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Flows of viscoplastic fluids

By

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A dissertation submitted to the University of Bristol in accordance with the requirements of the degree of DOCTOR OF PHILOSOPHY in the Faculty of Science.

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Abstract

Viscoplastic fluids are a class of material which behave as a rigid solid at stresses below a threshold, “yield”, stress, but flow like a fluid at larger stresses. This property is common in pastes, slurries and gels and hence the theory of viscoplastic fluid flow has wide ranging applications to environmental and industrial flows. In this thesis we analyse a number of viscoplastic flow problems through the techniques of matched-asymptotic methods, shallow-layer theory, direct numerical simulation, and laboratory experiments.

A particular focus is on the inclusion of a yield stress in classical similarity solutions for viscous fluids undergoing non-trivial two-dimensional motion in wedge geometries, for which we detail the existence and structure of unyielded “plugs” and viscoplastic boundary layers as functions of the non-dimensional yield stress. These features are typical of yield-stress flows and represent qualitatively different phenomenology from the corresponding Newtonian solutions. Static unyielded regions are of particular importance to applications. For example, in the food processing industry, stagnant material can spoil and contaminate the product. We quantify the size of unyielded regions of viscoplastic fluid in recirculating corner flows and squeezing flows between hinged plates, and make general conclusions about the unyielded region that forms at a stagnation point in a planar flow.

Finally, we explore the evolution of a thin layer of viscoplastic material scraped by a translating scraper. Such a flow occurs in applications including the removal of excess plaster from a wall or of mud from a road following a mudslide. We employ shallow layer theory to derive the transient evolution of the mound of fluid in front of the scraper and of the residual material behind the scraper, and perform laboratory experiments using a commercial hair gel to test the validity of the predicted surface profiles.
Dedication and acknowledgements

I would like to thank my supervisor, Andrew Hogg, for his untiring support in all things academic and administrative. The research in this thesis would not have been possible without our regular discussions and his enthusiastic guidance. I would also like to thank Alison Rust, Charles Clapham, and Gerard Mwale for facilitating the experimental work reported in Chapter 7, and in particular allowing me to occupy their milling machine for the duration.

On a personal note, I would like to dedicate this thesis to my grandparents, John and Julie Taylor, whose mathematical expertise and enthusiasm was likely instrumental in developing my own love for the subject, and to thank my parents for their ongoing interest in my work. Finally, I would like to thank Violet, for her companionship and support throughout the PhD.

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Author’s declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University’s *Regulations and Code of Practice for Research Degree Programmes* and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate’s own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED: .................................................. DATE: ..........................................
# Table of Contents

**List of Tables**

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>xi</td>
</tr>
</tbody>
</table>

**List of Figures**

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>xiii</td>
</tr>
</tbody>
</table>

## 1 Introduction

1.1 Viscoplastic fluids .................................................. 1
1.2 Models of viscoplastic fluids .......................................... 2
1.3 Outline of thesis ...................................................... 4

## 2 Background Theory

2.1 Classical methods for the solution of viscous and plastic flow problems . 7
  2.1.1 Stokes flow .................................................... 8
  2.1.2 Plasticity and slipline theory .................................. 9
2.2 Viscoplastic boundary layers ........................................... 10
2.3 Shallow-layer equations for a viscoplastic fluid ...................... 15
2.4 Numerical Methods .................................................... 18
  2.4.1 Regularisation approaches ....................................... 19
  2.4.2 The augmented Lagrangian algorithm ............................. 20

## 3 Converging flow of a viscoplastic fluid in a wedge or cone

3.1 Introduction ............................................................ 25
3.2 Problem definition: Bingham fluid in a planar wedge .................. 29
  3.2.1 An outline of the solution ....................................... 31
3.3 Plastic regime ($Bi \gg 1$) .............................................. 32
  3.3.1 The bulk solution ............................................... 33
  3.3.2 Leading order bulk flow ......................................... 35
  3.3.3 The intermediate layer ......................................... 35
TABLE OF CONTENTS

3.3.4 Boundary layer solution ........................................... 38
3.3.5 Higher orders in the bulk ...................................... 40
3.3.6 Composite solutions ............................................. 43
3.3.7 Numerical simulations ........................................... 43
3.4 The viscous regime .................................................. 46
3.4.1 Asymptotic solution ............................................ 48
3.4.2 Numerical simulations ........................................... 50
3.5 Herschel-Bulkley flow ............................................. 51
3.6 Bingham fluid in a cone ........................................... 55
3.6.1 The plastic regime ............................................. 58
3.6.2 The viscous regime ............................................. 62
3.7 Discussion and conclusions ........................................ 64
3.7.1 Asymptotic analysis of the plastic solution in a wedge .... 66
3.7.2 Higher orders in the bulk for a Herschel-Bulkley fluid .... 66
3.7.3 Local solutions in the plastic regime for Bingham fluid in a cone ... 68
3.7.4 Plane and pipe Poiseuille flow of a Bingham fluid .......... 69
3.8 Viscoplastic corner eddies .......................................... 71
4.1 Introduction ......................................................... 71
4.2 Problem definition .................................................. 73
4.3 Numerical method .................................................. 76
4.4 Results and key scalings .......................................... 77
4.4.1 The critical Bingham number .................................. 78
4.4.2 Flow fields when 0 < Bi_c - Bi ≪ 1 ......................... 83
4.5 Comparison with flow past triangular inclusion ................. 90
4.6 Conclusion .......................................................... 92
4.6.1 Viscoplastic flow in corners with α > α_c .................. 93
4.6.2 Torque induced yielding of fluid in a wedge ............... 94
4.6.3 Flow within a parallel-sided channel: α → 0 limit ......... 97
5.1 Introduction .......................................................... 101
5.2 Similarity equations ................................................ 106
5.3 Numerical integration .............................................. 108
5.4 Viscoplastic boundary layers: Bi ≫ 1 ........................... 111
5.4.1 Below the critical angle: 0 < π/4 - α = O(1) ............... 113
# Table of Contents

5.4.2 Near the critical angle: $\frac{\pi}{4} - \alpha = O(Bi^{-1/2})$ 114
5.5 Comparison to full numerical simulations 118
5.6 Discussion and conclusions 119
5.6.1 The Newtonian regime: $Bi \ll 1, N = 1$ 121

6 Stagnation point flow of a viscoplastic fluid 123
6.1 Introduction 123
6.2 Problem definition 125
6.2.1 Non-dimensionalisation and governing equations 126
6.3 Asymptotic solution far from the stagnation point 127
6.4 Numerical Simulations 131
6.5 Plug geometry 133
6.5.1 Vertex angles 133
6.5.2 Determining $\phi_L$ 134
6.5.3 Relative dimensions of the plug 136
6.5.4 Plug width 137
6.6 Embedding in global flow 138
6.6.1 Corner Eddies 139
6.6.2 Flow around a cylinder 141
6.7 Conclusions 144
6.8 Details of embedding 145

7 Scraping of a thin layer of viscoplastic fluid 149
7.1 Introduction 149
7.2 Problem definition 151
7.3 Leakage flux model 153
7.4 The full system 156
7.5 Constant leakage flux 157
7.5.1 Early time 158
7.5.2 Late time 160
7.6 Variable leakage flux 164
7.6.1 Intermediate-time and approach to steady state 165
7.7 Scraping of Herschel-Bulkley fluid 170
7.8 Comparison to experiments 172
7.8.1 Methodology 172
7.8.2 Accounting for slip 179
# TABLE OF CONTENTS

7.8.3 Dimensionless parameters ........................................ 184  
7.8.4 Results and discussion ............................................ 186  
7.9 Conclusions .......................................................... 189  
7.A Early time ODE ....................................................... 190  
7.B Numerical scheme for integrating (7.3)-(7.4) ..................... 191  
7.C Leakage flux for a Herschel-Bulkley fluid ....................... 192  

8 Conclusions ........................................................... 197  

Bibliography ............................................................ 203
List of Tables

Table                                                                 Page

5.1 Table of constants in the asymptotic prediction for the critical angle, (5.49), 117
for different values of the flow index, $N$. The magnitude of the radial pressure
gradient when $\alpha \geq \alpha_c$ is given asymptotically by $2aBi^{1/2}/r$ when $Bi \gg 1$.

7.1 Table of dimensionless parameters and typical length and time scales for the
five scraping experiments. The dimensionless parameters are the Bingham
number, $Bi$, the dimensionless gap height, $\hat{h}_a$, the dimensionless slip length,
$L_s$, and the aspect ratio, $\epsilon = h_\infty / L$. 186
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>11</td>
</tr>
<tr>
<td>2.2</td>
<td>12</td>
</tr>
<tr>
<td>2.3</td>
<td>20</td>
</tr>
<tr>
<td>3.1</td>
<td>29</td>
</tr>
<tr>
<td>3.2</td>
<td>33</td>
</tr>
<tr>
<td>3.3</td>
<td>42</td>
</tr>
<tr>
<td>3.4</td>
<td>44</td>
</tr>
</tbody>
</table>
3.5 The asymptotic (dashed line) and numerical (circles) radial and angular velocities as function of angle, $\theta$, for $\alpha = \pi/4$, $Bi = 1$ and $r = 100$. The numerical data points shown are only a small sub-sample of the full numerical solutions, which are of a much higher resolution. a) and b) plot the radial and angular velocities respectively, across the domain. c) and d) are close-ups of the boundary layers for a) and b) respectively. The position of the fake yield surface, $\theta_Y$, is depicted by a red dotted line in all panels (and is almost indistinguishable from the boundary, $\theta = \pi/4$, in the upper two panels).... 45

3.6 The difference between the leading order asymptotic and numerical solutions for the velocity fields, evaluated as $\Delta u$ and $\Delta v$, as a function of radial position, for $\alpha = \pi/6$, $\pi/4$ and $\pi/3$. The slope markers show the predicted slopes of -2 (top) and -8/3 (bottom).... 47

3.7 a) A density plot of log strain rate from the $\alpha = \pi/3$ numerical simulation. 

b) The scaled radial velocity, $ru$, as a function of angle, $\theta$, for $\alpha = \pi/3$ and $r = 1$ (dotted), 4 (dashed), and 20 (solid). The $\theta$ axis is reversed so that the no-slip boundary is located at the left of the figure.... 47

3.8 The first order corrections to the velocities, $u$ and $v$, in the viscous regime, as functions of the scaled angle, $\theta/\alpha$, for a wedge of half-angle $\alpha = \pi/6$ (solid), $\pi/4$ (dashed), $\pi/3$ (dash-dotted), and $\pi/2$ (dotted).... 49

3.9 Features of the perturbation solution for the planar converging flow of a viscoplastic in the low Bingham number regime: a) the additional wall shear stress, $\tau_{r\theta}^{(1)}|_{\theta=\alpha}$, and b) the scaled additional radial pressure gradient, $r\partial p_1/\partial r = C$, as functions of wedge half-angle, $\alpha$.... 50

3.10 The asymptotic (dashed line) and numerical (circles) radial and angular velocities as functions of angle, $\theta$, for $Bi = 0.01$ and $r = 1$. a) and b) show the radial and angular profiles, respectively, for a wedge of half-angle $\alpha = \pi/4$. 51

3.11 First order corrections to the velocity and stress fields in the bulk as functions of the rescaled angle, $\Theta$, for a Herschel-Bulkley fluid with flow index $N$ (shown in legend). A solid line depicts the results for the Bingham model, $N = 1$. All solutions are for $\alpha = \pi/4$.... 55

3.12 The first order corrections to the velocities, $u$ and $v$, and the stress orientation functions, $\psi$ and $\chi$, for viscoplastic flow in a cone in the plastic regime, as functions of the scaled polar coordinate $\Theta$, for $\alpha = \pi/6$ (solid), $\pi/4$ (dashed), $\pi/3$ (dash-dotted), and $\pi/2$ (dotted).... 61
3.13 The first order corrections to the velocities, $u$ and $v$, for viscoplastic flow in a cone in the viscous regime, as functions of the scaled polar coordinate $\theta/\alpha$, for $\alpha = \pi/6$ (solid), $\pi/4$ (dashed), $\pi/3$ (dash-dotted), and $\pi/2$ (dotted).

