RATIONAL POINTS ON CUBIC HYPERSURFACES OVER $\mathbb{F}_q(t)$

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Abstract. The Hasse principle and weak approximation is established for non-singular cubic hypersurfaces X over the function field $\mathbb{F}_q(t)$, provided that $\mathrm{char}(\mathbb{F}_q) > 3$ and X has dimension at least 6.

Contents

1. Introduction

Let K be a global field and let $X \subset \mathbb{P}_K^{n-1}$ be a cubic hypersurface defined over K. A "folklore" conjecture predicts that the set $X(K)$ of K-rational points on X is non-empty as soon as $n \geqslant 10$. When $K = \mathbb{F}_q(C)$ is the function field of a smooth and projective curve C over the finite field $\mathbb{F}_q$ this conjecture follows from the Lang–Tsen theorem (see [12, Thm. 3.6]), since K has transcendence degree 1 over a C_1-field. Alternatively, when K is a number field, it follows from recent work of the authors [2] provided that X is assumed to be non-singular. We record this observation as follows.

Theorem 1.1. *Let K be a global field and let $X \subseteq \mathbb{P}_K^{n-1}$ be a non-singular cubic hypersurface defined over K. If $n \geqslant 10$ then $X(K) \neq \emptyset$.*

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The main goal of this paper is to improve this result in the special case $K=\mathbb{F}_q(t)$. Compared to the situation over number fields, there are relatively few results in the literature which deal with the Hasse principle and weak approximation for cubic hypersurfaces defined over K. One notable exception is found in work of Colliot-Thélène [4, §3], which establishes the Hasse principle for the diagonal threefolds

$$a_1x_1^3+\cdots+a_5x_5^3=0, \quad (a_1,\dots,a_5\in K^*),$$

provided that q is odd and $q\equiv 2 \bmod 3$. Furthermore, subject to a collection of explicit constraints on the coefficients, he is able to draw the same conclusion for diagonal cubic surfaces in $\mathbb{P}^3_K$. These results are established by adapting to K work of Swinnerton-Dyer [35] on this problem over number fields. It is worth highlighting that Swinnerton-Dyer's approach relies on a delicate analysis of certain Selmer groups and this leads to a final result which is conditional on the conjecture that the Tate–Shafarevich group of an elliptic curve is finite. The advantage of working over the function field K is that the analagous statements can be made *unconditional* — a feature that will resurface in the present investigation.

Turning to weak approximation, in the setting $n=4$ of non-singular cubic surfaces it follows from work of Hu [21, Thm. 5] that X satisfies weak approximation at the places of good reduction, provided that $\mathrm{char}(\mathbb{F}_q)>3$ and $q>47$. For larger values of n a suitable variant of the Hardy–Littlewood circle method can be brought to bear on this problem. Let $X\subset\mathbb{P}^{n-1}_K$ be a non-singular cubic hypersurface defined over K. Assuming that $\mathrm{char}(\mathbb{F}_q)>3$ it follows from work of Lee (see [27] and his 2013 PhD thesis [28]) that weak approximation holds for X over K provided that $n\geqslant 14$. Note that the Hasse principle is trivial for n in this range by Theorem 1.1.

By developing an alternative version of the circle method, we shall establish the following improvement.

Theorem 1.2. *Let $K=\mathbb{F}_q(t)$ with $\mathrm{char}(\mathbb{F}_q)>3$. Let $X\subset\mathbb{P}^{n-1}_K$ be a non-singular cubic hypersurface defined over K, with $n\geqslant 8$. Then X satisfies the Hasse principle and weak approximation over K.*

The restriction on the characteristic of $\mathbb{F}_q$ in this result is unfortunate but intrinsic to the method. The same restriction appears in Lee's work [27, 28], where it stems from the use of Weyl differencing in the analysis of certain cubic exponential sums, which produces factors of 3! within the argument of the resulting sums. In our case, the restriction on the characteristic comes from the need to find an auxiliary point on the hypersurface X at which the associated Hessian does not vanish. For diagonal forms over $\mathbb{F}_q(t)$, Liu and Wooley [29] have shown how to handle arbitrary characteristic. Their approach uses the large sieve to act as a substitute for Weyl differencing and it would be interesting to see whether this innovation could be adapted to general forms.

It is natural to compare Theorem 1.2 with the situation of non-singular cubic hypersurfaces over function fields $K = k(C)$ of a curve C over an algebraically closed field k of characteristic 0. In this setting Lang–Tsen theory confirms that $X(K) \neq \emptyset$ for $n \geqslant 4$. On the other hand, Hassett and Tschinkel [14, Thm. 1] have shown that X satisfies weak approximation over K provided that $n \geqslant 7$.

Theorem 1.2 is the $\mathbb{F}_q(t)$-analogue of recent work by Hooley [19] about non-singular cubic hypersurfaces $X \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ over the rational numbers. Hooley's main result establishes the Hasse principle for X, provided that $n \geqslant 8$, conditionally under a certain unproved "Hypothesis HW" about the analytic properties of Hasse–Weil L-functions associated to a family of 5-dimensional cubic hypersurfaces. Over the last century the theory of the Hardy–Littlewood circle method has become heavily industrialised in its application to cubic forms over $\mathbb{Q}$, reaching a zenith in Hooley's work on octonary cubic forms. The igniting spark in his work is the smooth δ-function technology that was introduced by Duke, Friedlander and Iwaniec [8]. This paves the way to getting non-trivial averaging over the approximating fractions a/q that appear in the associated cubic exponential sums. Note that Hooley requires non-trivial averaging over both numerators and denominators to handle cubic forms in 8 variables. This is usually termed a "double Kloosterman refinement", with the usual "Kloosterman refinement" connoting non-trivial averaging over the numerators only. The ordinary Kloosterman refinement is only capable of handling cubic forms in $n \geqslant 9$ variables (see pioneering work of Heath-Brown [15] and Hooley [17]), but when it works it produces completely unconditional results. The use of a double Kloosterman refinement over $\mathbb{Q}$ leads to the analysis of global L-functions associated to cubic hypersurfaces of dimension 5. Since our knowledge about such L-functions is extremely scarce in dimension > 1, any progress is dependent on Hypothesis HW, which describes the meromorphic continuation and location of zeros of these L-functions. The significance of Theorem 1.2 is that working over $K = \mathbb{F}_q(t)$ affords a completely unconditional result.

The proof of Theorem 1.2 is long and complicated and we proceed to outline some of the key ingredients. Our approach is based on estimating the number $N(d)$ of suitably weighted vectors $(x_1, \dots, x_n) \in \mathbb{F}_q[t]^n$, with $\max_i \deg x_i < d$, for which $[x_1, \dots, x_n] \in X(K)$. The principal result of this paper is Theorem 7.1, which provides an asymptotic formula for $N(d)$ when $n = 8$, as $d \to \infty$. This will suffice to prove Theorem 1.2 when $n = 8$. For $n \geqslant 9$ we will deduce the result via a fibration $X \to \mathbb{P}^1_K$ in §7.1. Theorem 7.1 is established using the circle method.

As one might expect, parts of the circle method machinery become greatly simplified when transported to the function field K. The first simplification comes in the analogue of the smooth δ-function that lies at the heart of Hooley's work. Indeed, the absolute values of K satisfy the ultrametric inequality. This allows us to cover the analogue of the unit interval with non-overlapping arcs using nothing more sophisticated than a version of Dirichlet's theorem on Diophantine approximation over K (see Remark 4.3). Thus we are immediately placed in the position of being able to carry out a double Kloosterman refinement. This appears to be the first attempt to extract non-trivial savings, à la Kloosterman, over function fields.

The process of non-trivial averaging leads us to consider the global L-function $L(H_\ell^m(Y), s)$ which is affiliated to the middle ℓ-adic cohomology group $H_\ell^m(Y) = H_{\text{ét}}^m(Y \otimes_K \overline{K}, \mathbb{Q}_\ell)$ of a non-singular cubic hypersurface $Y \subset \mathbb{P}_K^{m+1}$ of dimension m. In §3 we will relate these L-functions to a very general class of global L-functions that were associated to arbitrary lisse ℓ-adic sheaves by Grothendieck [13]. The second major advantage of working over K is that, thanks to Grothendieck and Deligne [7, 13], we know that these L-functions are actually rational functions of q^{-s} that satisfy the Riemann hypothesis. Thus for $k \in \{0, 1, 2\}$ there are polynomials $P_k = P_{k,m} \in \mathbb{Z}[T]$, with inverse roots having absolute value $q^{(k+m)/2}$, such that

$$L(H_\ell^m(Y), s) = \frac{P_1(q^{-s})}{P_0(q^{-s})P_2(q^{-s})}.$$

In §8, for even $n \geqslant 8$, this information will allow us to execute an unconditional double Kloosterman refinement by getting savings in the treatment of the relevant cubic exponential sums with square-free modulus. This is the most novel part of our investigation.

In order to make use of the analytic properties of $L(H_\ell^m(Y), s)$, we shall also need to contend with an issue that represents a much greater challenge in the function field setting than in the classical one. Over $\mathbb{Q}$, partial summation is widely used as a means of transforming the summation of products of sequences into easier summations, but this device is not readily available over K. The underlying obstacle comes from the fact that there are only two rational integers with a given absolute value, but q^{d+1} elements of $\mathbb{F}_q[t]$ with given degree d (and all of these will have equal absolute value). We will circumvent this difficulty by introducing Dirichlet characters on $(\mathbb{F}_q[t^{-1}]/t^{-J}\mathbb{F}_q[t^{-1}])^*$, for a positive integer J, and then showing that the analytic properties enjoyed by $L(H_\ell^m(Y), s)$ continue to hold when $H_\ell^m(Y)$ is twisted by the Galois representation induced by these characters.

There remains the not insignificant task of handling cubic exponential sums with square-full modulus. Unfortunately, the passage to function fields doesn't offer any simplification of this task and the necessary arguments are mostly direct analogues of the corresponding treatment over $\mathbb{Q}$ found in [15] and [19]. Given the length of the paper we will capitalise on the inherent similarities by not providing a complete treatment of all the estimates that are recorded in §5 and §6. Instead we shall content ourselves with proving the function field analogues of the key ideas that underpin the arguments over $\mathbb{Q}$. One ingredient that we require is a non-trivial bound for the number of K-rational points on a geometrically irreducible hypersurface $V \subset \mathbb{P}_K^{n-1}$ which is not a hyperplane. Let $H : \mathbb{P}_K^{n-1}(K) \to \mathbb{R}$ be the standard exponential height function. Then, as a special case of Lemma 2.10, it follows that

$$\#\left\{x \in V(K) : H(x) \leqslant q^B\right\} = O_{\varepsilon,V}\left(q^{B(n-3/2+\varepsilon)}\right),$$

for any $B \geqslant 1$ and any $\varepsilon > 0$. There are very few results of this sort in the literature over function fields and it would be interesting to see whether the rapid recent advances involving the "determinant method" over number fields could be adapted to improve this upper bound.

Finally, suppose that $X \subset \mathbb{P}_{\mathbb{F}_q}^{n-1}$ is a non-singular cubic hypersurface defined over a finite field $\mathbb{F}_q$, with $\mathrm{char}(\mathbb{F}_q) > 3$. There is a correspondence between the counting function $N(d)$ for $\mathbb{F}_q(t)$-points on X of bounded height and the cardinality of $\mathbb{F}_q$-points on the moduli space $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)$, which parameterises the rational maps of degree d on X. Following an idea of Ellenberg and Venkatesh it is possible to exploit the Lang–Weil estimate to make deductions about the basic geometry of this moduli space via an asymptotic formula for $N(d)$, provided that sufficient uniformity is achieved in the q-aspect. Using the present investigation as a base, we have produced a short companion paper [3] which carries out this plan.

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2. Auxiliary facts about function fields

2.1. **Notation.** In this section we collect together some notation and basic facts concerning the function field $K = \mathbb{F}_q(t)$. To begin with, for any real number R we will always write $\widehat{R} = q^R$.

Let $\mathscr{O} = \mathbb{F}_q[t]$ be the ring of integers of K and let Ω be the set of places of K. These correspond to either monic irreducible polynomials ϖ in $\mathscr{O}$, which we call the *finite primes*, or the *prime at infinity* t^{-1} which we usually denote by ∞. The associated absolute value $|\cdot|_v$ is either $|\cdot|_\varpi$ for some prime $\varpi \in \mathscr{O}$ or $|\cdot|_\infty$, according to whether v is a finite or infinite place, respectively. These are given by

$$|a/b|_\varpi = \left(\frac{1}{q^{\deg \varpi}}\right)^{\mathrm{ord}_\varpi(a/b)} \qquad \text{and} \qquad |a/b|_\infty = q^{\deg a - \deg b},$$

for any $a/b \in K^*$. We extend these definitions to K by taking $|0|_\varpi = |0|_\infty = 0$. We will usually just write $|\cdot| = |\cdot|_\infty$.

For $v \in \Omega$ we let K_v denote the completion of K at v with respect to $|\cdot|_v$. We put $\mathscr{O}_v = \{a \in K_v : |a|_v \leqslant 1\}$ for the maximal compact subring and $\mathscr{O}_v^* = \{a \in K_v : |a|_v = 1\}$ for the unit group. Furthermore, we let $\mathbb{F}_v$ denote its residue field. We have $\mathbb{F}_\infty = \mathbb{F}_q$ and $\mathbb{F}_\varpi = \mathbb{F}_{q^{\deg(\varpi)}}$ for any finite prime ϖ. The elements of $\mathscr{O}_\infty$ are power series expansions in t^{-1}.

We may identify K_∞ with the set

$$\mathbb{F}_q((1/t)) = \left\{ \sum_{i \leqslant N} a_i t^i : \text{for } a_i \in \mathbb{F}_q \text{ and some } N \in \mathbb{Z} \right\}$$

and put

$$\mathbb{T} = \{\alpha \in K_\infty : |\alpha| < 1\} = \left\{ \sum_{i \leqslant -1} a_i t^i : \text{for } a_i \in \mathbb{F}_q \right\}.$$

Let $\delta \in \mathbb{T}$. Then $\mathbb{T}/\delta\mathbb{T}$ is the set of cosets $\alpha + \delta\mathbb{T}$, of which there are $|\delta|$.

We can extend the absolute value at the infinite place to K_∞ to get a non-archimedean absolute value $|\cdot| : K_\infty \to \mathbb{R}_{\geqslant 0}$ given by $|\alpha| = q^{\mathrm{ord}\,\alpha}$, where $\mathrm{ord}\,\alpha$ is the largest $i \in \mathbb{Z}$ such that $a_i \neq 0$ in the representation $\alpha = \sum_{i \leqslant N} a_i t^i$. In this context we adopt the convention $\mathrm{ord}\,0 = -\infty$ and $|0| = 0$. We extend this to vectors by setting $|\mathbf{x}| = \max_{1 \leqslant i \leqslant n} |x_i|$, for any $\mathbf{x} \in K_\infty^n$.

Since $\mathbb{T}$ is a locally compact additive subgroup of K_∞ it possesses a unique Haar measure $\mathrm{d}\alpha$, which is normalised so that $\int_{\mathbb{T}} \mathrm{d}\alpha = 1$. We can extend $\mathrm{d}\alpha$ to a (unique) translation-invariant measure on K_∞ in such a way that

$$\int_{\{\alpha \in K_\infty : |\alpha| < \widehat{N}\}} \mathrm{d}\alpha = \widehat{N},$$

for any $N \in \mathbb{Z}_{>0}$. These measures also extend to $\mathbb{T}^n$ and K_∞^n, for any $n \in \mathbb{Z}_{>0}$.

For given $\mathbf{x}, \mathbf{b} \in \mathscr{O}^n$ and $M \in \mathscr{O}$ we will sometimes write $\mathbf{x} \equiv \mathbf{b} \bmod M$ to mean that $\mathbf{x} = \mathbf{b} + M\mathbf{y}$ for some $\mathbf{y} \in \mathscr{O}^n$.

2.2. **Characters.** There is a non-trivial additive character $e_q : \mathbb{F}_q \to \mathbb{C}^*$ defined for each $a \in \mathbb{F}_q$ by taking $e_q(a) = \exp(2\pi i\,\mathrm{Tr}(a)/p)$, where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ denotes the trace map. This character induces a non-trivial (unitary) additive character $\psi : K_\infty \to \mathbb{C}^*$ by defining $\psi(\alpha) = e_q(a_{-1})$ for any $\alpha = \sum_{i \leqslant N} a_i t^i$

in K_∞. In particular it is clear that $\psi|_{\mathscr{O}}$ is trivial. More generally, given any $\gamma \in K_\infty$, the map $\alpha \mapsto \psi(\alpha\gamma)$ is an additive character on K_∞. We have the basic orthogonality property

$$\sum_{\substack{b\in\mathscr{O}\\ |b|<\widehat{N}}} \psi(\gamma b) = \begin{cases} \widehat{N}, & \text{if } |\gamma| < \widehat{N}^{-1},\\ 0, & \text{otherwise.}\end{cases}$$

for any $\gamma \in K_\infty$ and any integer $N \geqslant 0$ (see Lemma 7 of [26]).

We will also need standard characters at the finite places (we follow Ex. 7.5 of [30] for their construction). Let K_ϖ be the completion of K at the place corresponding to finite prime $\varpi \in \mathscr{O}$ of degree $d \geqslant 1$, with corresponding ring of integers $\mathscr{O}_\varpi$. According to [30, Ex. 7.5(c)], any element $x \in K_\varpi$ can be written as $x = y/\varpi^N + z$ for some integer $N \geqslant 0$ and $z \in \mathscr{O}_\varpi$, where $y = a_1 t^{dN-1} + a_2 t^{dN-2} + \cdots + a_{dN}$, with all coefficients $a_i \in \mathbb{F}_q$. With this representation one defines the non-trivial additive character $\psi_\varpi : K_\varpi \to \mathbb{C}^*$ to be given by

$$\psi_\varpi(x) = e_q(a_1).$$

Letting $\mathbf{A}_K$ denote the adèles over K, we may now define the *standard adelic character* $\psi_K : \mathbf{A}_K \to \mathbb{C}^*$ to be

$$\psi_K(x) = \psi(x_\infty)\prod_\varpi \psi_\varpi(x_\varpi),$$

for any $x = (x_v) \in \mathbf{A}_K$. It follows from [30, Ex. 7.6] that ψ_K is a non-trivial additive character of $\mathbf{A}_K$ which is trivial on K.

2.3. **Fourier analysis on non-archimedean local fields.** The material we summarise here is found in [30, §7], but has its genesis in work of Schmid and Teichmüller [31]. (The authors are grateful to Ivan Fesenko for this reference.) We first fix a non-trivial additive character $\varphi : F \to \mathbb{C}^*$ on a non-archimedean local field F. A function $f : F \to \mathbb{C}$ is said to be *smooth* if it is locally constant (that is, $f(x) = f(x_0)$ for all x sufficiently close to x_0). A *Schwartz–Bruhat function* is a smooth function $f : F \to \mathbb{C}$ with compact support. We denote by $S(F)$ the set of all such functions. Then for any $f \in S(F)$ we may define the Fourier transform of f by

$$\widehat{f}(y) = \int_F f(x)\varphi(xy)\mathrm{d}x,$$

where $\mathrm{d}x$ is Haar measure. This function also belongs to $S(F)$.

Let $K = \mathbb{F}_q(t)$. We define $S(\mathbf{A}_K)$ to be the space of functions given by

$$f(x) = \prod_\nu f_\nu(x_\nu),$$

for $x=(x_\nu)\in \mathbf{A}_K$. Here, $f_\nu\in S(K_\nu)$ for every place ν and $f_\varpi|_{\mathscr{O}_\varpi}=1$ for almost all primes ϖ. The *adelic Fourier transform* of any $f\in S(\mathbf{A}_K)$ is given by

$$\widehat{f}(y)=\int_{\mathbf{A}_K} f(x)\psi_K(xy)\mathrm{d}x,$$

where ψ_K is the standard adelic character on $\mathbf{A}_K$ and $\mathrm{d}x$ is Haar measure on $\mathbf{A}_K$ (normalised to be the self-dual measure for ψ_K). With this notation the Poisson summation formula (see [30, Thm. 7.7], for example) states that

$$\sum_{x\in K} f(x)=\sum_{x\in K}\widehat{f}(x),$$

for any $f\in S(\mathbf{A}_K)$. This extends to a summation over $\mathbf{x}\in K^n$ in the obvious way.

We will need to introduce some weight functions on K^n. For a prime ϖ define $w_\varpi: K_\varpi\to\{0,1\}$ via

$$w_\varpi(x)=\begin{cases}1, & \text{if } |x|_\varpi\leqslant 1,\\ 0, & \text{otherwise.}\end{cases}$$

This gives an indicator function for the ring of integers $\mathscr{O}_\varpi$. It is easy to check that $\widehat{w}_\varpi=w_\varpi$. Next let $w_\infty: K_\infty\to\{0,1\}$ be the indicator function for $\mathbb{T}$, defined via

$$w_\infty(x)=\begin{cases}1, & \text{if } |x|<1,\\ 0, & \text{otherwise.}\end{cases}$$

We proceed to define weight functions $w_f, w: K^n\to\{0,1\}$ via

$$w_f(\mathbf{x})=\prod_{1\leqslant i\leqslant n}\prod_{\varpi} w_\varpi(x_i),\quad w(\mathbf{x})=\prod_{1\leqslant i\leqslant n} w_\infty(x_i). \tag{2.1}$$

Let $\mathbf{z}\in K^n$. Then $|\mathbf{z}|<\widehat{P}$ if and only if $w(\mathbf{z}/t^P)=1$ and $\mathbf{z}\in\mathscr{O}$ if and only if $w_f(\mathbf{z})=1$.

We will use the n-dimensional Poisson summation formula to prove the following result.

Lemma 2.1. *Let $f\in K_\infty[x_1,\dots,x_n]$ be a polynomial and let $v\in S(K_\infty^n)$. Then we have*

$$\sum_{\mathbf{z}\in\mathscr{O}^n}\psi(f(\mathbf{z}))v(\mathbf{z})=\sum_{\mathbf{c}\in\mathscr{O}^n}\int_{K_\infty^n} v(\mathbf{u})\psi(f(\mathbf{u})+\mathbf{c}.\mathbf{u})\mathrm{d}\mathbf{u}.$$

Proof. Recalling the definitions of the weight functions w_f and w, we may write

$$\sum_{\mathbf{z}\in\mathscr{O}^n}\psi(f(\mathbf{z}))v(\mathbf{z})=\sum_{\mathbf{z}\in K^n} g(\mathbf{z}),$$

where $g(\mathbf{z}) = \psi(f(\mathbf{z}))w_f(\mathbf{z})v(\mathbf{z})$. It is clear that $g \in S(\mathbf{A}_K^n)$ and so we are free to apply the n-dimensional version of Poisson summation to conclude that

$$\sum_{\mathbf{z}\in\mathscr{O}^n} \psi(f(\mathbf{z}))v(\mathbf{z}) = \sum_{\mathbf{c}\in K^n} \widehat{w}_f(\mathbf{c}) \int_{K_\infty^n} v(\mathbf{u})\psi(f(\mathbf{u}) + \mathbf{c}.\mathbf{u})\mathrm{d}\mathbf{u}.$$

Since $\widehat{w}_f(\mathbf{c}) = w_f(\mathbf{c})$, the lemma follows. □

2.4. **Some integral formulae.** In this section we collect some basic facts and estimates concerning multi-dimensional integrals over K_∞^n. Recall the definition of the additive character $\psi : K_\infty \to \mathbb{C}^*$ from §2.2. We begin by recording the following fact (see Lemma 1(f) of [26]).

Lemma 2.2. *Let $Y \in \mathbb{Z}$ and $\gamma \in K_\infty$. Then*

$$\int_{|\alpha|<\widehat{Y}} \psi(\alpha\gamma)\mathrm{d}\alpha = \begin{cases} \widehat{Y}, & \textit{if } |\gamma| < \widehat{Y}^{-1}, \\ 0, & \textit{otherwise.} \end{cases}$$

Taking $Y = 0$, it follows from this result that

$$\int_{\mathbb{T}} \psi(\alpha x)\mathrm{d}\alpha = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \in \mathscr{O} \setminus \{0\}. \end{cases} \tag{2.2}$$

We also have the following change of variables formula, which readily follows from Igusa [22, Lemma 7.4.2].

Lemma 2.3. *Let $\Gamma \subset K_\infty^n$ be a box defined by the inequalities $|x_i| < \widehat{R}_i$, for some real numbers $R_1, \dots, R_n$. Let $f : \Gamma \to \mathbb{C}$ be a continuous function. Then for any $M \in \mathrm{GL}_n(K_\infty)$ we have*

$$\int_\Gamma f(\boldsymbol{\alpha})\mathrm{d}\boldsymbol{\alpha} = |\det M| \int_{M\boldsymbol{\beta}\in\Gamma} f(M\boldsymbol{\beta})\mathrm{d}\boldsymbol{\beta}.$$

It will be convenient to reserve some notation for the *height* of a polynomial $f \in K_\infty[x_1, \dots, x_n]$. Assuming that $f(\mathbf{x}) = \sum_{\mathbf{i}} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, for coefficients $a_{\mathbf{i}} \in K_\infty$, we define

$$H_f = \max_{\mathbf{i}} |a_{\mathbf{i}}|.$$

We proceed to establish the following result.

Lemma 2.4. *Let $f \in K_\infty[x_1, \dots, x_n]$ be a polynomial and let $\mathbf{w} \in K_\infty^n$. Assume that $|\mathbf{w}| \geqslant 1$ and $|\mathbf{w}| > H_f$. Then*

$$\int_{\mathbb{T}^n} \psi\left(f(\mathbf{x}) + \mathbf{w}.\mathbf{x}\right)\mathrm{d}\mathbf{x} = 0.$$

Proof. Suppose without loss of generality that $|\mathbf{w}| = |w_1| = \widehat{N}$ for some integer $N \geqslant 0$. We concentrate on the one-dimensional integral

$$I = \int_{\mathbb{T}} \psi\left(f(x, x_2, \dots, x_n) + w_1 x\right) \mathrm{d}x,$$

for fixed $x_2, \dots, x_n \in \mathbb{T}$. Write $g(x) = f(x, x_2, \dots, x_n)$. We may suppose that $g(x) = c_0 x^d + \dots + c_{d-1} x$ for appropriate coefficients $c_i = c_i(x_2, \dots, x_n) \in K_\infty$. Our hypothesis implies that

$$H_g \leqslant H_f < \widehat{N}. \tag{2.3}$$

It is clear that $\psi(g(x) + w_1 x) = 1$ if $|x| < q^{-N-1}$. But then, using the definition of integration over $\mathbb{T}$, we find that

$$\begin{aligned} I &= \lim_{m \to \infty} q^{-m} \sum_{a_{-m}, \dots, a_{-1} \in \mathbb{F}_q} \psi\left(h(a_{-m} t^{-m} + \dots + a_{-1} t^{-1})\right) \\ &= q^{-N-1} \sum_{a_{-N-1}, \dots, a_{-1} \in \mathbb{F}_q} \psi\left(h(a_{-N-1} t^{-N-1} + \dots + a_{-1} t^{-1})\right), \end{aligned}$$

where $h(x) = g(x) + w_1 x$. The coefficient of t^{-1} in $w_1 x$ is a_{-N-1} Moreover, (2.3) implies that $|g(a_{-N-1} t^{-N-1} + y) - g(y)| < |t^{-1}|$ for any $y \in \mathbb{T}$. This implies that the coefficient of t^{-1} in $g(x)$ is a polynomial in $a_{-N}, \dots, a_{-1}$ alone, from which it follows that $I = 0$, since

$$\sum_{a_{-N-1} \in \mathbb{F}_q} e_q(a_{-N-1}) = 0.$$

This completes the proof of the lemma. $\square$

As an easy consequence of Lemma 2.4 we get the following result.

