \{1, 2\}$,
- (B) $1 \leq m \leq 5$ and $n \in \{0, 3\}$,
- (C) $m = 0$ and $n = 3$.

We adopt a different strategy for each class. For systems of type A, we obtain an asymptotic formula for $R(P)$. For systems of type B we instead count the number $R^*(P)$ of solutions of (1.1) subject to (4.1), and in addition constrained by the condition $y_j \in \mathcal{A}(P, R)$ ($1 \leq j \leq m$). Finally, for systems of type C we count the number $R^\dagger(P)$ of solutions of (1.1) subject to (4.1), but now constrained by the condition $x_l \in \mathcal{A}(P, R)$.

Plainly, one has $R(P) \geq R^*(P)$ and $R(P) \geq R^\dagger(P)$, and so if we show in the respective cases that $R(P)$, or $R^*(P)$, or $R^\dagger(P)$, grows in proportion to P^{s-3} as $P \rightarrow \infty$, then in all cases we will be able to conclude that $R(P) \gg P^{s-3}$. This establishes the conclusion of Theorem 1.2, and hence, in view of our earlier discussion, also the conclusion of Theorem 1.1.

Next, we must describe the apparatus required for our application of the Hardy-Littlewood method. Write

$$\mathcal{F}(\boldsymbol{\alpha}) = \prod_{i=1}^l f_i(\alpha_2, \alpha_3) \prod_{j=1}^m g_j(\alpha_3) \prod_{k=1}^n h_k(\alpha_2).$$

Also, denote by $\mathcal{F}^*(\boldsymbol{\alpha})$ the analogous generating function in which g_j is decorated with a tilde for $1 \leq j \leq m$, and $\mathcal{F}^\dagger(\boldsymbol{\alpha})$ that in which instead f_l is decorated with a tilde. Then it follows from orthogonality that

$$R(P) = \oint \mathcal{F}(\boldsymbol{\alpha}) d\boldsymbol{\alpha},$$

and likewise when this relation is adorned with asterisks or obelisks.

Finally, we describe the Hardy-Littlewood dissection. Let Q be a real number with $1 \leq Q \leq P$, and put $\Xi_i = 18tP^i$ ($i = 2, 3$). Then, when $0 \leq r_i \leq q \leq Q$ ($i = 2, 3$) and $(q, r_2, r_3) = 1$, we denote a typical major arc by

$$\mathfrak{M}(q, \mathbf{r}; Q) = \{\boldsymbol{\alpha} \in [0, 1)^2 : |q\alpha_i - r_i| \leq Q\Xi_i^{-1} \ (i = 2, 3)\}.$$

Note that the arcs $\mathfrak{M}(q, \mathbf{r})$ are disjoint whenever $1 \leq Q \leq P$. Let $\mathfrak{M}(Q)$ be the union of these arcs $\mathfrak{M}(q, \mathbf{r}; Q)$, and put $\mathfrak{m}(Q) = [0, 1)^2 \setminus \mathfrak{M}(Q)$. For the sake of convenience, we put $\mathfrak{M} = \mathfrak{M}(P^{3/4})$ and $\mathfrak{m} = \mathfrak{m}(P^{3/4})$. Also, when $0 \leq r_i \leq q \leq Q$ ($i = 2, 3$) and $(q, r_2, r_3) = 1$, we denote a typical inhomogeneous major arc by

$$\mathfrak{P}(q, \mathbf{r}; Q) = \{\boldsymbol{\alpha} \in [0, 1)^2 : |\alpha_i - r_i/q| \leq Q\Xi_i^{-1} \ (i = 2, 3)\}.$$

Let $\mathfrak{P}(Q)$ be the union of these arcs $\mathfrak{P}(q, \mathbf{r}; Q)$, and put $\mathfrak{p}(Q) = [0, 1)^2 \setminus \mathfrak{P}(Q)$. With $\delta = 10^{-6}$, we then put $\mathfrak{P} = \mathfrak{P}(P^{30\delta})$ and $\mathfrak{p} = \mathfrak{p}(P^{30\delta})$.

Henceforth, implicit constants in the notations of Landau and Vinogradov will depend at most on $s, \varepsilon, \eta, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\boldsymbol{\theta}$, unless stated otherwise.

5. AUXILIARY ESTIMATES FOR SYSTEMS OF TYPE A

Our initial focus is on estimating $R(P)$ when $m = n = 0$ or $n \in \{1, 2\}$. We begin in this section by introducing several auxiliary mean value estimates, as well as an estimate of Weyl-type, useful both in estimating the minor arc contribution for systems of type A, as well as in later sections.

Lemma 5.1. *For all i, j, k , one has*

- (i) $\oint f_i^{32/3} d\boldsymbol{\alpha} \ll P^{17/3+\varepsilon},$
- (ii) $\oint f_i^8 h_k^2 d\boldsymbol{\alpha} \ll P^{5+\varepsilon},$
- (iii) $\oint f_i^4 g_j^4 h_k^2 d\boldsymbol{\alpha} \ll P^{5+\varepsilon},$
- (iv) $\oint f_i^4 g_j^8 d\boldsymbol{\alpha} \ll P^{7+\varepsilon},$
- (v) $\oint g_j^8 h_k^4 d\boldsymbol{\alpha} \ll P^{7+\varepsilon}.$

Proof. We begin with the estimate (i). The definition of $f_i(\boldsymbol{\alpha})$ implies that

$$f_i(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha}; 2\xi_i P) - f(\boldsymbol{\alpha}; \xi_i P/2),$$

and so whenever $w > 16/3$, it follows from Theorem 1.3 that

$$\oint f_i^{2w} d\boldsymbol{\alpha} \ll T_w(2\xi_i P) + T_w(\xi_i P/2) \ll P^{2w-5}.$$

Thus, we deduce from Hölder's inequality that when $\nu = \varepsilon/5$, one has

$$\oint f_i^{32/3} d\boldsymbol{\alpha} \ll \left(\oint f_i^{32(1+\nu)/3} d\boldsymbol{\alpha} \right)^{1/(1+\nu)} \ll P^{17/3+\varepsilon}.$$

We turn next to the estimate (ii). By orthogonality, the mean value I in question is bounded above by the number of integral solutions of the system

$$\left. \begin{aligned} a_i \sum_{u=1}^4 (x_u^3 - y_u^3) &= 0 \\ b_i \sum_{u=1}^4 (x_u^2 - y_u^2) &= d_k(x_0^2 - y_0^2) \end{aligned} \right\}, \quad (5.1)$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq P$. Denote by I_0 the number of these solutions with $x_0 = y_0$, and I_1 the corresponding number with $x_0 \neq y_0$. Then, on considering the underlying systems of Diophantine equations, it follows via Theorem 1.3 that $I_0 \leq PT_4(P) \ll P^{5+\varepsilon}$. Suppose next that \mathbf{x}, \mathbf{y} is a solution of (5.1) counted by I_1 . By applying Hua's lemma¹ to the cubic equation in (5.1), one sees that the number I_2 of choices for x_u, y_u ($1 \leq u \leq 4$) satisfies $I_2 = O(P^{5+\varepsilon})$. Fix any one such choice. Since $x_0 \neq y_0$, the integer

$$N = b_i \sum_{u=1}^4 (x_u^2 - y_u^2)$$

is fixed and non-zero. Both $x_0 - y_0$ and $x_0 + y_0$ are divisors of N , and so there are $O(N^\varepsilon)$ possible choices for x_0 and y_0 . Hence $I_1 \ll N^\varepsilon I_2 \ll P^{5+4\varepsilon}$. On combining these estimates, we obtain the desired bound $I = I_0 + I_1 \ll P^{5+\varepsilon}$.

The argument for part (iii) is similar. By orthogonality, the mean value I in question is bounded above by the number of integral solutions of the system

$$\left. \begin{aligned} a_i \sum_{u=1}^2 (x_u^3 - y_u^3) + c_j \sum_{u=3}^4 (x_u^3 - y_u^3) &= 0 \\ b_i \sum_{u=1}^2 (x_u^2 - y_u^2) &= d_k(x_0^2 - y_0^2) \end{aligned} \right\}, \quad (5.2)$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq P$. Denote by I_3 the number of these solutions with $x_0 = y_0$, and I_4 the corresponding number with $x_0 \neq y_0$. Suppose first that \mathbf{x}, \mathbf{y} is a solution of (5.2) counted by I_3 , so that $x_0 = y_0$. By applying Hua's lemma to the quadratic equation in (5.2), one finds that the number I_5 of possible choices for x_1, x_2, y_1, y_2 satisfies $I_5 = O(P^{2+\varepsilon})$. Fix any one such choice. Then it follows from the cubic equation in (5.2) that $c_j(x_3^3 + x_4^3 - y_3^3 - y_4^3) = N$, where N is now the fixed integer $-a_i(x_1^3 + x_2^3 - y_1^3 - y_2^3)$. Hence, by the triangle inequality in combination with Hua's lemma, one sees that the number I_6 of choices for x_3, x_4, y_3, y_4 satisfies $I_6 = O(P^{2+\varepsilon})$. Thus we conclude that $I_3 \ll P(P^{2+\varepsilon})^2 \ll P^{5+2\varepsilon}$. The argument required to bound I_4 is identical with that applied in case (ii), and thus we discern that $I_4 \ll P^{5+\varepsilon}$. By combining these estimates, we obtain the bound $I = I_3 + I_4 \ll P^{5+\varepsilon}$ asserted in case (iii).