3.14 Features of the perturbation solution for the conical converging flow of a viscoplastic in the low Bingham number regime: a) the additional wall shear stress, $\tau_{rr}^{(1)}|_{\theta=\alpha}$, and b) the scaled additional radial pressure gradient, $r\partial p_1/\partial r = C$, as functions of cone half-angle, $\alpha$.

4.1 Strain-rate factor, $S_2 = e^{-(\lambda_r-2)\pi/\lambda_i}$, as a function of corner half-angle, $\alpha$.

4.2 Schematic of viscoplastic eddies in a wedge. Black regions represent unyielded fluid, and only half of the domain is shown, with the lower half determined by anti-symmetry under vertical reflection. No eddies are present in region a), where the fluid is unyielded, and the eddies in region b) are essentially unchanged from the corresponding viscous eddies described by Moffatt [92].

4.3 Contours of the modulus of the strain rate, $|\dot{\gamma}|$, (gray-scale) and streamlines (red) for $\alpha = 20^\circ$ and a) $Bi = 3$, b) $Bi = 1$, c) $Bi = 0.25$, and d) $Bi = 0.018$. The unyielded regions are shown in black. The critical Bingham number at which a new eddy forms, $Bi_c$, lies somewhere between the value of $Bi$ for panels c) and d). Note the logarithmic scale for the strain rate.

4.4 Extent of static plug in corner of wedge, $d$, as a function of $Bi$. Symbols show numerical results while the dotted lines show our heuristic predictions. a) $\alpha = 20^\circ$ on linear-linear scale, showing variation of $d$ with $Bi$. Log-log plots across a larger range of $Bi$ are given for b) $\alpha = 5^\circ$, c) $\alpha = 20^\circ$, d) $\alpha = 45^\circ$, and e) $\alpha = 60^\circ$, showing the jumps at critical values of $Bi$ where a new eddy forms and the self-similarity of consecutive generations of eddies. The red points $A$, $B$, $C$ and $D$ in panel c) indicate the four points derived in the heuristic approximation, as detailed in §4.4.1.

4.5 Examples of solutions before (top row) and after (bottom row) a new eddy has formed. The dotted line shows the dividing streamline $\psi_V = 0$ from the corresponding Moffatt solutions while the red lines show the semi-circles considered in §4.4.1.
4.6 (left) Torque, $G$, acting on the vertical radius of the semi-circle considered in §4.4.1 using the corresponding Moffatt solution (dotted) and from the viscoplastic numerical simulations shown in the top row of figure 4.5 (stars), as a function of wedge half-angle, $\alpha$. (right) The corresponding critical Bingham number, $Bi_c$, calculated from (4.15) (solid line), as a function of wedge half-angle, $\alpha$. The red dotted line shows the divergent behaviour as $\alpha \to 0$, given by $Bi_c \sim 0.0022/\alpha$, while the stars indicate the smallest Bingham numbers of numerical simulations in which the new eddy has not yet opened up (and hence represent numerical upper bounds for $Bi_c$). 83

4.7 Schematic of boundary layer geometry shortly after a new eddy has formed. The grey regions are unyielded fluid, and the central plug is in clockwise solid body rotation around the point $O$ with rotation rate $\Omega$. 85

4.8 (left) Rotation rate $\Omega$ as a function of $Bi$ for $\alpha = 20^\circ$, measured in numerical simulations (stars) and the scaling relationship $\Omega^{(2/3)} \sim Bi_c - Bi$. The horizontal dashed line shows $\Omega_V^{(2/3)}$ where $\Omega_V$ is the rotation rate at the point where strain rate vanishes in the viscous solution. (right) Pressure, $p$, as a function of the coordinate along the boundary-layer, $s$, for three values of the Bingham number deficit, $\Delta Bi$, indicated in the legend. The black dotted line shows the constant gradient $-2Bi_c/R$ predicted in the thicker section of the boundary layer for small $\Delta Bi$, and the vertical grey dashed line marks the narrowest point of the boundary layer. 89

4.9 (left) Boundary layer width as a function of streamwise coordinate, $s$, from the numerical solution for $\alpha = 20^\circ$ and $Bi = 0.02$ (black stars) and the asymptotic solutions (4.36) (cyan dotted) and (4.45) (red dashed). (right) A contour plot of log strain rate from the same numerical simulation (black regions represent unyielded plugs) and the predicted boundaries of the shear-layer from these asymptotic solutions (colours correspond to left panel). 90

4.10 Domain (grey), streamlines (red), and unyielded zones (black) for a lid-driven disturbance in a wedge. The motion is driven by a translating boundary, moving with velocity $U$. In this example $\alpha = 20^\circ$ and $\dot{Bi} = 0.006$. 91
4.11 Comparison of unyielded regions for the idealised problem (top row) and the lid-driven problem (bottom row) with $\alpha = 20^\circ$. The black regions represent unyielded fluid while grey regions represent yielded fluid. The Bingham numbers were chosen to be equivalent after scaling for the velocity and length scales of the first eddy in the lid-driven problem. Only a portion of the lid-driven domain is shown.

4.12 Schematic of expected streamlines (red) and vertex plug (solid black) for viscoplastic corner flow in corner of half-angle, $\alpha_c < \alpha < 90^\circ$. Only the upper half of domain is shown, with the lower half given by (anti)-symmetry.

4.13 General geometry for a potential yield surface in the static corner plug.

4.14 Contours of the components of stress, $\sigma_{xx}$ and $\sigma_{xy}$, in the eddy adjacent to the static corner plug from the numerical simulation for $\alpha = 20^\circ$ and $\Bi = 0.022$. The red dashed line shows a streamline in the eddy, while the white circular arc shows an example of a potential yield surface in the static plug and indicates the parametrisation of these arcs via $Y$ and $\delta$.

4.15 Viscoplastic eddies between parallel plates, driven by a rotating cylinder located at the origin (outline in blue). Plots show strain rate on a logarithmic scale (gray-scale) and streamlines (red dashed lines) for $\tilde{\Bi} = 0.1$ (top), $\tilde{\Bi} = 0.001$ (middle), and $\tilde{\Bi} = 0.0003$ (bottom).

5.1 Schematic of problem geometry. Only half of the geometry is shown, with the other half given by symmetry in $\theta = 0$. The shaded region indicates unyielded material.

5.2 Strain-rate (colour-plot) and streamlines (black) from numerical integration as described in §5.3, with $\alpha = 60^\circ$, $\Bi = 1000$, and $N = 1$. The solid blue region shows the unyielded plug.

5.3 (a,c) Radial and (b,d) azimuthal velocities as functions of polar angle for (a,b) $\Bi = 1/\sqrt{3}$ and (c,d) $\Bi = 10^4$ at various values of $\alpha$ (legend). Solid lines are for $N = 1$ and dotted lines for $N = 0.5$ (often indistinguishable from the $N = 1$ curve).

5.4 (a) $\alpha_c$ and (b) $A_c$ as functions of $\Bi$ from numerical integration for $N = 1$ (solid blue) and $N = 0.5$ (solid red). The corresponding asymptotic predictions for $\Bi \gg 1$ are given by dotted lines. The inset shows a close up of the region $10^5 < \Bi < 10^6$ with the numerically determined values shown as stars. The black dashed line in (a) shows the asymptote $\alpha_c = \pi/4$. 

xvii
5.5 The numerically computed solution, $F = 2u/(r\Omega)$, as a function of the polar angle, $\theta$, (solid line) and the asymptotic composite, $C\{F\}$ as a function of $C\{\theta\}$, (dotted) for $Bi = 10^4$, $\alpha = \pi/6$ (a,c) and $\alpha = \alpha_c$ (b,d), and $N = 1$ (a,b) and $N = 0.5$ (c,d). The curves are plotted parametrically via the independent variable, $\psi$, as $(\theta(\psi), F(\psi))$. .................................................. 115

5.6 a) Dimensionless strain rate, $\dot{\gamma}$, (colour-plot) and streamlines (black) from a numerical simulation with $\alpha = 60^\circ$ and $Bi = 1000$. c) $\log_{10}(\dot{\gamma})$ (colour-plot) and streamlines (white) from a numerical simulation with $\alpha = 30^\circ$ and $Bi = 1000$. b,d) Scaled radial velocity, $F = 2u/(r\Omega)$, as a function of $\theta$ for the numerical simulations shown in a) and c) respectively, at different radial distances from the vertex (see legend) compared against the similarity solution detailed in §5.3 (black). .................................................. 120

5.7 a) $F$ as a function of $\theta$ for $\alpha = \pi/2$ and $Bi = 10^{-3}$, determined by asymptotic predictions (black/red dotted) and numerical integration (blue solid). The asymptotic solution in black retains only the leading order terms in $F$ and $\alpha_c$, while the solution in red retains terms up to $O(Bi)$. The inset shows the thin unyielded region ($F = F' = 0$) near the plate, predicted by the first order asymptotic solution. b) $A_c$ and c) $\alpha_c$ as functions of $Bi$ from the asymptotic predictions (5.61) (red), and numerical integration (blue). .................................................. 122

