A Online Appendix

A.1 Multiself consistency and Proof of Theorem 2.1

We first extend the modified multi-self consistency to the setup here. Recall the expression $f(q,\theta)$ given by (6) and beliefs $p(q)$ and $p(q,x)$ given by (9).

Definition A.1. A SFSA M satisfies *modified multi-self consistency* under $\mathbf{P}_0$ if

1. for each memory state $q \in Q$ with $\sum_\theta f(q,\theta) > 0$, each signal x, and any q' such that $\sigma(q,x)(q') > 0$,

$$p(q,x)V_{q'}(H)+[1-p(q,x)]V_{q'}(L) \geq p(q,x)V_{q''}(H)+[1-p(q,x)]V_{q''}(L) \text{ for all } q'' \in Q; \tag{31}$$

2. for each memory state $q \in Q$ with $\sum_\theta f(q,\theta) > 0$ and $a = d(q)$,

$$p(q)u(a,H) + [1-p(q)]u(a,L) \geq p(q)u(a',H) + [1-p(q)]u(a',L) \text{ for all } a' \in A. \tag{32}$$

The following result is crucial for the proof of Theorem 2.1.

Proposition A.1. *Suppose that M is an optimal SFSA under prior $\mathbf{P}_0$ among those with $|Q| \leq K$.*

1. *(Modified Multi-self Consistency) It satisfies modified multi-self consistency under prior $\mathbf{P}_0$.*

2. *(Revelation Principle) For any $q, q' \in Q$,*

$$p(q)V_q(H) + [1-p(q)]V_q(L) \geq p(q)V_{q'}(H) + [1-p(q)]V_{q'}(L). \tag{33}$$

Proof. For any pairs of states of nature and memory states, (θ,q) and (θ',q'), define the set

$$W_{(\theta,q),(\theta',q')} = \bigcup_{n=1}^{\infty} W^n_{(\theta,q),(\theta',q')},$$

where for each $n = 1, 2, ...,$

$$W^n_{(\theta,q),(\theta',q')} = \{[(\theta,q),x_1;(\theta_1,q_1),x_2;....;(\theta_{n-1},q_{n-1}),x_n;(\theta',q')] : x_i \in X, q_i \in Q, \theta_i \in \Theta\},$$

that is, the set of possible state transitions from q to q'. Given a state of nature θ and $\mathbf{w} \in W^n_{(\theta,q),(\theta',q')}$, define

$$\mathbb{P}(\mathbf{w}) = \eta(1-\eta)^{n-1} \times \prod_{i=1}^{n} \nu^{\theta_{i-1}}_{\theta_i} \mu^{\theta_i}_{x_i} \sigma(q_{i-1}, x_i)(q_i),$$

where $(\theta_0, q_0) = (\theta, q)$ and $(\theta_n, q_n) = (\theta', q')$. The expected payoff from the SFSA is then

$$V = \sum_{\theta,\theta',q} \mathbf{P}_0(\theta) \sum_{\mathbf{w} \in W_{(\theta,q^o),(\theta',q)}} \mathbb{P}(\mathbf{w}) u[d(q), \theta']. \tag{34}$$

We now prove (31) and (32).

First, consider (32). Suppose, by contradiction, that for some memory state $\hat{q}$ with $f(\hat{q}, \theta) > 0$ such that (32) does not hold, and hence there are actions $a = d(q)$ and $a' \in A$ with the inequality in (32) reversed with a strict inequality. By (6), $f(\hat{q}, \theta) = \sum_{\theta'} \sum_{\mathbf{w} \in W_{(\theta',q^o),(\theta,\hat{q})}} \mathbf{P}_0(\theta')\mathbb{P}(\mathbf{w})$, this then implies that

$$\sum_{\theta',\theta} \mathbf{P}_0(\theta') \sum_{\mathbf{w} \in W_{(\theta',q^o),(\theta,\hat{q})}} \mathbb{P}(\mathbf{w}) u(a, \theta) < \sum_{\theta',\theta} \mathbf{P}_0(\theta) \sum_{\mathbf{w} \in W_{(\theta',q^o),(\theta,\hat{q})}} \mathbb{P}(\mathbf{w}) u(a', \theta). \tag{35}$$

Now, consider the alternative SFSA M', which differs from M only in that $d'(\hat{q}) = a'$. From (34) and (35) it follows that M' gives a strictly higher expected payoff than M, a contradiction to the optimality of M.