Lemma 2.5. *Let $f \in K_\infty[x_1, \dots, x_n]$ be a polynomial. Suppose that there exists $\mathbf{u} \in \mathbb{T}^n$ and $\lambda \geqslant 1$ such that $|\nabla f(\mathbf{u})| \geqslant \lambda$ and $|\partial^{\boldsymbol{\beta}} f(\mathbf{u})| < \lambda$, for all $|\boldsymbol{\beta}| \geqslant 2$. Then*

$$\int_{\mathbb{T}^n} \psi(f(\mathbf{x})) \mathrm{d}\mathbf{x} = 0.$$

Proof. Make the change of variables $\mathbf{x} = \mathbf{u} + \mathbf{y}$ and note that

$$f(\mathbf{x}) = f(\mathbf{u}) + \mathbf{y}.\nabla f(\mathbf{u}) + \tfrac{1}{2} \mathbf{y}^T \nabla^2 f(\mathbf{u}) \mathbf{y} + \dots.$$

The conclusion is now a direct consequence of Lemma 2.4 with $\mathbf{w} = \nabla f(\mathbf{u})$. $\square$

Given a non-zero polynomial $F \in K_\infty[x_1, \dots, x_n]$, integrals of the form

$$J_F(\gamma; \mathbf{w}) = \int_{\mathbb{T}^n} \psi\left(\gamma F(\mathbf{x}) + \mathbf{w}.\mathbf{x}\right) \mathrm{d}\mathbf{x} \tag{2.4}$$

will feature prominently in our work, for given $\gamma \in K_\infty$ and $\mathbf{w} \in K_\infty^n$. On noting that $H_{\gamma F} = |\gamma| H_F$, the following result is a trivial consequence of Lemma 2.4.

Lemma 2.6. *We have $J_F(\gamma; \mathbf{w}) = 0$ if $|\mathbf{w}| > \max\{1, |\gamma| H_F\}$.*

The following result will be useful when $|\mathbf{w}|$ is not too large.

Lemma 2.7. *We have*

$$J_F(\gamma; \mathbf{w}) = \int_\Omega \psi\left(\gamma F(\mathbf{x}) + \mathbf{w}.\mathbf{x}\right) \mathrm{d}\mathbf{x},$$

where $\Omega = \left\{\mathbf{x} \in \mathbb{T}^n : |\gamma \nabla F(\mathbf{x}) + \mathbf{w}| \leqslant H_F \max\{1, |\gamma|^{1/2}\}\right\}$.

Proof. Let $\Omega_0 = \mathbb{T}^n \setminus \Omega$. We break the integral over Ω_0 into a sum of integrals over smaller regions. Let $\delta \in K_\infty$ be such that $|\delta| = \min\{1, |\gamma|^{-1/2}\}$. Introducing a dummy sum over $\mathbf{y} \in (\mathbb{T}/\delta\mathbb{T})^n$ and then using Lemma 2.3 to make the change of variables $\mathbf{x} = \mathbf{y} + \delta\mathbf{z}$, we obtain

$$\begin{aligned}\int_{\Omega_0} \psi\left(\gamma F(\mathbf{x}) + \mathbf{w}.\mathbf{x}\right) \mathrm{d}\mathbf{x} &= |\delta|^{-n} \sum_{\mathbf{y} \in (\mathbb{T}/\delta\mathbb{T})^n} \int_{\Omega_0} \psi\left(\gamma F(\mathbf{x}) + \mathbf{w}.\mathbf{x}\right) \mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{y} \in (\mathbb{T}/\delta\mathbb{T})^n} \int_{\{\mathbf{z} \in \mathbb{T}^n : \mathbf{y} + \delta\mathbf{z} \in \Omega_0\}} \psi\left(f(\mathbf{z})\right) \mathrm{d}\mathbf{z},\end{aligned}$$

where $f(\mathbf{z}) = \gamma F(\mathbf{y} + \delta\mathbf{z}) + \mathbf{w}.(\mathbf{y} + \delta\mathbf{z})$. We want to show that the inner integral vanishes, to which end we claim that $\mathbf{y} + \delta\mathbf{z} \in \Omega_0$ if and only if $\mathbf{y}$ satisfies

$$|\gamma \nabla F(\mathbf{y}) + \mathbf{w}| > H_F \max\{1, |\gamma|^{1/2}\}. \tag{2.5}$$

Using Taylor expansion and observing that $|\delta\gamma| = \min\{1, |\gamma|^{1/2}\}$, we deduce that there is a vector $\mathbf{u}$ depending on $\mathbf{z}$, with $|\mathbf{u}| < H_F \min\{1, |\gamma|^{1/2}\}$, such that $\gamma \nabla F(\mathbf{y} + \delta\mathbf{z}) + \mathbf{w} = \gamma \nabla F(\mathbf{y}) + \mathbf{w} + \mathbf{u}$. Put $A = H_F \max\{1, |\gamma|^{1/2}\}$. If $\mathbf{y} + \delta\mathbf{z} \in \Omega_0$ then

$$A < \max\{|\gamma \nabla F(\mathbf{y}) + \mathbf{w}|, |\mathbf{u}|\} < \max\{|\gamma \nabla F(\mathbf{y}) + \mathbf{w}|, A\},$$

which implies that (2.5) holds. Conversely, if (2.5) holds then

$$A < |\gamma \nabla F(\mathbf{y}) + \mathbf{w} + \mathbf{u}| < |\gamma \nabla F(\mathbf{y} + \delta\mathbf{z}) + \mathbf{w}|.$$

This therefore establishes the claim. Hence we have

$$\int_{\Omega_0} \psi\left(\gamma F(\mathbf{x}) + \mathbf{w}.\mathbf{x}\right) \mathrm{d}\mathbf{x} = \sum_{\substack{\mathbf{y} \in (\mathbb{T}/\delta\mathbb{T})^n \\ |\gamma \nabla F(\mathbf{y}) + \mathbf{w}| > H_F \max\{1, |\gamma|^{1/2}\}}} \int_{\mathbb{T}^n} \psi\left(f(\mathbf{z})\right) \mathrm{d}\mathbf{z}.$$

Now all the partial derivatives of $f(\mathbf{z})$ of order $k \geqslant 2$ are strictly less than $H_F |\gamma| |\delta|^k \leqslant H_F \min\{1, |\gamma|\}$. Moreover, our preceding argument shows that $|\nabla f(\mathbf{z})| > H_F \max\{1, |\gamma|\}$ for every $\mathbf{z} \in \mathbb{T}^n$. An application of Lemma 2.5

therefore shows that the inner integral vanishes, as required to complete the proof. □

2.5. **Density of integer points on affine hypersurfaces.** Let $V \subset \mathbb{A}_K^n$ be an affine variety defined over $\mathscr{O}$ of degree $d \geqslant 1$ and dimension $m \geqslant 1$. Using a version of the large sieve inequality over function fields due to Hsu [20], our main goal in this section is to establish a pair of estimates for the number of $\mathscr{O}$-points on V with bounded absolute value.

Lemma 2.8. *We have $\#\{\mathbf{x} \in V(\mathscr{O}) : |\mathbf{x}| \leqslant \widehat{N}\} = O_{d,n}(q^{(N+1)m})$, where the implied constant only depends on d and n.*

This result is optimal whenever V contains a linear component of dimension m. Alternatively, we will obtain the following improvement.

Lemma 2.9. *Assume that V is absolutely irreducible and $d \geqslant 2$. Then we have*
$$\#\{\mathbf{x} \in V(\mathscr{O}) : |\mathbf{x}| \leqslant \widehat{N}\} = O_{d,n}(q^{(N+1)(m-1/2)} N \log q),$$
where the implied constant only depends on d and n.

Now let $G \in \mathscr{O}[X_1, \dots, X_n]$ be a homogeneous polynomial, which is absolutely irreducible over K and has degree $d \geqslant 2$. The following result is now a trivial consequence of Lemma 2.9 applied to the absolutely irreducible hypersurface $g = 0$, where $g(\mathbf{X}) = G(\mathbf{a} + k\mathbf{X})$.

Lemma 2.10. *Let $k \in \mathscr{O}$ and let $\mathbf{a} \in \mathscr{O}^n$. Then for any any $\varepsilon > 0$ we have*
$$\#\left\{\mathbf{x} \in \mathscr{O}^n : \begin{array}{l} |\mathbf{x}| \leqslant \widehat{N},\ G(\mathbf{x}) = 0, \\ \mathbf{x} \equiv \mathbf{a} \bmod k \end{array}\right\} \ll_{d,n,\varepsilon} \left(1 + \frac{\widehat{N}}{|k|}\right)^{n-3/2+\varepsilon}.$$

The implied constant in this estimate depends at most on n, the degree of G and on the choice of ε. Lemma 2.10 is an extension of [15, Lemma 15] to function fields.

We proceed with the proof of Lemmas 2.8 and 2.9, which are based on Serre's proof of the analogous result for number fields (see Serre [33, Chapter 13]). We select coordinates on $\mathbb{A}_K^n$ such that the projection $\pi : V \to \mathbb{A}_K^m$ onto the first m coordinates induces a finite morphism. Let $Z = \pi(V)$ be the corresponding (thin) subset of $\mathbb{A}_K^m$ and let $Z(N) = \#\{\mathbf{x} \in Z \cap \mathscr{O}^m : |\mathbf{x}| \leqslant \widehat{N}\}$. Since the fibre of each point under π has at most d points, it will be enough to prove the bound
$$Z(N) = \begin{cases} O_{d,n}(q^{(N+1)(m-1/2)} N \log q), & \text{if } V \text{ is abs. irred. and } d \geqslant 2, \\ O_{d,n}(q^{(N+1)m}), & \text{otherwise.} \end{cases} \tag{2.6}$$
Our key tool in proving these bounds will be the following large sieve inequality over K due to Hsu [20, Theorem 3.2].

Lemma 2.11. *Let $M, N, m \in \mathbb{Z}_{>0}$ and let X be a subset of $\mathscr{O}^m$. For each prime ϖ suppose that there exists a real number $\alpha_\varpi \in (0,1]$ such that*

$$\#X_\varpi \leqslant \alpha_\varpi |\varpi|^m,$$

where X_ϖ denotes the canonical image of X in $(\mathscr{O}/\varpi\mathscr{O})^m$. Then

$$\#\{\mathbf{x} \in X : |\mathbf{x}| \leqslant \widehat{N}\} \leqslant q^{m(\max\{N, 2M-1\}+1)}/L(M),$$

where

$$L(M) = 1 + \sum_{\substack{b \in \mathscr{O} \text{ monic} \\ |b| \leqslant \widehat{M}}} \prod_{\varpi | b} \left(\frac{1-\alpha_\varpi}{\alpha_\varpi} \right).$$

Taking $\alpha_\varpi = 1$ for every ϖ and $L(M) \geqslant 1$ we easily arrive at the second part of (2.6) by taking $M = (N+1)/2$. For the first part, for any prime $\varpi \in \mathscr{O}$, we let $Z(N)_\varpi$ denote the canonical image of Z in $(\mathscr{O}/\varpi\mathscr{O})^m$. The following result is proved in exactly the same way as the number field version [33, Thm. 5 in Chapter 13].

Lemma 2.12. *Assume that V is absolutely irreducible and has degree $d \geqslant 2$. There is a finite Galois extension K_π/K of degree at most $d!$ and a number $c_\pi \in (0, 1-1/d!]$ such that if ϖ splits completely in K_π then*

$$\#Z(N)_\varpi \leqslant c_\pi |\varpi|^m + O_{d,n}(|\varpi|^{m-1/2}).$$

We may now use this result to deduce the first part of (2.6). Let K_π be as in Lemma 2.12. Then if a prime ϖ splits completely in K_π it follows that $Z(N)_\varpi \leqslant c_\varpi |\varpi|^{n-1}$ for some constant $0 < c_\varpi \leqslant 1 - 1/d!$. We now apply Lemma 2.11 with $M = (N+1)/2$, invoking the prime number theorem to deduce that

$$L(M) \geqslant \sum_{\substack{\varpi \in \mathscr{O} \text{ monic and irreducible} \\ |\varpi| \leqslant \widehat{M} \\ \varpi \text{ splits completely in } K_\pi}} (1 - c_\varpi) \geqslant \frac{1}{d!} \frac{q^M}{[K_\pi : K] M} + O(q^{M/2}),$$

This completes the proof of (2.6), and so the proof of Lemma 2.9.

3. Global L-functions and ℓ-adic sheaves

In §3.1 we review some facts about ℓ-adic sheaves on affine curves and in §3.2 we recall the construction of their associated L-functions. In §3.3 we record the statement of the Weil conjectures as established by Deligne. Next, in §3.4, we recall the construction of the Hasse–Weil L-function of a smooth and projective variety over a global field of positive characteristic and state some of its fundamental properties. Finally, in §3.5, we discuss the analagous properties of a global L-function obtained through twisting by a character of finite order.

3.1. **Review of ℓ-adic sheaves and ℓ-adic cohomology.** The main references for this section are Deligne [7] and Katz (see [24, Chapter 2] and [25]). Let us assume that $j : U \hookrightarrow C$ is a non-empty affine open subset of a smooth proper geometrically connected curve C over the finite field $\mathbb{F}_q$. For any prime $\ell \nmid q$, suppose that we are given a lisse $\mathbb{Q}_\ell$-sheaf $\mathscr{F}$ on U and let V be the $\mathbb{Q}_\ell$-vector space associated to $\mathscr{F}$ by the monodromy action. For any integer $i \geqslant 0$ we have both ordinary and compact cohomology groups

$$H^i(\overline{U}, \mathscr{F}) \quad \text{and} \quad H^i_c(\overline{U}, \mathscr{F}).$$

These are finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces on which $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts continuously and which vanish for $i > 2$. There is a natural "forget supports" map $H^i_c(\overline{U}, \mathscr{F}) \to H^i(\overline{U}, \mathscr{F})$, which need not be an isomorphism (since U is not proper). We have

$$H^0_c(\overline{U}, \mathscr{F}) = H^2(\overline{U}, \mathscr{F}) = 0.$$

Let $\eta = \mathrm{Spec}(\mathbb{F}_q(U))$ be the generic point and $\overline{\eta} = \mathrm{Spec}(\overline{\mathbb{F}_q(U)})$ the geometric point above it. We denote by $\pi_1^{\mathrm{geom}} = \pi_1(\overline{U}, \overline{\eta})$ the "geometric" fundamental group. Then

$$H^0(\overline{U}, \mathscr{F}) = V^{\pi_1^{\mathrm{geom}}}$$

is the subspace of invariants of π_1^{geom} acting on V, and

$$H^2_c(\overline{U}, \mathscr{F}) = V(-1)_{\pi_1^{\mathrm{geom}}}$$

is the space of coinvariants of π_1^{geom} acting on the Tate twist $V(-1)$ of V.

We will require information about the dimensions of these cohomology groups. The Euler characteristic of $\overline{U}$ is

$$\chi(\overline{U}) = 2 - 2g - \sum_{x \in |C \setminus U|} \deg(x), \tag{3.1}$$

where g is the genus of C and the sum is taken over the closed points $x \in C \setminus U$, with $\deg(x)$ being the degree of its residue field over $\mathbb{F}_q$. Next, the Swan conductor of $\mathscr{F}$ takes the shape

$$\mathrm{swan}(\mathscr{F}) = \sum_{x \in |C \setminus U|} \deg(x)\, \mathrm{swan}_x(\mathscr{F}).$$

It measures the wild ramification of the sheaf. With this notation to hand, the Euler–Poincaré formula (see [24, §2.3.1]) states that

$$\sum_{i=0}^{2} (-1)^i \dim H^i_c(\overline{U}, \mathscr{F}) = \mathrm{rank}(\mathscr{F})\chi(\overline{U}) - \mathrm{swan}(\mathscr{F}),$$

whence

$$\dim H^1_c(\overline{U}, \mathscr{F}) = \dim H^2_c(\overline{U}, \mathscr{F}) - \mathrm{rank}(\mathscr{F})\chi(\overline{U}) + \mathrm{swan}(\mathscr{F}).$$

We may also form a constructible $\mathbb{Q}_\ell$-sheaf $j_*\mathscr{F}$ on C, where we recall that $j : U \hookrightarrow C$ is the inclusion map. Its cohomology groups are related to the above groups via the identities

$$H^i(\overline{C}, j_*\mathscr{F}) = \begin{cases} H^0(\overline{U}, \mathscr{F}), & \text{if } i = 0, \\ \mathrm{Im}(H^1_c(\overline{U}, \mathscr{F}) \to H^1(\overline{U}, \mathscr{F})), & \text{if } i = 1, \\ H^2_c(\overline{U}, \mathscr{F}), & \text{if } i = 2. \end{cases}$$

Fix an embedding $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ and suppose that $\mathscr{F}$ is ι-pure of weight w. Then, by the fundamental work of Deligne [7, Thm. 3.2.3], it follows that $H^i(\overline{C}, j_*\mathscr{F})$ is ι-pure of weight $w + i$ for each integer $0 \leqslant i \leqslant 2$.

It follows from the facts above that

$$\dim H^i(\overline{C}, j_*\mathscr{F}) \leqslant \mathrm{rank}(\mathscr{F}) \quad \text{for } i = 0, 2 \tag{3.2}$$

and

$$\dim H^1(\overline{C}, j_*\mathscr{F}) \leqslant \mathrm{rank}(\mathscr{F})(1 - \chi(\overline{U})) + \mathrm{swan}(\mathscr{F}). \tag{3.3}$$

Now suppose that $\mathscr{F}_1$ and $\mathscr{F}_2$ are lisse $\mathbb{Q}_\ell$-sheaves on U, with $r_i = \mathrm{rank}(\mathscr{F}_i)$ for $i = 1, 2$. Arguing as in the work of Fouvry, Kowalski and Michel (see the proof of [9, Prop. 8.2(2)]), one finds that

$$\mathrm{swan}(\mathscr{F}_1 \otimes \mathscr{F}_2) \leqslant r_1 r_2 \left(\mathrm{swan}(\mathscr{F}_1) + \mathrm{swan}(\mathscr{F}_2)\right).$$

It therefore follows from (3.3) that

$$\dim H^1(\overline{C}, j_*(\mathscr{F}_1 \otimes \mathscr{F}_2)) \leqslant r_1 r_2 \left\{1 - \chi(\overline{U}) + \mathrm{swan}(\mathscr{F}_1) + \mathrm{swan}(\mathscr{F}_2)\right\}. \tag{3.4}$$

3.2. **Global L-functions.** Let $\mathscr{F}$ be a ι-pure lisse $\mathbb{Q}_\ell$-sheaf of weight w on an open subset $j : U \hookrightarrow C$. In the 1960s, Grothendieck [13] associated a global L-function $L(C, j_*\mathscr{F}, T)$ to the constructible $\mathbb{Q}_\ell$-sheaf $j_*\mathscr{F}$. It follows from the correspondence of 30/09/64 in [5] (see also [32, Conj. C_9]), that this L-function is a rational function, with

$$L(C, j_*\mathscr{F}, T) = \frac{P_1(T)}{P_0(T)P_2(T)}, \tag{3.5}$$

where $P_0, P_1, P_2 \in \mathbb{Z}[T]$ are polynomials given by

$$P_i(T) = \det\left(1 - T\,\mathrm{Fr}_q \mid H^i(\overline{C}, j_*\mathscr{F})\right)$$

for $0 \leqslant i \leqslant 2$. Here Fr_q is the Frobenius endomorphism acting on $H^i(\overline{C}, j_*\mathscr{F})$. It follows from Deligne [7] that the inverse roots of P_i have modulus $q^{(w+i)/2}$.

3.3. **The Weil conjectures.** Let V be a smooth and projective variety of dimension m which is defined over a finite field $\mathbb{F}_q$. Then V is also defined over any extension $\mathbb{F}_{q^r}$ of $\mathbb{F}_q$ and we may define the zeta function

$$Z(V,T) = \exp\left(\sum_{r=1}^{\infty} \frac{\#V(\mathbb{F}_{q^r})T^r}{r}\right).$$

According to Deligne [6] and his resolution of the Weil conjectures, the zeta function can be expressed as a rational function

$$Z(V,T) = \frac{P_1(V,T)P_3(V,T)\dots P_{2m-1}(V,T)}{P_0(V,T)P_2(V,T)\dots P_{2m}(V,T)}, \tag{3.6}$$

where

$$P_i(V,T) = \det\left(1 - T\,\mathrm{Fr}_q \mid H^i_{\text{ét}}(\overline{V}, \mathbb{Q}_\ell)\right), \tag{3.7}$$

for $i \in \{0,\dots,2m\}$ and any prime $\ell \nmid q$. Here Fr_q is the Frobenius endomorphism acting on $H^i_{\text{ét}}(\overline{V},\mathbb{Q}_\ell)$ induced by the Frobenius map on $\overline{V}$. Note that if one takes $\mathscr{F} = \mathbb{Q}_\ell$ to be the trivial sheaf in §3.2 then $Z(C,T) = L(C, j_*\mathscr{F}, T)$.

There is a factorisation

$$P_i(V,T) = \prod_{j=1}^{b_{i,\ell}} (1 - \omega_{i,j}T), \tag{3.8}$$

where $b_{i,\ell} = \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(\overline{V},\mathbb{Q}_\ell)$. Deligne shows that each $\omega_{i,j}$ is an algebraic integer with the property that $|\omega_{i,j}| = q^{i/2}$, for $1 \leqslant j \leqslant b_{i,\ell}$ and $0 \leqslant i \leqslant 2m$. A formal consequence of (3.6) and (3.8) is the identity

$$\#V(\mathbb{F}_{q^r}) = \sum_{i=0}^{2m} (-1)^i \sum_{j=1}^{b_{i,\ell}} \omega_{i,j}^r, \tag{3.9}$$

to which we will return in due course.

3.4. **Global L-functions once again.** Let X be a smooth and projective variety of dimension m defined over $K = \mathbb{F}_q(C)$ and let $\overline{X} = X \otimes_K \overline{K}$. We will need to work with models for X over the ring of integers $\mathscr{O}$ of K. Let $S \subset \Omega$ denote the finite set of places outside of which X has good reduction. The smooth projective morphism $X \to \mathrm{Spec}(K)$ extends to a smooth projective morphism $p : \mathscr{X} \to U$, for a suitable open subset U of C. (This corresponds to choosing a specific equation over $\mathscr{O}$ which has good reduction at the primes outside of S.) For any $v \in \Omega \setminus S$, we let $\mathscr{X}_v$ be the special fibre at v of $\mathscr{X}$ over $\mathscr{O}_v$. Then $\mathscr{X}_v$ is a smooth and projective $\mathscr{O}_v$-scheme such that $\mathscr{X}_v \otimes_{\mathscr{O}_v} K_v$ can be identified with $X \otimes_K K_v$. We denote by $X_v = \mathscr{X}_v \otimes_{\mathscr{O}_v} \mathbb{F}_v$ the reduction at v. This is a smooth and projective variety defined over the finite field $\mathbb{F}_v$.

For any prime $\ell \nmid q$ it will be convenient to put $H^i_\ell(X) = H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ for the geometric ℓ-adic cohomology group.

In this section, following Serre [32], we define some global L-functions associated to X and discuss their analytic properties. Let $i \in \{0, \dots, 2m\}$. For $v \in \Omega$, Serre defines the local factor

$$L_v(H^i_\ell(X), s) = \det\left(1 - \#\mathbb{F}_v^{-s}\,\mathrm{Fr}_v \mid H^i_\ell(X)^{I_v}\right)^{-1},$$

where I_v is the inertia group of v and Fr_v is the geometric Frobenius endomorphism at v. Let $P_{i,v}(T) = \det\left(1 - T\,\mathrm{Fr}_v \mid H^i_\ell(X)^{I_v}\right)$. When $v \notin S$ this coincides with the polynomial $P_i(X_v, T)$ that we met in (3.7). For arbitrary $v \in \Omega$, it follows from Terasoma [36] that $P_{i,v}(T)$ is independent of the choice of ℓ and from Deligne [7, Thm. 1.8.4] that its inverse roots have absolute value at most $q^{i/2}$. When $v \notin S$ we then have

$$Z(X_v, \#\mathbb{F}_v^{-s}) = \prod_{i=0}^{2m} L_v(H^i_\ell(X), s)^{(-1)^i}.$$

For any $i \in \{0, \dots, 2m\}$, Serre [32] defines the global L-function

$$L(H^i_\ell(X), s) = \prod_{v \in \Omega} L_v(H^i_\ell(X), s).$$

This L-function satisfies a functional equation. Associated to X is a smooth model $p : \mathscr{X} \to U$, for a suitable open subset U of C. If $j_U : U \hookrightarrow C$ is the corresponding immersion then we obtain a lisse $\mathbb{Q}_\ell$-sheaf

$$j_* H^i_\ell(X) = (j_U)_* R^i p_* \mathbb{Q}_\ell,$$

where $j : \mathrm{Spec}(K) \to C$ is the inclusion of the generic point. According to Grothendieck [5] (see also [23, §5.5]) we then have

$$L(H^i_\ell(X), s) = L(C, j_* H^i_\ell(X), q^{-s}),$$

in the notation of §3.2. Hence it follows from (3.5) that

$$L(H^i_\ell(X), s) = \frac{P_{1,i}(q^{-s})}{P_{0,i}(q^{-s}) P_{2,i}(q^{-s})}, \tag{3.10}$$

where for $k \in \{0, 1, 2\}$ one has

$$P_{k,i}(T) = \det\left(1 - T\,\mathrm{Fr}_q \mid H^k(\overline{C}, j_* H^i_\ell(X))\right) \in \mathbb{Z}[T], \tag{3.11}$$

with inverse roots having absolute value $q^{(i+k)/2}$. In particular, any poles or zeros of $L(H^i_\ell(X), s)$ must have $\Re(s) = (i+k)/2$ for $k \in \{0, 1, 2\}$.

We now specialise the previous discussion to the case of a smooth hypersurface $X \subset \mathbb{P}^{m+1}_K$ of degree d. As before, let S be the finite set of places outside of which X has good reduction and choose a smooth model $p : \mathscr{X} \to U$, for a suitable open subset U in C, which we consider fixed once and for all.

We define the discriminant Δ_X of X to be the classical discriminant of the degree d form $F \in \mathscr{O}[x_0, \dots, x_m]$ that defines $\mathscr{X}$. (See Example 4.15 of [10, Chap. 1] for its construction.) Thus Δ_X is a (non-zero) polynomial of degree $(m+1)3^m$ in the coefficients of F. In particular $\Delta_X \in \mathscr{O}$ and its prime divisors correspond to the finite places in S.