6.1 A diagram of the flow in the neighbourhood of a stagnation point ................. 125
6.2 $G_p(\theta)$ as a function of $\theta$, for values of $\theta_0$ given in the legend. ............... 129
6.3 The constants $A$ and $C$ as functions of the stagnation angle $\theta_0$. .................. 130
6.4 Stagnation point plugs (blue) and streamlines (black) from numerical simulations with a) $\theta_0 = 90^\circ$, b) $\theta_0 = 60^\circ$, c) $\theta_0 = 45^\circ$, d) $\theta_0 = 30^\circ$. The red dotted lines indicate an angle of $\theta_0/2$ from the horizontal, which is found to be a good approximation for the slope of the upper-left yield surface at the vertex of the plug. .................................................. 132
6.5 Schematic of geometry for stagnation point plug ........................................ 133
6.6 The inclination of the vertex of the stagnation point plug, $\phi_L$, as a function of stagnation angle, $\theta_0$, from numerical simulations (blue dots) and the heuristic approximation $\phi_L = \theta_0/2$ (dotted). .................................................. 135
6.7 Schematic of circular arc approximation to the yield surfaces of the stagnation point plug. .................................................. 136
6.8 The geometrical ratios of the stagnation-point plugs as functions of $\theta_0$. The left panel shows a measure of symmetry, namely the ratio of widths to the right and left of the vertex, $(x_R - x_V)/(x_V - x_L)$, and the right panel shows the aspect ratio of height to width, $y_V/(x_R - x_L)$. The blue dots are determined from numerical simulations while the dotted lines indicate the predictions from the expressions (6.48) and (6.49) using the approximation $\phi_L = \theta_0/2$. 137

6.9 The plug width, $x_R - x_L$, and area, $\int_{x_L}^{x_R} y_Y(x) \, dx$ (where $y = y_Y(x)$ is the equation of the yield surface), as functions of stagnation angle, $\theta_0$. Blue dots are determined from numerical simulations, while the dotted lines show the curves proportional to $\csc(\theta_0)$ (left) and $\csc^2(\theta_0)$ (right) fitted to pass through the numerical data at $\theta_0 = \pi/2$. 138

6.10 Stagnation angle, $\theta_0$, against wedge half-angle, $\alpha$, for the stagnation points on the boundary of the corner-eddy problem. 141

6.11 Comparison of stagnation point plugs from numerical simulations of viscoplastic corner eddies in wedges of half-angle $\alpha = 60^\circ$ (b) and $\alpha = 20^\circ$ (d) and $Bi=1$, with the stagnation point plugs evaluated using the method described in §§6.3-6.4 (rotated to match the angle of the boundary), with $\theta_0 = 45.4^\circ$ (a), and $\theta_0 = 57.6^\circ$ (c), as determined to three significant figures from (6.64). 142

6.12 Example of stagnation point plug on a cylinder in a horizontal uniform flow. (left and middle) Contours of $\log \|\dot{\gamma}\|$ over subsets of the domain, black indicates unyielded fluid while the red box in the left panel shows the region magnified in the second panel. (right) The symmetric stagnation point plug predicted by the method detailed in §§6.3,6.4 for the idealised stagnation problem. 144

7.1 Schematic of flow geometry for scraping of a viscoplastic fluid by a scraper of rectangular cross-section. 151

7.2 Schematic of the non-dimensional velocity profiles in the gap under the scraper for the three different regimes of the non-dimensional pressure gradient, $\hat{G}$. Unyielded plug regions are shaded grey. 154
LIST OF FIGURES

7.3 Dimensionless leakage flux, $Q_\alpha$, as a function of the dimensionless pressure gradient under the scraper, $\hat{G}$, for $\hat{h}_\alpha = 0.1$ and $Bi = 1$ (a), $Bi = 100$ (b), and $Bi = 10^4$ (c). The transitions between different flow regimes are marked by vertical dotted lines (the two transitions are essentially indistinguishable in the first panel) while the approximations for large $\hat{G}$ are shown as red dashed line (the linear approximation, (7.15), is used in panel (a) since $Bi = O(1)$, whereas the general approximation, (7.14), is used in (b) and (c) where $Bi \gg 1$). . . 156

7.4 Early time solutions, $\mathcal{H}$ and $\mathcal{Y}$, for $Bi = 0.5$ (left column), $Bi = 5$ (middle column), and $Bi = 100$ (right column) and various $Q_\alpha$ (values shown in legend). Profiles are plotted against $x_N\xi$ to indicate the differing values of $x_N$, and both $x$ and $y$-axes are shared where not indicated. The black dotted lines in the top right panel shows the yield-stress dominated solution (7.30). 159

7.5 Early time similarity solution (black solid lines) compared with numerical solutions from the interval $0 < t \leq 10^{-2}$ (coloured dotted lines) for $Bi = 5$, $Q_\alpha = 0$ (left column) and $Bi = 0.5$, $Q_\alpha = 0.5$ (right column). Solutions are scaled according to the similarity solution derived in §7.5.1. . . . . . . . . . 161

7.6 Example profiles of free-surface height (top row) and yield-surface height (bottom row) from numerical simulations at $t = 10^2, 10^4, 10^6$ and $10^8$ (colored dotted lines, often coincident with the corresponding dashed lines), the similarity solutions given by (7.34) and (7.37) (black solid lines), and the composite solution given by (7.38) and (7.40) (colored dashed lines). The solutions are scaled according to the similarity solution (7.34)-(7.37). . . . . 163

7.7 Profiles of layer height, $h(x)$, at $t = 0.01, 1,$ and $10^8$, from a numerical solution for the variable leakage flux model described in §7.6 with $\hat{h}_\alpha = 0.2$, $\hat{L}_\alpha = 0.5$, and $Bi = 5$. The outline of the scraper is shown in black. Panels (a) and (b) show all regions of the layer on a linear scale while panel (c) shows only the upstream mound and the unyielded residual layer behind the scraper at late times, using a logarithmic scale for the vertical axis. The horizontal scale of panel (c) is non-uniform, with the different scales upstream and downstream of the scraper indicated on the axis, and the scale linear between the indicated points. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 165
7.8 Numerical solution for the variable leakage flux model described in §7.6 with $\hat{h}_\alpha = 0.2$, $\hat{L}_\alpha = 0.5$, and $Bi = 5$. (left) Free-surface height upstream of the scraper relative to the initial layer thickness, $\hat{h}_0 - 1$, as a function of time, $t$, on a log-log scale. (middle) length of disturbance upstream of scraper, $L$, as a function of time, $t$, on a log-log scale. (right) Leakage flux, $Q_\alpha$, as a function of time, $t$, on a semi-log scale and detail on a linear scale (inset). Dotted red lines show the transition between the different regimes in (7.9).

7.9 Numerical solutions (solid blue) for dimensionless maximum free-surface height, $\hat{h}_0$, as a function of time, $t$, on a log-log scale. Parameters are $\hat{h}_\alpha = 0.1$, $\hat{L}_\alpha = 0.5$, and $Bi = 10$ (a), $Bi = 0.01$ (b) and $Bi = 0.001$ (c). The red dashed line shows the quasi-static prediction given by (7.44), and the gray dashed lines give the predicted time scales for the approach to steady state in the yield-stress dominated regime, $T_y$ (panel a), the transition between viscously and yield-stress dominated behaviour, $T_t$ (panel b), and the approach to steady state in the viscously dominated regime, $T_v$ (panel c).

7.10 Numerical solutions (solid blue) for dimensionless maximum free-surface height, $\hat{h}_0$, as a function of time, $t$, on a log-log scale for a Herschel-Bulkley fluid. Parameters are $\hat{h}_\alpha = 0.1$, $\hat{L}_\alpha = 0.5$, $N = 0.5$, and $Bi = 10$ (a) and $Bi = 0.001$ (c). The slope indicators show the predicted scalings at early times, $t \ll 1$, and intermediate times, $t \gg 1$ with $Q_\alpha \ll 1$, for the yield-stress and viscously dominated behaviours (in (a) and (b) respectively).

7.11 Schematic of experiment configuration. The cuboidal scraper is fixed to the stationary milling machine head, while the tank containing a layer of hair gel is fixed to the bed of the machine, which can translate at uniform velocity, $U$. The scraper fits snugly in the channel, with the gaps on either side being approximately 0.1mm.
7.12 Hair gel rheometry. a) Steady state flow curve for the commercial hair gel used in the experiments. Data (symbols) is shown for two up and down, shear-rate stepped tests, separated by a 100 s rest period. The solid black line indicates the flow curve for a Herschel-Bulkley constitutive law with $\tau_c = 70$ Pa, $K = 80$ Pa $\cdot$ s$^N$, and $N = 0.25$. b) Storage modulus, $G'$ (blue stars), and loss modulus, $G''$ (red circles), as functions of strain. The dashed blue and red lines indicate the values 813 Pa and 52.5 Pa, respectively, obtained by averaging $G'$ and $G''$ over low strains ($\gamma < 10^{-2}$). The vertical grey line indicates the critical strain, $\gamma = 0.12$, at which the stress attains the yield stress, $G'\gamma = \tau_c = 70$ Pa, as obtained from the Herschel-Bulkley fit. 175

7.13 Typical images from a scraping experiment (Test I). The dial indicator behind the scraper is used to test for any deflection of the scraper. 176

7.14 Comparison of images before and after scaling and transformation as described in §7.8.1.3. Note the exaggerated vertical scale. The red crosses in the first panel indicate the key landmarks used to transform the image, and the dashed line in the second panel indicates the parabola fitted to the bottom of the layer, which we use to adjust for the remaining curvature after the first transformation.179

7.15 Example of extraction of the free surface by contouring according to saturation value. The panels show two snapshots from Test I. 180

7.16 Evidence of wall slip in the scraping experiments. Two images from Test IV, taken 50 seconds apart. The residual layer behind the scraper has a uniform thickness set by the gap height, suggesting that the fluid is slipping against the underside of the scraper. The red circles indicate two bubbles that are close to the bottom of the tank but have moved relative to the tank (the red arrows indicate the distance of the bubbles to the bolts, which are moving with the tank), indicative of slip at the base. 180

7.17 a) Free-surface profiles, $h$ (solid), and yield-surface profiles, $Y$ (dashed), from numerical solutions at $t = 20$ with $Bi = 1$, $N = 1$, $\hat{h}_a = 0$, and a selection of dimensionless slip-lengths, $L_a$ (see legend). The yield surface for $L_a = 1$ and $L_a = 1.5$ are both at $Y = 0$. b) Increase in height immediately upstream of the scraper, $h_0 - 1$, as a function of time, $t$, for the same parameters shown in a). 184
7.18 Comparison of surface profiles from experiments (red) and shallow-layer theory (blue). The panels show profiles at a selection of times from four different experiments, for which the dimensionless parameters are given in table 7.1. The profiles shown correspond to the following dimensionless times (increasing from bottom to top as indicated in panel b): $t = 1.4, 4.1, 9.4, 20$ (Test I); $t = 0.55, 2.5, 5.1, 10$ (Test II); $t = 0.041, 0.21, 0.50, 0.99$ (Test III); $t = 0.045, 0.11, 0.24, 0.60$ (Test IV). The vertical dashed line indicates $x = 2h_\infty$, to the left of which we anticipate the shallow-layer theory to fail.