Now consider (31). Suppose, by contradiction, that $\sigma(q, x)(q') > 0$ and that for some $q'' \neq q'$,

$$p(q, x)V_{q'}(H) + [1 - p(q, x)]V_{q'}(L) < p(q, x)V_{q''}(H) + [1 - p(q, x)]V_{q''}(L). \tag{36}$$

We denote $p' = \sigma(q, x)(q')$ and $p'' = \sigma(q, x)(q'')$. Now, fix all other transition probabilities other than p' and p'', each term $\mathbb{P}(\mathbf{w})$ in V given by (34) is a polynomial of (p', p'') and, since $\eta \in (0, 1)$, V is differentiable w.r.t. (p', p''). Since M is optimal and $p' = \tau(q, x)(q') > 0$, the FOCs require that $\frac{\partial}{\partial p'} V \geq \frac{\partial}{\partial p''} V$. However, we show below that (36) implies that

$$\frac{\partial}{\partial p''} V > \frac{\partial}{\partial p'} V, \tag{37}$$

a contradiction to the optimality of M.

To prove (37), it is straightforward to verify that

$$\frac{\partial}{\partial p'}V = \sum_{\theta,\theta',\hat{q}} \mathbf{P}_0(\theta) \sum_{\mathbf{w}\in W_{(\theta,q^o),(\theta',\hat{q})}(q,x;q')} \varphi_{(q,x;q')}(\mathbf{w})\frac{\mathbb{P}(\mathbf{w})}{p'}u[d(\hat{q}),\theta'], \tag{38}$$

where

$$W_{(\theta,q^o),(\theta',\hat{q})}(q,x;q') = \{\mathbf{w}\in W_{(\theta,q^o),(\theta',\hat{q})} : (q,x,q') \text{ occurs in } \mathbf{w}\}$$

and $\varphi_{(q,x;q')}(\mathbf{w})$ is the number of repetitions of the transition $(q,x;q')$ within $\mathbf{w}$.

Now, we show that $\frac{\partial}{\partial p'}V$ is proportional to $p(q,x)V_{q'}(H) + [1-p(q,x)]V_{q'}(L)$:

$$\begin{aligned}
&\left[\sum_{\theta,\theta'} f(q,\theta)\nu_{\theta'}^{\theta}\mu_x^{\theta'}\right][p(q,x)V_{q'}(H) + [1-p(q,x)]V_{q'}(L)] \\
=\ & \sum_{\theta,\theta'} f(q,\theta)\nu_{\theta'}^{\theta}\mu_x^{\theta'}V_{q'}(\theta') \\
=\ & \sum_{\theta_0,\theta,\theta',\theta''} \mathbf{P}_0(\theta_0)\sum_{\hat{q}\in Q}\left\{\left[\sum_{\mathbf{w}_q\in W_{(\theta_0,q^o),(\theta,q)}} \mathbb{P}(\mathbf{w}_q)\right]\nu_{\theta'}^{\theta}\mu_x^{\theta'}\left[\sum_{\mathbf{w}_{q'}\in W_{(\theta',q'),(\theta'',\hat{q})}} \mathbb{P}(\mathbf{w}_{q'})\right]\right\}u[d(\hat{q}),\theta''] \\
=\ & \sum_{\theta_0,\theta,\theta',\theta''} \mathbf{P}_0(\theta_0)\sum_{\hat{q}\in Q}\left\{\sum_{\mathbf{w}_q\in W_{(\theta_0,q^o),(\theta,q)},\mathbf{w}_{q'}\in W_{(\theta',q'),(\theta'',\hat{q})}} \frac{\mathbb{P}[(\mathbf{w}_q,x,\mathbf{w}'_q)]}{\sigma(q,x)(q')}\right\}u[d(\hat{q}),\theta''] \\
=\ & \sum_{\theta_0,\theta''} \mathbf{P}_0(\theta_0)\sum_{\hat{q}\in Q}\left\{\sum_{\mathbf{w}\in W_{(\theta_0,q^o),(\theta'',\hat{q})}} \varphi_{(q,x;q')}(\mathbf{w})\frac{\mathbb{P}(\mathbf{w})}{p'}\right\}u[d(\hat{q}),\theta''] = \frac{\partial}{\partial p'}V,
\end{aligned}$$

where the last equality follows from (38) and the second last equality follows from $p' = \sigma(q,x)(q')$ and the fact that for any $\mathbf{w}_q \in W_{(\theta,q^o),(\theta,q)}$ and any $\mathbf{w}_{q'} \in W_{(\theta',q'),(\theta'',\hat{q})}$, $(\mathbf{w}_q, x; \mathbf{w}_{q'}) \in W_{(\theta,q^o),(\theta'',\hat{q})}(q,x;q')$ and that each $\mathbf{w} \in W_{(\theta,q^o),(\theta'',\hat{q})}(q,x;q')$ is counted $\varphi_{(q,x;q')}(\mathbf{w})$ times in that list. We have analogous expression for $\frac{\partial}{\partial p''}V$, and hence (36) implies that (37).