Let X_v be the reduction of X at any $v \in \Omega \setminus S$. The middle cohomology group $H_\ell^m(X)$ is the only one of interest to us, since

$$H^i_{\text{ét}}(\overline{X}_v, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(-i/2), & \text{if } i \text{ is even and } i \neq m, \\ 0, & \text{if } i \text{ is odd and } i \neq m, \end{cases}$$

for any $v \in \Omega \setminus S$ and $i \in \{0, \dots, 2m\}$ (see Ghorpade and Lachaud [11, §3], for example). It then follows from (3.9) that

$$\#X_v(\mathbb{F}_v) = \#\mathbb{F}_v^m + \#\mathbb{F}_v^{m-1} + \cdots + 1 + (-1)^m \sum_{j=1}^{b_m} \omega_{m,j}, \tag{3.12}$$

where $b_m = \dim_{\mathbb{Q}_\ell} H_\ell^m(X)$ is a positive integer that depends only on d and m (it does not depend on the choice of ℓ), and $\omega_{m,j}$ are the eigenvalues of the Frobenius endomorphism at v on $H_\ell^m(X)$, satisfying $|\omega_{m,j}| = \#\mathbb{F}_v^{m/2}$ for $1 \leqslant j \leqslant b_m$.

Taking $i = m$ we will need to control the degrees of the polynomials $P_{k,m}(T)$ appearing in (3.11). The closed points $x \in |C \setminus U|$ correspond to the prime divisors ϖ of the discriminant Δ_X that was defined above. Hence (3.1) yields

$$-\chi(\overline{U}) = 2g - 2 + O_{d,m}(\log |\Delta_X|),$$

where g is the genus of C. Moreover, as is implicit in work of Hooley [16, §6], we have $\operatorname{swan}(H_\ell^m(X)) = O_{d,m}(\log |\Delta_X|)$, since $\operatorname{swan}_x(H_\ell^m(X))$ can be bounded uniformly in terms of d and m for any closed point $x \in |C \setminus U|$. Combining (3.2) and (3.3), we deduce that

$$\deg P_{0,m} \leqslant b_m, \quad \deg P_{1,m} = O_{d,m,g}\left(1 + \log |\Delta_X|\right), \quad \deg P_{2,m} \leqslant b_m.$$

3.5. **Twisting by a character.** For the sake of simplicity, in this section we shall restrict attention to the case $K = \mathbb{F}_q(t)$ (so that $C = \mathbb{P}^1$ and $g = 0$). We continue to assume that $X \subset \mathbb{P}_K^{m+1}$ is a smooth hypersurface of degree d with associated set $S \subset \Omega$ of places outside of which X has bad reduction. We let $p : \mathscr{X} \to U$ be a smooth model, which is fixed once and for all, and we let $\Delta_X \in \mathscr{O}$ denote the corresponding discriminant. We need to consider the effect of twisting the middle cohomology group $H_\ell^m(X)$ by a fixed character of finite order.

Let $N \in \mathbb{Z}_{>0}$ and let

$$\chi_{\text{Dir}} : (\mathscr{O}_\infty / t^{-N} \mathscr{O}_\infty)^* \to \mathbb{C}^*$$

be a Dirichlet character. Putting $x = t^{-1}$ and $A = \mathbb{F}_q[x]$, we note that $(\mathscr{O}_\infty/t^{-N}\mathscr{O}_\infty)^* \cong (A/x^N A)^*$. This lifts to a character $\chi_{\mathrm{Dir}} : \mathscr{O}_\infty^* \to \mathbb{C}^*$. Given any idèle $y = (y_\varpi) \in I_K$, we may suppose that $y_\varpi = u_\varpi \varpi^{e_\varpi}$ for $u_\varpi \in \mathscr{O}_\varpi^*$ and $e_\varpi \in \mathbb{Z}$ such that $e_\varpi = 0$ for almost all ϖ. Putting $a = \prod_\varpi \varpi^{-e_\varpi} \in K^*$, we then have a unique representation $y = au$ where $u = (u_\varpi) \in \prod_\varpi \mathscr{O}_\varpi^*$ for every prime ϖ. We may now define a Hecke character $\chi_{\mathrm{Hecke}} : I_K \to \mathbb{C}^*$ via

$$\chi_{\mathrm{Hecke}}(au) = \chi_{\mathrm{Dir}}(u_\infty).$$

It is constant on K^* and gives a character on the idèle class group I_K/K^*.

There are two relevant multiplicative characters in our investigation. The first is $\eta : \mathscr{O} \to \mathbb{C}^*$, given by

$$\eta(r) = \chi_{\mathrm{Dir}}(r/t^{\deg r})$$

for any $r \in \mathscr{O}$. Note that $r/t^{\deg r} \in \mathscr{O}_\infty^*$ for any $r \in \mathscr{O}$. The second is a Dirichlet character $\eta' : (\mathscr{O}/M\mathscr{O})^* \to \mathbb{C}^*$ modulo M, for given $M \in \mathscr{O}$ which in our application will have bounded absolute value. By class field theory one can view η and η' as lisse $\mathbb{Q}_\ell$-sheafs on U of rank 1, both of which are ι-pure of weight 0. The character η is ramified only at infinity and η' is ramified only at the primes dividing M. One has $\mathrm{swan}(\eta) = O(N)$ and $\mathrm{swan}(\eta') = O_{|M|}(1)$.

We may now define the global L-function $L(\eta \otimes \eta' \otimes H_\ell^m(X), s)$, with local factors

$$L_v(\eta \otimes \eta' \otimes H_\ell^m(X), s) = \det\left(1 - \#\mathbb{F}_v^{-s}\,\mathrm{Fr}_v \mid \eta \otimes \eta' \otimes H_\ell^m(X)^{I_v}\right)^{-1},$$

for $v \in \Omega$. As before, we have

$$L(\eta \otimes \eta' \otimes H_\ell^m(X), s) = L(\mathbb{P}^1, j_*(\eta \otimes \eta' \otimes H_\ell^m(X)), q^{-s}),$$

where if $j : \mathrm{Spec}(K) \to \mathbb{P}^1$ is the inclusion of the generic point then

$$j_*(\eta \otimes \eta' \otimes H_\ell^m(X)) = (j_U)_*(\eta \otimes \eta' \otimes R^i p_* \mathbb{Q}_\ell).$$

Moreover, the analogue of (3.10) holds true. Thus

$$L(\eta \otimes \eta' \otimes H_\ell^m(X), s) = \frac{P_{1,m}(q^{-s})}{P_{0,m}(q^{-s})P_{2,m}(q^{-s})}, \tag{3.13}$$

where $P_{k,m} \in \mathbb{Z}[T]$ for $k \in \{0,1,2\}$, with inverse roots having absolute value $q^{(m+k)/2}$. Finally, using (3.4) and noting that $\mathrm{rank}(\eta \otimes \eta' \otimes H_\ell^m(X)) \leqslant b_m$, we have

$$\deg P_{0,m} \leqslant b_m, \ \ \deg P_{1,m} = O_{d,m,|M|}\left(\log|\Delta_X| + N\right), \ \ \deg P_{2,m} \leqslant b_m. \tag{3.14}$$

In our work it is the reciprocal of $L(\eta \otimes \eta' \otimes H_\ell^m(X), s)$ that features and so the location of its poles is dictated by the zeros of $P_{1,m}(q^{-s})$ in (3.13).

4. Activation of the circle method over function fields

We suppose that we are given a form $F \in \mathscr{O}[x_1, \dots, x_n]$ of degree $d \geqslant 2$ together with a vector $\mathbf{b} \in \mathscr{O}^n$ and an element $M \in \mathscr{O}$ such that $M \mid F(\mathbf{b})$. Let $\omega \in S(K_\infty^n)$ be a weight function. Then, for $P \in \mathscr{O}$ we consider the counting function

$$N(P) = \sum_{\substack{\mathbf{x} \in \mathscr{O}^n \\ F(\mathbf{x})=0 \\ \mathbf{x} \equiv \mathbf{b} \bmod M}} \omega(\mathbf{x}/P).$$

We are interested in the behaviour of this as $|P| \to \infty$, for fixed M and $\mathbf{b}$. According to (2.2) we may write

$$N(P) = \int_{\mathbb{T}} S(\alpha) \mathrm{d}\alpha,$$

where

$$S(\alpha) = \sum_{\substack{\mathbf{x} \in \mathscr{O}^n \\ \mathbf{x} \equiv \mathbf{b} \bmod M}} \psi(\alpha F(\mathbf{x})) \omega(\mathbf{x}/P).$$

We would like to dissect $\mathbb{T}$ into a disjoint union of intervals in order to try and use non-trivial averaging in our estimation of $S(\alpha)$. The starting point for this is the following analogue of Dirichlet's approximation theorem (as proved in [26, Lemma 3] or [27, Lemma 5.1], for example).

Lemma 4.1. *Let $\alpha \in K_\infty$ and let $Q > 1$. Then there exists coprime $a, r \in \mathscr{O}$, with r monic, such that $|a| < |r| \leqslant \widehat{Q}$ and*

$$|r\alpha - a| < \widehat{Q}^{-1}.$$

For any $Q > 1$ this result allows one to partition $\mathbb{T}$ into a union of intervals centred at rationals a/r. The non-archimedean nature of K ensures that the intervals are actually non-overlapping, as follows.

Lemma 4.2. *For any $Q > 1$ we have a disjoint union*

$$\mathbb{T} = \bigsqcup_{\substack{r \in \mathscr{O} \\ |r| \leqslant \widehat{Q} \\ r \text{ monic}}} \bigsqcup_{\substack{a \in \mathscr{O} \\ |a| < |r| \\ (a,r)=1}} \left\{ \alpha \in \mathbb{T} : |r\alpha - a| < \widehat{Q}^{-1} \right\}.$$

Proof. Suppose that there exists $\alpha \in \mathbb{T}$ belonging to two distinct intervals associated to $a/r \neq a'/r'$, say. Then by the ultrametric inequality we have

$$\left| \frac{a}{r} - \frac{a'}{r'} \right| \leqslant \max \left\{ \left| \frac{a}{r} - \alpha \right|, \left| \frac{a'}{r'} - \alpha \right| \right\} < \frac{1}{\widehat{Q} \min\{|r|, |r'|\}}.$$

On the other hand, since $ar'-a'r$ is a non-zero element of $\mathscr{O}$, we have

$$\left|\frac{a}{r}-\frac{a'}{r'}\right| \geqslant \frac{1}{|rr'|} \geqslant \frac{1}{\widehat{Q}\min\{|r|,|r'|\}}.$$

This is a contradiction, which thereby establishes the lemma. □

It follows from Lemma 4.2 that

$$N(P)=\sum_{\substack{r\in\mathscr{O}\\ |r|\leqslant\widehat{Q}\\ r \text{ monic}}}\ \sideset{}{^*}\sum_{|a|<|r|}\int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}} S\left(\frac{a}{r}+\theta\right)\mathrm{d}\theta, \tag{4.1}$$

where we henceforth put

$$\sideset{}{^*}\sum_{|a|<|r|} = \sum_{\substack{a\in\mathscr{O}\\ |a|<|r|\\ (a,r)=1}}.$$

Remark 4.3. The reader will note that there is no division into major and minor arcs in our expression for $N(P)$. In the classical setting over $\mathbb{Q}$ this would correspond to the opening steps of a Kloosterman refinement, a device which is rendered essentially trivial over function fields.

We may write

$$S\left(\frac{a}{r}+\theta\right)=\sum_{\substack{\mathbf{y}\in\mathscr{O}^n\\ |\mathbf{y}|<|r_M|\\ \mathbf{y}\equiv\mathbf{b} \bmod M}}\psi\left(\frac{aF(\mathbf{y})}{r}\right)\sum_{\mathbf{z}\in\mathscr{O}^n}\psi(\theta F(\mathbf{y}+r_M\mathbf{z}))\omega\left(\frac{\mathbf{y}+r_M\mathbf{z}}{P}\right),$$

where $r_M=rM/(r,M)$ is the least common multiple of r and M. We evaluate the inner sum over $\mathbf{z}$ using Poisson summation. Thus Lemma 2.1 implies that

$$\begin{aligned}\sum_{\mathbf{z}\in\mathscr{O}^n}&\psi(\theta F(\mathbf{y}+r_M\mathbf{z}))\omega\left(\frac{\mathbf{y}+r_M\mathbf{z}}{P}\right)\\ &=\sum_{\mathbf{c}\in\mathscr{O}^n}\int_{K_\infty^n}\omega\left(\frac{\mathbf{y}+r_M\mathbf{u}}{P}\right)\psi(\theta F(\mathbf{y}+r_M\mathbf{u})+\mathbf{c}.\mathbf{u})\mathrm{d}\mathbf{u}.\end{aligned}$$

Making the change of variables $\mathbf{x}=(\mathbf{y}+r_M\mathbf{u})P$, it follows from Lemma 2.3 (together with the fact that the measure on K_∞^n is translation invariant) that the right hand side is

$$\left|\frac{P}{r_M}\right|^n\sum_{\mathbf{c}\in\mathscr{O}^n}\psi\left(\frac{-\mathbf{c}.\mathbf{y}}{r_M}\right)\int_{K_\infty^n}\omega(\mathbf{x})\psi\left(\theta P^dF(\mathbf{x})+\frac{P\mathbf{c}.\mathbf{x}}{r_M}\right)\mathrm{d}\mathbf{x}.$$

Putting everything together in (4.1), we may now establish the following result.

Lemma 4.4. *We have*

$$N(P)=|P|^n\sum_{\substack{r\in\mathscr{O}\\ |r|\leqslant\widehat{Q}\\ r\text{ monic}}}|r_M|^{-n}\int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}}\sum_{\mathbf{c}\in\mathscr{O}^n}S_{r,M,\mathbf{b}}(\mathbf{c})I_{r_M}(\theta;\mathbf{c})\mathrm{d}\theta,$$

where $r_M=rM/(r,M)$ *and*

$$S_{r,M,\mathbf{b}}(\mathbf{c})=\sideset{}{^*}\sum_{|a|<|r|}\sum_{\substack{\mathbf{y}\in\mathscr{O}^n\\ |\mathbf{y}|<|r_M|\\ \mathbf{y}\equiv\mathbf{b}\bmod M}}\psi\left(\frac{aF(\mathbf{y})}{r}\right)\psi\left(\frac{-\mathbf{c}.\mathbf{y}}{r_M}\right),$$

$$I_s(\theta;\mathbf{c})=\int_{K_\infty^n}\omega(\mathbf{x})\psi\left(\theta P^dF(\mathbf{x})+\frac{P\mathbf{c}.\mathbf{x}}{s}\right)\mathrm{d}\mathbf{x}.$$

The exponential integrals can be estimated using the results in §2.4 provided that the weight function ω is chosen suitably. The exponential sums $S_{r,M,\mathbf{b}}(\mathbf{c})$ satisfy the following basic multiplicativity property.

Lemma 4.5. *Let $r=r_1r_2$ for coprime $r_1,r_2\in\mathscr{O}$. Let $M=M_1M_2M_3$ for $M_1,M_2,M_3\in\mathscr{O}$ such that $M_1\mid r_1^\infty$, $M_2\mid r_2^\infty$ and $(M_3,r)=1$. Then there exists $\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\in(\mathscr{O}/M\mathscr{O})^n$, depending on $\mathbf{b},M$ and the residue of r_1,r_2 modulo M, such that*

$$S_{r,M,\mathbf{b}}(\mathbf{c})=S_{r_1,M_1,\mathbf{b}_1}(\mathbf{c})S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})\psi\left(\frac{-\mathbf{c}.\mathbf{b}_3}{M_3}\right).$$

Proof. Let us put $s_i=r_iM_i/(r_i,M_i)$ for $i=1,2$. Then s_1,s_2,M_3 are pairwise coprime and we have a factorisation $r_M=s_1s_2M_3$. As $\mathbf{y}_1$ ranges over vectors modulo s_1, $\mathbf{y}_2$ ranges modulo s_2, and $\mathbf{y}_3$ ranges modulo M_3, so the vector

$$\mathbf{y}=s_2M_3\mathbf{y}_1+s_1M_3\mathbf{y}_2+s_1s_2\mathbf{y}_3$$

ranges over a complete set of residues modulo r_M. Likewise, as a_1 (resp. a_2) ranges over elements of $\mathscr{O}$ modulo r_1 (resp. modulo r_2), which are coprime to r_1 (resp. r_2), so $a=r_2a_1+r_1a_2$ ranges over a complete set of residues modulo r, which are coprime to r. It is now clear that

$$\psi\left(\frac{aF(\mathbf{y})}{r}\right)=\psi\left(\frac{a_1s_2^3M_3^3F(\mathbf{y}_1)}{r_1}\right)\psi\left(\frac{a_2s_1^3M_3^3F(\mathbf{y}_2)}{r_2}\right)$$

and

$$\psi\left(\frac{-\mathbf{c}.\mathbf{y}}{r_M}\right)=\psi\left(\frac{-\mathbf{c}.\mathbf{y}_1}{s_1}\right)\psi\left(\frac{-\mathbf{c}.\mathbf{y}_2}{s_2}\right)\psi\left(\frac{-\mathbf{c}.\mathbf{y}_3}{M_3}\right).$$

Choose $t_1,t_2,t_3\in\mathscr{O}$ such that $t_1M_3s_2\equiv1\bmod M_1$, $t_2M_3s_1\equiv1\bmod M_2$ and $t_3s_1s_2\equiv1\bmod M_3$. Then it is clear that the statement of the lemma holds with $\mathbf{b}_i=t_i\mathbf{b}\bmod M$, for $1\leqslant i\leqslant3$. $\square$

The importance of Lemma 4.5 is that it allows us to factorise the exponential sum in which we are interested, so that it suffices to examine the sum at the prime power moduli. When piecing these together it will be important to bear in mind the following convention that will henceforth be adopted.

Definition 4.6. Associated to any $r \in \mathscr{O}$ and $i \in \mathbb{Z}_{>0}$ are the elements

$$b_i = \prod_{\varpi^i \| r} \varpi^i, \quad k_i = \prod_{\varpi^i \| r} \varpi, \quad r_i = \prod_{\substack{\varpi^e \| r \\ e \geqslant i}} \varpi^e,$$

in $\mathscr{O}$. In particular, for any $j \in \mathbb{Z}_{>0}$ we have the factorisation

$$r = r_{j+1} \prod_{i=1}^{j} b_i = r_{j+1} \prod_{i=1}^{j} k_i^i, \quad \text{(with } (j+1)\text{-full } r_{j+1}).$$

5. Cubic exponential sums: basic estimates

We now specialise to the case of non-singular cubic forms $F \in \mathscr{O}[x_1, \dots, x_n]$ under the hypothesis that $\mathrm{char}(\mathbb{F}_q) > 3$. We define the associated Hessian matrix

$$\mathbf{H}(\mathbf{x}) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{1 \leqslant i,j \leqslant n}. \tag{5.1}$$

Our assumption on the characteristic of $\mathbb{F}_q$ ensures that this matrix doesn't vanish identically. Of special importance to us will be the dual form

$$F^* \in \mathscr{O}[x_1, \dots, x_n],$$

whose zero locus parameterises the set of hyperplanes whose intersection with the cubic hypersurface $F = 0$ produce a singular variety. It is well-known that F^* is absolutely irreducible and has degree $3 \cdot 2^{n-2}$.

This section is devoted to a suite of estimates for the complete cubic exponential sum

$$S_{r,M,\mathbf{b}}(\mathbf{c}) = \sideset{}{^*}\sum_{|a|<|r|} \sum_{\substack{\mathbf{y} \in \mathscr{O}^n \\ |\mathbf{y}|<|r_M| \\ \mathbf{y} \equiv \mathbf{b} \bmod M}} \psi\left(\frac{aF(\mathbf{y})}{r} \right) \psi\left(\frac{-\mathbf{c}.\mathbf{y}}{r_M} \right),$$

both pointwise and on average over $\mathbf{c}$. We recall that $r, M \in \mathscr{O}$ and $\mathbf{b}, \mathbf{c} \in \mathscr{O}^n$, with $M \mid F(\mathbf{b})$ and $r_M = rM/(r, M)$.

We begin by focusing our attention on the exponential sum $S_{\varpi^\alpha,M,\mathbf{b}}(\mathbf{c})$ for a prime ϖ and an integer $\alpha \geqslant 1$. We will typically do so for large primes. In particular we will have $\varpi \nmid M$ for all of the primes considered in this section, so that

$$S_{\varpi^\alpha,M,\mathbf{b}}(\mathbf{c}) = S_{\varpi^\alpha,1,\mathbf{0}}(\mathbf{c}) = S_{\varpi^\alpha}(\mathbf{c}),$$

say.

The cases $\alpha \in \{1,2\}$. Suppose that $\varpi \nmid M$ and $\alpha \in \{1,2\}$. Then

$$S_{\varpi^\alpha}(\mathbf{c}) = \sideset{}{^*}\sum_{|a|<|\varpi^\alpha|} \sum_{|\mathbf{y}|<|\varpi^\alpha|} \psi\left(\frac{aF(\mathbf{y}) - \mathbf{c}.\mathbf{y}}{\varpi^\alpha}\right).$$

When $\alpha = 1$ it follows from Heath-Brown [15, Lemma 12] and Hooley [18, Lemma 60] that there is a constant $A(n, |\Delta_F|) > 0$ depending only on n and $|\Delta_F|$ such that

$$S_\varpi(\mathbf{c}) \leqslant A(n, |\Delta_F|)|\varpi|^{(n+1)/2}|(\varpi, \nabla F^*(\mathbf{c}))|^{1/2}. \tag{5.2}$$

These estimates are founded on the work of Deligne [6].

Suppose next that $\alpha = 2$. We write $a = a_1 + \varpi a_2$ and $\mathbf{y} = \mathbf{y}_1 + \varpi \mathbf{y}_2$, for $a_i, \mathbf{y}_i$ running modulo ϖ. Then

$$\begin{aligned} S_{\varpi^2}(\mathbf{c}) = \sideset{}{^*}\sum_{|a_1|<|\varpi|} \sum_{|a_2|<|\varpi|} \sum_{|\mathbf{y}_1|<|\varpi|} \psi\left(\frac{a_1F(\mathbf{y}_1) - \mathbf{c}.\mathbf{y}_1}{\varpi^2}\right) \\ \times \sum_{|\mathbf{y}_2|<|\varpi|} \psi\left(\frac{(a_1\nabla F(\mathbf{y}_1) - \mathbf{c}).\mathbf{y}_2 + a_2F(\mathbf{y}_1)}{\varpi}\right). \end{aligned}$$

The inner sum over $\mathbf{y}_2$ vanishes unless $a_1\nabla F(\mathbf{y}_1) \equiv \mathbf{c} \bmod \varpi$. Likewise, the sum over a_2 vanishes unless $\varpi \mid F(\mathbf{y}_1)$. It follows that

$$|S_{\varpi^2,M,\mathbf{b}}(\mathbf{c})| \leqslant |\varpi|^{n+1}N$$

where N is the number of $a_1, \mathbf{y}_1 \bmod \varpi$ such that $a_1\nabla F(\mathbf{y}_1) \equiv \mathbf{c} \bmod \varpi$ and $(a_1, \varpi) = 1$ and $\varpi \mid F(\mathbf{y}_1)$. But this is now a problem about point counting over finite fields and the argument used by Hooley [18, Lemma 11] yields $N = 0$ if $\varpi \nmid F^*(\mathbf{c})$ and $N = O(|\varpi|)$ otherwise. This therefore shows that there is a constant $A(n, |\Delta_F|) > 0$ such that

$$S_{\varpi^2}(\mathbf{c}) \leqslant A(n, |\Delta_F|)|\varpi|^{n+1}|(\varpi, F^*(\mathbf{c}))|. \tag{5.3}$$

Recalling the notation in Definition 4.6, we may now combine Lemma 4.5 with (5.2) and (5.3) to deduce the following result.

Lemma 5.1. *There is a constant $A = A(n, |\Delta_F|) > 0$ such that*

$$|S_{b_1b_2,M,\mathbf{b}}(\mathbf{c})| \leqslant A^{\omega(b_1b_2)}|b_1b_2|^{(n+1)/2}|(b_1, \nabla F^*(\mathbf{c}))|^{1/2}|(k_2, F^*(\mathbf{c}))|,$$

uniformly in $\mathbf{b} \in \mathscr{O}^n$.

The case $\alpha > 2$. Suppose that $\varpi \nmid M$ and that $\alpha > 2$ is an integer. Evaluating the sum over a, we begin by noting that

$$\begin{aligned} S_{\varpi^\alpha}(\mathbf{c}) &= \sum_{\substack{\mathbf{y}\in\mathscr{O}^n \\ |\mathbf{y}|<|\varpi|^\alpha}} \psi\left(\frac{-\mathbf{c}.\mathbf{y}}{\varpi^\alpha}\right)\left(\sum_{|a_1|<|\varpi|^\alpha}\psi\left(\frac{a_1F(\mathbf{y})}{\varpi^\alpha}\right) - \sum_{|a_2|<|\varpi|^{\alpha-1}}\psi\left(\frac{a_2F(\mathbf{y})}{\varpi^{\alpha-1}}\right)\right) \\ &= |\varpi|^\alpha \sum_{\substack{\mathbf{y}\in\mathscr{O}^n \\ |\mathbf{y}|<|\varpi|^\alpha \\ F(\mathbf{y})\equiv 0 \bmod \varpi^\alpha}} \psi\left(\frac{\mathbf{c}.\mathbf{y}}{\varpi^\alpha}\right) - |\varpi|^{\alpha-1} \sum_{\substack{\mathbf{y}\in\mathscr{O}^n \\ |\mathbf{y}|<|\varpi|^\alpha \\ F(\mathbf{y})\equiv 0 \bmod \varpi^{\alpha-1}}} \psi\left(\frac{\mathbf{c}.\mathbf{y}}{\varpi^\alpha}\right) \\ &= |\varpi|^\alpha S_1(\varpi^\alpha,\mathbf{c}) - |\varpi|^{\alpha-1}S_2(\varpi^\alpha,\mathbf{c}), \end{aligned}$$

say. Substituting $\mathbf{y} = \mathbf{y}_1 + \varpi^{\alpha-1}\mathbf{y}_2$, we get

$$S_2(\varpi^\alpha,\mathbf{c}) = \sum_{\substack{\mathbf{y}_1\in\mathscr{O}^n \\ |\mathbf{y}_1|<|\varpi|^{\alpha-1} \\ F(\mathbf{y}_1)\equiv 0 \bmod \varpi^{\alpha-1}}} \psi\left(\frac{\mathbf{c}.\mathbf{y}_1}{\varpi^\alpha}\right) \sum_{\substack{\mathbf{y}_2\in\mathscr{O}^n \\ |\mathbf{y}_2|<|\varpi|}} \psi\left(\frac{\mathbf{c}.\mathbf{y}_2}{\varpi}\right).$$

This term clearly vanishes if $\varpi \nmid \mathbf{c}$. Therefore

$$S_{\varpi^\alpha}(\mathbf{c}) = |\varpi|^\alpha S_1(\varpi^\alpha,\mathbf{c}), \quad \text{if } \alpha > 1 \text{ and } \varpi \nmid \mathbf{c}.$$

For $\alpha > 1$ and $\varpi \nmid \mathbf{c}$, the argument in [19, §6] goes through to give

$$S_{\varpi^\alpha}(\mathbf{c}) = \frac{|\varpi|}{|\varpi|-1}\left\{|\varpi|^\alpha\nu_1(\varpi^\alpha,\mathbf{c}) - |\varpi|^{\alpha-1}\nu_2(\varpi^\alpha,\mathbf{c})\right\},$$

where $\nu_1(\varpi^\alpha,\mathbf{c})$ denotes the number of incongruent solutions modulo ϖ^α of the conditions

$$F(\mathbf{y}) \equiv 0 \bmod \varpi^\alpha, \quad \mathbf{c}.\mathbf{y} \equiv 0 \bmod \varpi^\alpha, \quad \mathbf{y} \not\equiv \mathbf{0} \bmod \varpi,$$

whereas $\nu_2(\varpi^\alpha,\mathbf{c})$ is the number of solutions modulo ϖ^α of

$$F(\mathbf{y}) \equiv 0 \bmod \varpi^\alpha, \quad \mathbf{c}.\mathbf{y} \equiv 0 \bmod \varpi^{\alpha-1}, \quad \mathbf{y} \not\equiv \mathbf{0} \bmod \varpi.$$

We may now conclude as follows.