7.19 The instability observed behind the scraper for slow speeds and narrow gaps. The images are from Tests II (left) and V (right), with dimensional gap sizes, $h_\alpha = 3$ mm and 0.6 mm, respectively, and speed, $U \approx 0.4$ mm/s for both. The red lines and text indicate a typical scale for the wavelength of the instability.

7.20 Schematic of flow in thin gap under scraper.

7.21 Dimensionless leakage flux, $Q_\alpha$, as a function of the dimensionless pressure gradient under the scraper, $\hat{G}$, for $\hat{h}_\alpha = 0.1$, $N = 0.5$ and $Bi = 1$ (a) and $Bi = 100$ (b). The transitions between different flow regimes are marked by vertical dotted lines.
1.1 Viscoplastic fluids

The study of fluids dates back to antiquity with Archimedes’ work on buoyancy in Greece and Frontinus’ study of water flow in Roman aqueducts [115]. The mathematical developments required for quantitative modelling of fluid flow arose throughout the 17th, 18th, and 19th centuries, from the works of Newton, d’Alembert and Euler to the full Navier-Stokes equations [115]. These historical studies concerned themselves with fluids which have a constant viscosity (Newtonian fluids) such as water, however a majority of real-world fluids have a viscosity that depends on the state of flow of the material, in particular the stress and strain rate. For instance, tomato ketchup becomes runnier upon shaking and toothpaste requires an imposed force to be squeezed out of its tube. The study of the flow of materials under stress was termed “rheology” by Eugene Bingham in the 1920s [29], and, continuing to develop over the last century, is a relatively modern subject in the study of fluids. Due to the non-linearity of the relationship between stress and strain rate, non-Newtonian fluids pose a mathematical and computational challenge to study, often precluding the existence of exact analytical solutions and the application of standard computational techniques. Nonetheless, effective modelling of these flows is of great import, with non-Newtonian fluids ubiquitous in environmental flows and industrial applications.

Viscoplastic fluids, or yield-stress fluids, are a particular class of non-Newtonian fluid which act as a solid or flow as a viscous fluid, depending on whether the stress is
CHAPTER 1. INTRODUCTION

less than or exceeds a critical yield stress, respectively. This behaviour is common for slurries and suspensions, and the viscoplastic model hence has wide ranging applications in geophysics (e.g. mud flows, avalanches and lava flows), industry (e.g. concrete, wood pulp for paper production, and chocolate processing), and the every-day (e.g. toothpaste, cosmetic creams, peanut-butter and blood) [12, 19, 61].

The physical origin of a yield stress in particulate suspensions is attributed to an equilibrium arrangement of particles and network of interactions that must be broken before the suspension can flow in a homogeneous manner [43]. Historically, there has been some debate regarding the existence of a true yield stress. Notably, in Barnes and Walters [24] and Barnes [23], the authors argued that the appearance of a yield stress was the result of experimental limitations in measuring very low strain rates, suggesting that, rather than behaving as a solid at low stresses, these materials were undergoing very slow “creep” deformation, acting as a Newtonian fluid with very large viscosities under these conditions. However, more recent rheological experiments have indicated that the deformation at low stresses is neither steady, nor homogeneous, with very low shear rates that decay over time and the occurrence of shear-banding, both of which could give the impression of viscous flow at low stresses but are more reminiscent of creep deformation in solids [19]. In either case, deformation rates are extremely low at stresses below the threshold stress and become significant at larger stresses, so we can usefully consider the yield stress as a threshold above which the strain rate becomes noticeable over typical experimental timescales. In practice, the yield stress is typically defined by the intersection with the stress-axis of an extrapolated stress/strain rate curve measured under steady-state conditions. Whether this determines an “apparent” or “true” yield stress, this definition can be incorporated into functional constitutive laws for viscoplastic fluids which are used to model the flow of such materials under more general conditions.

1.2 Models of viscoplastic fluids

The first constitutive law for a viscoplastic fluid is due to Bingham [28] in 1916, when he measured the pressure required to drive a clay suspension through a capillary tube and observed a threshold pressure below which the flow rate vanished, and above which the flow rate was approximately linear in the excess pressure. The resulting constitutive law thus assumes that the rate-of-strain is directly proportional to the additional stress
1.2. MODELS OF VISCOPLASTIC FLUIDS

above the yield stress. In simple shear we therefore write

\[
\begin{cases}
\tau = \tau_c + \mu \dot{\gamma} & \text{when } \tau > \tau_c, \\
\dot{\gamma} = 0 & \text{otherwise},
\end{cases}
\] (1.1)

where \( \tau \) and \( \dot{\gamma} \) are the shear stress and strain rate, respectively, \( \tau_c \) is the yield stress, and \( \mu \) is a viscosity. For two- or three-dimensional flows we require a tensorial constitutive law, relating the deviatoric-stress tensor, \( \tau \), defined in terms of the Cauchy stress tensor, \( \sigma \), in \( n \)-dimensions (where \( n = 2 \) or \( 3 \)) by

\[
\sigma = -pI + \tau; \quad p = -\frac{1}{n} \text{tr } (\sigma),
\] (1.2)
to the symmetric rate-of-strain tensor,

\[
\dot{\gamma} = \nabla \mathbf{u} + (\nabla \mathbf{u})^T.
\] (1.3)

Throughout this thesis we shall employ the convention (1.3) for the strain-rate tensor, but note that the definition of \( \dot{\gamma} \) in the literature often includes a factor of \( 1/2 \), such that it corresponds to the symmetric part of the tensor \( \partial u_i / \partial x_j \), namely

\[
E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).
\] (1.4)

Correspondingly, when using this alternative convention, above yielding the constitutive law is written as \( \tau = \tau_c + 2\mu E \) and the physical results of the model are unchanged. The tensorial form of the Bingham constitutive law assumes that the fluid yields when the second invariant of the deviatoric-stress tensor exceeds the yield stress and that, above yielding, the deviatoric-stress tensor is aligned with the strain-rate tensor with a magnitude consistent with the scalar constitutive law \[42\]. Thus we write

\[
\begin{cases}
\tau = \left( \mu + \frac{\tau_c}{\dot{\gamma}} \right) \dot{\gamma} & \text{when } \tau > \tau_c, \\
\dot{\gamma} = 0 & \text{otherwise},
\end{cases}
\] (1.5)

where \( \tau \) and \( \dot{\gamma} \) are the second invariants of the deviatoric-stress and strain-rate tensors,

\[
\tau = \sqrt{\tau_{ij} \tau_{ij}/2}, \quad \dot{\gamma} = \sqrt{\dot{\gamma}_{ij} \dot{\gamma}_{ij}/2}.
\] (1.6)

The Bingham model is convenient due to its relative simplicity, though it nonetheless represents a highly non-linear constitutive law, resulting in analytical and numerical
challenges. In this thesis we will conduct the majority of analysis for a Bingham constitutive law, providing the advantage of clarity of exposition and the ability to highlight solution phenomenology that is intrinsic to the existence of a yield stress and not other rheological effects. It should be acknowledged, however, that the simplicity of the model also precludes its accurate quantitative representation of most real-world yield-stress fluids, which are known to also exhibit shear-thinning, elasticity, and thixotropy to various degrees. One step towards a more realistic model is to allow for shear-thinning in the constitutive law by introducing a power-law dependence of the viscosity on the strain rate. This results in the Herschel-Bulkley constitutive law for a viscoplastic fluid [68]. The physical explanation for this shear-thinning is that the inter-particle interactions that give rise to the yield stress are not entirely broken at yielding, and so subsequent increases in stress further break the micro-structure, decreasing the viscosity of the fluid. The yield condition for the Herschel-Bulkley model is the same as the Bingham model, while above yielding the constitutive law is given in tensorial form by (see [43])

$$\tau = \left( K\dot{\gamma}^{N-1} + \frac{\tau_c}{\dot{\gamma}} \right) \dot{\gamma},$$  \hspace{1cm} (1.7)

where $K$ is the consistency and $N$ is the flow-index. By taking $N > 1$ the constitutive law can also model shear-thickening behaviour although this is not commonly observed in viscoplastic fluids (apart from some dense suspensions of particles).

Other models of viscoplastic fluids are also available, including the Casson model [36] which is used commonly in the food industry and in modelling blood flow, and elastoviscoplastic models such as the Saramito model [122] which account for elastic deformation prior to yielding and viscoelastic behaviour after yielding. We will not consider these models in this thesis, appealing to the popular use of the Bingham and Herschel-Bulkley models in the fluid dynamics literature in the case of the Casson model, and in order to focus on the manner in which a yield stress expresses itself in complex fluid flow, without the inclusion of additional rheological effects, in the case of the elastoviscoplastic models.

### 1.3 Outline of thesis

In this thesis we apply the Bingham and Herschel-Bulkley viscoplastic models to the solution of a collection of flow problems, using asymptotic and numerical methods to illuminate and detail the features that are characteristic to fluids with a yield stress, in particular the occurrence of viscoplastic boundary layers and rigid unyielded regions.
In §2 we cover the important background theory required for the thesis; §3 concerns the converging flow of a viscoplastic fluid in a wedge or cone; §4 concerns recirculating flow in a wedge; §5 concerns the boundary-layer structure that arises when viscoplastic fluid is compressed in a closing wedge; §6 concerns the unyielded plug that forms at a stagnation point in a flow of viscoplastic fluid; and §7 concerns the scraping of a thin layer of viscoplastic fluid by a vertical scraper. Finally, we make some general conclusions in §8.
Before detailing the novel results presented in this thesis, the current chapter introduces some key background theory from the literature relevant to viscoplastic flows. In §2.1 we describe classical approaches to solving problems involving the flow of viscous and plastic materials; in §2.2 we detail the theory of viscoplastic boundary layers in which thin, high shear rate layers arise when the yield stress is large compared to a typical viscous stress; in §2.3 we detail how the governing equations for free-surface flow of a viscoplastic fluid can be asymptotically reduced under the assumption of a shallow layer of fluid; and in §2.4 we detail the regularisation and augmented Lagrangian approaches to the numerical solution of viscoplastic flow problems.