Now we prove (33). By modified multi-self consistency, for any $x \in X$ and any q_1, q_2 with $\sigma(q,x)(q_1) > 0$ and $\sigma(q,x)(q_2) > 0$ and any $q_3 \in Q$,

$$\begin{aligned}
[p(q,x)V_{q_1}(H) + [1-p(q,x)]V_{q_1}(L)] &= [p(q,x)V_{q_2}(H) + [1-p(q,x)]V_{q_2}(L)] \\
&\geq [p(q,x)V_{q_3}(H) + [1-p(q,)]V_{q_3}(L)],
\end{aligned}$$

By (9), this implies that for all $x \in X$,

$$\sum_{\theta,\theta'} f(q,\theta)\nu_{\theta'}^{\theta}\mu_x^{\theta'}V_{q_1}(\theta') = \sum_{\theta,\theta'} f(q,\theta)\nu_{\theta'}^{\theta}\mu_x^{\theta'}V_{q_2}(\theta') \geq \sum_{\theta,\theta'} f(q,\theta)\nu_{\theta'}^{\theta}\mu_x^{\theta'}V_{q_3}(\theta'). \tag{39}$$

Thus, (here we assume that the decision rule is deterministic, with no loss of generality because of (32))

$$
\begin{aligned}
& p(q)V_q(H) + [1-p(q)]V_q(L) \\
= \; & p(q)\left\{\eta u[d(q),H] + (1-\eta)\left[\sum_{x\in X, q''\in Q, \theta'} \nu_{\theta'}^{H}\mu_x^{\theta'}\sigma(q,x)(q'')V_{q''}(\theta')\right]\right\} \\
+ \; & [1-p(q)]\left\{\eta u[d(q),L] + (1-\eta)\left[\sum_{x\in X, q''\in Q, \theta'} \nu_{\theta'}^{L}\mu_x^{\theta'}\sigma(q,x)(q'')V_{q''}(\theta')\right]\right\} \\
= \; & \eta\{p(q)u[d(q),H] + [1-p(q)]u[d(q),L]\} \\
+ \; & (1-\eta)\sum_{x\in X}\left\{\sum_{q''\in Q}\frac{\sum_{\theta,\theta'} f(q,\theta)\nu_{\theta'}^{\theta}\mu_x^{\theta'}V_{q''}(\theta')}{f(q,H)+f(L,q)}\sigma(q,x)(q'')\right\} \\
\geq \; & \eta\{p(q)u[d(q'),H] + [1-p(q)]u[d(q'),L]\} \\
+ \; & (1-\eta)\sum_{x\in X}\left\{\sum_{q''\in Q}\frac{\sum_{\theta,\theta'} f(q,\theta)\nu_{\theta'}^{\theta}\mu_x^{\theta'}V_{q''}(\theta')}{f(q,H)+f(L,q)}\sigma(q',x)(q'')\right\} \\
= \; & p(q)V_{q'}(H) + [1-p(q)]V_{q'}(L),
\end{aligned}
$$

where the first equality follows from the recursive equation for $V_q(\theta)$ for each $\theta = H, L$, the second follows from (9), the inequality follows term by term, first the terms starting with η follow from (32), the terms starting with $(1-\eta)$ follows from (39), again term by term for each x: any term with q'' with $\sigma(q,x)(q'') > 0$ has the same value in the inequality above, and that value is no less than that for the corresponding term with $\sigma(q',x)(q'') > 0$, and the last equality follows from the recursive equation for $V_{q'}(\theta)$. $\square$

Now we are ready to prove Theorem 2.1.

(1) Let $i < j$ be given. By (33),

$$p(q_j)\Delta V_{j,i}^H + [1-p(q_j)]\Delta V_{j,i}^L \geq 0, \text{ and } p(q_i)\Delta V_{i,j}^H + [1-p(q_i)]\Delta V_{i,j}^L \geq 0. \tag{40}$$

Since there are no equivalent states, either $\Delta V_{i,j}^H > 0$ or $\Delta V_{i,j}^H < 0$. By our convention it must be $\Delta V_{j,i}^H > 0$. By the second inequality in (40), $\Delta V_{i,j}^L \geq 0$. Now, if this last inequality is an equality, then we can replace all the transition to q_i to transition to q_j and obtain a higher ex ante payoff, which is a contradiction to the optimality of the SFSA. Now, let $i < j < k$. Again, by (33), we have

$$p(q_j)\Delta V_{j,i}^H + [1-p(q_j)]\Delta V_{j,i}^L \geq 0, \text{ and } p(q_j)\Delta V_{j,k}^H + [1-p(q_j)]\Delta V_{j,k}^L \geq 0, \tag{41}$$

and hence

$$\frac{\Delta V_{i,j}^L}{\Delta V_{j,i}^H} \leq \frac{p(q_j)}{1-p(q_j)} \leq \frac{\Delta V_{j,k}^L}{\Delta V_{k,j}^H}.$$

(2) For part (a), (33) implies that

$$p(q_i)V_{q_i}(H) + [1-p(q_i)]V_{q_i}(L) \geq p(q_i)V_{q_{i+1}}(H) + [1-p(q_i)]V_{q_{i+1}}(L)$$

and hence, by rearranging terms, we have $\xi[p(q_i)] \leq \bar{\xi}_i$. A similar argument holds for $\xi[p(q_i)] \geq \bar{\xi}_{i-1}$.