¹See [15, Lemma 2.5].

By orthogonality, the mean value I in case (iv) is bounded above by the number of integral solutions of the system

$$\left. \begin{aligned} a_i \sum_{u=1}^2 (x_u^3 - y_u^3) &= c_j \sum_{u=3}^6 (x_u^3 - y_u^3) \\ b_i \sum_{u=1}^2 (x_u^2 - y_u^2) &= 0 \end{aligned} \right\}, \quad (5.3)$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq P$. Suppose that \mathbf{x}, \mathbf{y} is a solution of (5.3) counted by I . By applying Hua's lemma to the quadratic equation in (5.3), one sees that the number I_7 of possible choices for x_1, x_2, y_1, y_2 satisfies $I_7 = O(P^{2+\varepsilon})$. Fix any one such choice. We now apply the triangle inequality in combination with Hua's lemma to the cubic equation of (5.3) in a manner similar to that of the previous case. Thus, the number I_8 of possible choices for x_u, y_u ($3 \leq u \leq 6$) satisfies $I_8 = O(P^{5+\varepsilon})$. We therefore conclude that $I \ll (P^{2+\varepsilon})(P^{5+\varepsilon}) = P^{7+2\varepsilon}$, confirming the estimate asserted in case (iv) of the lemma.

The mean value I in case (v) counts the number of integral solutions of

$$d_k \sum_{u=1}^2 (x_u^2 - y_u^2) = c_j \sum_{u=3}^6 (x_u^3 - y_u^3) = 0,$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq P$. A comparison with (5.3) reveals that an argument identical to that applied in case (iv) may be used to confirm the bound $I \ll P^{7+\varepsilon}$. This completes the proof of the lemma. \square

Finally, we recall a Weyl estimate for $f_i(\boldsymbol{\alpha})$ sensitive to both α_2 and α_3 .

Lemma 5.2. *Suppose that $Q \leq P^{3/4}$. Then for all i , one has*

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{p}(Q)} |f_i(\boldsymbol{\alpha})| \leq \sup_{\boldsymbol{\alpha} \in \mathfrak{m}(Q)} |f_i(\boldsymbol{\alpha})| \ll P^{1+\varepsilon} Q^{-1/3}.$$

Proof. Let $\tau > 0$, so that $(P/(P^{1+\tau}Q^{-1/3}))^{4+\varepsilon} < P$. If $|f_i(\boldsymbol{\alpha})| \geq P^{1+\tau}Q^{-1/3}$, then we see from [1, Theorem 5.1] that there exist $q \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{Z}^2$ such that $(q, r_2, r_3) = 1$, $q < P^{-\tau}Q$ and $|q\alpha_j - r_j| < QP^{-\tau-j}$ ($j = 2, 3$). Thus $\boldsymbol{\alpha} \in \mathfrak{M}(Q)$, and so it follows that whenever $\boldsymbol{\alpha} \in \mathfrak{m}(Q)$, then $|f_i(\boldsymbol{\alpha})| < P^{1+\tau}Q^{-1/3}$. Since $\mathfrak{p}(Q) \subseteq \mathfrak{m}(Q)$, the conclusion of the lemma follows. \square

6. THE MINOR ARC CONTRIBUTION FOR SYSTEMS OF TYPE A

We now estimate the contribution of the minor arcs \mathfrak{p} within the integral giving $R(P)$ for systems of type A. Thus we may suppose that either $m = n = 0$ or else $0 \leq m \leq 5$ and $n \in \{1, 2\}$. Here and later we make use of the inequality

$$|z_1 z_2 \dots z_n| \leq |z_1|^n + \dots + |z_n|^n. \quad (6.1)$$

Lemma 6.1. *One has $\int_{\mathfrak{p}} |\mathcal{F}(\boldsymbol{\alpha})| d\boldsymbol{\alpha} \ll P^{s-5-\delta}$.*

Proof. On recalling the definition of $\mathcal{F}(\boldsymbol{\alpha})$ and applying (6.1), we deduce that for some integers i, j and k , one has

$$\int_{\mathfrak{p}} |\mathcal{F}(\boldsymbol{\alpha})| \, d\boldsymbol{\alpha} \ll \int_{\mathfrak{p}} f_i^l g_j^m h_k^n \, d\boldsymbol{\alpha}. \quad (6.2)$$

In view of our hypotheses concerning m and n , by repeated application of (6.1), as in the proof of [17, Lemma 7.3], one obtains the bound

$$\int_{\mathfrak{p}} f_i^l g_j^m h_k^n \, d\boldsymbol{\alpha} \ll \int_{\mathfrak{p}} (f_i^s + f_i^{s-2} h_k^2 + f_i^{s-7} g_j^5 h_k^2 + f_i^{s-6} g_j^5 h_k) \, d\boldsymbol{\alpha}.$$

Thus it follows from (6.2) that

$$\int_{\mathfrak{p}} |\mathcal{F}(\boldsymbol{\alpha})| \, d\boldsymbol{\alpha} \ll \left(\sup_{\boldsymbol{\alpha} \in \mathfrak{p}} f_i \right)^{s-65/6} (I_{0,0} + I_{0,2} + I_{5,2} + I_{5,1}), \quad (6.3)$$

where

$$I_{a,b} = \oint f_i^{65/6-a-b} g_j^a h_k^b \, d\boldsymbol{\alpha}.$$

The trivial estimate $f_i \leq P$ combines with Lemma 5.1(i) to give

$$I_{0,0} \leq P^{1/6} \oint f_i^{32/3} \, d\boldsymbol{\alpha} \ll P^{35/6+\varepsilon}.$$

In a similar manner, one finds from Lemma 5.1(ii) that

$$I_{0,2} \leq P^{5/6} \oint f_i^8 h_k^2 \, d\boldsymbol{\alpha} \ll P^{35/6+\varepsilon}.$$

Next, by applying first Hölder's inequality, and then Lemma 5.1(i), (iii), (iv) and (v), one obtains the bound

$$\begin{aligned} I_{5,2} &\leq \left(\oint f_i^{32/3} \, d\boldsymbol{\alpha} \right)^{1/8} \left(\oint f_i^4 g_j^4 h_k^2 \, d\boldsymbol{\alpha} \right)^{1/2} \left(\oint f_i^4 g_j^8 \, d\boldsymbol{\alpha} \right)^{1/8} \left(\oint g_j^8 h_k^4 \, d\boldsymbol{\alpha} \right)^{1/4} \\ &\ll P^\varepsilon (P^{17/3})^{1/8} (P^5)^{1/2} (P^7)^{1/8} (P^7)^{1/4} \ll P^{35/6+\varepsilon}. \end{aligned}$$

Similarly, but now using Lemma 5.1(i),(iii) and (iv), one finds that

$$\begin{aligned} I_{5,1} &\leq \left(\oint f_i^{32/3} \, d\boldsymbol{\alpha} \right)^{1/8} \left(\oint f_i^4 g_j^4 h_k^2 \, d\boldsymbol{\alpha} \right)^{1/2} \left(\oint f_i^4 g_j^8 \, d\boldsymbol{\alpha} \right)^{3/8} \\ &\ll P^\varepsilon (P^{17/3})^{1/8} (P^5)^{1/2} (P^7)^{3/8} \ll P^{35/6+\varepsilon}. \end{aligned}$$

Finally, Lemma 5.2 supplies the bound

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{p}} f_i \ll P^{1+\varepsilon} (P^{30\delta})^{-1/3} \ll P^{1-9\delta}.$$

Thus, we conclude from (6.3) that

$$\int_{\mathfrak{p}} |\mathcal{F}(\boldsymbol{\alpha})| \, d\boldsymbol{\alpha} \ll (P^{1-9\delta})^{s-65/6} P^{35/6+\varepsilon} \ll P^{s-5-\delta},$$

and the proof of the lemma is complete. \square

7. THE MAJOR ARC CONTRIBUTION FOR SYSTEMS OF TYPE A

We next estimate the contribution of the major arcs within $R(P)$ for systems of type A, beginning with some additional notation. For each i, j, k , we write

$$\begin{aligned} S_{f,i}(q, \mathbf{r}) &= \sum_{u=1}^q e_q(a_i r_3 u^3 + b_i r_2 u^2), & v_{f,i}(\boldsymbol{\beta}) &= \int_{\xi_i P/2}^{2\xi_i P} e(a_i \beta_3 \gamma^3 + b_i \beta_2 \gamma^2) d\gamma, \\ S_{g,j}(q, \mathbf{r}) &= \sum_{u=1}^q e_q(c_j r_3 u^3), & v_{g,j}(\boldsymbol{\beta}) &= \int_{\eta_j P/2}^{2\eta_j P} e(c_j \beta_3 \gamma^3) d\gamma, \\ S_{h,k}(q, \mathbf{r}) &= \sum_{u=1}^q e_q(d_m r_2 u^2), & v_{h,k}(\boldsymbol{\beta}) &= \int_{\zeta_k P/2}^{2\zeta_k P} e(d_m \beta_2 \gamma^2) d\gamma, \end{aligned}$$

where, as usual, we write $e_q(z)$ for $e^{2\pi iz/q}$. We then define

$$T(q, \mathbf{r}) = q^{-s} \prod_{i=1}^l S_{f,i}(q, \mathbf{r}) \prod_{j=1}^m S_{g,j}(q, \mathbf{r}) \prod_{k=1}^n S_{h,k}(q, \mathbf{r})$$

and

$$V(\boldsymbol{\beta}) = \prod_{i=1}^l v_{f,i}(\boldsymbol{\beta}) \prod_{j=1}^m v_{g,j}(\boldsymbol{\beta}) \prod_{k=1}^n v_{h,k}(\boldsymbol{\beta}).$$

We recall some estimates for these generating functions recorded in [17].