2.1 Classical methods for the solution of viscous and plastic flow problems

Since viscoplastic fluids exhibit both viscous and plastic behaviour, it is informative to first detail some of the mathematical techniques employed for the solution of viscous and plastic flow problems. In particular, existing solutions to viscous and plastic flow problems can be utilised as the leading order solution to the equivalent viscoplastic flow problem in the regime of low or high yield stress, respectively, and this approach is used in §§3-5. Here, we first outline the solution of the Stokes equation for slow viscous flow, and then introduce the method of sliplines for rigid plastic flow.
CHAPTER 2. BACKGROUND THEORY

2.1.1 Stokes flow

Many flows of viscoplastic fluids are slow, and hence allow for an approximation that neglects inertial stresses on the fluid in comparison to viscous and yield stresses. The ratio of a typical inertial stress to a typical viscous stress is given by the dimensionless Reynolds number,

\[ Re = \frac{\rho \mathcal{L} \mathcal{U}}{\mu}, \]  
(2.1)

where \( \mathcal{L} \) and \( \mathcal{U} \) are typical length and velocity scales and \( \rho \) and \( \mu \) are the density and viscosity. Thus we can neglect inertial stresses when \( Re \ll 1 \). This requires that the forces acting on any parcel of fluid must be in instantaneous balance, resulting in the Stokes equations

\[ \nabla \cdot \sigma = f, \]  
(2.2)

where \( \sigma \) is the stress tensor, defined by (1.2), and \( f \) includes any body forces acting on the fluid. Assuming the fluid is incompressible, we also have the conservation of mass,

\[ \nabla \cdot u = 0, \]  
(2.3)

for the velocity field, \( u \). For a Newtonian fluid, the deviatoric stress tensor is given by

\[ \tau = \mu \left( \nabla u + \nabla u^T \right), \]  
(2.4)

and so the force balance can be expressed as

\[ \mu \nabla^2 u = \nabla p + f. \]  
(2.5)

For two-dimensional problems the conservation of mass condition can be satisfied automatically through the use of a streamfunction, \( \Psi(x,y) \), such that

\[ u = \nabla \times (\Psi \hat{z}). \]  
(2.6)

Taking the curl of (2.5) eliminates pressure, and the body force in the case of a potential force \( f = \nabla \phi \), resulting in the biharmonic equation,

\[ \nabla^4 \Psi = 0. \]  
(2.7)

When consistent with the boundary conditions, the linearity of this equation permits the search for separable solutions that can be superimposed to form more general solutions. Some particular such separable solutions in a wedge or conical geometry will be detailed in future chapters.
2.1. CLASSICAL METHODS FOR THE SOLUTION OF VISCOUS AND PLASTIC FLOW PROBLEMS

2.1.2 Plasticity and slipline theory

In the case of plastic flow, it is also often assumed that the flow is sufficiently slow to neglect inertia, which is valid when

$$\frac{\rho U^2}{\tau_c} \ll 1,$$

for density, $\rho$, typical velocity, $U$, and yield stress, $\tau_c$. Rather than deforming everywhere, a rigid plastic only deforms in regions in which the stress state meets, and must be held at, some “yield criterion”. For an isotropic material, this yield criterion must be independent of direction, and hence must be a function only of the invariants of the stress tensor [12]. In two dimensions there are two invariants, namely (for example) the trace, which is proportional to the mean stress and the magnitude of the stress deviator,

$$\tau = \sqrt{\tau_{ij}\tau_{ij}/2} = \sqrt{\frac{1}{2}\text{tr}(\tau^2)}.$$

(2.9)

In three dimensions there is an additional invariant, often chosen to be the “phase”, $\frac{1}{3}\text{tr}(\tau^3)$. It is observed that yielding occurs only under non-isotropic (shear) stress states, and so the yield criterion cannot be a function only of the first invariant. The simplest yield criterion that satisfies this requirement is the von Mises criterion which assumes yielding occurs when the second invariant of the stress tensor reaches some threshold yield stress, $\tau = \tau_c$. We will employ the von Mises yield criterion throughout this thesis.

Upon yielding, the resulting strain-rate tensor is assumed to be aligned with the deviatoric stress tensor since the material deforms in response to the stress (the coaxiality principle). We therefore have the constitutive rule (where yielded)

$$\frac{\tau}{\tau_c} = \frac{\dot{\gamma}}{\dot{\gamma}}.$$

(2.10)

For planar flows, one can deduce that the problem for the stress field is hyperbolic, and the characteristics correspond to the sliplines (the curves whose directions are everywhere aligned with the direction of maximum shear stress) [72]. In Cartesian coordinates we have

$$\tau^2 = \tau_{xx}^2 + \tau_{xy}^2 = \tau_c^2,$$

(2.11)

and so can write

$$\begin{pmatrix} \tau_{xx} \\ \tau_{xy} \end{pmatrix} = \tau_c \begin{pmatrix} -\sin 2\psi \\ \cos 2\psi \end{pmatrix},$$

(2.12)
where \( \psi \) is the angle between the directions of maximum shear stress and the coordinate axes. The balance of momentum, in the absence of body forces, leads to

\[
\frac{\partial p}{\partial x} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y},
\]

\[
\frac{\partial p}{\partial y} = \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{xx}}{\partial y},
\]

which, after substituting for the components of deviatoric stress, can be written in matrix form

\[
\left( \frac{\partial}{\partial x} + A \frac{\partial}{\partial y} \right) \begin{pmatrix} p \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

where the coefficient matrix, \( A \), is given by

\[
A = \begin{pmatrix} -\cot 2\psi & 2\tau_c \csc 2\psi \\ \frac{1}{2\tau_c} \csc 2\psi & -\cot 2\psi \end{pmatrix}.
\]

We then find the characteristics (termed \( \alpha \) and \( \beta \) sliplines) and Riemann invariants using the eigenvalues and eigenvectors of \( A \), giving

\[
\alpha \text{- lines: } \frac{dy}{dx} = \tan \psi, \quad p + 2\tau_c \psi = \text{const.},
\]

\[
\beta \text{- lines: } \frac{dy}{dx} = -\cot \psi, \quad p - 2\tau_c \psi = \text{const.},
\]

(as given in [20], for example). If the problem is statically determined (i.e. the stress field is fully determined by the stress boundary conditions), then these equations can be used to construct the net of orthogonal characteristics in the plastic region from the conditions on the boundaries, determining the stress state throughout. The velocity field can then be determined via the “Geiringer equations”, which encode (2.10) and incompressibility, given the velocity boundary conditions [72].

### 2.2 Viscoplastic boundary layers

To measure the relative significance of viscous and plastic stresses in a viscoplastic flow, it is convenient to define a non-dimensional Bingham number,

\[
Bi = \frac{\tau_c}{\mu U/L},
\]

given by the ratio of the yield stress to a typical viscous stress, \( \mu U/L \), where \( U \) and \( L \) are imposed velocity and length scales in the flow problem. When the Bingham number
2.2. VISCOPLASTIC BOUNDARY LAYERS

Figure 2.1: Experimental demonstration of viscoplastic boundary layers forming for a rigid plate penetrating through a viscoplastic fluid. The stationary fluid was prepared with coloured layers, which have been deformed in a thin region close to the plate. Reproduced with permission from Boujlel et al. [33].

is large, the yield stress is much larger than a typical viscous stress and the tendency of the fluid is to remain unyielded or undergo a near perfect plastic flow in the bulk of the solution. Where this unyielded or plastic flow solution is inconsistent with velocity boundary conditions, such as in a pressure driven flow through a domain with no slip boundary conditions, narrow viscoplastic boundary layers develop to enforce the boundary conditions [20]. In these boundary layers, the shear rate is large and hence viscous stresses become significant, allowing the solution to diverge from the perfectly plastic solution. Such regions in which the shear rate becomes large have been observed experimentally in the penetration of a plate into viscoplastic fluid by Boujlel et al. [33] (see figure 2.1) and viscoplastic flow through a sudden expansion by Chevalier et al. [39].

The dynamical balance in these viscoplastic boundaries can reside in two different regimes: one in which viscous stresses dominate the yield stress (and hence the “plastic” stresses) and are balanced by the along-layer pressure gradient, and another in which both viscous and plastic stresses are significant in the along-layer balance of momentum. These two regimes were proposed historically by Piau [106] and Oldroyd [99], respectively, but were rationalised and unified recently by Balmforth et al. [20]. We present the theory of Balmforth et al. [20] below for a planar flow using a curvilinear coordinate system based on a curve following the boundary layer (see figure 2.2). We assume lengths have been non-dimensionalised by a typical length scale, \( \mathcal{L} \), of the flow in which the boundary layer is embedded, and velocities by a typical velocity, \( \mathcal{U} \), of this flow. Strain rates are then non-dimensionalised by \( \mathcal{U}/\mathcal{L} \) and stresses by a typical viscous stress, \( \mu \mathcal{U}/\mathcal{L} \). In general the boundary layer may have non-zero curvature, which is non-dimensionalised.
CHAPTER 2. BACKGROUND THEORY

Figure 2.2: Schematic of viscoplastic boundary layer geometry. The boundary layer may be between two regions of unyielded (or almost unyielded) fluid, or between unyielded fluid and a rigid boundary.

by \(1/\mathcal{L}\). After non-dimensionalisation, the coordinates are denoted \((s, n)\), measured along and across the boundary layer, respectively. The velocity components in this coordinate system are denoted \((u, v)\) and the conservation of mass and momentum are given by (see [20])

\[
\begin{align*}
\frac{\partial u}{\partial s} + (1 - \kappa n) \frac{\partial v}{\partial n} - \kappa v &= 0, \\
\frac{\partial \tau_{ss}}{\partial s} + (1 - \kappa n) \frac{\partial \tau_{sn}}{\partial n} - 2\kappa \tau_{sn} &= \frac{\partial p}{\partial s}, \\
\frac{\partial \tau_{sn}}{\partial s} - (1 - \kappa n) \frac{\partial \tau_{ss}}{\partial n} + 2\kappa \tau_{ss} &= \frac{\partial p}{\partial n},
\end{align*}
\]

(2.20, 2.21, 2.22)

where \(\kappa\) is the dimensionless curvature of the boundary layer, and the components of the deviatoric stress tensor are given (for a Bingham fluid) by

\[
\begin{pmatrix}
\tau_{ss} \\
\tau_{sn}
\end{pmatrix} = \begin{pmatrix} 1 + Bi \dot{\gamma} \end{pmatrix} \begin{pmatrix} \dot{\gamma}_{ss} \\
\dot{\gamma}_{sn}\end{pmatrix} \quad \text{for} \quad \tau \equiv \sqrt{\dot{\gamma}_{ss}^2 + \dot{\gamma}_{sn}^2} > Bi,
\]

(2.23)

and \(\ddot{\gamma}_{ij} = 0\) otherwise. Here the components and magnitude of the strain-rate tensor are given by

\[
\dot{\gamma}_{ss} = \frac{2}{1 - \kappa n} \left( \frac{\partial u}{\partial s} - \kappa v \right), \quad \dot{\gamma}_{sn} = \frac{1}{1 - \kappa n} \left( \frac{\partial v}{\partial s} + \kappa u \right) + \frac{\partial u}{\partial n}, \quad \dot{\gamma} \equiv \sqrt{\dot{\gamma}_{ss}^2 + \dot{\gamma}_{sn}^2},
\]

(2.24)

and the Bingham number is given by (2.19).