For (b), let $q \in Q$ be given. By (31), $\sigma(q,x)(q_i) > 0$ only if

$$p(q,x)V_{q_i}(H) + [1-p(q,x)]V_{q_i}(L) \geq p(q,x)V_{q_j}(H) + [1-p(q,x)]V_{q_j}(L)$$

for both $j = i-1$ and $j = i+1$. This then implies (10). Conversely, it is straightforward to verify that if (10) holds, then

$$p(q,x)V_{q_i}(H) + [1-p(q,x)]V_{q_i}(L) \geq p(q,x)V_{q_j}(H) + [1-p(q,x)]V_{q_j}(L)$$

for any $j = 0, ..., K-1$, where $q_0 = q_L$ and $q_{K-1} = q_H$. Note that we need the fact that $\bar{\xi}_i$ increases with i for this, as proved in part (1). Moreover, if $\xi[p(q,x)] \in (\bar{\xi}_{i-1}, \bar{\xi}_i)$, then the above inequality is strict for any $j \neq i$ and hence $\sigma(q,x)(q_i) = 1$.

Finally, (c) follows from (32) and a similar argument.

A.2 Regime change with $\xi(p_0) < \xi^*$

Here we consider the regime change model as in Table 3 with $\xi(p_0) < \xi^*$. As mentioned in the main text, when ν is sufficiently small according to (12), one h-signal can bring the posterior across ξ^* and hence the availability heuristic M_2^a can implement the unconstrained optimum, but with $q^o = q_L$. In contrast, at the other extreme where $\nu = 1$, the DFSA that implements the unconstrained optimum is given by M_{N+2}^b as depicted in Figure 6, where N is given by (13) and where $q^o = q_1$ and $d(q_L) = d(q_i) = a^L$ for all $i = 1, ..., N$ and $d(q_H) = a^H$ (see Hu (2022) for detailed arguments). Intuitively, in the fixed-world environment, q_L represents the memory state in which the DM has received an ℓ-signal and hence is fully convinced of the state of the world being L and hence it is a self-absorbing memory state. In contrast, at q_i for $i \geq 1$, the DM has not

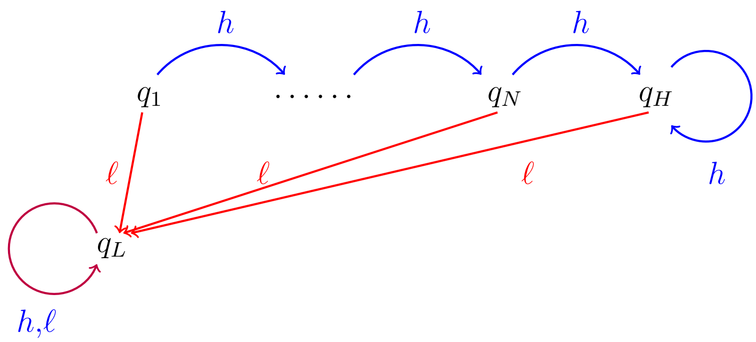

Figure 6: The DFSA, M^b_{N+2}, that implements unconstrained optimum when $p_0 < p^*$

received any ℓ-signal but have received $i-1$ h-signals, and hence the posterior on H has gone up but has not crossed ξ^* yet, which happens only at q_H.

Now we turn to the case where ν is below one and hence even after an ℓ-signal the state of the world can still change from L to H. In this case, to implement the unconstrained optimum, it requires an unbounded number of memory states. The reason is that, after one ℓ-signal it would require a large number of h-signals to bring the posterior to reach ξ^* again for ν close to one, a number that converges to infinity as ν approaches one. However, for any given constraint K, the following theorem shows that it is optimal to ignore the possibility of regime change when ν is sufficiently high.

Theorem A.1. *Suppose that $\Delta^H = 1$ and $\Delta^L = \nu \in [0,1]$ and that $\mu^H_h = 1 > \mu = \mu^L_\ell$, and that $\xi(p_0) < \xi^*$. If $K \geq N+2$ with N given by (13), then there exists $\tilde{\nu} < 1$ such that for all $\nu \geq \tilde{\nu}$, the optimal SFSA is M^b_{N+2} with $q^o = q_1$.*