Lemma 7.1. *Suppose that $q \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{Z}$ satisfy $(q, r_2, r_3) = 1$. Then*

$$\begin{aligned} S_{f,i}(q, \mathbf{r}) &\ll q^{2/3+\varepsilon}, & S_{g,j}(q, \mathbf{r}) &\ll q^{2/3+\varepsilon} (q, r_3)^{1/3}, \\ S_{h,k}(q, \mathbf{r}) &\ll q^{1/2+\varepsilon} (q, r_2)^{1/2}. \end{aligned}$$

When p is a prime number with $(p, r_2, r_3) = 1$, and $h \in \{1, 2\}$, moreover, then

$$\begin{aligned} S_{f,i}(p^h, \mathbf{r}) &\ll p^{h/2}, & S_{g,j}(p^h, \mathbf{r}) &\ll p^{h/2} (p^h, r_3)^{1/2}, \\ S_{h,k}(p^h, \mathbf{r}) &\ll p^{h/2} (p^h, r_2)^{1/2}. \end{aligned}$$

Proof. The first batch of estimates follow from [15, Theorem 7.1]. Meanwhile, the first of the second batch follows from [13, Corollary 2F of Chapter II] in the case $h = 1$, and from the argument of the proof of [15, Theorem 7.1] in the case $h = 2$. The final two estimates are immediate from [15, Lemmata 4.3 and 4.4]. Implicit constants here may depend on the coefficients a_i, b_i, c_j, d_k . \square

Lemma 7.2. *One has*

$$\begin{aligned} v_{f,i}(\boldsymbol{\beta}) &\ll P(1 + P^2|\beta_2| + P^3|\beta_3|)^{-1/3}, & v_{g,j}(\boldsymbol{\beta}) &\ll P(1 + P^2|\beta_3|)^{-1/3}, \\ v_{h,k}(\boldsymbol{\beta}) &\ll P(1 + P^2|\beta_2|)^{-1/2}. \end{aligned}$$

Proof. The respective estimates follow from [15, Theorem 7.3]. \square

We define the function $f_i^*(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in [0, 1]^2$ by taking

$$f_i^*(\boldsymbol{\alpha}) = q^{-1} S_{f,i}(q, \mathbf{r}) v_{f,i}(\boldsymbol{\alpha} - \mathbf{r}/q), \quad (7.1)$$

when $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{r}; P) \subseteq \mathfrak{M}(P)$, and by taking $f_i^*(\boldsymbol{\alpha}) = 0$ when $\boldsymbol{\alpha} \notin \mathfrak{M}(P)$. We define $g_j^*(\boldsymbol{\alpha})$ and $h_k^*(\boldsymbol{\alpha})$ in an analogous manner.

Lemma 7.3. *Suppose that $q \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{Z}$ satisfy $(q, r_2, r_3) = 1$. Then, when $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{r}; P) \subseteq \mathfrak{M}(P)$, one has*

$$\begin{aligned} f_i(\boldsymbol{\alpha}) - f_i^*(\boldsymbol{\alpha}) &\ll q^{2/3+\varepsilon}, & g_j(\alpha_3) - g_j^*(\boldsymbol{\alpha}) &\ll q^{2/3+\varepsilon}(q, r_3)^{1/3}, \\ h_k(\alpha_2) - h_k^*(\boldsymbol{\alpha}) &\ll q^{1/2+\varepsilon}(q, r_2)^{1/2}. \end{aligned}$$

Proof. This is immediate from [1, Lemma 4.4]. \square

Next we introduce the incomplete singular series and integral

$$\mathfrak{S}(Q) = \sum_{1 \leq q \leq Q} \sum_{\substack{r_2=1 \\ (q, r_2, r_3)=1}}^q \sum_{r_3=1}^q T(q, \mathbf{r}) \quad \text{and} \quad \mathfrak{J}(Q) = \int_{-QP^{-3}}^{QP^{-3}} \int_{-QP^{-2}}^{QP^{-2}} V(\boldsymbol{\beta}) \, d\boldsymbol{\beta},$$

and their completed counterparts

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{r_2=1 \\ (q, r_2, r_3)=1}}^q \sum_{r_3=1}^q T(q, \mathbf{r}) \quad \text{and} \quad \mathfrak{J} = \iint_{\mathbb{R}^2} V(\boldsymbol{\beta}) \, d\boldsymbol{\beta}.$$

The truncated singular integral $\mathfrak{J}(Q)$ is easily estimated via Lemma 7.2.

Lemma 7.4. *Provided that $s \geq 11$, $n \leq 3$ and $m \leq 5$, there is a positive constant \mathcal{C} with the property that $\mathfrak{J}(Q) = \mathcal{C}P^{s-5} + O(P^{s-5}Q^{-1/2})$.*

Proof. An application of Lemma 7.2 reveals that

$$V(\boldsymbol{\beta}) \ll P^s (1 + P^3|\beta_3|)^{-m/3} (1 + P^2|\beta_2|)^{-n/2} (1 + P^2|\beta_2| + P^3|\beta_3|)^{-l/3}.$$

Our hypotheses $s = l + m + n \geq 11$, $m \leq 5$ and $n \leq 3$ therefore ensure that

$$V(\boldsymbol{\beta}) \ll P^s (1 + P^3|\beta_3|)^{-11/6} (1 + P^2|\beta_2|)^{-11/6}.$$

When $\boldsymbol{\beta}$ lies outside the box $[-QP^{-2}, QP^{-2}] \times [-QP^{-3}, QP^{-3}]$, one has either $P^2|\beta_2| \geq Q$ or $P^3|\beta_3| \geq Q$. Hence we deduce that

$$\mathfrak{J} - \mathfrak{J}(Q) \ll P^s Q^{-1/2} \iint_{\mathbb{R}^2} (1 + P^3|\beta_3|)^{-4/3} (1 + P^2|\beta_2|)^{-4/3} \, d\boldsymbol{\beta},$$

so that $\mathfrak{J}(Q) = \mathfrak{J} + O(P^{s-5}Q^{-1/2})$.

In view of Lemma 4.1, we may assume that the equations $\Theta = \Phi = 0$ define an $(s-2)$ -dimensional manifold \mathcal{S} in the box $\mathcal{B} = [\frac{1}{2}\theta_1, 2\theta_1] \times \dots \times [\frac{1}{2}\theta_s, 2\theta_s]$, passing through the point $\boldsymbol{\theta}$ and having positive $(s-2)$ -volume. By making a change of variables one sees that

$$\mathfrak{J} = P^{s-5} \iint_{\mathbb{R}^2} \int_{\mathcal{B}} e(\beta_3\Theta(\boldsymbol{\gamma}) + \beta_2\Phi(\boldsymbol{\gamma})) \, d\boldsymbol{\gamma} \, d\boldsymbol{\beta}.$$

Applying Fourier's integral formula twice to the latter integral, in the form

$$\lim_{\lambda \rightarrow \infty} \int_{-\Gamma}^{\Gamma} \int_{-\lambda}^{\lambda} V(\gamma) e(\gamma \omega) \, d\omega \, d\gamma = V(0),$$

we therefore see that $\mathfrak{J} = \mathcal{C}P^{s-5}$, where $\mathcal{C} > 0$ is the $(s-2)$ -volume of \mathcal{S} within \mathcal{B} . Thus we deduce that $\mathfrak{J} = \mathcal{C}P^{s-5}$, and hence $\mathfrak{J}(Q) = \mathcal{C}P^{s-5} + O(P^{s-5}Q^{-1/2})$. This completes the proof of the lemma. \square

Next we consider the truncated singular series $\mathfrak{S}(Q)$.

Lemma 7.5. *Provided that $s \geq 11$, $n \leq 3$ and $m \leq 5$, the singular series \mathfrak{S} converges absolutely, one has $\mathfrak{S} > 0$, and $\mathfrak{S}(Q) = \mathfrak{S} + O(Q^{-1/6})$.*

Proof. We begin by investigating the quantity

$$A(q) = \sum_{\substack{r_2=1 \\ (q, r_2, r_3)=1}}^q \sum_{\substack{r_3=1 \\ (q, r_2, r_3)=1}}^q |T(q, \mathbf{r})|$$

when $q = p^h$ is a power of the prime number p . First, when $h \geq 2$, it follows from Lemma 7.1 and our hypotheses concerning s , m and n that

$$\begin{aligned} A(p^h) &\ll (p^h)^{\varepsilon-s/3} \sum_{r_2=1}^{p^h} \sum_{r_3=1}^{p^h} p^{-nh/6} (p^h, r_2)^{n/2} (p^h, r_3)^{m/3} \\ &\ll (p^h)^{\varepsilon-s/3} \sum_{a=0}^h \sum_{b=0}^h (p^{h-a} (p^a)^{n/3}) (p^{h-b} (p^b)^{m/3}) \\ &\ll (p^h)^{2+\varepsilon-s/3} \sum_{a=0}^h \sum_{b=0}^h p^{2b/3} \ll h(p^h)^{\varepsilon-1}. \end{aligned} \quad (7.2)$$