Assuming the boundary layer is thin, of typical width \(\epsilon \ll 1\), and the curvature is at most \(O(1)\) we define a rescaled cross-layer coordinate via \(n = \epsilon \eta\) and conservation of mass then implies

\[
\begin{pmatrix}
u \\
\end{pmatrix} = \begin{pmatrix} U(s, \eta) \\
\epsilon V(s, \eta)\end{pmatrix},
\]

(2.25)
and
\[ \partial_s U + \partial_n V = 0, \quad (2.26) \]

where the notation \( \partial_x \) represents the partial derivative with respect to \( x \) and we will later use \( \partial_{xy} \) to mean a second order partial derivative with respect to \( x \) and \( y \). The components of the strain-rate tensor are given asymptotically by
\[ \dot{\gamma}_{ss} = 2 \partial_s U + O(\epsilon), \quad \dot{\gamma}_{sn} = \frac{1}{\epsilon} \partial_n U + \kappa U + O(\epsilon), \quad (2.27) \]

and thus (with \( Bi \gg 1 \)) the deviatoric stress components are given by
\[ \tau_{ss} = 2 \partial_s U + 2 \sigma \epsilon Bi \partial_s U \partial_n U + \ldots, \quad \tau_{sn} = \sigma Bi + \frac{1}{\epsilon} \partial_n U - 2 \sigma \epsilon^2 Bi \left( \frac{\partial_s U}{\partial_n U} \right)^2 + \ldots, \quad (2.28) \]

where \( \sigma = \text{sgn}(\partial_n u) \) and we have included all non-constant terms that could be leading order, depending on the relative magnitudes of \( Bi \) and \( \epsilon \). The conservation of momentum in the \( s \) and \( n \) directions are given by
\[ \partial_s P = 2 \sigma Bi \partial_s \left( \frac{\partial_s U}{\partial_n U} \right) + \frac{1}{\epsilon^2} \partial_{n\eta} U - 2 \sigma \epsilon Bi \partial_n \left( \frac{\partial_s U}{\partial_n U} \right)^2 - \sigma \kappa Bi + \ldots, \quad (2.29) \]
\[ \frac{1}{\epsilon} \partial_n P = -\frac{1}{\epsilon} \partial_{sn} U - 2 \sigma Bi \partial_n \left( \frac{\partial_s U}{\partial_n U} \right) + \ldots. \quad (2.30) \]

We must have viscous stresses entering this leading order momentum balance, since the existence of the viscoplastic boundary layer was motivated by the inability of the purely plastic solution to match to particular velocity boundary conditions. This implies \( \epsilon^3 Bi \leq O(1) \) and a balance with the along-layer pressure gradient is achieved when (see [69])
\[ p = -\sigma Bi \partial(s) + \frac{1}{\epsilon^2} P(s, \eta) + \ldots, \quad (2.31) \]

where \( \partial(s) \) is the angle that the centerline of the boundary layer makes with a fixed coordinate axis, so that \( \partial \partial / \partial s = \kappa \). With this definition the (potentially) dominant terms in the conservation of momentum equations are
\[ \partial_s P = \partial_{n\eta} U - 2 \sigma \epsilon^3 Bi \partial_n \left( \frac{\partial_s U}{\partial_n U} \right)^2 + 2 \sigma \epsilon^3 Bi \partial_s \left( \frac{\partial_s U}{\partial_n U} \right), \quad (2.32) \]
\[ \partial_n P = -2 \sigma \epsilon^3 Bi \partial_n \left( \frac{\partial_s U}{\partial_n U} \right). \quad (2.33) \]

We start by assuming that \( \epsilon^3 Bi \ll 1 \), in which case at leading order the equations reduce to
\[ \partial_s P = \partial_{n\eta} U, \quad \partial_n P = 0, \quad (2.34) \]
and so

\[ P = P(s), \quad \text{and} \quad U = \frac{\partial_s P}{2} \left( \eta^2 + A\eta + B \right). \quad (2.35) \]

The width of the boundary layer is then set by the magnitude of the along-layer pressure gradient where the boundary layer meets the rigid plastic flow. Often, matching to the pressure in the outer flow will require the second term on the right-hand side of (2.31) to be \( O(Bi) \), and hence \( \epsilon^{-2} = Bi \implies \epsilon = Bi^{-1/2} \). This is the boundary layer scaling predicted first by Piau [106], in which the viscous shear stress balances the along-layer pressure gradient.

The failure of the above solution occurs when the boundary layer is located between two regions of unyielded fluid (as opposed to between fluid and a rigid boundary, such as in figure 2.1). This is because the quadratic velocity profile found in (2.35) can only have vanishing shearing rate, \( \partial_s U = 0 \), at a single location, while this shear rate must vanish at any location where the boundary layer meets unyielded fluid (where the strain rate is uniformly zero). Thus, for a boundary layer sandwiched between regions of fluid we must have \( \epsilon^3 Bi = 1 \implies \epsilon = Bi^{-1/3} \) which gives the scaling of Oldroyd [99], and all terms in (2.32)-(2.33) are of the same order. Integrating (2.33) once with respect to \( \eta \) gives

\[ P = -2\sigma \frac{\partial_s U}{\partial_\eta U} + H(s) \quad (2.36) \]

where \( H(s) \) is an arbitrary function of integration. Substitution into (2.32) then gives the boundary layer equation [20],

\[ \partial_\eta \left( \partial_s U - 2\sigma \left( \frac{\partial_s U}{\partial_\eta U} \right)^2 \right) + 4\sigma \partial_s \left( \frac{\partial_s U}{\partial_\eta U} \right) = G(s), \quad (2.37) \]

where \( G(s) = dH/ds \). When the boundary layer is sandwiched between two rigid plugs, so that \( U \) matches to constant values, \( U_+ \) and \( U_- \) at the top and bottom of the boundary layer respectively, this equation permits a particular similarity solution, which is symmetrical about the centerline of the layer, with a cubic velocity profile. If this self-similar boundary layer is closed at one end, the layer can either be infinite in extent, with a width that grows with arc-length like \( s^{2/3} \), or else the half-width of the boundary layer, \( Y \) (in scaled boundary-layer coordinates), can achieve a maximum, \( Y_E \), after a finite length. In the latter case the half-width of the boundary layer is related to the arc-length implicitly via

\[ Y_E^{3/2} \left( \tan^{-1} \sqrt{\frac{Y/Y_E}{1 - Y/Y_E}} \right) = \sqrt{3\Delta U \left( s - s_0 \right) / 2}, \quad (2.38) \]

where \( s = s_0 \) is the start of the boundary layer and \( \Delta U \) is the velocity difference \( |U_+ - U_-| \) (see [20]).
2.3 Shallow-layer equations for a viscoplastic fluid

Another asymptotic regime in which it is possible to make analytical headway in the study of viscoplastic flows is in the case of a small aspect ratio in the geometry of the flow. This includes both shallow, free-surface flows driven by gravity, with application to environmental flows of (for example) lava [64] and mud [89], and to lubrication flows between moving surfaces, with application to journal bearings [71] and the role of mucus in the locomotion of a snail [52]. The shallow-layer equations for a two-dimensional, planar flow of a Herschel-Bulkley fluid spreading under gravity on a horizontal surface are derived below as an illustrative case, while the application to different geometries and boundary conditions follows similarly (see for example [16, 71]).

Following the derivation given by Ancey and Cochard [13], we define coordinates, \((x, z)\), and velocities, \((u, w)\), in the horizontal and vertical directions, respectively. The governing equations are then given by

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (2.39)
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} - \frac{\partial p}{\partial x}, \quad (2.40)
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) = \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{xx}}{\partial z} - \frac{\partial p}{\partial z} - \rho g, \quad (2.41)
\]

\[
\tau_{xx} = \left( K \dot{\gamma}^{N-1} + \frac{\tau_c}{\dot{\gamma}} \right) \dot{\gamma}_{xx}, \quad \tau_{xz} = \left( K \dot{\gamma}^{N-1} + \frac{\tau_c}{\dot{\gamma}} \right) \dot{\gamma}_{xz} \quad \text{where } \sqrt{\tau_{xx}^2 + \tau_{xz}^2} > \tau_c, \quad (2.42)
\]

\[
\dot{\gamma}_{xx} = 2 \frac{\partial u}{\partial x}, \quad \dot{\gamma}_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \dot{\gamma} = \sqrt{\dot{\gamma}_{xx}^2 + \dot{\gamma}_{xz}^2}. \quad (2.43)
\]

Where the yield condition, \(\sqrt{\tau_{xx}^2 + \tau_{xz}^2} > \tau_c\), is not satisfied we have \(\dot{\gamma} = 0\). The boundary conditions are given by

\[
u = w = 0 \quad \text{on } z = 0, \quad (2.44)
\]

representing no slip and no penetration at the bottom boundary, and

\[
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = w \quad \text{and } (\mathbf{\tau} - p \mathbf{I}) \cdot \left( \begin{array}{c} \frac{\partial h}{\partial x} \\ 1 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \quad (2.45)
\]
on the free surface $z = h(x, t)$, representing the kinematic and stress-free boundary conditions at the free surface.