Proof. The proof follows the same steps as in Theorem 4.1. As there, for any given $K \geq N+2$, We know that M^b_{N+2} is the unique optimal SFSA when $\nu = 1$. Thus, for any such K, $V(M^b_{N+2}) > V(M')$ for other SFSA $M \in \mathcal{M}_K$ whose randomization probabilities are ϵ away from that in M^b_{N+2} and this inequality holds for a range of ν's below 1. Now we demonstrate local optimality of M^b_{N+2} for a range of ν's below 1 by appealing to Theorem 2.1. First we compute the value functions under M^b_{N+2}:

$$\begin{aligned}
V_{q_H}(H) &= u^H,\ V_{q_H}(L) = 0,\\
V_{q_L}(H) &= 0,\ V_{q_i}(H) = (1-\eta)^{N+1-i}u^H,\ i = 1, ..., N\\
V_{q_i}(L) &= -\frac{\eta\{[1-\nu(1-\mu)]u^L + (1-\nu)u^H\}}{[1-\nu(1-\mu)][1-(1-\eta)\nu(1-\mu)]}[(1-\eta)\nu(1-\mu)]^{N+1-i}\\
&+ \frac{\eta}{1-\nu(1-\eta)}u^L + \frac{(1-\nu)(1-\eta)^{N-i}}{1-\nu(1-\mu)}u^H,\ i =, 1..., N,\\
V_{q_L}(L) &= \frac{\eta}{1-(1-\eta)\nu}u^L.
\end{aligned}$$

Thus,

$$\begin{aligned}
\bar{\xi}_1 &= \frac{V_{q_L}(L) - V_{q_1}(L)}{V_{q_1}(H) - V_{q_L}(H)}\\
&= \left(\xi^* + \frac{(1-\nu)}{1-\nu(1-\mu)}\right)\frac{\eta[\nu(1-\mu)]^N}{1-(1-\eta)\nu(1-\mu)} - \frac{(1-\nu)}{1-\nu(1-\mu)},\\
\bar{\xi}_i &= \frac{V_{q_{i-1}}(L) - V_{q_i}(L)}{V_{q_i}(H) - V_{q_{i-1}}(H)}\\
&= \left(\xi^* + \frac{(1-\nu)}{1-\nu(1-\mu)}\right)[\nu(1-\mu)]^{N-i+1} - \frac{1-\nu}{1-\nu(1-\mu)},\ i = 2, ..., N,\\
\bar{\xi}_N &= \frac{V_{q_N}(L) - V_{q_H}(L)}{V_{q_H}(H) - V_{q_N}(H)} = \xi^*.
\end{aligned}$$

Now, the corresponding beliefs are given by

$$\begin{aligned}
\xi(q_L) &= \frac{(1-\eta)(1-\nu)}{\eta},\\
\xi(q_1) &= \xi(p_0),\\
\xi(q_i) &= \frac{1-\nu+\nu\mu p_0}{(1-p_0)[1-\nu(1-\mu)][\nu(1-\mu)]^{i-2}} - \frac{1-\nu}{1-\nu(1-\mu)},\ i = 2, ..., N,\\
\xi(q_H) &= \frac{[1-\nu+\nu\mu p_0][1-(1-\eta)\nu(1-\mu)]}{\eta(1-p_0)[1-\nu(1-\mu)][\nu(1-\mu)]^N} - \frac{1-\nu}{1-\nu(1-\mu)}.
\end{aligned}$$

Appealing to Theorem 2.1, local optimality requires

$$\xi(q_H) \geq \xi^*,\ \xi(q_{i-1}, h) \in [\bar{\xi}_i, \bar{\xi}_{i+1}] \text{ for } i = 2, ..., N, \text{ and } \xi(q_L, h) < \bar{\xi}_1. \tag{42}$$

Now, consider a SFSA M that is a replica of M^b_{N+2} with $K' \leq K$ memory states. We show that M^b_{N+2} is still locally optimal against small deviations from such replica, by showing the corresponding condition for (42). Note that the continuation value does not change with the replica memory states and hence the thresholds $\bar{\xi}_i$'s remain the same. Regarding beliefs, since for the replica state of each q_i with $i \geq 1$, it can be passed through for at most once, a simple induction argument shows that its belief coincides with the corresponding memory state in M^b_{N+2}. Moreover,

$$\xi(q_{i-1}, h) = \frac{1-\nu+\nu\mu p_0}{(1-p_0)[1-\nu(1-\mu)][\nu(1-\mu)]^{i-1}} - \frac{1-\nu}{1-\nu(1-\mu)} \in [\bar{\xi}_i, \bar{\xi}_{i+1}),$$

which is equivalent to

$$\begin{aligned} &\left(\xi^* + \frac{(1-\nu)}{1-\nu(1-\mu)}\right)[\nu(1-\mu)]^N \\ &\leq \xi(p_0) + \frac{(1-\nu)}{1-\nu(1-\mu)} \leq \left(\xi^* + \frac{(1-\nu)}{1-\nu(1-\mu)}\right)[\nu(1-\mu)]^{N-1}. \end{aligned} \tag{43}$$

When $\nu = 1$, the first inequality is weak and the second is strict, which follow from the definition of N given by (13). For $\nu < 1$, the first becomes strict, and the second is preserved for ν not too small.