When $h = 1$ we proceed similarly, applying Lemma 7.1 to obtain

$$\begin{aligned} A(p) &\ll p^{-s/2} \sum_{\substack{r_2=1 \\ (p, r_2, r_3)=1}}^p \sum_{\substack{r_3=1 \\ (p, r_2, r_3)=1}}^p (p, r_3)^{m/2} (p, r_2)^{n/2} \\ &\ll p^{-s/2} (p^2 + p^{1+m/2} + p^{1+n/2}) \ll p^{-2}. \end{aligned} \quad (7.3)$$

The standard theory associated with singular series shows that $A(q)$ is multiplicative (compare [17, Lemmata 10.4 and 10.5]). Then we deduce from (7.2) and (7.3) that for some constant C depending only on \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} , one has

$$\begin{aligned} \sum_{q=1}^{\infty} q^{1/6} A(q) &\leq \prod_p \left(1 + Cp^{-11/6} + C \sum_{h=2}^{\infty} hp^{-2h/3} \right) \\ &\leq \prod_p (1 + C^2 p^{-4/3}) \ll 1. \end{aligned}$$

Both the absolute convergence of \mathfrak{S} and the final conclusion of the lemma follow by applying this bound to show that

$$|\mathfrak{S} - \mathfrak{S}(Q)| \leq \sum_{q>Q} (q/Q)^{1/6} A(q) \ll Q^{-1/6}.$$

Furthermore, the argument underlying the proof of [17, Lemmata 10.8] combines with the bounds (7.2) and (7.3) to confirm that the quantity

$$\chi_p = \sum_{h=0}^{\infty} \sum_{\substack{r_2=1 \\ (p,r_2,r_3)=1}}^{p^h} \sum_{r_3=1}^{p^h} T(p^h, \mathbf{r})$$

satisfies $\chi_p \ll 1$, and that $\mathfrak{S} = \prod_p \chi_p$. In view of Lemma 4.2, we are at liberty to assume that, for each prime p , there is a natural number $w = w(p)$ with the property that $M(p^t) \geq p^{(t-w)(s-2)}$ for $t \geq w$. The conclusion of [17, Lemma 10.9] therefore establishes that $\chi_p > 0$ for each prime number p , and hence that $\mathfrak{S} = \prod_p \chi_p > 0$. This completes the proof of the lemma. \square

We now complete the analysis of the major arcs for systems of type A. From Lemma 7.3 we see that when $\boldsymbol{\alpha} \in \mathfrak{P}(q, \mathbf{r}) \subseteq \mathfrak{P}$, one has

$$f_i(\boldsymbol{\alpha}) - f_i^*(\boldsymbol{\alpha}) \ll P^{31\delta}, \quad g_j(\alpha_3) - g_j^*(\boldsymbol{\alpha}) \ll P^{31\delta}, \quad h_k(\alpha_2) - h_k^*(\boldsymbol{\alpha}) \ll P^{31\delta},$$

and hence

$$\mathcal{F}(\boldsymbol{\alpha}) - T(q, \mathbf{r})V(\boldsymbol{\alpha} - \mathbf{r}/q) \ll P^{s-1+31\delta}.$$

The measure of the set of arcs \mathfrak{P} is $O((P^{30\delta})^5 P^{-5})$, and thus we conclude that

$$\int_{\mathfrak{P}} \mathcal{F}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} - \mathfrak{S}(P^{30\delta})\mathfrak{J}(P^{30\delta}) \ll (P^{s-1+31\delta})(P^{150\delta-5}) \ll P^{s-5-\delta}.$$

We therefore infer from Lemmata 7.4 and 7.5 that

$$\int_{\mathfrak{P}} \mathcal{F}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} = \mathfrak{C}\mathfrak{S}P^{s-5} + O(P^{s-5-\delta}).$$

Since $[0, 1]^2$ is the union of \mathfrak{P} and \mathfrak{p} , we conclude from Lemma 6.1 that

$$R(P) = \iint_{[0,1]^2} \mathcal{F}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} = \mathfrak{C}\mathfrak{S}P^{s-5} + O(P^{s-5-\delta}). \quad (7.4)$$

Since $\mathfrak{C}\mathfrak{S} > 0$, we deduce that $R(P) \gg P^{s-5}$. The conclusion of Theorem 1.2, and hence also Theorem 1.1, therefore follows for systems of type A.

8. AUXILIARY ESTIMATES FOR SYSTEMS OF TYPE B

In the next phase of our argument, we focus on estimating $R^*(P)$ when $1 \leq m \leq 5$ and $n \in \{0, 3\}$. We again begin by introducing several auxiliary mean value estimates useful both here and later. The value of η is chosen in accordance with the following lemma.

Lemma 8.1. *Suppose that $\eta > 0$ is sufficiently small. Then for all j , one has*

$$\int_0^1 \tilde{g}_j^6 d\beta \ll P^{13/4-3\delta} \quad \text{and} \quad \int_0^1 \tilde{g}_j^{31/4} d\beta \ll P^{19/4}.$$

Proof. The definition of $\tilde{g}_j(\beta)$ implies that

$$\tilde{g}_j(\beta) = \tilde{g}(\beta; 2\eta_j P) - \tilde{g}(\beta; \eta_j P/2),$$

and so it follows from [20, Theorem 1.2] and [4, Theorem 2] that

$$\int_0^1 \tilde{g}_j^u d\beta \ll \max_{1 \leq X \leq P} \int_0^1 |\tilde{g}(\beta; X)|^u d\beta \ll P^{\mu_u},$$

where $\mu_6 = 3.2495$, and $\mu_u = u - 3$ whenever $u \geq 7.691$. The conclusion of the lemma follows at once. \square

Lemma 8.2. *For all i, j, k , one has*

- (i) $\oint f_i^4 \tilde{g}_j^6 d\alpha \ll P^{21/4-2\delta}$,
- (ii) $\oint \tilde{g}_j^6 h_k^4 d\alpha \ll P^{21/4-2\delta}$,
- (iii) $\oint \tilde{g}_j^{31/4} h_k^{17/4} d\alpha \ll P^7$.

Proof. We begin with the estimate (i). By orthogonality, the mean value I in question is bounded above by the number of integral solutions of the system

$$\left. \begin{aligned} a_i \sum_{u=1}^2 (x_u^3 - y_u^3) &= c_j \sum_{u=3}^5 (x_u^3 - y_u^3) \\ b_i \sum_{u=1}^2 (x_u^2 - y_u^2) &= 0 \end{aligned} \right\}, \quad (8.1)$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq P$ and $x_u, y_u \in \mathcal{A}(P, R)$ ($3 \leq u \leq 5$). Suppose that \mathbf{x}, \mathbf{y} is a solution of (8.1) counted by I . By applying Hua's lemma to the quadratic equation in (8.1), one sees that the number I_1 of possible choices for x_1, x_2, y_1, y_2 satisfies $I_1 = O(P^{2+\varepsilon})$. Fix any one such choice. Then it follows from the cubic equation in (8.1) that $c_j(x_3^3 + x_4^3 + x_5^3 - y_3^3 - y_4^3 - y_5^3) = N$, where N is the fixed integer $a_i(x_1^3 + x_2^3 - y_1^3 - y_2^3)$. Hence, by the triangle inequality in combination with Lemma 8.1, one sees that the number I_2 of choices for x_u, y_u ($3 \leq u \leq 5$) satisfies $I_2 = O(P^{13/4-3\delta})$. Thus $I \ll (P^{2+\varepsilon})(P^{13/4-3\delta}) \ll P^{21/4-2\delta}$, confirming the estimate (i) asserted in the lemma.

We turn next to the estimate (ii). Here, Lemma 8.1 combines with Hua's lemma to show that the mean value in question is equal to

$$\int_0^1 \tilde{g}_j^6 d\alpha_3 \int_0^1 h_k^4 d\alpha_2 \ll (P^{13/4-3\delta})(P^{2+\varepsilon}),$$

and hence the estimate (ii) follows.

In order to establish the estimate (iii), we begin by noting that a straightforward application of the circle method shows that whenever $u > 4$, one has

$$\int_0^1 h_k^u d\alpha_2 \ll P^{u-2}. \quad (8.2)$$

Thus, Lemma 8.1 shows that the mean value now in question is equal to

$$\int_0^1 \tilde{g}_j^{31/4} d\alpha_3 \int_0^1 h_k^{17/4} d\alpha_2 \ll (P^{19/4})(P^{9/4}) \ll P^7.$$

This completes the proof of the estimate (iii). \square

We define the multiplicative function $\kappa(q) = \kappa_C(q)$ for prime powers p^h by taking $\kappa(p^h)$ to be $Cp^{-h/2}$, when $h \in \{1, 2\}$, and to be $Cp^{\varepsilon-h/3}$, when $h > 2$.

Lemma 8.3. *For all i, j, k , one has*

- (i) $\sup_{\alpha \in \mathfrak{M}(P) \setminus \mathfrak{M}(Q)} |f_i^*(\alpha)| \ll PQ^{-1/4} \quad (Q \leq P),$
- (ii) $\oint |f_i^*|^u d\alpha \ll P^{u-5} \quad (u > 7),$
- (iii) $\oint |f_i^*|^u \tilde{g}_j^{31/4} d\alpha \ll P^{u+11/4} \quad (u > 4).$
- (iv) $\oint |f_i^* h_k|^u d\alpha \ll P^{2u-5} \quad (u > 4).$

Proof. We begin with the estimate (i). Suppose that $\alpha \in \mathfrak{M}(q, \mathbf{r}) \subseteq \mathfrak{M}(P)$. Then on recalling (7.1), we find from Lemmata 7.1 and 7.2 that

$$f_i^*(\alpha) \ll P(q + P^2|q\alpha_2 - r_2| + P^3|q\alpha_3 - r_3|)^{\varepsilon-1/3}.$$

When $\alpha \notin \mathfrak{M}(Q)$, one of the lower bounds $q > Q$, or $|q\alpha_2 - r_2| > QP^{-2}$, or $|q\alpha_3 - r_3| > QP^{-3}$ must hold, and hence $|f_i^*(\alpha)| \ll PQ^{-1/4}$.