Assuming that the vertical scale of the flow, $H$, is significantly smaller than the horizontal scale, $L$, we define the aspect ratio $\epsilon = H/L \ll 1$ (noting that $\epsilon$ thus changes its meaning from the previous section, §2.2, but continues to represent the relevant small parameter in our asymptotic expansion). We further introduce a typical horizontal velocity scale, $U = (\rho g H^2 + N/K L)^{1/N}$ (or equivalently the velocity and vertical length scale could be imposed and the horizontal length scale defined by this relation), and introduce non-dimensional variables via

\[
(x, z) = L (\hat{x}, \epsilon \hat{z}), \quad (u, w) = U (\tilde{u}, \epsilon \tilde{w}), \quad h = \epsilon L \tilde{h}, \quad t = \frac{L}{U} \tilde{t},
\]

and scale the stresses and strain rates by

\[
p = \rho g H \tilde{p}, \quad (\tau_{xx}, \tau_{xz}) = K \left( \frac{U}{H} \right)^N (\tilde{\tau}_{xx}, \tilde{\tau}_{xz}), \quad (\dot{\gamma}_{xx}, \dot{\gamma}_{xz}) = \frac{U}{H} (\tilde{\dot{\gamma}}_{xx}, \tilde{\dot{\gamma}}_{xz}).
\]

Substituting into the governing equations, and dropping tildes for the dimensionless variables gives

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
\]

\[
\epsilon Re \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = \epsilon \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} - \frac{\partial p}{\partial x},
\]

\[
\epsilon^3 Re \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) = \epsilon^2 \frac{\partial \tau_{xz}}{\partial x} - \epsilon \frac{\partial \tau_{xx}}{\partial z} - \frac{\partial p}{\partial z} - 1,
\]

\[
\tau_{xx} = \left( \dot{\gamma}^{N-1} + \frac{Bi}{\dot{\gamma}} \right) \dot{\gamma}_{xx}, \quad \tau_{xz} = \left( \dot{\gamma}^{N-1} + \frac{Bi}{\dot{\gamma}} \right) \dot{\gamma}_{xz} \quad \text{where} \quad \sqrt{\tau_{xx}^2 + \tau_{xz}^2} > Bi,
\]

\[
\dot{\gamma}_{xx} = 2 \epsilon \frac{\partial u}{\partial x}, \quad \dot{\gamma}_{xz} = \frac{\partial u}{\partial z} + \epsilon^2 \frac{\partial w}{\partial x}, \quad \dot{\gamma} = \sqrt{\dot{\gamma}_{xx}^2 + \dot{\gamma}_{xz}^2},
\]

where the Reynolds number, $Re$, and the Bingham number, $Bi$, are given by

\[
Re = \frac{\rho U^2}{K (U/H)^N}, \quad Bi = \frac{\tau_c}{K (U/H)^N}.
\]

From (2.51) and (2.52) we further note that $\tau_{xx} = O(\epsilon)$ while $\tau_{xz} = O(1)$, and hence define $\tau_{xx} = \epsilon \tilde{\tau}_{xx}$ where $\tilde{\tau}_{xx}$ is of order unity. The boundary conditions become

\[
u = w = 0 \quad \text{on} \quad z = 0,
\]

and, on the free surface, $z = h$,

\[
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = w,
\]

From (2.51) and (2.52) we further note that $\tau_{xx} = O(\epsilon)$ while $\tau_{xz} = O(1)$, and hence define $\tau_{xx} = \epsilon \tilde{\tau}_{xx}$ where $\tilde{\tau}_{xx}$ is of order unity. The boundary conditions become

\[
u = w = 0 \quad \text{on} \quad z = 0,
\]
and
\[ \tau_{xz} + p \frac{\partial h}{\partial x} = \epsilon^2 \sigma_{xx} \frac{\partial h}{\partial x}, \quad p = -\epsilon \left( \sigma_{xx} + \tau_{xz} \frac{\partial h}{\partial x} \right). \]  

(2.56)

Thus, assuming \( \epsilon \text{Re} \ll 1 \), to leading order we find
\[ \frac{\partial p}{\partial z} = -1 \quad \Rightarrow \quad p = p_0(x,t) - z, \]  

(2.57)

then the free-surface boundary conditions imply \( p_0(x,t) = h(x,t) \). (2.49) then gives
\[ \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial p_0(x,t)}{\partial x} \quad \Rightarrow \quad \tau_{xz} = (z - h) \frac{\partial h}{\partial x}, \]  

(2.58)

using the free surface boundary conditions. The yield condition, to leading order is given by \( |\tau_{xz}| > Bi \), and thus we find a yield surface at
\[ z = Y(x,t) = \max \left( 0, h - \frac{Bi}{\frac{\partial h}{\partial x}} \right), \]  

(2.59)

with the fluid above this surface being unyielded (with \( \partial u/\partial z = 0 \)) to leading order. Note that this is actually only a “pseudo yield surface”, since the higher orders in the stresses actually result in the magnitude of the stress being held slightly above the yield stress in this region. To rigorously match the solution in \( z > Y \) to the shearing flow below requires a careful expansion to higher orders in \( z > Y \) (see [16]). However, to obtain the evolution equation for the free-surface height, it is sufficient to take the leading order approximation in which the fluid above \( z = Y \) is unyielded. Thus, from the constitutive equations we have
\[ \begin{cases} \frac{\partial}{\partial z} \left( \left| \frac{\partial u}{\partial z} \right| \right)^{N-1} \frac{\partial u}{\partial z} \frac{\partial h}{\partial x} = \frac{\partial h}{\partial x}, & \text{for } z < Y, \\ \frac{\partial u}{\partial z} = 0, & \text{otherwise}, \end{cases} \]  

(2.60)

which can be integrated, making use of the boundary condition at \( z = 0 \) and continuity at \( z = Y \), to obtain
\[ u = \begin{cases} -\frac{\sigma N}{N+1} \frac{\partial h}{\partial x} \left( Y^{1+\frac{1}{N}} - (Y - z)^{1+\frac{1}{N}} \right), & \text{for } z < Y, \\ -\frac{\sigma N}{N+1} \frac{\partial h}{\partial x} Y^{1+\frac{1}{N}}, & \text{otherwise}, \end{cases} \]  

(2.61)

where \( \sigma = \text{sgn}(\partial h/\partial x) \). Vertical integration of (2.48) over the layer then gives
\[ w\big|_{z=h} = u\big|_{z=h} \frac{\partial h}{\partial x} - \frac{\partial}{\partial x} \left( \int_0^h udz \right) = -\frac{\partial Q}{\partial x} + u\big|_{z=h} \frac{\partial h}{\partial x}. \]  

(2.62)
where
\[ Q = -\frac{\sigma N}{(N + 1)(2N + 1)} Y^{1 + \frac{1}{N}} \left( (2N + 1)h - NY \right) \left| \frac{\partial h}{\partial x} \right|^{\frac{1}{N}}. \] (2.63)

Combining this with the kinematic condition for the free-surface, (2.55), gives the evolution equation for the free surface,
\[ \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\sigma N}{(N + 1)(2N + 1)} Y^{1 + \frac{1}{N}} \left( (2N + 1)h - NY \right) \left| \frac{\partial h}{\partial x} \right|^{\frac{1}{N}} \right), \] (2.64)
as given by Ancey and Cochard [13].

The result for a Bingham fluid is retrieved by setting \( N = 1 \) to find
\[ \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{6} Y^2 (3h - Y) \left| \frac{\partial h}{\partial x} \right|^{\frac{1}{2}} \right). \] (2.65)

Similarly, for the 3D problem with coordinates \((x, y, z)\) and free-surface \( z = h(x, y) \) the Herschel-Bulkley result generalises to
\[ \frac{\partial h}{\partial t} = \nabla_H \cdot \left( \frac{N}{(N + 1)(2N + 1)} Y^{1 + \frac{1}{N}} \left( (2N + 1)h - NY \right) \left| \nabla_H h \right|^{\frac{1}{N}} \nabla_H h \right) \] (2.66)
(see for example [18]), where \( \nabla_H \) represents the horizontal gradient operator \((\partial_z, \partial_y, 0)\).

Thus, for a Bingham fluid we have
\[ \frac{\partial h}{\partial t} = \nabla_H \cdot \left( \frac{1}{6} Y^2 (3h - Y) \nabla_H h \right). \] (2.67)

The planar results, (2.64) and (2.65), are applied in §7 where they are used to describe the evolution of a mound of viscoplastic fluid in front of an infinitely wide translating scraper.

### 2.4 Numerical Methods

The numerical solution of viscoplastic flow equations presents a challenge due to the non-linear and non-smooth constitutive law, and the undefined stress state and infinite viscosity in the unyielded regions [124]. This difficulty has been addressed via two general approaches: regularisation, in which the infinite viscosity at low stresses is replaced by large but finite viscosities; and the augmented Lagrangian algorithm, in which additional tensor fields are introduced to decouple the stress and strain rate from the velocity field, and the variational form of the problem is solved via the solution of a related saddle-point problem using optimisation techniques. These two approaches are reviewed extensively by Frigaard and Nouar [60] and Glowinski and Wachs [63], but a brief summary is given below.
2.4. Regularisation approaches

The approach of regularisation is to write the constitutive law via an effective viscosity, $\mu$, such that

$$\tau = \mu (\dot{\gamma}; \varepsilon) \dot{\gamma},$$

(2.68)

where $\varepsilon \ll 1$ (which again differs from $\epsilon$ defined previously in this chapter) is a regularisation parameter such that when $\varepsilon = 0$ the constitutive law reduces to the true viscoplastic constitutive law, given by

$$\mu (\dot{\gamma}) = 1 + \frac{Bi}{\dot{\gamma}} \quad \text{when } \dot{\gamma} > 0,$$

(2.69)

in non-dimensional form for a Bingham fluid. The parameter $\varepsilon$ can be interpreted as a small strain-rate scale, above which the solution can be trusted to accurately represent the true Bingham solution, and below which the solution cannot be trusted to accurately represent the Bingham solution. Examples of regularisations of this constitutive law include the smooth regularisations,

$$\mu (\dot{\gamma}; \varepsilon) = 1 + \frac{Bi}{\sqrt{\dot{\gamma}^2 + \varepsilon^2}},$$

(2.70)

and

$$\mu (\dot{\gamma}; \varepsilon) = 1 + Bi \left( \frac{1 - e^{-\dot{\gamma}/\varepsilon}}{\dot{\gamma}} \right),$$

(2.71)

of Bercovier and Engelman [26] and Papanastasiou [101], respectively, and the non-smooth “bi-viscosity” model,

$$\mu (\dot{\gamma}; \varepsilon) = \begin{cases} \frac{1}{\varepsilon} & \text{where } \dot{\gamma} \leq \frac{\varepsilon}{1-\varepsilon} Bi, \\ 1 + \frac{Bi}{\dot{\gamma}} & \text{where } \dot{\gamma} > \frac{\varepsilon}{1-\varepsilon} Bi. \end{cases}$$

(2.72)

Plots of $\mu$ against $\dot{\gamma}$ are given in figure 2.3 for these different choices of regularisation.