Now we consider the replica states of q_H and q_L. Let equivalent states of q_H be denoted by $q_H^1, ..., q_H^I$, and q_L be denoted by $q_L^1, ..., q_L^J$, with

$$\begin{aligned} &\sigma(q_H^i, h)(q_H^j) = \alpha_{hij},\ \sigma(q_H^i, \ell)(q_L^j) = \alpha_{\ell ij}, \\ &\sigma(q_{L,1}^i, h)(q_L^j) = \beta_{hij},\ \sigma(q_L^i, \ell)(q_L^j) = \beta_{\ell ij}, \\ &\sigma(q_N, h)(q_H^j) = \gamma_{hj},\ \sigma(q_{n-1}, \ell)(q_L^j) = \gamma_{\ell nj}. \end{aligned} \tag{44}$$

Then, we have the following recursive equations:

$$\begin{aligned} f(q_H^i, H) =&(1-\eta)\left\{[f(q_N, H) + f(q_N, L)(1-\nu)]\gamma_{hi}\right\} \\ &+(1-\eta)\left\{\sum_{j=1}^{I}[f(q_H^j, L)(1-\nu) + f(q_H^j, H)]\alpha_{hji}\right\}, \\ f(q_H^i, L) =&(1-\eta)\left\{\sum_{j=1}^{I} f(q_H^j, L)\nu(1-\mu)\alpha_{hji} + f(q_N, L)\nu(1-\mu)\gamma_{hi}\right\}, \\ f(q_L^i, H) =&(1-\eta)\left\{\sum_{j=1}^{J}[f(q_L^j, H) + f(q_L^j, L)(1-\nu)]\beta_{hji}\right\}, \\ f(q_L^i, L) =&(1-\eta)\left\{\sum_{j=1}^{J} f(q_L^j, L)\nu[(1-\mu)\beta_{hji} + \mu\beta_{\ell ji}] + \sum_{k=1}^{I} f(q_H^k, L)\nu\mu\alpha_{\ell ki}\right\} \\ &+(1-\eta)\left\{\sum_{n=2}^{N+1} f(q_{n-1}, L)\nu\mu\gamma_{\ell ni}\right\}, \end{aligned} \tag{45}$$

Now we show that $\xi(q_H^i) \geq \xi^*$ for all i. Using the same methodology as in the proof of Theorem 3.1 (2), i.e., we take the equations in (45) as simultaneous equations and use the contractions mapping theorem, it suffices to show that

$$\frac{f(q_N, H) + f(q_N, L)(1-\nu)]}{f(q_N, L)\nu(1-\mu)} \geq \xi^*,$$

which follows from the earlier result that $\xi(q_N, h) \geq \xi^*$ for ν close to one.

Finally, the result that $\xi(q_L^i, h) < \bar{\xi}_1$ follows a similar argument to that in the proof of Theorem 4.1 (1) and is omitted. □

Theorem A.1 then extends Theorem 4.1 (1) to the case where $\xi(p_0) < \xi^*$. As Theorem 4.1 (1), this is a less-is-more result, as the optimal SFSA is M_{N+2}^b for any given $K \geq N+2$ for a range of ν's. Moreover, in that case, the DM behaves as if she ignores the possibility of regime change under the constrained optimal rule and is stuck to action a^L after receiving an ℓ-signal, while an unconstrained DM will continue to update her belief and will eventually be fully convinced of state of the world H. However, different from Theorem 4.1 (1), the result in Theorem A.1 requires $K \geq N+2$, with $N \geq 1$ given by (13).

Now we turn to the case where $K < N+2$. In the fixed-worlds environment, Hu (2022) has shown that randomization is optimal, with the optimal SFSA taking the same form with $Q = \{q_L, q_1, ..., q_{K-2}, q_H\}$, and that optimal randomization occurs at every q_i for $i = 1, ..., K-2$ with $\sigma(q_i, h)(q_i) = \alpha_i = 1 - \sigma(q_i, h)(q_{i+1}) \in (0,1)$ with $q_{K+1} = q_H$, and we denote this SFSA by $M_K^b(\alpha_1, ..., \alpha_{K-2})$; see Figure 7 for a graphical representation of $M_K^b(\alpha_1, ..., \alpha_3)$ with $K = 5$. In fact, Hu (2022) shows that it is optimal to set $\alpha_1 = \alpha_2 = ... = \alpha_{K-2}$. This optimal SFSA features randomization in all intermediate memory states, a feature in contrast to the characterization in Wilson (2014) where randomization occurs at the extreme memory states.

The following result extends this characterization result of randomization to changing worlds.