We next turn to the estimate (ii), and assume throughout that $u > 7$. On recalling (7.1), we deduce from Lemma 7.2 that

$$\oint |f_i^*|^u d\alpha \ll \mathcal{I}_u(P)W_u(P), \quad (8.3)$$

where

$$\mathcal{I}_u(P) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} P^u (1 + P^2|\beta_2| + P^3|\beta_3|)^{\varepsilon-u/3} d\beta,$$

$$W_u(P) = \sum_{1 \leq q \leq P} A_u(q) \quad \text{and} \quad A_u(q) = \sum_{\substack{r_2=1 \\ (q, r_2, r_3)=1}}^q \sum_{r_3=1}^q |q^{-1} S_{f,i}(q, \mathbf{r})|^u.$$

On the one hand,

$$\mathcal{I}_u(P) \leq P^u \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + P^2|\beta_2|)^{\varepsilon-u/6} (1 + P^3|\beta_3|)^{\varepsilon-u/6} d\beta \ll P^{u-5}.$$

Meanwhile, when p is prime, it follows from Lemma 7.1 that for $h \geq 2$, one has $A_u(p^h) \ll p^{2h}(p^h)^{\varepsilon-u/3} \ll p^{-h/3-\nu}$, for some $\nu > 0$, and for $h \in \{1, 2\}$, instead

$A_u(p^h) \ll p^{2h}(p^h)^{-u/2} \ll p^{-3h/2}$. Since the standard theory of singular series shows that $A(q)$ is multiplicative, we deduce that for some $C > 0$, one has

$$W_u(P) \ll \prod_{p \leq P} (1 + Cp^{-3/2} + Cp^{-1-\nu}) \ll 1.$$

The estimate (ii) is confirmed by substituting these estimates into (8.3).

Considering next the estimate (iii), we may assume that $u > 4$. From [17, Lemma 10.4], one finds that $S_{f,i}(q, \mathbf{r})$ possesses the usual quasi-multiplicative property. Hence it follows from Lemma 7.1 that there is a number $C > 0$ with the property that whenever $(q, r_2, r_3) = 1$, then $S_{f,i}(q, \mathbf{r}) \ll \kappa(q)$. Thus, we deduce from (7.1) via Lemma 7.2 that whenever $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{r}) \subseteq \mathfrak{M}(P)$, then

$$|f_i^*(\boldsymbol{\alpha})| \ll \kappa(q)P(1 + P^2|\alpha_2 - r_2/q|)^{-1/3}.$$

Hence

$$\oint |f_i^*|^u \tilde{g}_j^{31/4} d\boldsymbol{\alpha} \ll \mathcal{J}_u(P) \sum_{1 \leq q \leq P} \sum_{r_2=1}^q \kappa(q)^u, \quad (8.4)$$

where

$$\mathcal{J}_u(P) = P^u \int_0^1 \int_{-1/2}^{1/2} (1 + P^2|\beta_2|)^{-u/3} |\tilde{g}_j(\alpha_3)|^{31/4} d\beta_2 d\alpha_3.$$

Since $u > 4$, on the one hand, one finds by means of Lemma 8.1 that

$$\mathcal{J}_u(P) \ll P^{u-2} \int_0^1 \tilde{g}_j^{31/4} d\alpha_3 \ll P^{u+11/4}.$$

Provided that C is chosen sufficiently large, on the other hand, it follows from the multiplicative property of $\kappa(q)$ that there is a number $\nu > 0$ such that

$$\sum_{1 \leq q \leq P} q\kappa(q)^u \ll \prod_p (1 + C^2p^{-1-\nu}) \ll 1. \quad (8.5)$$

The estimate (iii) follows by substituting these bounds into (8.4).

Finally, we turn to the estimate (iv), and again assume that $u > 4$. Then, as in the discussion leading to (8.4), one sees that whenever $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{r}) \subseteq \mathfrak{M}(P)$, there is a number $C > 0$ such that

$$|f_i^*(\boldsymbol{\alpha})| \ll \kappa(q)P(1 + P^3|\alpha_3 - r_3/q|)^{-1/3}.$$

Thus we find that whenever $u > 4$, one has

$$\oint |f_i^*|^u h_k^u d\boldsymbol{\alpha} \ll \mathcal{J}_u(P) \sum_{1 \leq q \leq P} \sum_{r_3=1}^q \kappa(q)^u, \quad (8.6)$$

where

$$\mathcal{J}_u(P) = P^u \int_{-1/2}^{1/2} \int_0^1 |h_k(\alpha_2)|^u (1 + P^3|\beta_3|)^{-u/3} d\alpha_2 d\beta_3.$$

On the one hand, an estimate of the shape (8.2) shows that

$$\mathcal{J}_u(P) \ll P^{u-3} \int_0^1 h_k^u d\alpha_2 \ll P^{2u-5}.$$

Provided that C is chosen sufficiently large, on the other hand, it follows from the multiplicative property of $\kappa(q)$ that the estimate (8.5) again holds. The conclusion of the lemma follows by substituting these estimates into (8.6). \square

9. THE MINOR ARC CONTRIBUTION FOR SYSTEMS OF TYPE B

Owing to the presence of smooth Weyl sums within the generating function $\mathcal{F}^*(\boldsymbol{\alpha})$, our treatment of the minor arcs for systems of type B involves some pruning exercises. We begin by examining a set of minor arcs of large height.

Lemma 9.1. *One has $\int_{\mathfrak{m}} |\mathcal{F}^*(\boldsymbol{\alpha})| d\boldsymbol{\alpha} \ll P^{s-5-\delta}$.*

Proof. We divide into two cases according to whether $n = 0$ or $n = 3$. Suppose first that $n = 0$. Then on recalling the definition of $\mathcal{F}^*(\boldsymbol{\alpha})$ and applying (6.1), we deduce that for some integers i and j , one has

$$\int_{\mathfrak{m}} |\mathcal{F}^*(\boldsymbol{\alpha})| d\boldsymbol{\alpha} \ll \int_{\mathfrak{m}} f_i^{s-m} \tilde{g}_j^m d\boldsymbol{\alpha}. \quad (9.1)$$

In view of our hypotheses concerning m , by repeated application of (6.1), as in the proof of [17, Lemma 7.3], one obtains the bound

$$\int_{\mathfrak{m}} f_i^{s-m} \tilde{g}_j^m d\boldsymbol{\alpha} \ll \int_{\mathfrak{m}} (f_i^s + f_i^{s-5} \tilde{g}_j^5) d\boldsymbol{\alpha}. \quad (9.2)$$

Thus it follows from (9.1) that

$$\int_{\mathfrak{m}} |\mathcal{F}^*(\boldsymbol{\alpha})| d\boldsymbol{\alpha} \ll \left(\sup_{\boldsymbol{\alpha} \in \mathfrak{m}} f_i \right)^{s-32/3} I_1 + \left(\sup_{\boldsymbol{\alpha} \in \mathfrak{m}} f_i \right)^{s-91/9} I_2, \quad (9.3)$$

where

$$I_1 = \oint f_i^{32/3} d\boldsymbol{\alpha} \quad \text{and} \quad I_2 = \oint f_i^{46/9} \tilde{g}_j^5 d\boldsymbol{\alpha}.$$

Lemma 5.1(i) shows that $I_1 \ll P^{17/3+\varepsilon}$. Meanwhile, by applying Hölder's inequality in combination with Lemmata 5.1(i) and 8.2(i), one sees that

$$I_2 \leq \left(\oint f_i^{32/3} d\boldsymbol{\alpha} \right)^{1/6} \left(\oint f_i^4 \tilde{g}_j^6 d\boldsymbol{\alpha} \right)^{5/6} \ll (P^{17/3+\varepsilon})^{1/6} (P^{21/4-2\delta})^{5/6}.$$

Lemma 5.2 shows that $f_i = O(P^{3/4+\varepsilon})$ for $\boldsymbol{\alpha} \in \mathfrak{m}$, and thus (9.3) yields

$$\int_{\mathfrak{m}} |\mathcal{F}^*(\boldsymbol{\alpha})| d\boldsymbol{\alpha} \ll P^\varepsilon \left((P^{3/4})^{s-32/3} P^{17/3} + (P^{3/4})^{s-91/9} P^{383/72} \right) \ll P^{s-5-\delta}.$$

This completes the proof of the lemma when $n = 0$.