Lemma 5.2. *We have $S_{\varpi^\alpha}(\mathbf{c}) = 0$ if $\alpha > 1$ and $\varpi \nmid MF^*(\mathbf{c})$.*

The following result is the function field analogue of the union of Lemmas 12–15 in [19] . The desired estimates are established in exactly the same manner and the necessary arguments will not be repeated here.

Lemma 5.3. *Let $\varpi \nmid \mathbf{c}$ be a prime such that $|\varpi| \gg 1$ and $\varpi \mid F^*(\mathbf{c})$. Let r denote the minimal value of the rank modulo ϖ of the Hessian $\mathbf{H}(\mathbf{y})$, where $\mathbf{y}$*

runs over the vectors which contribute to $\nu_1(\varpi^2,\mathbf{c})$. Then we have

$$S_{\varpi^\alpha}(\mathbf{c}) \ll \begin{cases} |\varpi|^{2n+3-r/2}, & \text{if } \alpha=3,\\ |\varpi|^{(\alpha-1)n+4-r}, & \text{if } \alpha\geqslant 4,\\ |\varpi|^{(\alpha-1)n+6-2r}, & \text{if } \alpha\geqslant 6,\end{cases}$$

with $r\geqslant 2$. If $\varpi \mid \mathbf{c}$, then for $\alpha=3$ or 4 we have

$$S_{\varpi^\alpha}(\mathbf{c}) \ll |\varpi|^{(\alpha-1)n+3}.$$

The estimates in this result are true for a given value of $r\geqslant 2$ which depends on the value of $\mathbf{c}$. According to Hooley (see [19, Eq. (56)]), associated to each prime ϖ is an affine algebraic variety $V_\varpi\subset \mathbb{A}^n_{\mathbb{F}_\varpi}$, with dimension

$$D(\varpi)\leqslant \begin{cases} r-1, & \text{if } r=n-1 \text{ or } n,\\ r, & \text{if } 2\leqslant r\leqslant n-2,\end{cases} \tag{5.4}$$

such that the estimates in Lemma 5.3 are true for a given value of $r\geqslant 2$ when the reduction of $\mathbf{c}$ modulo ϖ is constrained to lie in V_ϖ.

6. Cubic exponential sums: averages

Recall Definition 4.6 and the attendant notation b_i,k_i,r_i associated to an element $r\in\mathscr{O}$. Throughout this section $M\in\mathscr{O}$ will denote a generic fixed integer and $\mathbf{b}$ mod M such that $M\mid F(\mathbf{b})$ will also be regarded as fixed. In particular, the implied constant in any estimate is allowed to depend on $|\mathbf{b}|$ and $|M|$. The purpose of this section is to estimate $|S_{r_3,M,\mathbf{b}}(\mathbf{c})|$ on average over $\mathbf{c}$. We shall follow the strategy in [15] and [19], although several of our arguments are closer in spirit to those found in [1, §5].

We begin by recording the trio of estimates that we shall require, before moving onto a discussion of their proofs. The first result we need is the analogue of [19, Lemma 16]

Lemma 6.1. *For any $C\geqslant 1$ and any $\varepsilon>0$ we have*

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ |\mathbf{c}|<\widehat{C}}} |S_{r_3,M,\mathbf{b}}(\mathbf{c})| \ll |r_3|^{n/2+1+\varepsilon}\left(|r_3|^{n/3}+\widehat{C}^n\right)$$

and

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ |\mathbf{c}|<\widehat{C}}} |S_{b_3,M,\mathbf{b}}(\mathbf{c})| \ll |b_3|^{n/2+2/3+\varepsilon}\left(|b_3|^{n/3}+\widehat{C}^n\right).$$

Our remaining results concern averages of $|S_{r_3,M,\mathbf{b}}(\mathbf{c})|$ over sparser sets of $\mathbf{c}$. The following result is a slight sharpening of the analogous results in [15, Lemma 16] and [17, Lemma 12].

$\gamma_{j,1}$	$\gamma_{j,2}$	Conditions on j
$2n+\frac{13}{2}$	$3n+4$	$j=4$
$4n+\frac{7}{2}$	$6n$	$j=6$
$\frac{j}{2}(n+\frac{7}{2})-\frac{3}{4}$	$\frac{j}{2}(2n+1)-\frac{n}{2}+\frac{1}{2}$	j odd
$\frac{j}{2}(n+\frac{7}{2})-1$	$\frac{j}{2}(2n+1)-1$	$j\notin\{4,6\}$ even

TABLE 1. Value of the exponents $\gamma_{j,i}$ in (6.1)

Lemma 6.2. *For any $C\geqslant 1$ and any $\varepsilon>0$ we have*

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\setminus\{\mathbf{0}\}\\ |\mathbf{c}|<\widehat{C}\\ F^*(\mathbf{c})=0}} |S_{r_3,M,\mathbf{b}}(\mathbf{c})| \ll |r_3|^{\varepsilon}\widehat{C}^{\varepsilon}\left(|b_3|^{5n/6+2/3}|r_4|^{n+1/2}+\widehat{C}^{n-3/2}|r_3|^{n/2+4/3}\right).$$

The final bound involves a summation over an even sparser set of vectors $\mathbf{c}$. In order to proceed we recall the definition of the functions $G_1(r)$ and $G_2(r)$ that appear in Hooley's work. For any $r\in\mathscr{O}$ and $i=1,2$, let

$$G_i(r)=\prod_{\varpi^j\|r}\varpi^{\gamma_{j,i}}, \tag{6.1}$$

where the values of $\gamma_{j,i}$ are given in Table 1 and are extracted from [19, Eqs. (83), (84)]. We are now ready to record the following result, which is the analogue of [19, Lemma 21].

Lemma 6.3. *Let $n=8$. For any $C\geqslant 1$ and any $\varepsilon>0$ we have*

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\setminus\mathbf{0}\\ |\mathbf{c}|<\widehat{C}\\ \nabla F^*(\mathbf{c})=\mathbf{0}}} |S_{r_3,M,\mathbf{b}}(\mathbf{c})| \ll |r_3|^{\varepsilon}\widehat{C}^{\varepsilon}\left(|G_1(r_3)|\widehat{C}^{n-5/2}+|G_2(r_3)|\right).$$

With reference to Table 1, when $n=8$ we easily deduce that

$$\frac{|G_1(r_3)|}{|r_3|^{n/2+2}}\leqslant\frac{1}{|b_3|^{1/2}|b_4|^{3/8}|b_5|^{2/5}|b_6|^{1/12}|b_7|^{5/14}|r_8|^{1/4}} \tag{6.2}$$

and

$$\frac{|G_2(r_3)|}{|r_3|^{n/2+2}}\leqslant|b_3|^{4/3}|b_4||b_5|^{9/5}|b_6|^{2}|b_7|^{2}|b_8|^{19/8}|r_9|^{5/2}. \tag{6.3}$$

In particular, it follows from these bounds and Lemma 6.3 that

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\setminus\mathbf{0}\\ |\mathbf{c}|<\widehat{C}\\ \nabla F^*(\mathbf{c})=\mathbf{0}}} \frac{|S_{r_3,M,\mathbf{b}}(\mathbf{c})|}{|r_3|^{n/2+2}} \ll |r_3|^{\varepsilon}\widehat{C}^{\varepsilon}\left(\frac{\widehat{C}^{n-5/2}}{|b_3|^{1/2}}+|b_3|^{4/3}|r_4|^{5/2}\right), \qquad (6.4)$$

when $n=8$.

We will provide reasonably detailed proofs of Lemma 6.1 and Lemma 6.2, but the proof of Lemma 6.3 will not be given here. The latter is closely based on ideas already present in the proofs of the preceding lemmas, with the added information about the behaviour at small prime powers that is provided by Lemma 5.3. The changes required for the function field analogue of [19, Lemma 21] are tedious, routine and do not merit repetition here.

6.1. **Proof of Lemma 6.1.** We begin by establishing the second part of the lemma. It follows from multiplicativity and Lemmas 5.2 and 5.3 that

$$|S_{b_3,M,\mathbf{b}}(\mathbf{c})| \ll |b_3|^{\varepsilon}\prod_{\varpi\mid b_3}|\varpi|^{2n+3-r(\varpi,\mathbf{c})/2+R(\varpi,\mathbf{c})}, \qquad (6.5)$$

where $R(\varpi,\mathbf{c})=0$ if $r(\varpi,\mathbf{c})>1$ and $R(\varpi,\mathbf{c})=1/2$ if $r(\varpi,\mathbf{c})=1$. Here we stress that the value of $r(\varpi,\mathbf{c})$ depends only on the value of $\mathbf{c}$ modulo ϖ^2. Recall from Definition 4.6 the notation $b_3=k_3^3$ and note that there are at most $(\widehat{C}/|k_3|+1)^n$ choices of $\mathbf{c}\in\mathscr{O}^n$ for which $|\mathbf{c}|<\widehat{C}$ and $\mathbf{c}\equiv\mathbf{a} \bmod k_3$. But then, on invoking (5.4) and the remark after Lemma 5.3, we easily deduce that

$$\begin{aligned}\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ |\mathbf{c}|<\widehat{C}}}|S_{b_3,M,\mathbf{b}}(\mathbf{c})| &\ll |b_3|^{\varepsilon}\left(\frac{\widehat{C}}{|k_3|}+1\right)^n\sum_{\mathbf{a} \bmod k_3}\prod_{\varpi\mid b_3}|\varpi|^{2n+3-r(\varpi,\mathbf{a})/2+R(\varpi,\mathbf{a})}\\ &\ll |k_3|^{2n+3}|b_3|^{\varepsilon}\left(\frac{\widehat{C}}{|k_3|}+1\right)^n\sum_{2\leqslant r\leqslant n}\prod_{\varpi\mid b_3}|\varpi|^{D(\varpi)-r/2}\\ &\ll |k_3|^{5n/2+2}|b_3|^{\varepsilon}\left(\frac{\widehat{C}}{|k_3|}+1\right)^n\end{aligned}$$

This completes the proof of the second part of Lemma 6.1.

We now turn to the first part of the lemma. In fact, future work will deem it convenient to establish the following more general version, in which the implied constant is made more explicit.

Lemma 6.4. *Let $\mathbf{r} \in K_\infty^n$, let $C \geqslant 1$ and let $\varepsilon > 0$. Then there exists a constant $c_{n,\varepsilon} > 0$, depending only on n and ε, such that*

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ |\mathbf{c}-\mathbf{r}|<\widehat{C}}} |S_{r_3,M,\mathbf{b}}(\mathbf{c})| \leqslant c_{n,\varepsilon}|M|^n|\Delta_F|^{2n}H_F^{n/2}|r_3|^{n/2+1+\varepsilon}\left(|r_3|^{n/3}+\widehat{C}^n\right).$$

The statement of Lemma 6.1 easily follows on taking $\mathbf{r} = \mathbf{0}$ in this result. During the proof of Lemma 6.4 we will reserve c_n (resp. $c_{n,\varepsilon}$) for a generic positive constant that depends only on n (resp. n and ε). Recall the definition (5.1) of the Hessian matrix $\mathbf{H}(\mathbf{x})$ associated to the cubic form F. For any $m \in \mathscr{O}$ and any $\mathbf{k} \in \mathscr{O}^n$ let

$$N_m(\mathbf{k}) = \#\{\mathbf{y} \bmod m : \mathbf{H}(\mathbf{k})\mathbf{y} \equiv \mathbf{0} \bmod m\}. \tag{6.6}$$

We will need the following result, which is an analogue of [1, Lemma 13].

Lemma 6.5. *For any $m \in \mathscr{O}$ and $R \geqslant 1$ there exists a constant $c_n > 0$ such that*

$$\sum_{\substack{\mathbf{k}\in\mathscr{O}^n\\ |\mathbf{k}|<\widehat{R}}} N_m(\mathbf{k})^{1/2} \leqslant c_n H_F^{n/2}|m|^{n/2}\left(1+\frac{\widehat{R}^3}{|m|}\right)^{n/2}.$$

Proof. Let D denote the degree of m. Given $K \geqslant 1$, let

$$S_K = \{\mathbf{y} \in \mathscr{O}^n : |\mathbf{y}| < \widehat{D-K}\} \quad \text{and} \quad S_K^1 = \{t^{D-K}\mathbf{y} : |\mathbf{y}| < \widehat{K}\}.$$

For any $\mathbf{y} \in \mathscr{O}^n$ such that $|\mathbf{y}| < |m|$, we write $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1 \in S_K$ and $\mathbf{y}_2 \in S_K^1$. Thus

$$N_m(\mathbf{k}) = \sum_{\mathbf{y}_2\in S_K^1} \sum_{\substack{\mathbf{y}_1\in S_K\\ \mathbf{H}(\mathbf{k})(\mathbf{y}_1+\mathbf{y}_2)\equiv\mathbf{0} \bmod m}} 1 \leqslant \widehat{K}^n\#\{\mathbf{y} \in S_K : \mathbf{H}(\mathbf{k})\mathbf{y} \equiv \mathbf{0} \bmod m\},$$

since if $\mathbf{y}_1 + \mathbf{y}_2$ and $\mathbf{y}_1' + \mathbf{y}_2$ are both counted by the inner sum then we have $\mathbf{y}_3 = \mathbf{y}_1 - \mathbf{y}_1' \in S_K$ and $\mathbf{H}(\mathbf{k})\mathbf{y}_3 \equiv \mathbf{0} \bmod m$.

Choosing K such that $\widehat{K} = H_F\widehat{R}$, we find that

$$|\mathbf{H}(\mathbf{k})\mathbf{y}| < \frac{H_F\widehat{R}|m|}{\widehat{K}} = |m|$$

for any $\mathbf{y} \in S_K$. Thus, for $\mathbf{y} \in S_K$ we have $\mathbf{H}(\mathbf{k})\mathbf{y} \equiv \mathbf{0} \bmod m$ if and only if $\mathbf{H}(\mathbf{k})\mathbf{y} = \mathbf{0}$. It follows that

$$N_m(\mathbf{k}) \leqslant \widehat{K}^n\#\{\mathbf{y} \in S_K : \mathbf{H}(\mathbf{k})\mathbf{y} = \mathbf{0}\} = (H_F\widehat{R})^n\left(\frac{|m|}{H_F\widehat{R}}\right)^{n-\varrho(\mathbf{k})},$$

where $\varrho(\mathbf{k}) = \operatorname{rank} \mathbf{H}(\mathbf{k})$. Hence

$$\sum_{|\mathbf{k}|<\widehat{R}} N_m(\mathbf{k})^{1/2} \leqslant H_F^{n/2} \widehat{R}^{n/2} \sum_{r=0}^{n} \left(\frac{|m|}{\widehat{R}}\right)^{(n-r)/2} \#\{|\mathbf{k}| < \widehat{R} : \varrho(\mathbf{k}) = r\}.$$

According to [1, Lemma 2], the condition $\varrho(\mathbf{k}) \leqslant r$ forces $\mathbf{k}$ to lie in an affine variety $T_r \subset \mathbb{A}_K^n$ of dimension at most r and degree $O_n(1)$. Hence Lemma 2.8 implies that there is a positive constant $c_n > 0$ such that

$$\#\{|\mathbf{k}| < \widehat{R} : \varrho(\mathbf{k}) = r\} \leqslant \#\{\mathbf{k} \in T_r(\mathscr{O}) : |\mathbf{k}| < \widehat{R}\} \leqslant c_n \widehat{R}^r.$$

It follows that

$$\begin{aligned}\sum_{|\mathbf{k}|<\widehat{R}} N_m(\mathbf{k})^{1/2} &\leqslant c_n H_F^{n/2} \widehat{R}^{n/2} \sum_{r=0}^{n} \left(\frac{|m|}{\widehat{R}}\right)^{(n-r)/2} \widehat{R}^r \\ &\leqslant (n+1) c_n H_F^{n/2} |m|^{n/2} \left(1 + \frac{\widehat{R}^3}{|m|}\right)^{n/2}.\end{aligned}$$

The statement of the lemma is now clear. $\square$

It will be convenient to relate $S_{r_3,M,\mathbf{b}}(\mathbf{c})$ to the exponential sum

$$T(a,s;\mathbf{c}) = \sum_{\substack{\mathbf{z}\in\mathscr{O}^n \\ |\mathbf{z}|<|s|}} \psi\left(\frac{ag(\mathbf{z}) - \mathbf{c}.\mathbf{z}}{s}\right),$$

for appropriate $g \in \mathscr{O}[x_1, \dots, x_n]$, $a, s \in \mathscr{O}$ with $(a,s) = 1$ and $\mathbf{c} \in \mathscr{O}^n$. These sums satisfy the following multiplicativity property.

Lemma 6.6. *Suppose that $s_1, s_2 \in \mathscr{O}$ are coprime and let $\overline{s}_1, \overline{s}_2 \in \mathscr{O}$ be chosen so that $s_1\overline{s}_1 + s_2\overline{s}_2 = 1$. Then $T(a, s_1s_2; \mathbf{c}) = T(a\overline{s}_2, s_1; \overline{s}_2\mathbf{c})T(a\overline{s}_1, s_2; \overline{s}_1\mathbf{c})$.*

Proof. As $\mathbf{z}_1$ ranges over vectors in $\mathscr{O}^n$ modulo s_1 and $\mathbf{z}_2$ ranges over vectors modulo s_2, so $\mathbf{z} = s_2\overline{s}_2\mathbf{z}_1 + s_1\overline{s}_1\mathbf{z}_2$ ranges over a complete set of residues modulo s_1s_2. Moreover, we clearly have

$$ag(\mathbf{z}) - \mathbf{c}.\mathbf{z} \equiv s_2\overline{s}_2 \{ag(\mathbf{z}_1) - \mathbf{c}.\mathbf{z}_1\} + s_1\overline{s}_1 \{ag(\mathbf{z}_2) - \mathbf{c}.\mathbf{z}_2\} \bmod s_1s_2,$$

since $(s_i\overline{s}_i)^j \equiv s_i\overline{s}_i \bmod s_1s_2$ for $i \in \{1,2\}$ and all $j \geqslant 1$. The desired result now follows easily. $\square$

Making the change of variables $\mathbf{y} = \mathbf{b} + M\mathbf{z}$ we obtain

$$S_{r_3,M,\mathbf{b}}(\mathbf{c}) = \psi\left(\frac{-\mathbf{c}.\mathbf{b}}{r_3M/(r_3,M)}\right) \sideset{}{^*}\sum_{|a|<|r_3|} T\left(a, \frac{r_3}{(r_3,M)}; \mathbf{c}\right), \tag{6.7}$$

with underlying polynomial

$$g(\mathbf{z}) = \frac{1}{(r_3, M)} F(M\mathbf{z} + \mathbf{b}). \tag{6.8}$$

This is a cubic polynomial with coefficients in $\mathscr{O}$ since $M \mid F(\mathbf{b})$. Moreover it has non-singular homogeneous cubic part $g_0(\mathbf{z}) = (r_3, M)^{-1} M^3 F(\mathbf{z})$. We now factorise $r_3/(r_3, M)$ into a cube-free part and a cube-full part. Since r_3 is cube-full it follows that the cube-free part has absolute value at most $|M|$. Applying Lemma 6.6 and estimating the contribution from the cube-free part trivially it follows from (6.7) that

$$|S_{r_3,M,\mathbf{b}}(\mathbf{c})| \leqslant |M|^n \sideset{}{^*}\sum_{|a|<|r_3|} |T(\bar{b}a, s; \bar{b}\mathbf{c})|$$

for some cube-full $s \in \mathscr{O}$ with $s \mid r_3$, together with some element $\bar{b} \in \mathscr{O}$ with $|\bar{b}| \leqslant |M|$ and $(\bar{b}, s) = 1$. To prove Lemma 6.4, it will therefore suffice to show that there is a constant $c_{n,\varepsilon} > 0$ depending only on n and ε such that

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n \\ |\mathbf{c}-\mathbf{r}|<\widehat{C}}} |T(a, s; \mathbf{c})| \leqslant c_{n,\varepsilon} |\Delta_F|^{2n} H_F^{n/2} |s|^{n/2+\varepsilon} \left(\widehat{C}^n + |s|^{n/3}\right), \tag{6.9}$$

for any cube-full $s \in \mathscr{O}$, any $a \in \mathscr{O}$ which is coprime to s and any $C \geqslant 1$.

We henceforth write $s = c^2 d$, where $d \mid c$ and

$$d = \prod_{\substack{\varpi^e \| s \\ e\geqslant 3,\ 2\nmid e}} \varpi. \tag{6.10}$$

Following the opening argument in [1, Lemma 11] more or less verbatim, we easily conclude that

$$|T(a, s; \mathbf{c})| \leqslant |c^2 d|^{n/2} \sum_{\substack{|\mathbf{u}|<|c| \\ a\nabla g(\mathbf{u})-\mathbf{c}\equiv\mathbf{0} \bmod c}} M_d(\mathbf{u})^{1/2},$$

where

$$M_m(\mathbf{u}) = \#\{\mathbf{y} \bmod m : \nabla^2 g(\mathbf{u})\mathbf{y} \equiv \mathbf{0} \bmod m\}. \tag{6.11}$$

Let us denote the left hand side of (6.9) by $\mathscr{M}(C)$. Then our work so far shows that

$$\mathscr{M}(C) \leqslant |c^2 d|^{n/2} \sum_{|\mathbf{c}-\mathbf{r}|<\widehat{C}} \sum_{\substack{|\mathbf{u}|<|c| \\ a\nabla g(\mathbf{u})-\mathbf{c}\equiv\mathbf{0} \bmod c}} M_d(\mathbf{u})^{1/2}.$$

Let $\varepsilon > 0$. Then it follows from [1, Lemma 14] that there is a constant $c_{n,\varepsilon} > 0$ depending only on n and ε such that

$$\begin{aligned} \sum_{|\mathbf{u}|<|d|} M_d(\mathbf{u}) &= \#\{\mathbf{u}, \mathbf{y} \bmod d : \nabla^2 g(\mathbf{u})\mathbf{y} \equiv 0 \bmod d\} \\ &\leqslant c_{n,\varepsilon} |\Delta_F|^{2n} |d|^{n+\varepsilon}. \end{aligned} \tag{6.12}$$

Our argument now differs according to whether $|c| < \widehat{C}$ or $|c| \geqslant \widehat{C}$. Beginning with the former case, we have

$$\begin{aligned} \mathscr{M}(C) &\leqslant |c^2 d|^{n/2} \sum_{|\mathbf{u}|<|c|} M_d(\mathbf{u})^{1/2} \#\{|\mathbf{c}-\mathbf{r}| < \widehat{C} : a\nabla g(\mathbf{u}) - \mathbf{c} \equiv \mathbf{0} \bmod c\} \\ &= |c^2 d|^{n/2} \left(\frac{\widehat{C}}{|c|}\right)^n \sum_{|\mathbf{u}|<|c|} M_d(\mathbf{u})^{1/2} \\ &\leqslant |c^2 d|^{n/2} \left(\frac{\widehat{C}}{|c|}\right)^n \left(\frac{|c|}{|d|}\right)^n \sum_{|\mathbf{u}|<|d|} M_d(\mathbf{u}). \end{aligned}$$

This is at most $c_{n,\varepsilon}|\Delta_F|^{2n}|c^2 d|^{n/2+\varepsilon}\widehat{C}^n$, by (6.12).

Next, suppose that $|c| \geqslant \widehat{C}$. Starting as above we note that

$$\#\{|\mathbf{c}-\mathbf{r}| < \widehat{C} : a\nabla g(\mathbf{u}) - \mathbf{c} \equiv \mathbf{0} \bmod c\} = \sum_{\mathbf{h}\in\mathscr{O}^n} w\left(\frac{a\nabla g(\mathbf{u}) - \mathbf{r} - c\mathbf{h}}{t^C}\right),$$

where w is given by (2.1). Now it follows from Lemma 2.1 that

$$\sum_{\mathbf{h}\in\mathscr{O}^n} w\left(\frac{a\nabla g(\mathbf{u}) - \mathbf{r} - c\mathbf{h}}{t^C}\right) = \sum_{\mathbf{k}\in\mathscr{O}^n} \int_{K_\infty^n} w\left(\frac{a\nabla g(\mathbf{u}) - \mathbf{r} - c\mathbf{x}}{t^C}\right) \psi(\mathbf{k}.\mathbf{x}) \mathrm{d}\mathbf{x}.$$

But this is equal to

$$\left(\frac{\widehat{C}}{|c|}\right)^n \sum_{\mathbf{k}\in\mathscr{O}^n} \psi\left(\frac{a\mathbf{k}.\nabla g(\mathbf{u}) - \mathbf{r}.\mathbf{k}}{c}\right) \int_{\mathbb{T}^n} \psi\left(\frac{t^C \mathbf{k}.\mathbf{y}}{c}\right) \mathrm{d}\mathbf{y},$$

whence an application of Lemma 2.2 yields

$$\mathscr{M}(C) \leqslant \frac{|c^2 d|^{n/2}\widehat{C}^n}{|c|^n} \sum_{\substack{\mathbf{k}\in\mathscr{O}^n \\ |\mathbf{k}|<|c|/\widehat{C}}} \sigma_{\mathbf{k}},$$

where

$$\sigma_{\mathbf{k}} = \sum_{|\mathbf{y}|<|d|} M_d(\mathbf{y})^{1/2} \sum_{\substack{|\mathbf{u}|<|c| \\ \mathbf{u}\equiv\mathbf{y} \bmod d}} \psi\left(\frac{a\mathbf{k}.\nabla g(\mathbf{u}) - \mathbf{r}.\mathbf{k}}{c}\right).$$

We proceed with an application of Cauchy's inequality and (6.12), to obtain

$$\begin{aligned}|\sigma_{\mathbf{k}}|^2 &\leqslant c_{n,\varepsilon}|\Delta_F|^{2n}|d|^{n+\varepsilon}\sum_{|\mathbf{y}|<|d|}\left|\sum_{\substack{|\mathbf{u}|<|c|\\ \mathbf{u}\equiv\mathbf{y} \bmod d}}\psi\left(\frac{a\mathbf{k}.\nabla g(\mathbf{u})}{c}\right)\right|^2\\ &\leqslant c_{n,\varepsilon}|\Delta_F|^{2n}|d|^{n+\varepsilon}\sum_{\substack{|\mathbf{u}_1|,|\mathbf{u}_2|<|c|\\ \mathbf{u}_1\equiv\mathbf{u}_2 \bmod d}}\psi\left(\frac{a\mathbf{k}.(\nabla g(\mathbf{u}_1)-\nabla g(\mathbf{u}_2))}{c}\right).\end{aligned}$$

Writing $\mathbf{u}_1=\mathbf{u}_2+d\mathbf{z}$ and recalling (6.8), we see that

$$\nabla g(\mathbf{u}_1)-\nabla g(\mathbf{u}_2)=d(r_3,M)^{-1}M^3\mathbf{H}(\mathbf{z})\mathbf{u}_2$$

plus a term which in independent of $\mathbf{u}_2$. Hence there exists $m\in\mathscr{O}$, with $|m|\leqslant|c/d|$, such that $|\sigma_{\mathbf{k}}|^2\leqslant c_{n,\varepsilon}|\Delta_F|^{2n}|d|^{n+\varepsilon}|c|^nN_m(\mathbf{k})$, in the notation of (6.6). It now follows from Lemma 6.5 that

$$\begin{aligned}\mathscr{M}(C) &\leqslant c_{n,\varepsilon}^{1/2}|\Delta_F|^n\frac{|c^2d|^{n/2}\widehat{C}^n}{|c|^n}\sum_{\substack{\mathbf{k}\in\mathscr{O}^n\\ |\mathbf{k}|<|c|/\widehat{C}}}|d|^{n/2+\varepsilon}|c|^{n/2}N_m(\mathbf{k})^{1/2}\\ &\leqslant c_nc_{n,\varepsilon}^{1/2}|\Delta_F|^nH_F^{n/2}\frac{|c^2d|^{n/2}\widehat{C}^n|d|^{n/2+\varepsilon}|c|^{n/2}}{|c|^n}\left(|m|+\frac{|c|^3}{\widehat{C}^3}\right)^{n/2}\\ &\leqslant c_nc_{n,\varepsilon}^{1/2}|\Delta_F|^{2n}H_F^{n/2}|c^2d|^{n/2+\varepsilon}\widehat{C}^n\left(1+\frac{|c^2d|}{\widehat{C}^3}\right)^{n/2}.\end{aligned}$$

In view of our earlier work this bound is also valid when $|c|<\widehat{C}$.