Theorem A.2. *Suppose that $\Delta^H = 1$ and $\Delta^L = \nu \in [0,1]$ and that $\mu_h^H = 1 > \mu = \mu_\ell^L$, and that $\xi(p_0) < \xi^*$. If $3 \leq K < N+2$ with N given by (13), there exists $\tilde{\nu} < 1$ such that for all $\nu \geq \tilde{\nu}$, the optimal SFSA takes the form $M_K^b(\alpha)$.*

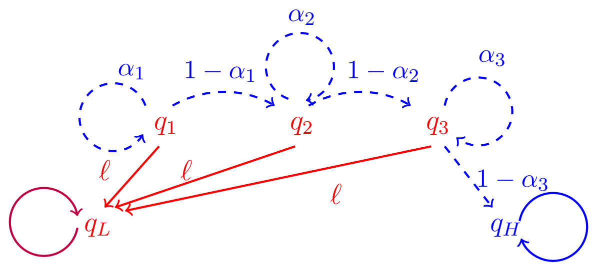

Figure 7: $M_5^b(\alpha)$

Proof. By Hu (2022), when $\nu = 1$ the optimal SFSA takes the form $M_K^b(\alpha_1, ..., \alpha_{K-2})$ with $\alpha_1 = \alpha_2 = ...\alpha_{K-2} = \alpha > 0$. Following the same arguments as in the proof of Theorem 4.1, for a range of ν's below $\nu = 1$ we only need to consider local optimality. Now we show that for any ν, the locally optimal SFSA of the form $M_K^b(\alpha_1, ..., \alpha_{K-2})$ has the form $\alpha_1 = \alpha_2 = ...\alpha_{K-2} = \alpha > 0$.

First we compute the value functions:

$$
\begin{aligned}
V_{q_H}(H) &= u^H,\ V_{q_L}(L) = \frac{\eta}{1-(1-\eta)\nu}u^L,\ V_{q_L}(H) = 0, \\
V_{q_H}(L) &= (1-\eta)\left\{\nu[\mu V_{q_L}(L) + (1-\mu)V_{q_H}(L)] + (1-\nu)V_{q_H}(H)\right\}, \\
&= \frac{1}{1-\nu(1-\eta)(1-\mu)}\left\{\frac{\mu(1-\eta)\nu\eta}{1-(1-\eta)\nu}u^L + (1-\eta)(1-\nu)u^H\right\}, \\
V_{q_i}(H) &= (1-\eta)\{\alpha_i V_{q_i}(H) + (1-\alpha_i)V_{q_{i+1}}(H)\}, i = 1, ..., K-2, \\
V_{q_i}(H) &= \left[\prod_{j=i}^{K-2}\frac{1-\alpha_j}{1-(1-\eta)\alpha_j}\right](1-\eta)^{K-1-i}u^H, \\
V_{q_i}(L) &= \eta u^L + (1-\eta)\{(1-\nu)[\alpha_i V_{q_i}(H) + (1-\alpha_i)V_{q_{i+1}}(H)] \\
&+ \nu[\mu V_{q_L}(L) + (1-\mu)[\alpha_i V_{q_i}(L) + (1-\alpha_i)V_{q_{i+1}}(L)]]\}, i = 1, ..., K-2,
\end{aligned}
$$

We get the solution for $V_{q_i}(L)$:

$$
\begin{aligned}
V_{q_i}(L) &= \frac{\eta u^L}{1-\nu(1-\eta)} + \frac{1-\nu}{1-\nu(1-\mu)}\left[\prod_{j=i}^{K-2}\frac{1-\alpha_j}{1-(1-\eta)\alpha_j}\right](1-\eta)^{K-1-i}u^H \\
&+ \tilde{C}V_{q_H}(L)[\nu(1-\mu)(1-\eta)]^{K-1-i}\left[\prod_{j=i}^{K-2}\frac{1-\alpha_j}{1-\nu(1-\mu)(1-\eta)\alpha_j}\right],
\end{aligned}
$$

where the constant $\tilde{C}$ can be found by meeting the "initial" condition, i.e. by equating the solution $V_{q_i}(L)$ in case $i = K-2$ to the expression $V_{q_{K-2}}(L)$ which can be found explicitly from the formula for $V_{q_i}(L)$, and hence

$$C \equiv \tilde{C}V_{q_H}(L) = \frac{-\eta}{1-\nu(1-\mu)(1-\eta)}\left[u^H + \frac{(1-\nu)u^L}{1-\nu(1-\mu)}\right].$$