Suppose next that $n = 3$. Then on recalling the definition of $\mathcal{F}^*(\boldsymbol{\alpha})$ and applying (6.1), we deduce that for some integers i, j, k , one has

$$\int_{\mathfrak{m}} |\mathcal{F}^*(\boldsymbol{\alpha})| d\boldsymbol{\alpha} \ll \int_{\mathfrak{m}} f_i^{s-m-3} \tilde{g}_j^m h_k^3 d\boldsymbol{\alpha}. \quad (9.4)$$

In view of our hypotheses concerning m , by repeated application of (6.1), as in the proof of [17, Lemma 7.3], one obtains the bound

$$\int_{\mathfrak{m}} f_i^{s-m-3} \tilde{g}_j^m h_k^3 \, d\alpha \ll \int_{\mathfrak{m}} (f_i^{s-4} \tilde{g}_j h_k^3 + f_i^{s-8} \tilde{g}_j^5 h_k^3) \, d\alpha. \quad (9.5)$$

Thus it follows from (9.4) that

$$\int_{\mathfrak{m}} |\mathcal{F}^*(\alpha)| \, d\alpha \ll \left(\sup_{\alpha \in \mathfrak{m}} f_i \right)^{s-32/3} I_1 + \left(\sup_{\alpha \in \mathfrak{m}} f_i \right)^{s-91/9} I_2, \quad (9.6)$$

where

$$I_1 = \oint f_i^{20/3} \tilde{g}_j h_k^3 \, d\alpha \quad \text{and} \quad I_2 = \oint f_i^{19/9} \tilde{g}_j^5 h_k^3 \, d\alpha.$$

Applying Hölder's inequality with Lemmata 5.1(ii) and 8.2(ii), one obtains

$$I_1 \leq \left(\oint f_i^8 h_k^2 \, d\alpha \right)^{5/6} \left(\oint \tilde{g}_j^6 h_k^8 \, d\alpha \right)^{1/6} \ll (P^{5+\varepsilon})^{5/6} (P^4(P^{21/4-2\delta}))^{1/6}.$$

Meanwhile, by applying Hölder's inequality now with Lemmata 5.1(i) and 8.2(i), (ii), we obtain the estimate

$$\begin{aligned} I_2 &\leq \left(\oint f_i^{32/3} \, d\alpha \right)^{1/6} \left(\oint f_i^4 \tilde{g}_j^6 \, d\alpha \right)^{1/12} \left(\oint \tilde{g}_j^6 h_k^4 \, d\alpha \right)^{3/4} \\ &\ll (P^{17/3+\varepsilon})^{1/6} (P^{21/4-2\delta})^{1/12} (P^{21/4-2\delta})^{3/4}. \end{aligned}$$

Lemma 5.2 shows that $f_i = O(P^{3/4+\varepsilon})$ for $\alpha \in \mathfrak{m}$, and thus (9.6) yields

$$\int_{\mathfrak{m}} |\mathcal{F}^*(\alpha)| \, d\alpha \ll (P^{3/4})^{s-32/3} P^{6-7/24} + (P^{3/4})^{s-91/9} P^{6-49/72} \ll P^{s-5-\delta}.$$

This completes the proof of the lemma when $n = 3$. \square

We next prune down to a narrow set of major arcs. Take $W = (\log \log P)^{100}$, and then put $\Omega = \mathfrak{P}(W)$ and $\mathfrak{q} = \mathfrak{p}(W)$. We seek to estimate the contribution of the set $\mathfrak{M} \setminus \Omega$ within the integral giving $R^*(P)$.

Lemma 9.2. *One has $\int_{\mathfrak{M} \setminus \Omega} |\mathcal{F}^*(\alpha)| \, d\alpha \ll P^{s-5} (\log \log P)^{-1}$.*

Proof. We again divide into two cases according to whether $n = 0$ or $n = 3$. Suppose first that $n = 0$. The procedure leading to (9.2) shows that for some i and j , one has

$$\int_{\mathfrak{M} \setminus \Omega} |\mathcal{F}^*(\alpha)| \, d\alpha \ll I_1 + I_2, \quad (9.7)$$

where

$$I_1 = \int_{\mathfrak{M} \setminus \Omega} f_i^s \, d\alpha \quad \text{and} \quad I_2 = \int_{\mathfrak{M} \setminus \Omega} f_i^{s-5} \tilde{g}_j^5 \, d\alpha.$$

On recalling (7.1), we see from Lemma 7.3 that whenever $\alpha \in \mathfrak{M}$, one has $f_i(\alpha) = f_i^*(\alpha) + O(P^{1/2+\varepsilon})$. Thus we deduce that

$$I_1 \ll \left(\sup_{\alpha \in \mathfrak{M} \setminus \Omega} |f_i^*(\alpha)| \right)^{s-8} \oint |f_i^*|^8 \, d\alpha + P^{s/2+\varepsilon}.$$

Since $s \geq 11$, we infer from Lemma 8.3(i), (ii) that

$$I_1 \ll (PW^{-1/4})^{s-8} P^3 + P^{s/2+\varepsilon} \ll P^{s-5} W^{-3/4} \ll P^{s-5} (\log \log P)^{-1}.$$

In a similar manner, we obtain the bound $I_2 \ll I_3 + P^{(s-5)/2+\varepsilon} I_4$, where

$$I_3 = \int_{\mathfrak{M} \setminus \Omega} |f_i^*|^{s-5} \tilde{g}_j^5 d\alpha \quad \text{and} \quad I_4 = \oint \tilde{g}_j^5 d\alpha.$$

Hölder's inequality leads via Lemma 8.3(i), (ii), (iii) to the bound

$$\begin{aligned} I_3 &\ll (PW^{-1/4})^{s-1339/124} \left(\oint |f_i^*|^{29/4} d\alpha \right)^{11/31} \left(\oint |f_i^*|^5 \tilde{g}_j^{31/4} d\alpha \right)^{20/31}, \\ &\ll (PW^{-1/4})^{s-1339/124} (P^{9/4})^{11/31} (P^{31/4})^{20/31} \ll P^{s-5} W^{-1/20}. \end{aligned}$$

Meanwhile, Hölder's inequality combines with Lemma 8.1 to give

$$I_4 \leq \left(\int_0^1 \tilde{g}_j^6 d\alpha_3 \right)^{5/6} \ll (P^{13/4-2\delta})^{5/6} \ll P^{11/4-\delta}.$$

Thus we deduce that

$$I_2 \ll P^{s-5} (\log \log P)^{-1} + P^{(2s+1)/4} \ll P^{s-5} (\log \log P)^{-1}.$$

When $n = 0$, the lemma now follows by substituting these estimates into (9.7).

Suppose next that $n = 3$. Then the procedure leading to (9.5) shows that for some i, j, k , one has

$$\int_{\mathfrak{M} \setminus \Omega} |\mathcal{F}^*(\alpha)| d\alpha \ll I_1 + I_2, \tag{9.8}$$

where

$$I_1 = \int_{\mathfrak{M} \setminus \Omega} f_i^{s-4} \tilde{g}_j h_k^3 d\alpha \quad \text{and} \quad I_2 = \int_{\mathfrak{M} \setminus \Omega} f_i^{s-8} \tilde{g}_j^5 h_k^3 d\alpha.$$

On recalling (7.1), we see from Lemma 7.3 that whenever $\alpha \in \mathfrak{M}$, one has $f_i(\alpha) = f_i^*(\alpha) + O(P^{1/2+\varepsilon})$. Thus we deduce that $I_1 \ll I_3 + P^{(s-4)/2+\varepsilon} I_4$, where

$$I_3 = \oint |f_i^*|^{s-4} \tilde{g}_j h_k^3 d\alpha \quad \text{and} \quad I_4 = \oint \tilde{g}_j h_k^3 d\alpha.$$

An application of Hölder's inequality yields

$$\begin{aligned} I_3 &\leq \left(\sup_{\alpha \in \mathfrak{M} \setminus \Omega} |f_i^*(\alpha)| \right)^{s-4640/527} \left(\oint |f_i^*|^8 d\alpha \right)^{5/17} \\ &\quad \times \left(\oint \tilde{g}_j^{31/4} h_k^{17/4} d\alpha \right)^{4/31} \left(\oint |f_i^* h_k|^{17/4} d\alpha \right)^{304/527}, \end{aligned}$$

and thus we deduce from Lemmata 8.2(iii) and 8.3(i), (ii), (iv) that

$$I_3 \ll (PW^{-1/4})^{s-4640/527} (P^3)^{5/17} (P^7)^{4/31} (P^{7/2})^{304/527} \ll P^{s-5} W^{-1/2}.$$

Meanwhile, Hua's lemma combines with Hölder's inequality to give

$$I_4 \leq \left(\int_0^1 \tilde{g}_j^4 d\alpha_3 \right)^{1/4} \left(\int_0^1 h_k^4 d\alpha_2 \right)^{3/4} \ll P^{2+\varepsilon}.$$

Thus we conclude that

$$I_1 \ll P^{s-5}(\log \log P)^{-1} + P^{s/2+\varepsilon} \ll P^{s-5}(\log \log P)^{-1}.$$

We find in like manner that $I_2 \ll I_5 + P^{(s-8)/2+\varepsilon} I_6$, where

$$I_5 = \int_{\mathfrak{M} \setminus \mathfrak{Q}} |f_i^*|^{s-8} \tilde{g}_j^5 h_k^3 d\alpha \quad \text{and} \quad I_6 = \oint \tilde{g}_j^5 h_k^3 d\alpha.$$

An application of Hölder's inequality yields

$$\begin{aligned} I_5 &\leq \left(\sup_{\alpha \in \mathfrak{M} \setminus \mathfrak{Q}} |f_i^*(\alpha)| \right)^{s-5592/527} \left(\oint |f_i^*|^8 d\alpha \right)^{5/17} \\ &\quad \times \left(\oint \tilde{g}_j^{31/4} h_k^{17/4} d\alpha \right)^{20/31} \left(\oint |f_i^* h_k|^{17/4} d\alpha \right)^{32/527}, \end{aligned}$$

and thus we deduce from Lemmata 8.2(iii) and 8.3(i), (ii), (iv) that

$$I_5 \ll (PW^{-1/4})^{s-5592/527} (P^3)^{5/17} (P^7)^{20/31} (P^{7/2})^{32/527} \ll P^{s-5} W^{-1/11}.$$

Since Lemma 8.1 and Hua's lemma combine with Hölder's inequality to give

$$I_6 \leq \left(\int_0^1 \tilde{g}_j^6 d\alpha_3 \right)^{5/6} \left(\int_0^1 h_k^4 d\alpha_2 \right)^{3/4} \ll (P^{13/4-2\delta})^{5/6} (P^{2+\varepsilon})^{3/4},$$

we thus conclude that

$$I_2 \ll P^{s-5}(\log \log P)^{-1} + P^{(2s+1)/4} \ll P^{s-5}(\log \log P)^{-1}.$$

When $n = 3$, the lemma now follows by inserting these estimates into (9.8). \square

Since $\mathfrak{q} = \mathfrak{m} \cup (\mathfrak{M} \setminus \mathfrak{Q})$, the estimates supplied by Lemmata 9.1 and 9.2 combine to give the following conclusion.