Let $D=\deg(c^2d)$. We therefore arrive at the desired bound (6.9) on noting that $\mathscr{M}(C)\leqslant\mathscr{M}(\max\{C,\frac{1}{3}D\})$.

6.2. **Proof of Lemma 6.2.** In addition to taking into account the sparsity of vectors $\mathbf{c}$ for which $F^*(\mathbf{c})=0$, in the proof of Lemma 6.2 we will also need to sum non-trivially over a in the definition of $S_{r_3,M,\mathbf{b}}(\mathbf{c})$.

To begin with we factorise $r_3=b_3r_4$ and use Lemma 4.5 to factorise the sum $S_{r_3,M,\mathbf{b}}(\mathbf{c})$. The sum corresponding to b_3 we estimate using (6.5). For the sum involving r_4 we return to (6.7) and relate the exponential sum to $T(a,s;\mathbf{c})$ for a quartic-full $s\in\mathscr{O}$. Abusing notation slightly, this leads to the preliminary estimate

$$\sum_{\substack{|\mathbf{c}|<\widehat{C}\\ F^*(\mathbf{c})=0}}|S_{r_3,M,\mathbf{b}}(\mathbf{c})|\ll|r_3|^\varepsilon\sum_{\substack{|\mathbf{c}|<\widehat{C}\\ F^*(\mathbf{c})=0}}\prod_{\varpi|b_3}|\varpi|^{2n+3-r(\varpi,\mathbf{c})/2+R(\varpi,\mathbf{c})}\left|\sideset{}{^*}\sum_{|a|<|r_4|}T(a,r_4;\mathbf{c})\right|.$$

The term involving b_3 only depends on $\mathbf{c}$ modulo k_3. Thus, arguing as in the proof of the second part of Lemma 6.1, we break the $\mathbf{c}$-sum into residue classes modulo k_3 and deduce that

$$\sum_{\substack{|\mathbf{c}|<\widehat{C}\\ F^*(\mathbf{c})=0}} |S_{r_3,M,\mathbf{b}}(\mathbf{c})| \ll |k_3|^{2n+3}|r_3|^{\varepsilon} \sum_{\mathbf{a} \bmod k_3} \Sigma(\mathbf{a}) \prod_{\varpi | b_3} |\varpi|^{-r(\varpi,\mathbf{a})/2+R(\varpi,\mathbf{a})},$$

where

$$\Sigma(\mathbf{a}) = \sum_{\substack{|\mathbf{c}|<\widehat{C}\\ F^*(\mathbf{c})=0\\ \mathbf{c}\equiv \mathbf{a} \bmod k_3}} \left| \sideset{}{^*}\sum_{|a|<|r_4|} T(a,r_4;\mathbf{c}) \right|.$$

We will show that

$$\Sigma(\mathbf{a}) \ll |r_4|^{n+1/2+\varepsilon} + \frac{|r_4|^{n/2+4/3+\varepsilon}\widehat{C}^{n-3/2+\varepsilon}}{|b_3|^{n/3-1/2}}. \tag{6.13}$$

Recollecting (5.4), we can insert this into the above estimate in order to conclude the proof of Lemma 6.2.

In order to prove (6.13), we write $r_4 = c^2 d$ as before, with d given by (6.10). The argument in [15, §7] now goes through more or less verbatim, leading to the bound

$$\sideset{}{^*}\sum_{|a|<|r_4|} T(a,r_4;\mathbf{c}) \ll |c|^{n+1}|d|^{n/2+1} \sideset{}{^*}\sum_{|a_1|<|c|} \sum_{\substack{|\mathbf{u}|<|c|\\ a_1\nabla g(\mathbf{u})-\mathbf{c}\equiv \mathbf{0} \bmod c\\ g(\mathbf{u})\equiv 0 \bmod c}} M_d(\mathbf{u})^{1/2},$$

in the notation of (6.11). Making the change of variables $\mathbf{h} = M\mathbf{u}+\mathbf{b}$, we deduce that there are elements c', d' with $d' \mid c'$ and $|c'|$ (resp. $|d'|$) of order $|c|$ (resp. $|d|$), such that

$$\sum_{\substack{|\mathbf{u}|<|c|\\ a_1\nabla g(\mathbf{u})-\mathbf{c}\equiv \mathbf{0} \bmod c\\ g(\mathbf{u})\equiv 0 \bmod c}} M_d(\mathbf{u})^{1/2} = \sum_{\substack{|\mathbf{h}|<|c'|\\ a_1\nabla F(\mathbf{h})-\mathbf{c}\equiv \mathbf{0} \bmod c'\\ F(\mathbf{h})\equiv 0 \bmod c'}} N_{d'}(\mathbf{h})^{1/2}.$$

Summing trivially over a_1, we now find that

$$\sum_{\substack{|\mathbf{c}|<\widehat{C}\\ F^*(\mathbf{c})=0\\ \mathbf{c}\equiv \mathbf{a} \bmod k_3}} \left| \sideset{}{^*}\sum_{|a|<|r_4|} T(a,r_4;\mathbf{c}) \right| \ll |c|^{n+2}|d|^{n/2+1}\mathscr{N} \sum_{\substack{|\mathbf{h}|<|c'|\\ F(\mathbf{h})\equiv 0 \bmod c'}} N_{d'}(\mathbf{h})^{1/2}, \tag{6.14}$$

where

$$\mathscr{N} = \max_{|\mathbf{r}|<|k_3c'|} \#\left\{ \mathbf{c}\in \mathscr{O}^n : |\mathbf{c}| < \widehat{C},\ F^*(\mathbf{c})=0,\ \mathbf{c}\equiv \mathbf{r} \bmod k_3c' \right\}.$$

The equation $F^*(\mathbf{c}) = 0$ cuts out an absolutely irreducible hypersurface in $\mathbb{A}^n$ of dimension $n-1$. Hence it follows from Lemma 2.10 that

$$\mathscr{N} \ll \left(\frac{\widehat{C}}{|k_3 c|} + 1 \right)^{n-3/2}. \tag{6.15}$$

It remains to analyse the sum

$$S(c,d) = \sum_{\substack{|\mathbf{h}|<|c| \\ F(\mathbf{h}) \equiv 0 \bmod c}} N_d(\mathbf{h})^{1/2},$$

for given $c, d \in \mathscr{O}$ such that d is square-free and $d \mid c$. We will show that

$$S(c,d) \ll |c|^{n-1+\varepsilon} |d|^{1/2}.$$

Once combined with (6.15) in (6.14), this gives the desired bound (6.13) on noting that $|d| \leqslant |c^2 d|^{1/3} = |r_4|^{1/3}$. The sum in question satisfies $S(c_1c_2, d_1d_2) = S(c_1, d_1)S(c_2, d_2)$ for any $c_i, d_i \in \mathscr{O}$ such that $(c_1d_1, c_2d_2) = 1$ and $d_i \mid c_i$. Hence it will suffice to show that

$$S_1 = S(\varpi^e, 1) \ll |\varpi|^{e(n-1)} \quad \text{and} \quad S_2 = S(\varpi^e, \varpi) \ll |\varpi|^{e(n-1)+1/2},$$

for any $e \in \mathbb{Z}_{>0}$ and any prime ϖ. This is achieved by closely following the argument of Heath-Brown [15, page 245]. The estimation of S_1 uses exponential sums and an application of Lemma 6.1 with $C = 1$. The main ingredient in the estimation of S_2 is (6.12). Given that the arguments of [15, page 245] carry over verbatim to the function field setting, they will not be repeated here.

7. Return to the main counting function

Recall our standing assumption that $\mathrm{char}(\mathbb{F}_q) > 3$, together with the definition (5.1) of the Hessian matrix associated to our non-singular cubic form $F \in \mathscr{O}[x_1, \dots, x_n]$. The proof of [17, Lemma 1] shows that there exists a point $\mathbf{x}_0 \in K_\infty^n$ satisfying

$$F(\mathbf{x}_0) = 0, \quad \det \mathbf{H}(\mathbf{x}_0) \neq 0, \quad |\mathbf{x}_0| < 1/H_F. \tag{7.1}$$

An inspection of the proof reveals that the result is false in characteristic 2 or 3 when F is cubic. Such a point will automatically satisfy $\nabla F(\mathbf{x}_0) \neq \mathbf{0}$, since F is non-singular.

Next, let $L \geqslant 1$ be an integer. We define the weight function $\omega : K_\infty^n \to \mathbb{R}_{\geqslant 0}$ via

$$\omega(\mathbf{x}) = w\left(t^L(\mathbf{x} - \mathbf{x}_0)\right), \tag{7.2}$$

where w is given by (2.1). Ultimately, L will be taken to be a large but fixed integer. For L large enough, it is clear that

$$|\mathbf{x}| < 1/H_F \quad \text{and} \quad |\det \mathbf{H}(\mathbf{x})| = |\det \mathbf{H}(\mathbf{x}_0)|, \tag{7.3}$$

for any $\mathbf{x} \in K_\infty^n$ such that $\omega(\mathbf{x}) \neq 0$.

Let $\mathbf{b} \in \mathscr{O}^n$ and let $M \in \mathscr{O}$ such that $M \mid F(\mathbf{b})$. It is clear that $\omega \in S(K_\infty^n)$ and we are interested in the asymptotic behaviour of the counting function

$$N(P) = \sum_{\substack{\mathbf{x} \in \mathscr{O}^n \\ F(\mathbf{x})=0 \\ \mathbf{x} \equiv \mathbf{b} \bmod M}} \omega(\mathbf{x}/P), \tag{7.4}$$

as $|P| \to \infty$. The quantities $\mathbf{x}_0, \mathbf{b}, M, L$ are to be considered fixed once and for all. Consequently, all our implied constants are allowed to depend on these quantities as well as on the height H_F of F.

Our main result concerning the behaviour of $N(P)$ is as follows.

Theorem 7.1. *Suppose that $n = 8$. Then there exists constants $c \geqslant 0$ and $\delta > 0$ such that*

$$N(P) = c|P|^{n-3} + O(|P|^{n-3-\delta}).$$

The constant c is a Hardy–Littlewood product of local densities, with $c > 0$ if for every finite prime ϖ there exists $\mathbf{x} \in \mathscr{O}_\varpi^n$ such that $F(\mathbf{x}) = 0$ and $|\mathbf{x} - \mathbf{b}|_\varpi < |M|_\varpi$.

In §7.1 we show how this result implies the statement of Theorem 1.2. Next, in §7.2, we initiate our analysis of $N(P)$ along the lines of §4. The outcome of this first phase of the argument is recorded in Lemma 7.2. The main contribution to $N(P)$ comes from the trivial characters, which is what we analyse in §7.3. It is here that the explicit value of the leading constant c is recorded. Finally, §7.4 is devoted to a preliminary analysis of the contribution from the non-trivial characters.

7.1. **Deduction of Theorem 1.2.** This section shows how Theorem 7.1 implies Theorem 1.2. Let $X \subset \mathbb{P}_K^{n-1}$ be a non-singular cubic hypersurface defined by a cubic form F over K with $n \geqslant 8$ variables. Assume that $X(K_v) \neq \emptyset$ for every place $v \in \Omega$. In order to establish the Hasse principle and weak approximation, we need to show that $X(K) \neq \emptyset$ and $X(K)$ is dense in $X(\mathbf{A}_K)$ under the product topology.

Using a familiar fibration argument, we use induction on the number of variables $n \geqslant 8$, supposing for the moment that it is has been verified when $n = 8$. Thus let $n \geqslant 9$ and let H_1, H_2 be generic hyperplanes in $\mathbb{P}_K^{n-1}$ defined over K. We consider the fibration $\pi : X \to \mathbb{P}_K^1$ with fibres $X_{\lambda,\mu} = X \cap H_{\lambda,\mu}$, where $H_{\lambda,\mu} = \lambda H_1 + \mu H_2$. By the Lefschetz hyperplane theorem $\mathrm{Pic}(X)$ is a free abelian group of rank 1 generated by the class of a hyperplane section Y.

All fibres of π are therefore geometrically integral. Indeed if a fibre $X_{\lambda,\mu}$ were reducible, say $X_{\lambda,\mu} = Y_1 + Y_2$, then Y_1, Y_2 would give independent elements of the Picard group of X which are not multiples of Y, which is impossible. Moreover $\mathbb{P}^1_K$ satisfies the Hasse principle and weak approximation, as do the smooth fibres by the inductive hypothesis. A standard argument (see Skorobogatov [34], for example) therefore yields the desired conclusion subject to a satisfactory treatment of the case $n = 8$.

Henceforth suppose that $n = 8$. Let S be a finite set of primes of K. Suppose that we are given points $x_\infty \in X(K_\infty)$ and $x_\varpi \in X(K_\varpi)$ for each $\varpi \in S$. We wish to prove that there exists a rational point $x \in X(K)$ which is simultaneously close to these local points in their respective topologies. Since the Hessian does not vanish identically on X, there is no loss of generality in assuming that x_∞ doesn't lie on the Hessian variety.

Let N_∞, N be positive integers. We choose representative coordinates so that $x_\infty = [\mathbf{x}_\infty]$ for $\mathbf{x}_\infty \in \mathbb{T}^n$ such that $|\mathbf{x}_\infty| < 1/H_F$ and $x_\varpi = [\mathbf{x}_\varpi]$ for $\mathbf{x}_\varpi \in \mathscr{O}_\varpi^n$, for each $\varpi \in S$. We need to show that there exists a non-zero vector $\mathbf{z} \in K^n$ such that $F(\mathbf{z}) = 0$, with

$$|\mathbf{z} - \mathbf{x}_\infty| < \widehat{N}_\infty^{-1} \quad \text{and} \quad |\mathbf{z} - \mathbf{x}_\varpi|_\varpi < |\varpi|_\varpi^{-N}, \text{ for all } \varpi \in S. \tag{7.5}$$

Combining weak approximation for $\mathscr{O}^n$ with the Chinese remainder theorem, we can find a vector $\mathbf{b} \in \mathscr{O}^n$ such that $\mathbf{b} \equiv \mathbf{x}_\varpi \bmod \varpi^N$ for every $\varpi \in S$. Let $M = \prod_{\varpi \in S} \varpi^N$ and let B run through elements of $\mathscr{O}$ for which $B \equiv 1 \bmod M$. For $|B|$ suitably large we will show that there is a vector $\mathbf{x} \in \mathscr{O}^n$ such that $F(\mathbf{x}) = 0$, with

$$|\mathbf{x} - B\mathbf{x}_\infty| < \widehat{N}_\infty^{-1}|B| \quad \text{and} \quad \mathbf{x} \equiv \mathbf{b} \bmod M.$$

We claim that the vector $\mathbf{z} = \mathbf{x}/B \in K^n$ will satisfy the conditions required to draw the desired conclusion. Now it is clear that $F(\mathbf{z}) = 0$ and that the restriction at the infinite place in (7.5) is satisfied. Moreover, for any $\varpi \in S$ we will have $|\mathbf{z} - \mathbf{x}_\varpi|_\varpi < |\varpi|_\varpi^{-N}$ if and only if $|\mathbf{x} - \mathbf{x}_\varpi|_\varpi < |\varpi|_\varpi^{-N}$, since $B \equiv 1 \bmod \varpi^N$. But this follows from the fact that

$$|\mathbf{x} - \mathbf{x}_\varpi|_\varpi \leqslant \max\{|\mathbf{x} - \mathbf{b}|_\varpi, |\mathbf{b} - \mathbf{x}_\varpi|_\varpi\} < |\varpi|_\varpi^{-N}.$$

It will therefore suffice to study the counting function $N(P)$ in (7.4), with $\mathbf{x}_0 = \mathbf{x}_\infty$ and $L = N_\infty$. Indeed, our arguments so far show that the Hasse principle and weak approximation hold when $n = 8$, if we are able to show that

$$N(P) > 0,$$

for $P \in \mathscr{O}$ such that $|P| \to \infty$. But this follows directly from the statement of Theorem 7.1.

7.2. **Preliminary analysis of $N(P)$.** Our starting point is Lemma 4.4, which gives

$$N(P)=|P|^n\sum_{\substack{r\in\mathscr{O}\\ |r|\leqslant\widehat{Q}\\ r \text{ monic}}}|r_M|^{-n}\int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}}\sum_{\mathbf{c}\in\mathscr{O}^n}S_{r,M,\mathbf{b}}(\mathbf{c})I_{r_M}(\theta;\mathbf{c})\mathrm{d}\theta, \tag{7.6}$$

where $r_M=rM/(r,M)$ and $S_{r,M,\mathbf{b}}(\mathbf{c}),I_{r_M}(\theta;\mathbf{c})$ are as in the statement of lemma. We proceed to use the results of §2.4 to study $I_r(\theta;\mathbf{c})$ for given $r\in\mathscr{O}$. In view of (7.2) and Lemma 2.3, we have

$$\begin{aligned}
I_r(\theta;\mathbf{c})&=\int_{K_\infty^n}w\left(t^L(\mathbf{x}-\mathbf{x}_0)\right)\psi\left(\theta P^3F(\mathbf{x})+\frac{P\mathbf{c}.\mathbf{x}}{r}\right)\mathrm{d}\mathbf{x}\\
&=\frac{1}{\widehat{L}^n}\psi\left(\frac{P\mathbf{c}.\mathbf{x}_0}{r}\right)\int_{K_\infty^n}w\left(\mathbf{y}\right)\psi\left(\theta P^3F(\mathbf{x}_0+t^{-L}\mathbf{y})+\frac{Pt^{-L}\mathbf{c}.\mathbf{y}}{r}\right)\mathrm{d}\mathbf{y}\\
&=\frac{1}{\widehat{L}^n}\psi\left(\frac{P\mathbf{c}.\mathbf{x}_0}{r}\right)J_G\left(\theta P^3;\frac{Pt^{-L}\mathbf{c}}{r}\right),
\end{aligned}\tag{7.7}$$

in the notation of (2.4), where $G(\mathbf{y})=F(\mathbf{x}_0+t^{-L}\mathbf{y})$. It is clear that G is a polynomial with coefficients in K_∞ and height $H_G\leqslant H_F$.

According to Lemma 2.6 we have $J_G(\theta P^3;Pt^{-L}\mathbf{c}/r)=0$ if

$$\frac{|P||\mathbf{c}|}{|r|}>\widehat{L}\max\{1,|P|^3|\theta|H_F\}.$$

Hence we may truncate the sum over $\mathbf{c}$ in (7.6) to arrive at the following result.

Lemma 7.2. *We have*

$$N(P)=|P|^n\sum_{\substack{r\in\mathscr{O}\\ |r|\leqslant\widehat{Q}\\ r \text{ monic}}}|r_M|^{-n}\int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}}\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ |\mathbf{c}|\leqslant\widehat{C}}}S_{r,M,\mathbf{b}}(\mathbf{c})I_{r_M}(\theta;\mathbf{c})\mathrm{d}\theta,$$

where $\widehat{C}=\widehat{L}H_F|r_M||P|^{-1}\max\{1,|\theta||P|^3\}$.

We will need a good upper bound for $I_{r_M}(\theta;\mathbf{c})$, for $r,\theta,\mathbf{c}$ appearing in the expression for $N(P)$ in this lemma. This need is met by the following result.

Lemma 7.3. *We have*

$$|I_{r_M}(\theta;\mathbf{c})|\ll H_F^n\max\{1,|\theta||P|^3\}^{-n/2}.$$

Proof. When $|\mathbf{c}|\leqslant\widehat{C}$ we put $\gamma=\theta P^3$ and $\mathbf{w}=Pt^{-L}\mathbf{c}/r_M$, for convenience. In particular we have

$$|\mathbf{w}|\leqslant H_F\max\{1,|\gamma|\}.$$

It then follows from Lemma 2.7 that

$$\begin{aligned}|I_{r_M}(\theta;\mathbf{c})| &\leqslant \frac{1}{\widehat{L}^n}\,|J_G(\gamma;\mathbf{w})|\\ &\leqslant \frac{1}{\widehat{L}^n}\operatorname{meas}\left\{\mathbf{y}\in\mathbb{T}^n : |\gamma\nabla G(\mathbf{y})+\mathbf{w}|\leqslant H_G\max\{1,|\gamma|^{1/2}\}\right\}\\ &\leqslant \operatorname{meas}(\mathscr{R}),\end{aligned}$$

where

$$\mathscr{R}=\left\{\mathbf{x}\in\mathbb{T}^n : |\mathbf{x}-\mathbf{x}_0|<\widehat{L}^{-1},\ |\gamma\nabla F(\mathbf{x})+\mathbf{w}|\leqslant H_F\max\{1,|\gamma|^{1/2}\}\right\}.$$

We would like to estimate the measure of this region. Recall that $\mathbf{x}_0$ satisfies (7.1). The parameter $L\geqslant 0$ is chosen large enough that (7.3) holds for all $\mathbf{x}\in K_\infty^n$ such that $|\mathbf{x}-\mathbf{x}_0|<\widehat{L}^{-1}$.

To begin with, if $|\gamma|\leqslant 1$ then we take the trivial bound $\operatorname{meas}(\mathscr{R})\leqslant 1$. Let us suppose instead that $|\gamma|>1$. If $\mathbf{x}$ and $\mathbf{x}+\mathbf{x}'$ are both in $\mathscr{R}$ then

$$|\nabla F(\mathbf{x}+\mathbf{x}')-\nabla F(\mathbf{x})|\leqslant H_F|\gamma|^{-1/2}.$$

But

$$|\nabla F(\mathbf{x}+\mathbf{x}')-\nabla F(\mathbf{x})-\mathbf{H}(\mathbf{x})\mathbf{x}'|\leqslant H_F|\mathbf{x}'|^2.$$

Using the inverse of $\mathbf{H}(\mathbf{x})$, whose entries each have absolute value $O(1)$, we find that

$$|\mathbf{x}'|\ll H_F\max\left\{|\gamma|^{-1/2},|\mathbf{x}'|^2\right\}.$$

This implies that $|\mathbf{x}'|\ll H_F|\gamma|^{-1/2}$, since $|\mathbf{x}'|<1/H_F$. We have therefore shown that

$$\operatorname{meas}(\mathscr{R})\ll H_F^n\min\{1,|\gamma|^{-n/2}\},\tag{7.8}$$

which concludes the proof of the lemma. □

7.3. **The main term.** In this section we investigate the contribution to $N(P)$ in Lemma 7.2 coming from $\mathbf{c}=\mathbf{0}$. Let us denote this term by $M(P)$. We will always assume that $n\geqslant 8$. Recalling the definition of $I_{r_M}(\theta;\mathbf{0})$ from Lemma 4.4, we find that

$$M(P)=|P|^n\sum_{\substack{r\in\mathscr{O}\\ |r|\leqslant\widehat{Q}\\ r\text{ monic}}}|r_M|^{-n}S_{r,M,\mathbf{b}}(\mathbf{0})K_r,$$

where

$$K_r=\int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}}\int_{K_\infty^n}\omega(\mathbf{x})\psi\left(\theta P^3F(\mathbf{x})\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\theta.$$

It follows from Lemma 7.3 that $K_r=O(|P|^{-3})$ for any r. Moreover, we recall from (7.2) that $\omega(\mathbf{x})=w(t^L(\mathbf{x}-\mathbf{x}_0))$ in K_r, where w is given by (2.1), L is a

large fixed integer and $\mathbf{x}_0$ satisfies (7.1). In particular $\nabla F(\mathbf{x}_0) \neq \mathbf{0}$ and we let $\xi \in \mathbb{Z}$ be such that
$$\widehat{\xi} = |\nabla F(\mathbf{x}_0)|.$$
In particular $|\xi| < 1$.

We begin with the following basic result.

Lemma 7.4. *For any $Y \geqslant 1$ and any $\varepsilon > 0$, we have*
$$\sum_{\substack{r \in \mathscr{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} |r_M|^{-n} |S_{r,M,\mathbf{b}}(\mathbf{0})| \ll \widehat{Y}^{5/4 - n/6 + \varepsilon}.$$

Proof. We factorise any r in the summation as $r = b_1 b_2 b_3 r_4$ and use the multiplicativity property Lemma 4.5 that is enjoyed by $S_{r,M,\mathbf{b}}(\mathbf{0})$. For the modulus $b_1 b_2$ we apply Lemma 5.1. For the modulus b_3 (resp. r_4) we use the second (resp. first) part of Lemma 6.1 with $C = 1$. This leads to the conclusion that
$$\begin{aligned} S_{r,M,\mathbf{b}}(\mathbf{0}) &\ll |r|^{\varepsilon} |b_1 b_2|^{n/2+1} |b_3|^{5n/6+2/3} |r_4|^{5n/6+1} \\ &\ll |r|^{n/2+1+\varepsilon} |b_3|^{n/3-1/3} |r_4|^{n/3}. \end{aligned}$$
Hence
$$\sum_{\substack{r \in \mathscr{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} |r_M|^{-n} |S_{r,M,\mathbf{b}}(\mathbf{0})| \ll \widehat{Y}^{1-n/6+\varepsilon} \sum_{|b_3 r_4| \leqslant \widehat{Y}} |b_3|^{-1/3} \sum_{|b_1 b_2| = \widehat{Y}/|b_3 r_4|} \frac{1}{|b_1 b_2|^{n/3}}.$$
The inner sum is absolutely convergent and there are $O(\widehat{R}^{1/j})$ elements $r_j \in \mathscr{O}$ such that $|r_j| \leqslant \widehat{R}$. Summing first over r_4 we see that the resulting sum over b_3 is absolutely convergent, which therefore completes the proof of the lemma. $\square$

Let us put $C = \widehat{L - \xi}$. Since $K_r = O(|P|^{-3})$ and $5/4 - n/6 < 0$ for $n \geqslant 8$, Lemma 7.4 implies that there exists $\delta > 0$ such that the overall contribution to $M(P)$ from r satisfying $C^{-1}\widehat{Q} \leqslant |r| \leqslant \widehat{Q}$ is $O(|P|^{n-3-\delta})$. On the remaining range for r we will actually show that K_r is independent of r. Let
$$J_\theta = \int_{K_\infty^n} \omega(\mathbf{x}) \psi\left(\theta P^3 F(\mathbf{x})\right) \mathrm{d}\mathbf{x}.$$
We then have
$$K_r = \int_{|\theta| < C|P|^{-3}} J_\theta \, \mathrm{d}\theta + \int_{C|P|^{-3} \leqslant |\theta| < |r|^{-1}\widehat{Q}^{-1}} J_\theta \, \mathrm{d}\theta.$$
The first integral is independent of r and the second integral is over a non-empty interval if and only if $|r| < C^{-1}\widehat{Q}^{-1}|P|^3 = C^{-1}\widehat{Q}$.