Note that the ex ante payoff $p_0V_{q_1}(H)+(1-p_0)V_{q_1}(L)$ is symmetric in $(\alpha_1, ..., \alpha_{K-2})$ and takes the form $M\prod_{k=1}^{K-2} A(\alpha_k) - \prod_{k=1}^{K-2} B(\alpha_k)$ with $A'(\alpha) < 0$, $B'(\alpha) < 0$ and $[A'(\alpha)B(\alpha)]/[B'(\alpha)A(\alpha)]$ is monotonic in $\alpha \in [0,1]$. This implies that any interior solution to the FOC's is symmetric and interior solution can be guaranteed because $3 \leq K < N+2$ with N given by (13). Hence it is optimal to set $\alpha = \alpha_i$ for all i. By doing so, the ex ante payoff is

$$\begin{aligned}
F(\alpha) &= p_0\left[\frac{1-\alpha}{1-(1-\eta)\alpha}\right]^{K-2}(1-\eta)^{K-2}u^H + (1-p_0)\frac{\eta u^L}{1-\nu(1-\eta)} \\
&+ (1-p_0)\frac{1-\nu}{1-\nu(1-\mu)}\left[\frac{1-\alpha}{1-(1-\eta)\alpha}\right]^{K-2}(1-\eta)^{K-2}u^H \\
&+ (1-p_0)C[\nu(1-\mu)(1-\eta)]^{K-2}\left[\frac{1-\alpha}{1-\nu(1-\mu)(1-\eta)\alpha}\right]^{K-2}
\end{aligned}$$

Now we show that $F'(\alpha)|_{\alpha=0} > 0 \Leftrightarrow \xi(p_0, h^{K-2}) < \xi^*$, that is if and only if $K-2 < N$ with N given by (13). Now,

$$\begin{aligned}
\frac{F'(\alpha)}{K-2} &= \left[p_0 + \frac{(1-p_0)(1-\nu)}{1-\nu(1-\mu)}\right]\left[\frac{1-\alpha}{1-(1-\eta)\alpha}\right]^{K-3}\frac{-\eta(1-\eta)^{K-2}}{[1-(1-\eta)\alpha]^2}u^H \\
&+ (1-p_0)(-C)[\nu(1-\mu)(1-\eta)]^{K-2}\left[\frac{1-\alpha}{1-\nu(1-\mu)(1-\eta)\alpha}\right]^{K-3}\frac{1-\nu(1-\mu)(1-\eta)}{[1-\nu(1-\mu)(1-\eta)\alpha]^2},
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{F'(\alpha)|_{\alpha=0}}{\eta(1-\eta)^{K-2}u^H(K-2)(1-p_0)} \\
&= -\left[\xi(p_0) + \frac{1-\nu}{1-\nu(1-\mu)}\right] + [\nu(1-\mu)]^{K-2}[\xi^* + \frac{1-\nu}{1-\nu(1-\mu)}].
\end{aligned}$$

Therefore, $F'(\alpha)|_{\alpha=0} > 0$ if and only if

$$\begin{aligned}
&-[\xi(p_0) + \frac{1-\nu}{1-\nu(1-\mu)}] + [\nu(1-\mu)]^{K-2}[\xi^* + \frac{1-\nu}{1-\nu(1-\mu)}] > 0 \\
\iff &\xi(p_0) + \frac{1-\nu}{1-\nu(1-\mu)} < [\nu(1-\mu)]^{K-2}[\xi^* + \frac{1-\nu}{1-\nu(1-\mu)}] \\
\iff &\frac{\xi(p_0) + \frac{1-\nu}{1-\nu(1-\mu)}}{[\nu(1-\mu)]^{K-2}} - \frac{1-\nu}{1-\nu(1-\mu)} < \xi^* \\
\iff &\xi(p_0, h^{K-2}) < \xi^*.
\end{aligned}$$

Note that The left-side of the second last inequality is indeed $\xi(p_0, h^{K-2})$, i.e the result of applying Bayes rule to $\xi(p_0)$ $K-2$ times in the breakthrough environment for arbitrary ν. □

Theorem A.2 extends the ignoring-regime-change heuristic identified in Theorem A.1 to include randomization when the constraint K is lower than $N+2$. Different from Theorem A.1, however, in this case more memory states can increase the payoff up to $K = N+2$, and hence less-is-more does not hold for $K < N+2$. As a result, there would be three regimes if we would introduce a convex cost function for the memory states. When the cost is exactly zero, then the optimal K would be unbounded. When the cost is sufficiently small, then the optimal $K = N+2$ and the optimal SFSA is deterministic. That is, we can exclude randomization by endogenously determining the memory constraint. When the cost is higher, optimal K can be below $N+2$ and randomization is optimal.

Finally, we remark that all the results in this section would hold if we relax the assumption that $\Delta^H = 1$ but it is close to one, and the assumption that $\mu_h^H = 1$ is not knife-edge either. If we consider $\Delta^H < 1$ but close, the only difference is that when p_0 is at the boundary according to (13) such that $K = N+2$ exactly holds then we need to discuss the optimal SFSA, which may be M_{N+k}^b for $k = 1, 2, 3$, depending on the relative values of Δ^H and Δ^L. Moreover, when $K < N+2$, randomization may take more complicated forms as well. However, the baseline result that the decision-maker behaves as if she ignores the underlying regime change will remain, that is, she is stuck to action a^L once an ℓ-signal is received. Similar situation holds for $\mu_h^H < 1$ but close.