Lemma 9.3. *One has $\int_{\mathfrak{q}} |\mathcal{F}^*(\alpha)| d\alpha \ll P^{s-5}(\log \log P)^{-1}$.*

10. THE MAJOR ARC CONTRIBUTION FOR SYSTEMS OF TYPE B

The discussion of the major arcs \mathfrak{Q} for systems of type B requires an asymptotic analysis of the generating function $\tilde{g}_j(\beta)$.

Lemma 10.1. *There is a positive number c_η with the property that, whenever $\alpha \in \mathfrak{M}(q, \mathbf{r}; R) \subseteq \mathfrak{M}(R)$, then*

$$\tilde{f}_i(\alpha) - c_\eta f_i^*(\alpha) \ll P(\log P)^{-1}(q + P^2|q\alpha_2 - r_2| + P^3|q\alpha_3 - r_3|) \quad (10.1)$$

and

$$\tilde{g}_j(\alpha_3) - c_\eta g_j^*(\alpha) \ll P(\log P)^{-1}(q + P^3|q\alpha_3 - r_3|). \quad (10.2)$$

Proof. On writing $\rho(t)$ for the Dickman function, one sees that

$$\int_{\xi_i P/2}^{2\xi_i P} \rho\left(\frac{\log \gamma}{\log R}\right) e(a_i \beta_3 \gamma^3 + b_i \beta_2 \gamma^2) d\gamma = \rho(1/\eta) v_{f,i}(\beta) + O(P/\log P).$$

Thus, by reference to [17, Lemma 8.5], we see that the relation (10.1) follows with $c_\eta = \rho(1/\eta) > 0$. The second conclusion (10.2) follows in like manner. \square

We now complete the analysis of the major arcs for systems of type B. From Lemmata 7.3 and 10.1, we see that when $\boldsymbol{\alpha} \in \mathfrak{Q}(q, \mathbf{r}) \subseteq \mathfrak{Q}$, one has

$$\begin{aligned} f_i(\boldsymbol{\alpha}) - f_i^*(\boldsymbol{\alpha}) &\ll W, & \tilde{g}_j(\alpha_3) - c_\eta g_j^*(\boldsymbol{\alpha}) &\ll PW^{-10}, \\ h_k(\alpha_2) - h_k^*(\boldsymbol{\alpha}) &\ll W, \end{aligned}$$

whence

$$\mathcal{F}^*(\boldsymbol{\alpha}) - c_\eta^m T(q, \mathbf{r}) V(\boldsymbol{\alpha} - \mathbf{r}/q) \ll P^s W^{-10}.$$

The measure of the set of arcs \mathfrak{Q} is $O(W^5 P^{-5})$, and thus we conclude that

$$\int_{\mathfrak{Q}} \mathcal{F}^*(\boldsymbol{\alpha}) d\boldsymbol{\alpha} - c_\eta^m \mathfrak{S}(W) \mathfrak{J}(W) \ll P^{s-5} W^{-5}.$$

We therefore deduce from Lemmata 7.4 and 7.5 that

$$\int_{\mathfrak{Q}} \mathcal{F}^*(\boldsymbol{\alpha}) d\boldsymbol{\alpha} = c_\eta^m \mathfrak{S} \mathcal{C} P^{s-5} + O(P^{s-5} W^{-1/6}).$$

But $[0, 1]^2$ is the union of \mathfrak{Q} and \mathfrak{q} , and so we infer from Lemma 9.3 that

$$R^*(P) = \iint_{[0,1]^2} \mathcal{F}^*(\boldsymbol{\alpha}) d\boldsymbol{\alpha} = c_\eta^m \mathfrak{S} \mathcal{C} P^{s-5} + O(P^{s-5} (\log \log P)^{-1}).$$

Since $\mathcal{C} \mathfrak{S} > 0$, we deduce that $R^*(P) \gg P^{s-5}$. The conclusion of Theorem 1.2, and hence also Theorem 1.1, therefore follows for systems of type B.

11. AUXILIARY ESTIMATES FOR SYSTEMS OF TYPE C

Finally, we estimate $R^\dagger(P)$ when $n = 3$ and $m = 0$. We begin with some auxiliary estimates.

Lemma 11.1. *For all i and k , one has*

$$\begin{aligned} \text{(i)} \quad & \oint \tilde{f}_i^6 h_k^4 d\boldsymbol{\alpha} \ll P^{21/4-2\delta}, \\ \text{(ii)} \quad & \oint \tilde{f}_i^8 h_k^{17/4} d\boldsymbol{\alpha} \ll P^{29/4}. \end{aligned}$$

Proof. We begin with the estimate (i). By orthogonality, the mean value I in question is bounded above by the number of integral solutions of the system

$$\left. \begin{aligned} a_i \sum_{u=1}^3 (x_u^3 - y_u^3) &= 0 \\ b_i \sum_{u=1}^3 (x_u^2 - y_u^2) &= d_k \sum_{u=4}^5 (x_u^2 - y_u^2) \end{aligned} \right\}, \quad (11.1)$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq P$ and $x_u, y_u \in \mathcal{A}(P, R)$ ($1 \leq u \leq 3$). Suppose that \mathbf{x}, \mathbf{y} is a solution of (11.1) counted by I . By applying Lemma 8.1 to the cubic equation in (11.1), one sees that the number I_1 of possible choices for x_u, y_u ($1 \leq u \leq 3$) satisfies $I_1 = O(P^{13/4-3\delta})$. Fix any one such choice. Then it follows from the quadratic equation in (11.1) that $d_k(x_4^2 + x_5^2 - y_4^2 - y_5^2) = N$. Hence, by the triangle inequality in combination with Hua's lemma, one sees

that the number I_2 of choices for x_4, x_5, y_5, y_6 satisfies $I_2 = O(P^{2+\varepsilon})$. Thus $I \ll (P^{13/4-3\delta})(P^{2+\varepsilon}) \ll P^{21/4-2\delta}$, confirming the estimate (i).

We turn next to the estimate (ii). By orthogonality combined with [14, Theorem 1], one obtains the bound

$$\int_0^1 |\tilde{f}_i(\alpha_2, \alpha_3)|^8 d\alpha_3 \leq \int_0^1 |g(\alpha; P)|^8 d\alpha \ll P^5.$$

We therefore deduce from an estimate of the type (8.2) that

$$\int_0^1 |h_k(\alpha_2)|^{17/4} \int_0^1 |\tilde{f}_i(\alpha)|^8 d\alpha_3 d\alpha_2 \ll P^5 \int_0^1 h_k^{17/4} d\alpha_2 \ll P^{29/4},$$

confirming the estimate (ii). \square

12. THE MINOR ARC ESTIMATE FOR SYSTEMS OF TYPE C

The presence of the smooth Weyl sum $\tilde{f}_i(\alpha)$ in the generating function $\mathcal{F}^\dagger(\alpha)$ involves us again in some pruning exercises. We adopt the notation of §9, and begin by examining a set of minor arcs of large height.

Lemma 12.1. *One has $\int_{\mathfrak{m}} |\mathcal{F}^\dagger(\alpha)| d\alpha \ll P^{s-5-\delta}$.*

Proof. Recall that in systems of type C, one has $m = 0$ and $n = 3$. Thus, on recalling the definition of $\mathcal{F}^\dagger(\alpha)$ and applying (6.1), we deduce that for some integers i and k , one has

$$\int_{\mathfrak{m}} |\mathcal{F}^\dagger(\alpha)| d\alpha \ll \int_{\mathfrak{m}} f_i^{s-4} \tilde{f}_i h_k^3 d\alpha. \quad (12.1)$$

Then it follows from Hölder's inequality and the trivial estimate $h_k \leq P$ that

$$\int_{\mathfrak{m}} |\mathcal{F}^\dagger(\alpha)| d\alpha \ll P^{2/3} \left(\sup_{\alpha \in \mathfrak{m}} f_i \right)^{s-32/3} \left(\oint f_i^8 h_k^2 d\alpha \right)^{5/6} \left(\oint \tilde{f}_i^6 h_k^4 d\alpha \right)^{1/6}.$$

Lemma 5.2 shows that $f_i = O(P^{3/4+\varepsilon})$ for $\alpha \in \mathfrak{m}$, and hence by applying Lemmata 5.1(ii) and 11.1(i), one obtains

$$\int_{\mathfrak{m}} |\mathcal{F}^\dagger(\alpha)| d\alpha \ll P^{2/3+\varepsilon} (P^{3/4})^{s-32/3} (P^{5+\varepsilon})^{5/6} (P^{21/4-2\delta})^{1/6} \ll P^{s-5-\delta}.$$

This completes the proof of the lemma. \square

We next prune down to the narrow set of arcs Ω .