Recalling (7.2), we obtain

$$\begin{aligned} J_\theta &= \int_{K_\infty^n} w\left(t^L(\mathbf{x}-\mathbf{x}_0)\right)\psi\left(\theta P^3 F(\mathbf{x})\right)\mathrm{d}\mathbf{x} \\ &= \frac{1}{\widehat{L}^n}\int_{\mathbb{T}^n}\psi\left(\theta P^3 F(\mathbf{x}_0+t^{-L}\mathbf{y})\right)\mathrm{d}\mathbf{y}. \end{aligned}$$

Let $f(\mathbf{y})=\theta P^3 F(\mathbf{x}_0+t^{-L}\mathbf{y})$. Then, provided L is sufficiently large, we will have

$$|\nabla f(\mathbf{y})| = |\theta P^3 t^{-L}\nabla F(\mathbf{x}_0+t^{-L}\mathbf{y})| = \frac{|\theta||P|^3\widehat{\xi}}{\widehat{L}}=\lambda,$$

say, for all $\mathbf{y}\in\mathbb{T}^n$. Likewise, we have $|\partial^{\boldsymbol{\beta}} f(\mathbf{y})|<\lambda$, for all $|\boldsymbol{\beta}|\geqslant 2$ and all $\mathbf{y}\in\mathbb{T}^n$. Hence Lemma 2.5 implies that $J_\theta=0$ if $\lambda\geqslant 1$. It therefore follows that

$$K_r=\int_{|\theta|<C|P|^{-3}}\int_{K_\infty^n}\omega(\mathbf{x})\psi\left(\theta P^3F(\mathbf{x})\right)\mathrm{d}\mathbf{x}\mathrm{d}\theta,$$

when $|r|<C^{-1}\widehat{Q}$, which is now independent of r.

Making the change of variables $\varphi=\theta P^3$, we conclude that

$$M(P)=|P|^{n-3}\mathfrak{S}(Q)\mathfrak{I}+O(|P|^{n-3-\delta}),$$

for $n\geqslant 8$, where

$$\mathfrak{S}(Q)=\sum_{\substack{r\in\mathscr{O}\\ |r|\leqslant\widehat{Q}\\ r\text{ monic}}}|r_M|^{-n}S_{r,M,\mathbf{b}}(\mathbf{0})$$

and

$$\mathfrak{I}=\int_{|\varphi|<\widehat{L-\xi}}\int_{K_\infty^n}\omega(\mathbf{x})\psi\left(\varphi F(\mathbf{x})\right)\mathrm{d}\mathbf{x}\mathrm{d}\varphi.$$

The latter quantity is (essentially) the "singular integral" for the problem and can be evaluated explicitly as follows.

Lemma 7.5. *We have*

$$\mathfrak{I}=\frac{1}{|\nabla F(\mathbf{x}_0)|\widehat{L}^{n-1}}\gg 1.$$

Proof. Opening up ω and making a change of variables as before, we see that

$$\begin{aligned} \mathfrak{I} &= \frac{1}{\widehat{L}^n}\int_{|\varphi|<\widehat{L-\xi}}\int_{\mathbb{T}^n}\psi\left(\varphi F(\mathbf{x}_0+t^{-L}\mathbf{y})\right)\mathrm{d}\mathbf{y}\mathrm{d}\varphi \\ &= \frac{\widehat{L-\xi}}{\widehat{L}^n}\,\mathrm{meas}\left\{\mathbf{y}\in\mathbb{T}^n : |F(\mathbf{x}_0+t^{-L}\mathbf{y})|<\frac{\widehat{\xi}}{\widehat{L}}\right\}, \end{aligned}$$

by Lemma 2.2. Put $f(\mathbf{y}) = F(\mathbf{x}_0 + t^{-L}\mathbf{y})$. Then Taylor's theorem yields

$$f(\mathbf{y}) = t^{-L}\mathbf{y}.\nabla F(\mathbf{x}_0) + \tfrac{1}{2}t^{-2L}\mathbf{y}^T\nabla^2 F(\mathbf{x}_0)\mathbf{y} + t^{-3L}F(\mathbf{y}),$$

since $F(\mathbf{x}_0) = 0$ by (7.1). Now the second and third terms here have absolute value $O(|\mathbf{y}|^2/\widehat{L}^2)$, whereas the first term has absolute value at most $|\mathbf{y}|\widehat{\xi}/\widehat{L}$. Assuming that L is large enough, it therefore follows from the ultrametric inequality that $|f(\mathbf{y})| < \widehat{\xi}/\widehat{L}$ for any $\mathbf{y} \in \mathbb{T}^n$. Hence the region in which we are interested has measure 1, which finally leads to the desired conclusion. $\square$

In view of Lemma 7.4 we can extend the summation over r in $\mathfrak{S}(Q)$ to infinity with acceptable error. Thus, for $n \geqslant 8$, there exists $\delta > 0$ such that

$$M(P) = |P|^{n-3}\mathfrak{S}\mathfrak{I} + O(|P|^{n-3-\delta}),$$

where $\mathfrak{I}$ is given by Lemma 7.5 and

$$\mathfrak{S} = \sum_{\substack{r\in\mathscr{O}\\ r \text{ monic}}} |r_M|^{-n} S_{r,M,\mathbf{b}}(\mathbf{0})$$

is the (absolutely convergent) "singular series". The analysis of $\mathfrak{S}$ is standard and will not be repeated here. It runs exactly as in Lee [27, 28], with the outcome that $\mathfrak{S} > 0$ if for every finite prime ϖ there exists $\mathbf{x} \in \mathscr{O}_\varpi^n$ such that $F(\mathbf{x}) = 0$ and $|\mathbf{x} - \mathbf{b}|_\varpi < |M|_\varpi$.

7.4. **Preparations for the error term.** It remains to show that overall contribution to $N(P)$ in Lemma 7.2 from $\mathbf{c} \neq \mathbf{0}$ is $O(|P|^{n-3-\delta})$ for some $\delta > 0$ if $n = 8$. The purpose of this section is to lay some groundwork furthering this aim. Now it is clear from (4.1) that there is a satisfactory overall contribution to $N(P)$ from values of θ such that $|\theta| < \widehat{Q}^{-5}$. This allows us to henceforth focus on the contribution from $|\theta| \geqslant \widehat{Q}^{-5}$.

Let $Y, \Theta \in \mathbb{Z}$ be such that

$$0 \leqslant Y \leqslant Q, \quad -5Q \leqslant \Theta < -(Y+Q). \tag{7.9}$$

The last inequality is equivalent to $\widehat{Q}^{-5} \leqslant \widehat{\Theta} < (\widehat{Y}\widehat{Q})^{-1}$ and one sees that there are at most $4Q = O(\log|P|)$ choices for Y, Θ. We will content ourselves with focusing on the overall contribution to $N(P)$ from $\mathbf{c} \neq \mathbf{0}$ and r, θ such that $|r| = \widehat{Y}$ and $|\theta| = \widehat{\Theta}$. Let us denote this contribution by $E(P) = E(P; Y, \Theta)$. Suppose that we are able to prove the existence of a positive constant $\eta > 0$ such that

$$E(P) = O(|P|^{n-3-\eta}) \tag{7.10}$$

for any $Y, \Theta \in \mathbb{Z}$ satisfying (7.9). Then this will lead to an asymptotic formula for $N(P)$, as $|P| \to \infty$, for the range of n that (7.10) is valid for.

In what follows it will be convenient to introduce the notation

$$J(\Theta) = \max\{1, \widehat{\Theta}|P|^3\}. \tag{7.11}$$

The constraint on $\mathbf{c}$ imposed in Lemma 7.2 now becomes $|\mathbf{c}| \ll \widehat{Y}|P|^{-1}J(\Theta)$, for a suitable implied constant. In particular, since $\mathbf{c} \neq \mathbf{0}$, we must have

$$\widehat{Y} \gg \frac{|P|}{J(\Theta)}. \tag{7.12}$$

Switching the order of summation we obtain

$$E(P) = |P|^n \sum_{\substack{\mathbf{c}\in\mathscr{O}^n \\ \mathbf{c}\neq\mathbf{0} \\ |\mathbf{c}|\ll\widehat{Y}|P|^{-1}J(\Theta)}} \sum_{\substack{r\in\mathscr{O} \\ |r|=\widehat{Y} \\ r \text{ monic}}} |r_M|^{-n} \int_{|\theta|=\widehat{\Theta}} S_{r,M,\mathbf{b}}(\mathbf{c}) I_{r_M}(\theta;\mathbf{c})\mathrm{d}\theta, \tag{7.13}$$

where $I_{r_M}(\theta;\mathbf{c}) \ll J(\Theta)^{-n/2}$. Let S be a set of finite primes to be decided upon in due course, but which contains all prime divisors of M. Any $r \in \mathscr{O}$ can be written $r = b_1' b_1 r_2$ where b_1' is square-free such that $\varpi \mid b_1' \Rightarrow \varpi \in S$ and b_1 is square-free and coprime to S. According to Lemma 4.5 there is a factorisation $M = M_1 M_2 M_3$ for $M_1, M_2, M_3 \in \mathscr{O}$ such that $M_1 \mid (b_1')^\infty$, $M_2 \mid r_2^\infty$ and $(M_3, r) = 1$, together with $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in (\mathscr{O}/M\mathscr{O})^n$ such that

$$S_{r,M,\mathbf{b}}(\mathbf{c}) = S_{b_1,1,\mathbf{0}}(\mathbf{c}) S_{b_1',M_1,\mathbf{b}_1}(\mathbf{c}) S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c}) \psi\left(\frac{-\mathbf{c}.\mathbf{b}_3}{M_3}\right). \tag{7.14}$$

The vectors $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$ depend only on the value of b_1 mod M. In §8 we will consider the effect of $S_{b_1,1,\mathbf{0}}(\mathbf{c})$ on $E(P)$. Later, in §9, we will consider the contribution from $S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})$.

8. Contribution from square-free moduli

It will be convenient to define

$$\mathscr{O}^\sharp = \{b \in \mathscr{O} : b \text{ is monic and square-free}\}.$$

Recalling the expression (7.13) and the subsequent factorisation (7.14) of the exponential sum involved, it follows that there exists $\mathbf{b}_1, \mathbf{b}_2 \in (\mathscr{O}/M\mathscr{O})^n$ and $b_0 \in (\mathscr{O}/M\mathscr{O})^*$ such that

$$\begin{aligned} E(P) \ll \frac{|P|^n}{\widehat{Y}^{(n-1)/2}} &\sum_{\substack{\mathbf{c}\in\mathscr{O}^n \\ \mathbf{c}\neq\mathbf{0} \\ |\mathbf{c}|\ll\widehat{Y}|P|^{-1}J(\Theta)}} \sum_{\substack{b_1'\in\mathscr{O}^\sharp \\ \varpi\mid b_1'\Rightarrow\varpi\in S}} \\ &\times \sum_{\substack{r_2\in\mathscr{O} \\ |b_1' r_2|\leqslant\widehat{Y}}} \frac{|S_{b_1',M_1,\mathbf{b}_1}(\mathbf{c}) S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})|}{|b_1' r_2|^{(n+1)/2}} \int_{|\theta|=\widehat{\Theta}} |\Sigma(Y,\theta)|\mathrm{d}\theta, \end{aligned} \tag{8.1}$$

where

$$\Sigma(Y,\theta)=\sum_{\substack{b_1\in\mathscr{O}^\sharp\\(b_1,S)=1\\|b_1'b_1r_2|=\widehat{Y}\\b_1\equiv b_0 \bmod M}}\frac{S_{b_1,1,\mathbf{0}}(\mathbf{c})I_{b_1(b_1'r_2)_M}(\theta;\mathbf{c})}{|b_1|^{(n+1)/2}}. \tag{8.2}$$

Here we have observed that $(b_1'b_1r_2)_M=b_1(b_1'r_2)_M$ since $(b_1,M)=1$. Our main job in this section is to estimate $\Sigma(Y,\theta)$ whenever $\mathbf{c}$ is suitably generic.

In what follows we will put $b=b_1$ and redefine $b_1'r_2$ to be d, for simplicity. Putting $S_b(\mathbf{c})=S_{b,1,\mathbf{0}}(\mathbf{c})$, we have

$$S_b(\mathbf{c})=\sideset{}{^*}\sum_{|a|<|b|}\sum_{\substack{\mathbf{y}\in\mathscr{O}^n\\|\mathbf{y}|<|b|}}\psi\left(\frac{aF(\mathbf{y})-\mathbf{c}.\mathbf{y}}{r}\right).$$

Let $\Delta_F\in\mathscr{O}$ denote the non-zero discriminant of F. Assuming that $F^*(\mathbf{c})\neq 0$ we shall take S to be the set of primes dividing $\Delta_F MF^*(\mathbf{c})$. Alternatively, if $F^*(\mathbf{c})=0$ but $\nabla F^*(\mathbf{c})\neq\mathbf{0}$, then we will take S to be the set of primes dividing $\Delta_F M\nabla F^*(\mathbf{c})$. Lemma 4.5 shows that the sum $S_b(\mathbf{c})$ is a multiplicative function of b. When $b=\varpi$ for a prime ϖ, the sum is a complete exponential sum over the finite field $\mathbb{F}_\varpi$. It then follows from (5.2) that $S_\varpi(\mathbf{c})\ll|\varpi|^{(n+1)/2}|(\varpi,\nabla F^*(\mathbf{c}))|^{1/2}$. Hence

$$\sum_{\substack{b\in\mathscr{O}^\sharp\\(b,S)=1\\|bd|=\widehat{Y}\\b\equiv b_0 \bmod M}}\frac{|S_b(\mathbf{c})|}{|b|^{(n+1)/2}}\ll\widehat{Y}^\varepsilon\sum_{\substack{b\in\mathscr{O}^\sharp\\(b,S)=1\\|bd|=\widehat{Y}}}|(b,\nabla F^*(\mathbf{c}))|^{1/2}$$

in (8.2). According to our definition of S this is $O(|d|^{-1}\widehat{Y}^{1+\varepsilon})$ if $\nabla F^*(\mathbf{c})\neq\mathbf{0}$ and $O(|d|^{-3/2}\widehat{Y}^{3/2+\varepsilon})$ if $\nabla F^*(\mathbf{c})=\mathbf{0}$. Recalling the bound in Lemma 7.3 for $I_{bd_M}(\theta;\mathbf{c})$ and the definition (7.11) of $J(\Theta)$, this leads to the following "easy" estimate for $\Sigma(Y,\theta)$.

Lemma 8.1. *Let $\varepsilon>0$. If $\nabla F^*(\mathbf{c})\neq\mathbf{0}$ and S is the set of primes dividing $\Delta_F M\nabla F^*(\mathbf{c})$, then*

$$\Sigma(Y,\theta)\ll\frac{\widehat{Y}^{1+\varepsilon}}{|d|J(\Theta)^{n/2}}.$$

If $\nabla F^*(\mathbf{c})=\mathbf{0}$ and S is the set of primes dividing $\Delta_F M$, then

$$\Sigma(Y,\theta)\ll\frac{\widehat{Y}^{3/2+\varepsilon}}{|d|^{3/2}J(\Theta)^{n/2}}.$$

The implied constants in these estimates are allowed to depend on the choice of ε, a convention that we shall henceforth adhere to. Lemma 8.1 does not take advantage of any cancellation in the sum over $b \in \mathscr{O}^\sharp$ coming from sign changes in the exponential sum $S_b(\mathbf{c})$. The following "hard" estimate does so under suitable hypotheses. Its proof will occupy the rest of this section.

Lemma 8.2. *Assume that n is even and that $F^*(\mathbf{c}) \neq 0$. Let S be the set of primes dividing $\Delta_F M F^*(\mathbf{c})$. Assume, furthermore, that $\widehat{Y} > \sqrt{q|d||P||\mathbf{c}|}$. Then for any $\varepsilon > 0$ we have*

$$\Sigma(Y,\theta) \ll \frac{(|\mathbf{c}|\widehat{Y})^\varepsilon}{|d|^{1/2}} \frac{\widehat{Y}^{1/2}}{J(\Theta)^{n/2-1/2}}.$$

At this stage it might be useful to compare Lemmas 8.1 and 8.2 for typical values of $Y, \Theta, \mathbf{c}$ satisfying (7.9), by which we mean that $Y \sim Q$, $\widehat{\Theta} \sim (\widehat{Y}\widehat{Q})^{-1}$ and $|\mathbf{c}| \sim |P|^{1/2}$. But then $J(\Theta) \sim 1$ and the bound in Lemma 8.2 is roughly of order $(\widehat{Q}/|d|)^{1/2}$, while that in Lemma 8.1 is of order $\widehat{Q}/|d|$.

We begin the proof of Lemma 8.2 by writing

$$\Sigma(Y,\theta) = \int_{K_\infty^n} w\left(t^L(\mathbf{x}-\mathbf{x}_0)\right) \psi\left(\theta P^3 F(\mathbf{x})\right) \Sigma(Y;\mathbf{x}) \mathrm{d}\mathbf{x},$$

where

$$\Sigma(Y;\mathbf{x}) = \sum_{\substack{b \in \mathscr{O}^\sharp \\ (b,S)=1 \\ |b|=\widehat{Y}/|d| \\ b \equiv b_0 \bmod M}} \frac{S_b(\mathbf{c})}{|b|^{(n+1)/2}} \psi\left(\frac{P\mathbf{c}.\mathbf{x}/d_M}{b}\right).$$

Here we recall that $|\mathbf{x}| < 1$ for $\mathbf{x} \in K_\infty^n$ such that $w(t^L(\mathbf{x}-\mathbf{x}_0)) \neq 0$. We detect the condition $b \equiv b_0 \bmod M$ by summing over Dirichlet characters $\eta_1 \bmod M$. Letting $D_1 = (\mathscr{O}/M\mathscr{O})^*$, this gives

$$\Sigma(Y;\mathbf{x}) = \frac{1}{\#D_1} \sum_{\eta_1 \bmod M} \overline{\eta_1(b_0)} \sum_{\substack{b \in \mathscr{O}^\sharp \\ (b,S)=1 \\ |b|=\widehat{Y}/|d|}} \frac{\eta_1(b) S_b(\mathbf{c})}{|b|^{(n+1)/2}} \psi\left(\frac{P\mathbf{c}.\mathbf{x}/d_M}{b}\right).$$

Next, we let $J \in \mathbb{Z}$ be such that

$$\widehat{J} = q^J = \max\left\{1, \frac{q|P||\mathbf{c}|}{\widehat{Y}}\right\}.$$

In particular $J \geqslant 0$. Typically we expect $\widehat{J}$ to be rather small. For any b arising in $\Sigma(Y;\mathbf{x})$ let us put $K = \deg(b)$, so that

$$\widehat{K} = q^K = \frac{\widehat{Y}}{|d|}.$$

Lemma 8.2 is stated under the assumption that $\widehat{Y} \geqslant \sqrt{q|d||P||\mathbf{c}|}$, which is equivalent to $J \leqslant K$. Let us put $x = t^{-1}$ for the prime at infinity and $A = \mathbb{F}_q[x] \subset \mathscr{O}_\infty$. Then, since b is monic, there exist $c_1, \dots, c_K \in \mathbb{F}_q$ such that

$$b = \underbrace{t^K + c_1 t^{K-1} + \cdots + c_{J-1} t^{K-J+1}}_{=t^K a} + \underbrace{c_J t^{K-J} + \cdots + c_K}_{=t^{K-J} b'},$$

where $a \in (A/x^J A)^*$ and $b' \in A$. Thus $b = t^K(a + x^J b')$ with $|b'| \leqslant 1$ and $|a| = 1$. But then it follows that

$$\begin{aligned}\psi\left(\frac{P\mathbf{c}.\mathbf{x}/d_M}{b}\right) &= \psi\left(\frac{P\mathbf{c}.\mathbf{x}}{d_M}\left\{\frac{1}{t^K(a+x^Jb')} - \frac{1}{t^K a}\right\}\right)\psi\left(\frac{P\mathbf{c}.\mathbf{x}/d_M}{t^K a}\right)\\ &= \psi\left(\frac{P\mathbf{c}.\mathbf{x}/d_M}{t^K a}\right),\end{aligned}$$

since

$$\begin{aligned}\left|\frac{P\mathbf{c}.\mathbf{x}}{d_M}\left\{\frac{1}{t^K(a+x^Jb')} - \frac{1}{t^K a}\right\}\right| &< \frac{|P||\mathbf{c}|}{|t^K a d_M|}\left|-\frac{x^J b'}{a} + \dots\right|\\ &\leqslant \frac{|P||\mathbf{c}|}{\widehat{J}\widehat{K}|d|}\\ &\leqslant q^{-1}.\end{aligned}$$

The conclusion of this is that the character ψ in $\Sigma(Y; \mathbf{x})$ only depends on the value of $b/t^K \bmod x^J$.

Putting $D_2 = (A/x^J A)^*$, it follows that

$$\Sigma(Y;\mathbf{x}) = \frac{1}{\#D_1}\sum_{\eta_1 \bmod M}\overline{\eta_1(b_0)}\sum_{a\in D_2}\psi\left(\frac{P\mathbf{c}.\mathbf{x}/d_M}{t^K a}\right)\sum_{\substack{b\in\mathscr{O}^\sharp\\ (b,S)=1\\ |b|=\widehat{Y}/|d|\\ t^{-K}b\equiv a \bmod x^J}}\frac{\eta_1(b)S_b(\mathbf{c})}{|b|^{(n+1)/2}}.$$

Introducing Dirichlet characters $\chi : D_2 \to \mathbb{C}^*$ to detect the congruence condition in the inner sum, we deduce that

$$\begin{aligned}\Sigma(Y;\mathbf{x}) = &\frac{1}{\#D_1\#D_2}\sum_{\eta_1 \bmod M}\sum_{\chi \bmod x^J}\\ &\times\sum_{a\in D_2}\psi\left(\frac{P\mathbf{c}.\mathbf{x}/d_M}{t^K a}\right)\overline{\eta_1(b_0)\chi(a)}\Sigma_0(\eta_1,\chi;Y),\end{aligned}$$

where

$$\Sigma_0(\eta_1,\chi;Y) = \sum_{\substack{b\in\mathscr{O}^\sharp\\ (b,S)=1\\ |b|=\widehat{Y}/|d|}}\frac{\eta_1(b)\chi(t^{-K}b)S_b(\mathbf{c})}{|b|^{(n+1)/2}}.$$

In conclusion, we have therefore established the identity

$$\begin{aligned}\Sigma(Y,\theta) = \frac{1}{\#D_1\#D_2} \sum_{\eta_1 \bmod M} \overline{\eta_1(b_0)} \sum_{\chi \bmod x^J} \Sigma_0(\eta_1,\chi;Y) \\ \times \sum_{a\in D_2} \overline{\chi(a)} I_{t^K a d_M}(\theta;\mathbf{c}).\end{aligned} \tag{8.3}$$

Our first concern is an estimate for the inner sum over a. It is easy to see that

$$\frac{1}{\#D_2} \sum_{\chi \bmod x^J} \left| \sum_{a\in D_2} \overline{\chi(a)} I_{t^K a d_M}(\theta;\mathbf{c}) \right| \ll \widehat{J} J(\Theta)^{-n/2} \ll J(\Theta)^{1-n/2}, \tag{8.4}$$

since the size constraint on $\mathbf{c}$ in (8.1) gives $\widehat{J} = \max\{1, |P||\mathbf{c}|/\widehat{Y}\} \ll J(\Theta)$. It turns out that this bound does not suffice for (7.10) when $n=8$ and it is necessary to produce a bound which takes advantage of non-trivial averaging over a. This is achieved in the following result.

Lemma 8.3. *We have*

$$\frac{1}{\#D_2} \sum_{\chi \bmod x^J} \left| \sum_{a\in D_2} \overline{\chi(a)} I_{t^K a d_M}(\theta;\mathbf{c}) \right| \ll J(\Theta)^{1/2-n/2}.$$

Proof. Let $\chi \bmod x^J$ be a Dirichlet character. Opening up $I_{t^K a d_M}(\theta;\mathbf{c})$, we deduce from (7.7) that

$$\sum_{a\in D_2} \chi(a) I_{t^K a d_M}(\theta;\mathbf{c}) = \frac{1}{\widehat{L}^n} \sum_{a\in D_2} \chi(a)\psi\left(\frac{P\mathbf{c}.\mathbf{x}_0}{t^K a d_M}\right) J_G\left(\theta P^3, \frac{P t^{-L}\mathbf{c}}{t^K a d_M}\right),$$

in the notation of (2.4), where $G(\mathbf{x}) = F(\mathbf{x}_0 + t^{-L}\mathbf{x})$. Lemma 2.7 implies that

$$J_G\left(\theta P^3, \frac{P t^{-L}\mathbf{c}}{t^K a d_M}\right) = \int_{\Omega_a} \psi\left(\theta P^3 G(\mathbf{x}) + \frac{P t^{-L}\mathbf{c}.\mathbf{x}}{t^K a d_M}\right) \mathrm{d}\mathbf{x},$$

where

$$\Omega_a = \left\{ \mathbf{x}\in\mathbb{T}^n : \left|\theta P^3 \nabla G(\mathbf{x}) + \frac{P t^{-L}\mathbf{c}}{t^K a d_M}\right| \ll J(\Theta)^{1/2} \right\}.$$

It follows from (7.8) that $\mathrm{meas}(\Omega_a) \ll J(\Theta)^{-n/2}$.

Let $\varepsilon>0$ and choose $J_0\in\mathbb{Z}$ such that $\widehat{J_0}$ has order of magnitude $J(\Theta)^{1/2+\varepsilon}$. If $J_0 > J$ then Lemma 8.3 follows from (8.4). Alternatively, we may proceed under the assumption that $J/2 \leqslant J_0 \leqslant J$. Recall that $x = t^{-1}$ and suppose that $a \equiv a' \bmod x^{J_0}$, for $a, a' \in D_2$. Then

$$\left| \frac{P t^{-L}\mathbf{c}}{t^K a d_M} - \frac{P t^{-L}\mathbf{c}}{t^K a' d_M} \right| \leqslant \widehat{J} \left| \frac{a-a'}{aa'} \right| \leqslant \frac{\widehat{J}}{\widehat{J_0}} \ll J(\Theta)^{1/2-\varepsilon}.$$

Hence the set Ω_a only depends on the value of a mod x^{J_0}.

Let us write $a = a_0 + x^{J_0}a_1$, where $a_0 \in (A/x^{J_0}A)^*$ and $a_1 \in A/x^{J-J_0}A$. Then

$$\sum_{a\in D_2} \chi(a) I_{t^K a d_M}(\theta;\mathbf{c}) = \sum_{a_0\in(A/x^{J_0}A)^*} \sum_{a_1\in A/x^{J-J_0}A} \chi(a_0+x^{J_0}a_1) \times \frac{1}{\widehat{L}^n}\int_{\Omega_{a_0}} \psi\left(\theta P^3 G(\mathbf{x})\right)\psi\left(\frac{P\mathbf{c}.(\mathbf{x}_0+t^{-L}\mathbf{x})}{t^K(a_0+x^{J_0}a_1)d_M}\right)\mathrm{d}\mathbf{x}.$$

For fixed $a_0 \in (A/x^{J_0}A)^*$ and $\mathbf{x}\in\Omega_{a_0}$ we proceed to examine the sum

$$S(\mathbf{x}) = \sum_{a_1\in A/x^{J-J_0}A} \psi\left(\frac{P\mathbf{c}.\mathbf{y}}{t^K(a_0+x^{J_0}a_1)d_M}\right)\chi(1+x^{J_0}a_1\overline{a_0}),$$

where $\mathbf{y} = \mathbf{x}_0 + t^{-L}\mathbf{x}$ and $\overline{a_0}$ denotes the multiplicative inverse of a_0 mod x^{J-J_0}.

Let φ_χ be the additive character defined on $A/x^{J-J_0}A$ via

$$\varphi_\chi(a) = \chi(1+x^{J_0}a).$$

This must be a twist of the standard additive character. Thus there exists an element $a_\chi \in A/x^{J-J_0}A$ such that

$$\varphi_\chi(a) = \psi\left(\frac{a_\chi a}{x^{J-J_0}}\right),$$

for any $a \in A/x^{J-J_0}A$. This gives a surjective homomorphism

$$\varphi : \mathrm{Hom}\left((A/x^J A)^*, \mathbb{C}^*\right) \to A/x^{J-J_0}A,$$

defined by $\varphi(\chi) = a_\chi$, with kernel isomorphic to $\mathrm{Hom}((A/x^{J_0}A)^*, \mathbb{C}^*)$. We conclude that

$$S(\mathbf{x}) = \sum_{a_1\in A/x^{J-J_0}A} \psi\left(\frac{P\mathbf{c}.\mathbf{y}}{t^K(a_0+x^{J_0}a_1)d_M}\right)\psi\left(\frac{a_\chi a_1\overline{a_0}}{x^{J-J_0}}\right). \tag{8.5}$$

Observe that $|P\mathbf{c}.\mathbf{y}/(t^K d_M)| \leqslant \widehat{J}$. Hence

$$\begin{aligned}\psi\left(\frac{P\mathbf{c}.\mathbf{y}}{t^K(a_0+x^{J_0}a_1)d_M}\right) &= \psi\left(\frac{P\mathbf{c}.\mathbf{y}}{t^K a_0(1+x^{J_0}a_1\overline{a_0})d_M}\right)\\ &= \psi\left(\frac{P\mathbf{c}.\mathbf{y}(1-a_1\overline{a_0}x^{J_0})}{t^K a_0 d_M}\right)\\ &= \psi\left(\frac{P\mathbf{c}.\mathbf{y}}{t^K a_0 d_M}\right)\psi\left(\frac{-P\mathbf{c}.\mathbf{y}a_1\overline{a_0}x^{J_0}}{t^K a_0 d_M}\right)\end{aligned}$$

and

$$\psi\left(\frac{-P\mathbf{c}.\mathbf{y}a_1\overline{a_0}x^{J_0}}{t^K a_0 d_M}\right) = \psi\left(\frac{a'' a_1\overline{a_0}^2}{x^{J-J_0}}\right),$$

for some $a'' \in A$. Applying this reasoning in (8.5), we are led to the identity

$$S(\mathbf{x}) = \psi\left(\frac{P\mathbf{c}.\mathbf{y}}{t^K a_0 d_M}\right) \sum_{a_1 \in A/x^{J-J_0}A} \psi\left(\frac{a_1\overline{a_0}(a_\chi + a''\overline{a_0})}{x^{J-J_0}}\right),$$

where a_χ and a'' are independent of the choices of a_0 and a_1.

For fixed a_0 we deduce that $S(\mathbf{x}) = 0$ unless $a_\chi \equiv a''' \bmod x^{J-J_0}$, where $a''' = -a''\overline{a_0} \bmod x^{J-J_0}$, in which case $|S(\mathbf{x})| \leqslant \widehat{J}/\widehat{J}_0$. However, for fixed $a''' \in A/x^{J-J_0}A$ we have $\#\{\chi \in \varphi^{-1}(a''')\} \leqslant \#\{\chi \in \varphi^{-1}(0)\} \leqslant \widehat{J}_0$, since φ is a homomorphism. Thus

$$\begin{aligned}
\frac{1}{\#D_2} \sum_{\chi \bmod x^J} \left| \sum_{a \in D_2} \chi(a) I_{t^K a d_M}(\theta; \mathbf{c}) \right| &\ll \frac{1}{\widehat{J}} \sum_{\chi \bmod x^J} \sum_{a_0 \in (A/x^{J_0}A)^*} \\
&\quad \times \left| \int_{\Omega_{a_0}} \psi\left(\theta P^3 G(\mathbf{x})\right) S(\mathbf{x}) \mathrm{d}\mathbf{x} \right| \\
&\ll \sum_{a_0 \in (A/x^{J_0}A)^*} \operatorname{meas}(\Omega_{a_0}) \\
&\ll \widehat{J}_0 J(\Theta)^{-n/2}.
\end{aligned}$$

This completes the proof of the lemma, since $\widehat{J}_0$ has order $J(\Theta)^{1/2+\varepsilon}$. $\square$

It is now time to start analysing the sum $\Sigma_0(\eta_1, \chi; Y)$ for fixed Dirichlet characters $\eta_1 : D_1 \to \mathbb{C}^*$ and $\chi : D_2 \to \mathbb{C}^*$. Let us define a further character $\eta_2 : \mathscr{O} \to \mathbb{C}^*$, given by $\eta_2(r) = \chi(r/t^{\deg r})$ for any $r \in \mathscr{O}$. This a multiplicative character of order at most $\widehat{J}$. We proceed to bound the sum

$$\sum_{\substack{b \in \mathscr{O}^\sharp \\ (b,S)=1 \\ |b| \leqslant \widehat{Z}}} \frac{\eta_1(b)\eta_2(b) S_b(\mathbf{c})}{|b|^{(n+1)/2}}$$

for any $Z \geqslant 1$, where S is the set of primes dividing $\Delta_F M F^*(\mathbf{c})$.

Let $X \subset \mathbb{P}^{n-1}_K$ denote the smooth and projective hypersurface $F = 0$ defined over K and let $X_{\mathbf{c}} \subset \mathbb{P}^{n-2}_K$ denote the projective hypersurface cut out from X by the hyperplane $\mathbf{c}.\mathbf{x} = 0$. Since $F^*(\mathbf{c}) \neq 0$ it follows that $X_{\mathbf{c}}$ is smooth. Moreover, we have $\dim(X) = n-2$ and $\dim(X_{\mathbf{c}}) = n-3$. We begin our analysis of $S_\varpi(\mathbf{c})$ with an application of Hooley [19, Lemma 7 and Eq. (86)]. This shows that

$$S_\varpi(\mathbf{c}) = |\varpi| \left\{ |\varpi| \#X_{\mathbf{c},\varpi}(\mathbb{F}_\varpi) - \#X_\varpi(\mathbb{F}_\varpi) + 1 \right\}, \tag{8.6}$$

for any prime ϖ. It now follows from (3.12) that

$$S_\varpi(\mathbf{c}) = (-1)^{n-3}|\varpi|^2 \sum_{j=1}^{b_{n-3}} \omega_{n-3,j} + O(|\varpi|^{n/2}),$$

for any finite prime $\varpi \notin S$, where for any prime $\ell \nmid q$ the number b_{n-3} is the dimension of the middle cohomology group $H_\ell^{n-3}(X_\mathbf{c}) = H_{\text{ét}}^{n-3}(\overline{X}_\mathbf{c}, \mathbb{Q}_\ell)$ (as a vector space over $\mathbb{Q}_\ell$) and $\omega_{n-3,j}$ are the eigenvalues of the Frobenius endomorphism acting on it. The dimension b_{n-3} is independent of the choice of ℓ and is bounded in terms of n. Moreover, $|\omega_{n-3,j}| = |\varpi|^{(n-3)/2}$ for each index $1 \leqslant j \leqslant b_{n-3}$.

We proceed to study the Dirichlet series

$$F(s) = \sum_{\substack{b \in \mathscr{O}^\sharp \\ (b,S)=1}} \frac{\eta_1(b)\eta_2(b)S_b(\mathbf{c})}{|b|^s} = \prod_{\varpi \notin S} \left(1 + \frac{\eta_1(\varpi)\eta_2(\varpi)S_\varpi(\mathbf{c})}{|\varpi|^s}\right), \tag{8.7}$$

which is defined for $\sigma = \Re(s) > (n+3)/2$. Let $\varpi \notin S$ and $\sigma > n/2+1$. Then, with §§3.3–3.5 to hand, (8.6) implies that

$$\begin{aligned} 1 + \frac{\eta_1(\varpi)\eta_2(\varpi)S_\varpi(\mathbf{c})}{|\varpi|^s} &= \left(1 + \eta_1(\varpi)\eta_2(\varpi)\frac{(-1)^{n-3}}{|\varpi|^{s-2}} \sum_{j=1}^{b_{n-3}} \omega_{n-3,j}\right) \\ &\quad \times \left(1 + O(|\varpi|^{n/2-\sigma} + |\varpi|^{n+1-2\sigma})\right) \\ &= L_\varpi(\eta \otimes H_\ell^{n-3}(X_\mathbf{c}), s-2)^{(-1)^{n-3}} \\ &\quad \times \left(1 + O(|\varpi|^{n/2-\sigma})\right), \end{aligned} \tag{8.8}$$

where we view $\eta = \eta_1 \otimes \eta_2$ as a Galois representation by class field theory.

We may now appeal to the contents of §3.5 where some of the analytic properties of the global L-function $L(\eta \otimes H_\ell^{n-3}(X_\mathbf{c}), s-2)$ are recorded. When $\varpi \in S$ it follows from our discussion in §3.4 that

$$L_\varpi(\eta \otimes H_\ell^{n-3}(X_\mathbf{c}), s-2) = 1 + O(|\varpi|^{(n+1)/2-\sigma}),$$

since the inverse roots have modulus at most $|\varpi|^{(n-3)/2}$. Hence, on recalling the definition of the associated global L-function, we finally obtain

$$F(s) = L(\eta \otimes H_\ell^{n-3}(X_\mathbf{c}), s-2)^{(-1)^{n-3}} E(s), \quad (\sigma > n/2+1), \tag{8.9}$$

where

$$E(s) = \prod_{\varpi \notin S} \left(1 + O(|\varpi|^{n/2-\sigma})\right) \prod_{\varpi \in S} \left(1 + O(|\varpi|^{(n+1)/2-\sigma})\right). \tag{8.10}$$

Note that $E(s)$ is holomorphic and bounded for $\sigma > n/2+1$.

We will need a decent bound for the absolute value of the function $F(s)$ well inside its domain of analytic continuation. This is achieved in the following result.

Lemma 8.4. *Assume that n is even and let $\varepsilon > 0$. Then for $\sigma \geqslant 1/2 + \varepsilon$ we have $|F(s + (n+1)/2)| \ll_\varepsilon |\mathbf{c}|^\varepsilon$.*

Proof. Recalling (8.9), the fact that n is even implies that $F(s + (n+1)/2) = G(s)E(s + (n+1)/2)$ for $\sigma > 1/2$, with

$$G(s) = L\left(\eta \otimes H_\ell^{n-3}(X_{\mathbf{c}}), s + \tfrac{n-3}{2}\right)^{-1},$$

where $\eta = \eta_1 \otimes \eta_2$. It follows from (3.13) that

$$G(s) = \frac{P_0(q^{-s-(n-3)/2})P_2(q^{-s-(n-3)/2})}{P_1(q^{-s-(n-3)/2})},$$

with $P_k = P_{k,n-3} \in \mathbb{Z}[T]$ as in §3.5 for $k \in \{0, 1, 2\}$. Furthermore, if we put $e_k = \deg P_k$ then it follows that $e_0, e_2 = O(1)$ and

$$e_1 \ll 1 + \log |F^*(\mathbf{c})| \ll 1 + \log |\mathbf{c}|, \tag{8.11}$$

by (3.14). Moreover, the inverse roots of P_k have absolute value $q^{(n-3+k)/2}$. It is now clear that $G(s)$ is holomorphic in the half-plane $\sigma > 1/2$ and that in this region its only zeros come from the zeros of $P_2(q^{-s-(n-3)/2})$, which are located on the line $\sigma = 1$. We have

$$F\left(s + \tfrac{n+1}{2}\right) = P_2(q^{-s-(n-3)/2})H(s),$$

with

$$H(s) = E\left(s + \tfrac{n+1}{2}\right) \frac{P_0(q^{-s-(n-3)/2})}{P_1(q^{-s-(n-3)/2})}.$$

Now it is obvious that $|P_2(q^{-s-(n-3)/2})| \leqslant (1 + q^{-\sigma+1})^{e_2} \ll 1$, for $\sigma > 1/2$. Hence it suffices to establish the bound in the lemma for $H(s)$.

We will produce a good bound when $\sigma > 1$ together with a weaker bound which is valid for $\sigma > 1/2$. In the familiar way (cf. Titchmarsh [37, Chapter XIV]), we will then use the Hadamard three circle theorem to establish the final bound recorded in the statement of the lemma. Our trivial bound is based on (8.7). Thus it follows from (8.8) that there is a constant $c > 0$ such that

$$\begin{aligned}\left|\log F\left(s + \tfrac{n+1}{2}\right)\right| &\leqslant \sum_{j=1}^{b_{n-3}} \sum_{\varpi} \sum_{\alpha \geqslant 1} \frac{1}{\alpha |\varpi|^{\alpha\sigma}} + \sum_{\varpi} \sum_{\alpha \geqslant 1} \frac{1}{\alpha} \left(\frac{c}{|\varpi|^{\sigma+1/2}}\right)^\alpha \\ &\ll \log Z(\sigma),\end{aligned}$$

for $\sigma > 1$, where $Z(s)$ is the ordinary zeta function of $K = \mathbb{F}_q(t)$. It easily follows that

$$|\log H(s)| \ll \log Z(\sigma), \quad (\sigma > 1). \tag{8.12}$$

Next, for $\sigma > 1/2$, it follows from (8.10) that $E(s+(n+1)/2) \ll Z(\sigma+1/2)^c$ for some absolute constant $c > 0$. Hence we obtain

$$|H(s)| \ll Z\left(\sigma + \tfrac{1}{2}\right)^c (1 - q^{1/2-\sigma})^{-e_1}$$

for $\sigma > 1/2$, whence

$$\Re \log H(s) = \log |H(s)| \ll \log Z\left(\sigma + \tfrac{1}{2}\right) + e_1$$

in this region. Note that $\log H(s)$ is analytic in the half-plane $\sigma > 1/2$. We apply the Borel–Carathéodory theorem to $\log H(s)$ with circles of centre $3/2 + it_0$ and radii $1 - \varepsilon/2$ and $1 - \varepsilon$. This leads to the conclusion that

$$|\log H(s)| \ll \frac{1}{\varepsilon}\left\{\log Z\left(\sigma + \tfrac{1}{2}\right) + e_1\right\}, \quad (\sigma \geqslant \tfrac{1}{2} + \varepsilon). \tag{8.13}$$

We now refine this bound by applying the Hadamard three circle theorem to $\log H(s)$. Let $\sigma_0 = \sigma_0(\varepsilon)$ and let $s = \sigma + it$ with $1/2 + \varepsilon \leqslant \sigma \leqslant 1 + \varepsilon/2$. We take circles with centre $\sigma_0 + it$ and radii $r_1 = \sigma_0 - 1 - \varepsilon/2$, $r_2 = \sigma_0 - \sigma$ and $r_3 = \sigma_0 - 1/2 - \varepsilon/2$. Combining (8.12) and (8.13), we deduce the existence of constants $c_1(\varepsilon), c_2(\varepsilon) > 0$ such that

$$|\log H(s)| \leqslant c_1(\varepsilon)^{1-\beta} \left(c_2(\varepsilon) e_1\right)^{\beta},$$

where

$$\beta = \frac{\log r_2/r_1}{\log r_3/r_1} = 2 - 2\sigma + \varepsilon + O\left(\frac{1}{\sigma_0}\right) \leqslant 1 - \varepsilon + O\left(\frac{1}{\sigma_0}\right).$$

We take σ_0 sufficiently large to ensure that $\beta \leqslant 1 - \varepsilon/2 < 1$. Recalling the bound (8.11) for e_1, all of this is now seen to give

$$|H(s)| \leqslant c(\varepsilon)^{1+(\log |\mathbf{c}|)^{\beta}}, \quad (\sigma \geqslant \tfrac{1}{2} + \varepsilon),$$

for an appropriate constant $c(\varepsilon) > 0$. The statement of the lemma easily follows. □

We are now ready to establish the following estimate, which once combined with (8.3) and Lemma 8.3, clearly completes the proof of Lemma 8.2.

Lemma 8.5. *Assume that n is even and $F^*(\mathbf{c}) \neq 0$. Then for any $\varepsilon > 0$ we have*

$$\sum_{\substack{b \in \mathscr{O}^\sharp \\ (b, \Delta_F M F^*(\mathbf{c}))=1 \\ |b| \leqslant \widehat{Z}}} \frac{\eta_1(b)\eta_2(b) S_b(\mathbf{c})}{|b|^{(n+1)/2}} \ll (|\mathbf{c}|\widehat{J})^{\varepsilon} \widehat{Z}^{1/2+\varepsilon}.$$

Proof. It follows from Perron's formula that the sum to be estimated is equal to

$$\sum_{k \leqslant \widehat{Z}} \frac{a_k}{k^{(n+1)/2}} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F\left(s + \frac{n+1}{2}\right) \frac{\widehat{Z}^s \mathrm{d}s}{s},$$

where

$$a_k = \sum_{\substack{b \in \mathscr{O}^\sharp,\ |b|=k \\ (b, \Delta_F M F^*(\mathbf{c}))=1}} \eta_1(b)\eta_2(b) S_b(\mathbf{c})$$

and $F(s)$ is the Dirichlet series (8.7). The latter is absolutely convergent and bounded for $\sigma > (n+3)/2$. Noting that

$$\frac{1}{2\pi i} \int_{2\pm iT}^{2\pm i\infty} \frac{u^s \mathrm{d}s}{s} = O\left(\frac{u^2}{T|\log u|}\right),$$

this may clearly be rewritten as

$$\frac{1}{2\pi i} \int_{2-iT}^{2+iT} F\left(s + \frac{n+1}{2}\right) \frac{\widehat{Z}^s \mathrm{d}s}{s} + O\left(\frac{\widehat{Z}^3}{T}\right).$$

Let $\varepsilon > 0$. According to (8.9), the function $F(s+(n+1)/2)$ has an analytic continuation to the half-plane $\sigma \geqslant 1/2 + \varepsilon$ on which it is holomorphic. We change the contour of integration so that it consists of the remaining three sides of the rectangle R with vertices $2 - iT, 1/2 + \varepsilon - iT, 1/2 + \varepsilon + iT$ and $2 + iT$. We will use Lemma 8.4 to estimate the contributions from the various contours. Thus, to begin with, the horizontal contours are seen to contribute

$$\ll \frac{(|\mathbf{c}|\widehat{J})^\varepsilon}{T} \int_{\frac{1}{2}+\varepsilon}^{2} \widehat{Z}^\sigma \mathrm{d}\sigma \ll \frac{(|\mathbf{c}|\widehat{J})^\varepsilon \widehat{Z}^2}{T}.$$

The remaining contour makes the overall contribution

$$\ll |\mathbf{c}|^\varepsilon \widehat{Z}^{1/2+\varepsilon} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{\mathrm{d}t}{1+|t|} \ll |\mathbf{c}|^\varepsilon \widehat{Z}^{1/2+\varepsilon} T^\varepsilon.$$

Combining our estimates and taking $T = \widehat{Z}^3$, we therefore arrive at the statement of the lemma. □

Remark 8.6. Let us put $m = n - 3 = \dim X_{\mathbf{c}}$. Our discussion so far has focussed on the case of even n (i.e. m odd). The purpose of this remark is to highlight the difficulty of dealing with odd n (i.e. m even). Returning to the proof of Lemma 8.4 and applying (3.13), when m is even we instead have

$F(s+(n+1)/2) = G(s)H(s)$ for $\sigma > 1/2$, with $H(s)$ holomorphic and bounded in this half-plane and where

$$G(s) = L\left(\eta_1 \otimes \eta_2 \otimes H_\ell^m(X_{\mathbf{c}}), s + \tfrac{m}{2}\right) = \frac{P_1(q^{-s-m/2})}{P_0(q^{-s-m/2})P_2(q^{-s-m/2})},$$

for suitable polynomials $P_0, P_1, P_2 \in \mathbb{Z}[T]$. (Recall that for odd m it was the reciprocal of this function that we needed to analyse.) In order to have an analogue of Lemma 8.5 for even m we need a holomorphic continuation of $G(s)$ to the left of the line $\sigma = 1$. However, any inverse root of P_2 has absolute value $q^{\frac{m}{2}+1}$ and it is therefore possible that $P_2(q^{-s-m/2})$ has a zero at $s = 1$ (which would imply that $G(s)$ has a pole there). Since we have been unsuccessful in our attempts to analyse this situation precisely, this prevents us from establishing a version of Theorem 7.1 when $n = 9$ using the methods of this paper. As pointed out to the authors by the anonymous referee, the location of the poles of $L(\eta_1 \otimes \eta_2 \otimes H_\ell^m(X_{\mathbf{c}}), s)$ is related to the Tate conjectures and it would be interesting to see what they have to say in this setting.

9. Contribution from square-full moduli

In what follows we will adhere to the notation introduced in Definition 4.6 regarding j-full numbers. Thus any $r \in \mathscr{O}$ admits a unique factorisation

$$r = r_{j+1} \prod_{i=1}^{j} b_i = r_{j+1} \prod_{i=1}^{j} k_i^i,$$

for any integer $j \geqslant 1$, with r_j being j-full. In particular it is easy to prove that

$$\sum_{|r_j| < \widehat{X}} 1 = O(\widehat{X}^{1/j}) \quad \text{and} \quad \sum_{|r_j| > \widehat{X}} |r_j|^{-\ell} = O(\widehat{X}^{1/j-\ell})$$

for any $X > 1$ and $\ell > 1/j$. We will make frequent use of these bounds without further comment.

In this section we complete our estimation of $E(P)$, which was initiated in (8.1), by using the bounds for $\Sigma(Y, \theta)$ derived in the preceding section together with the estimates for averages of complete exponential sums in §6. We begin by recalling (8.1), in which it follows from (5.2) that

$$S_{b_1', M_1, \mathbf{b}_1}(\mathbf{c}) \ll |b_1'|^{(n+1)/2+\varepsilon} |(b_1', F^*(\mathbf{c}))|^{1/2}.$$

Hence there exists $\mathbf{b}_1, \mathbf{b}_2 \in (\mathscr{O}/M\mathscr{O})^n$ and $b_0 \in (\mathscr{O}/M\mathscr{O})^*$ such that

$$\begin{aligned} E(P) \ll \frac{|P|^{n+\varepsilon}}{\widehat{Y}^{(n-1)/2}} &\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ \mathbf{c}\neq\mathbf{0}\\ |\mathbf{c}|\ll\widehat{Y}|P|^{-1}J(\Theta)}} \sum_{\substack{b_1'\in\mathscr{O}^\sharp\\ \varpi|b_1'\Rightarrow\varpi\in S}} |(b_1', F^*(\mathbf{c}))|^{1/2} \\ &\times \sum_{\substack{r_2\in\mathscr{O}\\ |b_1'r_2|\leqslant\widehat{Y}}} \frac{|S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})|}{|r_2|^{(n+1)/2}} \int_{|\theta|=\widehat{\Theta}} |\Sigma(Y,\theta)|\mathrm{d}\theta, \end{aligned} \tag{9.1}$$

for suitable $M_2 \mid M$, where $\Sigma(Y,\theta)$ is given by (8.2). Our treatment of this sum differs according to the value of $\mathbf{c}$ in the outer sum. It will be convenient to differentiate these contributions by writing

- $E_1(P)$ for the part coming from $\mathbf{c}$ such that $F^*(\mathbf{c}) \neq 0$,
- $E_2(P)$ for the part coming from $\mathbf{c}$ such that $\nabla F^*(\mathbf{c}) \neq \mathbf{0}$ but $F^*(\mathbf{c}) = 0$,
- $E_3(P)$ for the part coming from $\mathbf{c} \neq \mathbf{0}$ such that $\nabla F^*(\mathbf{c}) = \mathbf{0}$.

In our estimation of these quantities we will follow common convention and take $\varepsilon > 0$ to be a positive quantity whose value may change from one appearance to the next. Finally, it will be convenient to set

$$\widehat{C} = \widehat{Y}|P|^{-1}J(\Theta), \tag{9.2}$$

to ease notation. In particular we must have $\widehat{C} \gg 1$ in (9.1), which recovers the bound $\widehat{Y} \gg |P|/J(\Theta)$ that we recorded in (7.12). Throughout this section we will make frequent use of the inequalities (7.9) satisfied by Y and Θ.

9.1. **Treatment of $E_1(P)$.** In this section we will assume that $n \geqslant 8$ is even and we will take S to be the set of primes dividing $\Delta_F M F^*(\mathbf{c})$. In particular, it is worth emphasising that $|b_1'|$ can potentially be rather large.

Let $Y_1 \geqslant 0$ be such that

$$\widehat{Y}_1 = \frac{\widehat{Y}}{J(\Theta)}. \tag{9.3}$$

Our argument will differ according to the size of $|b_1'r_2|$. Let $E_{1,a}(P)$ be the contribution to $E_1(P)$ from $|b_1'r_2| \leqslant \widehat{Y}_1$ and write $E_{1,b}(P)$ for the corresponding contribution from $|b_1'r_2| \geqslant \max\{1, \widehat{Y}_1\}$. In the first scenario it will be more efficient to apply Lemma 8.2, whereas Lemma 8.1 is sharper in the second scenario.