Lemma 12.2. *One has $\int_{\mathfrak{m} \setminus \Omega} |\mathcal{F}^\dagger(\alpha)| d\alpha \ll P^{s-5} (\log \log P)^{-1}$.*

Proof. The procedure leading to (12.1) shows that for some i and k , one has

$$\int_{\mathfrak{m} \setminus \Omega} |\mathcal{F}^\dagger(\alpha)| d\alpha \ll \int_{\mathfrak{m} \setminus \Omega} f_i^{s-4} \tilde{f}_i h_k^3 d\alpha.$$

On recalling (7.1), we see from Lemma 7.3 that whenever $\alpha \in \mathfrak{M}$, one has $f_i(\alpha) = f_i^*(\alpha) + O(P^{1/2+\varepsilon})$. Thus we deduce that

$$\int_{\mathfrak{M} \setminus \Omega} |\mathcal{F}^\dagger(\alpha)| \, d\alpha \ll I_1 + P^{(s-4)/2+\varepsilon} I_2,$$

where

$$I_1 = \int_{\mathfrak{M} \setminus \Omega} |f_i^*|^{s-4} \tilde{f}_l h_k^3 \, d\alpha \quad \text{and} \quad I_2 = \oint \tilde{f}_l h_k^3 \, d\alpha.$$

An application of Hölder's inequality yields

$$\begin{aligned} I_1 &\leq \left(\sup_{\alpha \in \mathfrak{M} \setminus \Omega} |f_i^*(\alpha)| \right)^{s-4799/544} \left(\oint |f_i^*|^8 \, d\alpha \right)^{5/17} \\ &\quad \times \left(\oint |f_i^* h_k|^{17/4} \, d\alpha \right)^{79/136} \left(\oint \tilde{f}_l^8 h_k^{17/4} \, d\alpha \right)^{1/8}, \end{aligned}$$

and thus we deduce from Lemmata 8.3(i), (ii), (iv) and 11.1(ii) that

$$I_1 \ll (PW^{-1/4})^{s-4799/544} (P^3)^{5/17} (P^{7/2})^{79/136} (P^{29/4})^{1/8} \ll P^{s-5} W^{-1/2}.$$

A second application of Hölder's inequality leads via Hua's lemma to the bound

$$I_2 \ll \left(\int_0^1 h_k^4 \, d\alpha \right)^{3/4} \left(\oint \tilde{f}_l^4 \, d\alpha \right)^{1/4} \ll P^{2+\varepsilon}.$$

Then we may conclude that

$$\int_{\mathfrak{M} \setminus \Omega} |\mathcal{F}^\dagger(\alpha)| \, d\alpha \ll P^{s-5} (\log \log P)^{-1} + P^{s/2+\varepsilon} \ll P^{s-5} (\log \log P)^{-1}.$$

This completes the proof of the lemma. \square

Since $\mathfrak{q} = \mathfrak{m} \cup (\mathfrak{M} \setminus \Omega)$, the estimates supplied by Lemmata 12.1 and 12.2 combine to give the following conclusion.

Lemma 12.3. *One has $\int_{\mathfrak{q}} |\mathcal{F}^\dagger(\alpha)| \, d\alpha \ll P^{s-5} (\log \log P)^{-1}$.*

13. THE MAJOR ARC CONTRIBUTION FOR SYSTEMS OF TYPE C

The analysis of the major arcs may be completed for systems of type C by adapting the corresponding discussion of §10. First, by applying Lemmata 7.3 and 10.1, we see that when $\alpha \in \mathfrak{Q}(q, \mathbf{r}) \subseteq \mathfrak{Q}$, then

$$\mathcal{F}^\dagger(\alpha) - c_\eta T(q, \mathbf{r}) V(\alpha - \mathbf{r}/q) \ll P^s W^{-10},$$

and hence one deduces that

$$\int_{\mathfrak{Q}} \mathcal{F}^\dagger(\alpha) \, d\alpha = c_\eta \mathfrak{S} C P^{s-5} + O(P^{s-5} W^{-1/6}).$$

But $[0, 1]^2$ is the union of \mathfrak{Q} and \mathfrak{q} , and so we infer from Lemma 12.3 that

$$R^\dagger(P) = \iint_{[0,1]^2} \mathcal{F}^\dagger(\alpha) \, d\alpha = c_\eta \mathfrak{S} C P^{s-5} + O(P^{s-5} (\log \log P)^{-1}).$$

Since $\mathcal{C}\mathfrak{S} > 0$, we deduce that $R^\dagger(P) \gg P^{s-5}$. The conclusion of Theorem 1.2, and hence also Theorem 1.1, therefore follows for systems of the final type C.

14. APPENDIX: A TRANSFERENCE PRINCIPLE

We take the opportunity here of establishing a standard transference principle. The version of this principle described in Exercise 2 of [15, §2.8] restricts attention to the situation relevant to Weyl's inequality, and contains an additional condition on the relevant Diophantine approximation, and so it seems worthwhile to provide in the literature a complete account for future reference.

Lemma 14.1. *Let θ, X, Y, Z be positive real numbers. Suppose that $\Psi : \mathbb{R} \rightarrow \mathbb{C}$ satisfies the property that whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha - a/q| \leq q^{-2}$, then*

$$\Psi(\alpha) \ll X(q^{-1} + Y^{-1} + qZ^{-1})^\theta. \quad (14.1)$$

Then, whenever $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ satisfy $(b, r) = 1$, one has

$$\Psi(\alpha) \ll X(\lambda^{-1} + Y^{-1} + \lambda Z^{-1})^\theta, \quad (14.2)$$

where $\lambda = r + Z|r\alpha - b|$.

Proof. Suppose that $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ satisfy $(b, r) = 1$. By Dirichlet's theorem on Diophantine approximation, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $1 \leq q \leq 2r$ and $|q\alpha - a| \leq (2r)^{-1}$. Suppose in the first instance that $a/q \neq b/r$. Then

$$(qr)^{-1} \leq \left| \frac{a}{q} - \frac{b}{r} \right| \leq \left| \alpha - \frac{b}{r} \right| + \left| \alpha - \frac{a}{q} \right| \leq \left| \alpha - \frac{b}{r} \right| + (2qr)^{-1}.$$

Hence $q^{-1} \leq 2|r\alpha - b|$, and so we deduce from (14.1) that

$$\Psi(\alpha) \ll X(|r\alpha - b| + Y^{-1} + rZ^{-1})^\theta \ll X(Y^{-1} + \lambda Z^{-1})^\theta,$$

confirming the estimate (14.2).

If, on the other hand, one has $a/q = b/r$, then since $(a, q) = (b, r) = 1$, one has $q = r$ and $a = b$, and hence $|r\alpha - b| \leq (2r)^{-1}$. If $\alpha = b/r$, then $\lambda = r$, and the desired conclusion (14.2) is immediate from (14.1). When $\alpha \neq b/r$, meanwhile, one has $0 < |\alpha - b/r| \leq r^{-2}$. In this situation, by Dirichlet's theorem on Diophantine approximation, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $1 \leq q \leq 2|r\alpha - b|^{-1}$ and $|q\alpha - a| \leq \frac{1}{2}|r\alpha - b|$. If one were to have $a/q = b/r$, then since $(a, q) = (b, r) = 1$, one finds that $q = r$ and $a = b$, and hence $0 < |r\alpha - b| \leq \frac{1}{2}|r\alpha - b|$, giving a contradiction. Thus $a/q \neq b/r$, and one has

$$\begin{aligned} (qr)^{-1} &\leq \left| \frac{a}{q} - \frac{b}{r} \right| \leq \left| \alpha - \frac{b}{r} \right| + \left| \alpha - \frac{a}{q} \right| \\ &\leq \left| \alpha - \frac{b}{r} \right| + (2q)^{-1}|r\alpha - b| \leq \left| \alpha - \frac{b}{r} \right| + (2qr)^{-1}. \end{aligned}$$

We therefore see that $q^{-1} \leq 2|r\alpha - b|$, and we deduce from (14.1) that

$$\Psi(\alpha) \ll X((Z|r\alpha - b|)^{-1} + Y^{-1} + |r\alpha - b|)^\theta.$$

Alternatively, since $|\alpha - b/r| \leq r^{-2}$, one may apply (14.1) to give

$$\Psi(\alpha) \ll X(r^{-1} + Y^{-1} + rZ^{-1})^\theta.$$

Thus, in any case, one obtains the bound $\Psi(\alpha) \ll X(\lambda^{-1} + Y^{-1} + \lambda Z^{-1})^\theta$, where $\lambda = r + Z|r\alpha - b|$. This completes the proof of the lemma. \square

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SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, UNIVERSITY WALK, CLIFTON,
BRISTOL BS8 1TW, UNITED KINGDOM

E-mail address: matdw@bristol.ac.uk