The contribution from $|b_1'r_2| \leqslant \widehat{Y}_1$. We may suppose that $\widehat{Y}_1 \geqslant 1$ since otherwise there is nothing to prove. Note that $|b_1'r_2||P||\mathbf{c}| \ll \widehat{Y}^2$ for any $\mathbf{c}$ contributing to $E_{1,a}(P)$. Since $F^*(\mathbf{c}) \neq 0$, it therefore follows from Lemma 8.2

that

$$\Sigma(Y,\theta) \ll \frac{|P|^\varepsilon}{|b_1'r_2|^{1/2}} \frac{\widehat{Y}^{1/2}}{J(\Theta)^{n/2-1/2}},$$

for any $\varepsilon > 0$. There are $O(|P|^\varepsilon)$ choices of $b_1' \in \mathscr{O}^\sharp$ such that $\varpi \mid b_1' \Rightarrow \varpi \in S$. Employing our bound for $\Sigma(Y,\theta)$ in (9.1) we therefore obtain

$$E_{1,a}(P) \ll \frac{|P|^{n+\varepsilon}\widehat{Y}^{1-n/2}\widehat{\Theta}}{J(\Theta)^{n/2-1/2}} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ F^*(\mathbf{c})\neq 0\\ |\mathbf{c}|\ll \widehat{C}}} \sum_{\substack{r_2\in\mathscr{O}\\ |r_2|\leqslant \widehat{Y}_1}} \frac{|S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})|}{|r_2|^{n/2+1}}.$$

Decomposing r_2 as b_2r_3 it follows from Lemma 4.5 and (5.3) that

$$\sum_{b_2\leqslant \widehat{Y}_1} \frac{|S_{b_2,M_2,\mathbf{b}_2}(\mathbf{c})|}{|b_2|^{n/2+1}} \ll |P|^\varepsilon \sum_{|k_2|\leqslant \widehat{Y}_1^{1/2}} \frac{|(k_2,F^*(\mathbf{c}))|}{|k_2|} \ll |P|^\varepsilon. \tag{9.4}$$

Hence

$$E_{1,a}(P) \ll \frac{|P|^{n+\varepsilon}\widehat{Y}^{1-n/2}\widehat{\Theta}}{J(\Theta)^{n/2-1/2}} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ F^*(\mathbf{c})\neq 0\\ |\mathbf{c}|\ll \widehat{C}}} \sum_{\substack{r_3\in\mathscr{O}\\ |r_3|\leqslant \widehat{Y}_1}} \frac{|S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})|}{|r_3|^{n/2+1}},$$

for appropriate $M_3 \mid M$ and $\mathbf{b}_3 \bmod M$. The following result is devoted to estimating the inner sums over $\mathbf{c}$ and r_3.

Lemma 9.1. *Let $R \geqslant 1$. There exists a constant $\delta > 0$ depending only on n such that*

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ |\mathbf{c}|\ll \widehat{C}}} \sum_{\substack{r_3\in\mathscr{O}\\ |r_3|=\widehat{R}}} |S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})| \ll |P|^\varepsilon \widehat{R}^{n/2+4/3-\delta}\left(\widehat{R}^{n/3}+\widehat{C}^n\right).$$

Proof. To estimate this we write $r_3 = b_3r_4$ and we will need to argue differently according to the size of $|r_4|$. For a parameter $0 < Z \leqslant R$, to be defined in due course, the contribution to the inner sum from r_4 such that $|r_4| = \widehat{Z}$ is at most

$$\begin{aligned}\sum_{\substack{r_3=b_3r_4\in\mathscr{O}\\ |r_3|=\widehat{R}\\ |r_4|=\widehat{Z}}} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ |\mathbf{c}|\ll \widehat{C}}} |S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})| &\ll |P|^\varepsilon \sum_{\substack{r_3=b_3r_4\in\mathscr{O}\\ |r_3|=\widehat{R}\\ |r_4|=\widehat{Z}}} |r_3|^{n/2+1}\left(|r_3|^{n/3}+\widehat{C}^n\right)\\ &\ll \frac{|P|^\varepsilon \widehat{R}^{n/2+4/3}}{\widehat{Z}^{1/12}}\left(\widehat{R}^{n/3}+\widehat{C}^n\right),\end{aligned} \tag{9.5}$$

by Lemma 6.1. Alternatively, for appropriate $M_3', \mathbf{b}_3'$, the second part of this same result gives

$$\begin{aligned}
\sum_{\substack{r_3=b_3r_4\in\mathscr{O}\\|r_3|=\widehat{R}\\|r_4|=\widehat{Z}}}\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\|\mathbf{c}|\ll\widehat{C}}}|S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})| &\ll \sum_{\substack{r_3=b_3r_4\in\mathscr{O}\\|r_3|=\widehat{R}\\|r_4|=\widehat{Z}}}|r_4|^{n+1}\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\|\mathbf{c}|\ll\widehat{C}}}|S_{b_3,M_3',\mathbf{b}_3'}(\mathbf{c})|\\
&\ll |P|^{\varepsilon}\widehat{Z}^{n+1}\sum_{\substack{r_3=b_3r_4\in\mathscr{O}\\|r_3|=\widehat{R}\\|r_4|=\widehat{Z}}}|b_3|^{n/2+2/3}\left(|b_3|^{n/3}+\widehat{C}^n\right)\\
&\ll |P|^{\varepsilon}\widehat{R}^{n/2+1}\widehat{Z}^{n/2+1/4}\left(\widehat{R}^{n/3}+\widehat{C}^n\right).
\end{aligned}$$

Taking the minimum of these two estimates and summing over q-adic intervals for Z, we readily arrive at the statement of the lemma. □

Recalling the definitions (9.2), (9.3) of $\widehat{C}$ and $\widehat{Y}_1$, and applying Lemma 9.1 with q-adic ranges for $\widehat{R}\ll\widehat{Y}_1$, our work so far shows that

$$\begin{aligned}
E_{1,a}(P) &\ll \frac{|P|^{n+\varepsilon}\widehat{Y}^{4/3-n/2-\delta}\widehat{\Theta}}{J(\Theta)^{n/2-1/6-\delta}}\left\{\left(\frac{\widehat{Y}}{J(\Theta)}\right)^{n/3}+\left(\widehat{Y}|P|^{-1}J(\Theta)\right)^n\right\}\\
&\ll |P|^{\varepsilon}\left\{\frac{|P|^{n-3}\widehat{Y}^{4/3-n/6-\delta}\widehat{\Theta}|P|^3}{J(\Theta)^{5n/6-1/6-\delta}}+\widehat{Y}^{n/2+4/3-\delta}\widehat{\Theta}J(\Theta)^{n/2+1/6+\delta}\right\},
\end{aligned}$$

for a constant $\delta>0$ depending only on n. Note that $4/3-n/6-\delta<0$ for $n\geqslant 8$. Hence, in view of (7.12), there exists $\delta'>0$ such that the first term is

$$\ll |P|^{n-3-\delta'}\frac{\widehat{\Theta}|P|^3}{J(\Theta)}\ll |P|^{n-3-\delta'}. \tag{9.6}$$

This is clearly satisfactory. Applying (7.9), the second term is seen to be

$$\begin{aligned}
&\ll |P|^{\varepsilon}\widehat{\Theta}\widehat{Y}^{n/2+4/3-\delta}+|P|^{3n/2+1/2+3\delta+\varepsilon}\widehat{\Theta}^{n/2+7/6+\delta}\widehat{Y}^{n/2+4/3-\delta}\\
&\ll |P|^{\varepsilon}\widehat{Q}^{n/2-2/3-\delta}+\frac{|P|^{3n/2+1/2+3\delta+\varepsilon}\widehat{Y}^{1/6-2\delta}}{\widehat{Q}^{n/2+7/6+\delta}}\\
&\ll |P|^{3n/4-1-\delta'},
\end{aligned} \tag{9.7}$$

for an appropriate constant $\delta'>0$ depending on δ and ε. This is satisfactory for $n\geqslant 8$.

The contribution from $|b_1'r_2| \geqslant \max\{1,\widehat{Y}_1\}$. Let us put $\widehat{Y}_2=\max\{1,\widehat{Y}_1\}$ to ease notation. In this case we deduce from Lemma 8.1 that

$$\Sigma(Y,\theta) \ll \frac{|P|^\varepsilon}{|b_1'r_2|}\frac{\widehat{Y}}{J(\Theta)^{n/2}},$$

since $\nabla F^*(\mathbf{c})\neq \mathbf{0}$. Applying this bound in (9.1) we obtain

$$E_{1,b}(P) \ll \frac{|P|^{n+\varepsilon}\widehat{Y}^{(3-n)/2}\widehat{\Theta}}{J(\Theta)^{n/2}} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ F^*(\mathbf{c})\neq 0\\ |\mathbf{c}|\ll \widehat{C}}} \sum_{\substack{b_1'\in\mathscr{O}^\sharp\\ \varpi | b_1' \Rightarrow \varpi\in S}} \sum_{\substack{r_2\in\mathscr{O}\\ \widehat{Y}_2<|b_1'r_2|\leqslant \widehat{Y}}} \frac{|S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})|}{|b_1'|^{1/2}|r_2|^{(n+3)/2}}.$$

Decomposing r_2 as b_2r_3, we find that

$$\sum_{\substack{r_2\in\mathscr{O}\\ \widehat{Y}_2<|b_1'r_2|\leqslant \widehat{Y}}} \frac{|S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})|}{|r_2|^{(n+3)/2}} = \sum_{\substack{r_2=b_2r_3\in\mathscr{O}\\ \widehat{Y}_2<|b_1'b_2r_3|\leqslant \widehat{Y}}} \frac{|S_{b_2,M_2',\mathbf{b}_2'}(\mathbf{c})S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})|}{|b_2r_3|^{(n+3)/2}},$$

for appropriate $M_2', M_3, \mathbf{b}_2', \mathbf{b}_3$, Summing this over the relevant $\mathbf{c}$, we now apply Lemma 9.1 for q-adic values of $\widehat{R}$ in the interval $\widehat{Y}_2/|b_1'b_2| < \widehat{R} \leqslant \widehat{Y}/|b_1'b_2|$ to conclude that

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ F^*(\mathbf{c})\neq 0\\ |\mathbf{c}|\ll \widehat{C}}} \sum_{\substack{r_3\in\mathscr{O}\\ \widehat{Y}_2<|b_1'b_2r_3|\leqslant \widehat{Y}}} \frac{|S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})|}{|r_3|^{(n+3)/2}} \ll |P|^\varepsilon\left(\widehat{Y}^{n/3-1/6-\delta}+\frac{\widehat{C}^n}{(\widehat{Y}_2/|b_1'b_2|)^{1/6+\delta}}\right).$$

The sums over b_1' and b_2 are now easily estimated (with recourse to (9.4) for the latter). Hence, recalling (9.2), we obtain

$$\begin{aligned} E_{1,b}(P) &\ll \frac{|P|^{n+\varepsilon}\widehat{Y}^{(3-n)/2}\widehat{\Theta}}{J(\Theta)^{n/2}}\left(\widehat{Y}^{n/3-1/6-\delta}+\frac{(\widehat{Y}|P|^{-1}J(\Theta))^n}{\widehat{Y}_2^{1/6+\delta}}\right)\\ &\ll |P|^\varepsilon\widehat{\Theta}\left\{\frac{|P|^n\widehat{Y}^{4/3-n/6-\delta}}{J(\Theta)^{n/2}}+\frac{\widehat{Y}^{n/2+3/2}J(\Theta)^{n/2}}{\widehat{Y}_2^{1/6+\delta}}\right\} \end{aligned} \tag{9.8}$$

for some $\delta>0$. Note that $4/3-n/6-\delta<0$ for $n\geqslant 8$, as before. Hence, in view of (7.12), there exists $\delta'>0$ such that the first term is bounded by (9.6), which is satisfactory. On the other hand, taking $\widehat{Y}_2\geqslant \widehat{Y}_1=\widehat{Y}/J(\Theta)$, the second term is seen to be

$$\ll |P|^\varepsilon\widehat{\Theta}\widehat{Y}^{n/2+4/3-\delta}+|P|^{3n/2+1/2+3\delta+\varepsilon}\widehat{\Theta}^{n/2+7/6+\delta}\widehat{Y}^{n/2+4/3-\delta}.$$

But this is satisfactory for $n\geqslant 8$, by (9.7).

9.2. **Treatment of $E_2(P)$.** In this section we will assume that $n \geqslant 8$ (without any assumption on the parity) and we will take S to be the set of primes dividing $\Delta_F M \nabla F^*(\mathbf{c})$. There are $O(|P|^\varepsilon)$ choices for b_1' in (9.1). Applying Lemma 8.1, we therefore obtain the bound

$$E_2(P) \ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}}{\widehat{Y}^{(n-3)/2} J(\Theta)^{n/2}} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n \\ F^*(\mathbf{c})=0 \\ \nabla F^*(\mathbf{c})\neq \mathbf{0} \\ |\mathbf{c}|\ll \widehat{C}}} \sum_{\substack{r_2\in\mathscr{O} \\ |r_2|\leqslant \widehat{Y}}} \frac{|S_{r_2,M_2,\mathbf{b}_2}(\mathbf{c})|}{|r_2|^{(n+3)/2}}.$$

The argument used in (9.4) allows us to replace r_2 by r_3 in the inner sum, after adjusting the value of the parameter ε in the exponent of $|P|$. We will need the following analogue of Lemma 9.1.

Lemma 9.2. *Let $R \geqslant 1$ and put $\delta = \frac{1}{2(n-1)}$. Then*

$$\sum_{\substack{\mathbf{c}\in\mathscr{O}^n \\ F^*(\mathbf{c})=0 \\ |\mathbf{c}|\ll \widehat{C}}} \sum_{\substack{r_3\in\mathscr{O} \\ |r_3|=\widehat{R}}} \frac{|S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})|}{|r_3|^{(n+3)/2}} \ll |P|^\varepsilon \left(\frac{\widehat{R}^{n/3}}{\widehat{R}^{1/6+\delta/3}} + \widehat{C}^{n-1/2-\delta} + \widehat{R}^{1/6}\widehat{C}^{n-3/2} \right),$$

Proof. To estimate this we write $r_3 = b_3 r_4$ and we start by considering the contribution to the inner sum from r_4 such that $|r_4| = \widehat{Z}$, for a parameter $0 < Z \leqslant R$ to be defined in due course. The estimate (9.5) gives

$$\sum_{\substack{r_3=b_3r_4\in\mathscr{O} \\ |r_3|=\widehat{R} \\ |r_4|=\widehat{Z}}} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n \\ F^*(\mathbf{c})=0 \\ |\mathbf{c}|\ll \widehat{C}}} |S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})| \ll \frac{|P|^\varepsilon \widehat{R}^{n/2+4/3}}{\widehat{Z}^{1/12}} \left(\widehat{R}^{n/3} + \widehat{C}^n \right).$$

Alternatively, we invoke Lemma 6.2, which gives

$$\begin{aligned} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n \\ F^*(\mathbf{c})=0 \\ |\mathbf{c}|\ll \widehat{C}}} |S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})| &\ll \widehat{R}^{n/2+4/3+\varepsilon} \left(|b_3|^{n/3-2/3} \widehat{Z}^{n/2-5/6} + \widehat{C}^{n-3/2} \right) \\ &= \widehat{R}^{n/2+4/3+\varepsilon} \left(\widehat{R}^{n/3-2/3} \widehat{Z}^{n/6-1/6} + \widehat{C}^{n-3/2} \right), \end{aligned}$$

for any $r_3 = b_3 r_4 \in \mathscr{O}$ such that $|r_3| = \widehat{R}$ and $|r_4| = \widehat{Z}$. There are clearly $O(\widehat{R}^{1/3}\widehat{Z}^{-1/12})$ such choices for r_3. This therefore gives

$$\sum_{\substack{r_3=b_3r_4\in\mathscr{O}\\ |r_3|=\widehat{R}\\ |r_4|=\widehat{Z}}} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ F^*(\mathbf{c})=0\\ |\mathbf{c}|\ll\widehat{C}}} |S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})| \ll \frac{\widehat{R}^{n/2+5/3+\varepsilon}}{\widehat{Z}^{1/12}} \left(\widehat{R}^{n/3-2/3}\widehat{Z}^{n/6-1/6} + \widehat{C}^{n-3/2}\right)$$
$$\ll \widehat{R}^{n/2+4/3+\varepsilon} \left(\widehat{R}^{n/3-1/3}\widehat{Z}^{n/6-1/4} + \widehat{R}^{1/3}\widehat{C}^{n-3/2}\right).$$

Taking the minimum of these two estimates gives

$$\sum_{\substack{r_3=b_3r_4\in\mathscr{O}\\ |r_3|=\widehat{R}\\ |r_4|=\widehat{Z}}} \sum_{\substack{\mathbf{c}\in\mathscr{O}^n\\ F^*(\mathbf{c})=0\\ |\mathbf{c}|\ll\widehat{C}}} |S_{r_3,M_3,\mathbf{b}_3}(\mathbf{c})| \ll |P|^{\varepsilon}\widehat{R}^{n/2+4/3} \left(A + B + \widehat{R}^{1/3}\widehat{C}^{n-3/2}\right),$$

where

$$\begin{aligned} A &= \min\{\widehat{R}^{n/3-1/3}\widehat{Z}^{n/6-1/4}, \widehat{Z}^{-1/12}\widehat{R}^{n/3}\},\\ B &= \min\{\widehat{R}^{n/3-1/3}\widehat{Z}^{n/6-1/4}, \widehat{Z}^{-1/12}\widehat{C}^{n}\}. \end{aligned}$$

We take $\min\{X,Y\} \leqslant X^{\delta}Y^{1-\delta}$ in both of these, with $\delta = \frac{1}{2(n-1)}$, to find that $A \leqslant \widehat{R}^{n/3-\delta/3}$ and $B \leqslant \widehat{R}^{1/6}\widehat{C}^{n-1/2-\delta}$. Summing over q-adic intervals for $\widehat{Z}$, we quickly arrive at the statement of the lemma. □

Applying Lemma 9.2 in our earlier bound for $E_2(P)$, with $1 \leqslant \widehat{R} \leqslant \widehat{Y}$, we are led to the conclusion that

$$E_2(P) \ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}}{\widehat{Y}^{(n-3)/2}J(\Theta)^{n/2}} \left(\widehat{Y}^{n/3-1/6-\delta/3} + \widehat{C}^{n-1/2-\delta} + \widehat{Y}^{1/6}\widehat{C}^{n-3/2}\right),$$

where $\delta = \frac{1}{2(n-1)}$ The first term here is equal to the first term in the estimate (9.8) for $E_{1,b}(P)$, with a different value of δ, and so makes a satisfactory overall contribution for $n \geqslant 8$. Recalling the definition (9.2) of $\widehat{C}$, the second term contributes

$$\begin{aligned} &\ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}(\widehat{Y}|P|^{-1}J(\Theta))^{n-1/2-\delta}}{\widehat{Y}^{(n-3)/2}J(\Theta)^{n/2}}\\ &= |P|^{1/2+\delta+\varepsilon}\widehat{\Theta}\widehat{Y}^{n/2+1-\delta}J(\Theta)^{n/2-1/2-\delta}\\ &\ll |P|^{1/2+\delta+\varepsilon}\widehat{Q}^{n/2-1-\delta} + |P|^{3n/2-1-2\delta+\varepsilon}\widehat{Y}^{n/2+1-\delta}\Theta^{n/2+1/2-\delta}\\ &\ll |P|^{3n/4-1-\delta/2+\varepsilon}, \end{aligned}$$

which is satisfactory for $n \geqslant 8$. Similarly, the contribution from the third term is seen to be

$$\begin{aligned}
&\ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}(\widehat{Y}|P|^{-1}J(\Theta))^{n-3/2}}{\widehat{Y}^{n/2-5/3}J(\Theta)^{n/2}} \\
&= |P|^{3/2+\varepsilon}\widehat{\Theta}\widehat{Y}^{n/2+1/6}J(\Theta)^{n/2-3/2} \\
&\ll |P|^{3/2+\varepsilon}\widehat{Q}^{n/2-11/6} + |P|^{3n/2-3+\varepsilon}\widehat{Y}^{n/2+1/6}\Theta^{n/2-1/2} \\
&\ll |P|^{3n/4-5/4+\varepsilon},
\end{aligned}$$

which is also satisfactory for $n \geqslant 8$.

9.3. **Treatment of $E_3(P)$.** In this section we will assume that $n = 8$ and we take S to be the set of primes dividing $\Delta_F M$. We combine the second part of Lemma 8.1 with the argument used in (9.4) to replace r_2 by r_3, to get

$$E_3(P) \ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}}{\widehat{Y}^{n/2-2}J(\Theta)^{n/2}} \sum_{\substack{r_3 \in \mathscr{O} \\ |r_3| \leqslant \widehat{Y}}} \sum_{\substack{\mathbf{c} \in \mathscr{O}^n \\ \nabla F^*(\mathbf{c}) = \mathbf{0} \\ 0 < |\mathbf{c}| \ll \widehat{C}}} \frac{|S_{r_3, M_3, \mathbf{b}_3}(\mathbf{c})|}{|r_3|^{n/2+2}},$$

for appropriate $M_3 \mid M$ and $\mathbf{b}_3$ mod M_3. Our main tools to estimate the inner sum over $\mathbf{c}$ will be Lemma 6.3 and its corollary (6.4), together with Lemma 6.1. We begin with the following result.

Lemma 9.3. *Let $n = 8$ and let $\Delta > 0$. Then*

$$\frac{|P|^{n+\varepsilon}\widehat{\Theta}\widehat{C}^{6-\Delta}}{\widehat{Y}^{n/2-2}J(\Theta)^{n/2}} \ll |P|^{5-\Delta/2+\varepsilon} = |P|^{n-3-\Delta/2+\varepsilon}$$

Proof. Recalling the notation (9.2) for $\widehat{C}$, we take $n = 8$ and see that the left hand side is

$$\begin{aligned}
&\ll |P|^{2+\Delta+\varepsilon}\widehat{\Theta}\widehat{Y}^{4-\Delta}J(\Theta)^{2-\Delta} \\
&\ll |P|^{2+\Delta+\varepsilon}\widehat{\Theta}\widehat{Y}^{4-\Delta} + |P|^{8-2\Delta+\varepsilon}\widehat{\Theta}^{3-\Delta}\widehat{Y}^{4-\Delta} \\
&\ll |P|^{5-\Delta/2+\varepsilon},
\end{aligned}$$

as claimed. $\square$

To begin with we dispatch the contribution from r_3 for which $|b_3| > \widehat{Y}^{1-\delta}$, for some small value of $\delta > 0$ to be determined below. In particular we must have $|r_4| < \widehat{Y}^{\delta}$ in the decomposition $r_3 = b_3 r_4$. In this setting (6.4) gives the contribution

$$\begin{aligned}
&\ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}}{\widehat{Y}^{n/2-2}J(\Theta)^{n/2}} \sum_{\substack{r_3\in\mathscr{O}\\ |r_3|\leqslant \widehat{Y}\\ |b_3|>\widehat{Y}^{1-\delta}}} \left(\frac{\widehat{C}^{n-5/2}}{|b_3|^{1/2}}+|b_3|^{4/3}|r_4|^{5/2}\right)\\
&\ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}\widehat{Y}^{O(\delta)}}{\widehat{Y}^{n/2-2}J(\Theta)^{n/2}} \left(\frac{\widehat{C}^{n-5/2}}{\widehat{Y}^{1/6}}+\widehat{Y}^{5/3}\right),
\end{aligned}$$

since there are $O(\widehat{Y}^{1/3})$ available choices of r_3. Assuming that δ is sufficiently small, the first term makes a satisfactory contribution, by Lemma 9.3. On the other hand, taking $n=8$, the second term contributes

$$\ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}\widehat{Y}^{-1/3+O(\delta)}}{J(\Theta)^{n/2}} = |P|^{n-3+\varepsilon}\left(\frac{\widehat{\Theta}|P|^{3}\widehat{Y}^{-1/3+O(\delta)}}{J(\Theta)^{n/2}}\right).$$

This too is satisfactory, if δ is small enough, since $\widehat{Y}\gg |P|/J(\Theta)$.

We now turn to the contribution from $|r_3|\leqslant \widehat{Y}$ such that $|b_3|\leqslant \widehat{Y}^{1-\delta}$. There are clearly at most $O(\widehat{Y}^{1/3-\delta/12})$ choices for r_3. In fact the only place we will need to use this inequality is when dealing with the term $|r_3|^{5n/6+1}$ that appears in Lemma 6.1. Summing over the available r_3 the effect of this term is seen to be

$$\ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}}{\widehat{Y}^{n/2-2}J(\Theta)^{n/2}} \sum_{\substack{r_3\in\mathscr{O}\\ |r_3|\leqslant \widehat{Y}\\ |b_3|\leqslant\widehat{Y}^{1-\delta}}} |r_3|^{n/3-1} \ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}}{\widehat{Y}^{n/6-4/3+\delta/12}J(\Theta)^{n/2}}.$$

The exponent of $\widehat{Y}$ is strictly positive for $n=8$, which is enough to conclude that this term makes a satisfactory overall contribution.

Applying Lemmas 6.3 and 6.1, the remaining contribution is found to be at most

$$\ll \frac{|P|^{n+\varepsilon}\widehat{\Theta}}{\widehat{Y}^{n/2-2}J(\Theta)^{n/2}}\left(H_1+H_2\right),$$

where

$$H_1=\sum_{\substack{r_3\in\mathscr{O}\\ |r_3|\leqslant \widehat{Y}}} \min\left\{\frac{\widehat{C}^{n}}{|r_3|}\ ,\ \frac{|G_1(r_3)|\widehat{C}^{n-5/2}}{|r_3|^{n/2+2}}\right\},\ H_2=\sum_{\substack{r_3\in\mathscr{O}\\ |r_3|\leqslant \widehat{Y}}} \min\left\{\frac{\widehat{C}^{n}}{|r_3|}\ ,\ \frac{|G_2(r_3)|}{|r_3|^{n/2+2}}\right\}.$$

In the light of Lemma 9.3 it suffices to show the existence of positive constants $\Delta_1,\Delta_2>0$ such that $H_i\ll \widehat{C}^{6-\Delta_i}$, for $i=1,2$. Beginning with H_1, we take

$\min\{X,Y\} \leqslant X^{1/5-2\delta/5}Y^{4/5+2\delta/5}$ for a very small value of $\delta > 0$. Recalling that $n = 8$ and then appealing to (6.2), we therefore find that

$$H_1 \leqslant \widehat{C}^{6-\delta} \sum_{\substack{r_3 \in \mathscr{O} \\ |r_3| \leqslant \widehat{Y}}} \frac{|r_3|^{O(\delta)}}{|r_3|^{1/5}\left(|b_3|^{1/2}|b_4|^{3/8}|b_5|^{2/5}|b_6|^{1/12}|b_7|^{5/14}|r_8|^{1/4}\right)^{4/5}}.$$

This therefore gives $H_1 \ll \widehat{C}^{6-\delta}$, as required, since the sum over r_3 is absolutely convergent if δ is small enough. Turning to H_2, for a very small value of $\delta > 0$, we take $\min\{X,Y\} \leqslant X^{3/4-\delta/8}Y^{1/4+\delta/8}$. This time we appeal to (6.3), giving

$$H_2 \leqslant \widehat{C}^{6-\delta} \sum_{\substack{r_3 \in \mathscr{O} \\ |r_3| \leqslant \widehat{Y}}} \frac{|r_3|^{O(\delta)}\left(|b_3|^{4/3}|b_4||b_5|^{9/5}|b_6|^2|b_7|^2|b_8|^{19/8}|r_9|^{5/2}\right)^{1/4}}{|r_3|^{3/4}}.$$

This therefore gives $H_2 \ll \widehat{C}^{6-\delta}$, as required, since the sum over r_3 is absolutely convergent if δ is small enough.

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