This approach was popular in the early computation of viscoplastic flows between the 1980s and early 2000s, since the models are relatively simple to implement in existing numerical codes, including commercial fluid dynamics packages, and the solution of the resulting discretised non-linear equations can be done efficiently via a Newton method. However, as $\varepsilon \to 0$, the resulting numerical problem becomes increasingly ill-conditioned and difficult to solve [19]. Furthermore, since the fluid is never truly unyielded in a regularised problem, regularised methods have their limitations in accurately predicting the location of yield surfaces, and cannot be easily trusted to make conclusions regarding the stability of yield stress fluids or their approach to a static unyielded state [60].
particular, Putz et al. [111] show that one cannot guarantee the convergence of predicted yield surfaces to the true locations as \( \varepsilon \to 0 \), with regularised solutions sometimes giving erroneous conclusions regarding the location of unyielded regions in the particular flow geometry they considered. These issues motivated the development of alternative approaches to compute viscoplastic fluid flows that accurately account for the unyielded regions.

### 2.4.2 The augmented Lagrangian algorithm

The augmented Lagrangian algorithm for viscoplastic flow problems was first proposed by Glowinski [62] and is described comprehensively by (for example) Saramito [123]. We consider the solution in a domain \( \Omega \) of the momentum equation,

\[
- \nabla \cdot \tau + \nabla p = f, \tag{2.73}
\]

with body force, \( f \), subject to the incompressibility condition, \( \nabla \cdot u = 0 \), velocity boundary conditions, \( u = u_F \) on \( \partial \Omega \), and the constitutive law,

\[
\tau = \left( 1 + \frac{Bi}{\| \dot{\gamma}(u) \|} \right) \dot{\gamma}(u) \quad \text{when } \tau > Bi, \quad \dot{\gamma}(u) = 0 \quad \text{otherwise}. \tag{2.74}
\]
It is proved in [109] that among divergence-free fields satisfying the velocity boundary conditions, the solution for \( \mathbf{u} \) minimises the following energy functional

\[
\mathcal{J}(\mathbf{u}) = \int_\Omega \frac{1}{4} \| \dot{\gamma}(\mathbf{u}) \|^2 + \frac{1}{2} B_i \| \dot{\gamma}(\mathbf{u}) \| - \mathbf{f} \cdot \mathbf{u} \, d^3x. \tag{2.75}
\]

The approach of the augmented Lagrangian algorithm is to write \( \mathcal{J} = \mathcal{J}(\mathbf{u}, \dot{\gamma}) \) and introduce a new tensor field, \( \mathbf{D} \), representing the strain-rate tensor, so that we minimise \( \mathcal{J}(\mathbf{u}, \mathbf{D}) \) subject to the additional constraint \( \mathbf{D} = \dot{\gamma}(\mathbf{u}) \). Thus, introducing the pressure, \( p \), as a Lagrange multiplier for the constraint \( \nabla \cdot \mathbf{u} = 0 \) and the symmetric tensor field, \( \lambda \), as a Lagrange multiplier for the constraint \( \mathbf{D} = \dot{\gamma}(\mathbf{u}) \) we derive the Lagrangian functional

\[
\mathcal{L}(\mathbf{u}, \mathbf{D}; p, \lambda) = \int_\Omega \frac{1}{4} \| \mathbf{D} \|^2 + \frac{1}{2} B_i \| \mathbf{D} \| - \mathbf{f} \cdot \mathbf{u} - p \nabla \cdot \mathbf{u} + \frac{1}{2} \lambda : (\dot{\gamma}(\mathbf{u}) - \mathbf{D}) \, d^3x. \tag{2.76}
\]

Carrying out the variation explicitly, we write \( \mathbf{u} = \mathbf{u}_0 + \hat{\mathbf{u}}, \mathbf{D} = \mathbf{D}_0 + \hat{\mathbf{D}}, p = p_0 + \hat{p} \) and \( \lambda = \lambda_0 + \hat{\lambda} \), where the hatted perturbation terms are small and \( \hat{\mathbf{u}} = \mathbf{0} \) on \( \partial \Omega \), since we are assuming velocity boundary conditions. Expanding up to terms linear in the variations, gives

\[
\delta \mathcal{L} \equiv \mathcal{L}(\mathbf{u}, \mathbf{D}; p, \lambda) - \mathcal{L}(\mathbf{u}_0, \mathbf{D}_0; p_0, \lambda_0)
= \int_\Omega \frac{1}{2} D_{0ij} \dot{D}_{ij} + \frac{B_i}{2} \frac{D_{0ij} \dot{D}_{ij}}{\| \mathbf{D}_0 \|} - f_i \hat{u}_i - \hat{p} \frac{\partial u_{0i}}{\partial x_i} - p_0 \frac{\partial u_{0i}}{\partial x_i} + \frac{1}{2} \lambda_{0ij} \left( \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} - \hat{D}_{ij} \right) \, d^3x. \tag{2.77}
\]

Using symmetry of \( \lambda \), we can write

\[
\delta \mathcal{L} = \int_\Omega \frac{1}{2} D_{0ij} \dot{D}_{ij} + \frac{B_i}{2} \frac{D_{0ij} \dot{D}_{ij}}{\| \mathbf{D}_0 \|} - f_i \hat{u}_i - \hat{p} \frac{\partial u_{0i}}{\partial x_i} - p_0 \frac{\partial u_{0i}}{\partial x_i} + \frac{1}{2} \lambda_{0ij} \left( 2 \frac{\partial \hat{u}_i}{\partial x_j} - \hat{D}_{ij} \right) \, d^3x
+ \frac{1}{2} \lambda_{0ij} \left( \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} - D_{0ij} \right) \, d^3x, \tag{2.78}
\]

where we have used integration by parts and \( \hat{\mathbf{u}} = \mathbf{0} \) on \( \partial \Omega \) in the final line. Thus, requiring that the variation of the Lagrangian vanishes for all perturbations, we find

\[
- \nabla \cdot \mathbf{\lambda}_0 + \nabla p_0 = \mathbf{f}, \quad \nabla \cdot \mathbf{u}_0 = 0, \tag{2.81}
\]

\[
\mathbf{\lambda}_0 = \left( 1 + \frac{B_i}{\| \mathbf{D}_0 \|} \right) \mathbf{D}_0, \quad \mathbf{D}_0 = \nabla \mathbf{u}_0 + \nabla \mathbf{u}_0^T = \dot{\gamma}(\mathbf{u}_0), \tag{2.82}
\]
which we see is equivalent to the governing equations if we identify the Lagrange multiplier, \( \lambda_0 \), with the deviatoric stress tensor, \( \tau \).

This Lagrangian is further “augmented” by the inclusion of a quadratic term, \( \beta \| \dot{\gamma}(u) - D \|^2 / 4 \), which penalises the violation of the new constraint \( D = \dot{\gamma}(u) \). Here \( \beta \) is an augmentation parameter, of which the resulting solution is independent, but which can effect the rate of convergence of the algorithm. The resulting problem thus requires finding a saddle point of the augmented Lagrangian

\[
L_A = \int_{\Omega} \left( \frac{1}{4} \| D \|^2 + \frac{1}{2} B\tau \cdot u - p \cdot u + \frac{1}{2} \lambda : (\dot{\gamma}(u) - D) + \frac{\beta}{4} \| \dot{\gamma}(u) - D \|^2 \right) dx. (2.83)
\]

The most popular algorithm for finding this saddle point involves first minimising with respect to \( u \) and \( p \) for a fixed value of \( D \) and \( \lambda \), then solving the minimisation problem for \( D \) in a pointwise sense, before updating \( \lambda \) via a single step gradient ascent. This method is summarised in Algorithm 1.

**Algorithm 1: Augmented Lagrangian Algorithm**

initialise \( k = 1 \) and guess for \( \lambda_1, D_1 \)

with known \( \lambda_k, D_k \):

while \( \| \dot{\gamma}(u_{k-1}) - D_k \| > tol \) and iters < max_iters do

solve \( \beta \nabla^2 \dot{u}_k - \nabla p_k = -f + \nabla \cdot (\beta D_k - \lambda_k), \nabla \cdot \dot{u}_k = 0 \) for \( \dot{u}_k, p_k \)

update \( D_{k+1} = \begin{cases} 0 & \text{where } \| \lambda_k + \beta \dot{\gamma}(u_k) \| < Bi \\ \frac{\lambda_k + \beta \dot{\gamma}(u_k)}{\beta + 1} & \text{where } \| \lambda_k + \beta \dot{\gamma}(u_k) \| \geq Bi \end{cases} \)

update \( \lambda_{k+1} = \lambda_k + \beta (\dot{\gamma}(u_k) - D_{k+1}) \)

\( k \leftarrow k + 1 \)

end

The augmented Lagrangian algorithm addresses the main issues with the regularisation approaches detailed above, but has its own challenges. Notably the algorithm typically requires bespoke numerical code and is more computationally demanding than regularised computations, often with slow convergence to the solution. Additionally, when implementing the algorithm using the finite element method, interpolation errors can remain large on elements which are intersected by the yield surface, requiring a mesh refinement routine to concentrate mesh resolution near the yield surface [139].

Other similar algorithms exist for the solution of the unregularised viscoplastic problem. For example, Treskatis et al. [138] propose an accelerated version of the augmented Lagrangian algorithm, to improve the convergence rate. The acceleration is achieved via an additional extrapolation step whereby a “leading point” for the Lagrange
multiplier representing the deviatoric stress, here denoted $\tilde{\lambda}$, is calculated via a carefully chosen linear combination of the previous two iterates for $\lambda$. This leading point is then used in the minimisation with respect to $u$, $p$ and $D$ in the following iteration. The resulting ‘FISTA’ algorithm shares many of its steps with minor alterations of Algorithm 1. One version of the algorithm is detailed in Algorithm 2.

Algorithm 2: FISTA Algorithm

initialise $k = 1, t_1 = 1$, guess for $\lambda_0$, and $\tilde{\lambda}_1 = \lambda_0$
with known $\lambda_{k-1}$, $\tilde{\lambda}_k$:

while $\|\dot{\gamma}(u_{k-1}) - D_{k-1}\| > tol$ and iters < max_iters do

set $D_k = \begin{cases} 0 & \text{where } \|\tilde{\lambda}_k\| < Bi \\ \left(1 - \frac{Bi}{\|\tilde{\lambda}_k\|}\right)\tilde{\lambda}_k & \text{where } \|\tilde{\lambda}_k\| \geq Bi \end{cases}$

solve $\nabla^2 u_k - \nabla p_k = -f + \nabla \cdot (D_k - \tilde{\lambda}_k)$, $\nabla \cdot u_k = 0$ for $u_k, p_k$

set $\lambda_k = \tilde{\lambda}_k + (\dot{\gamma}(u_k) - D_k)$

update $t_{k+1} = \left(1 + \sqrt{1 + 4t_k^2}\right)/2$

update $\tilde{\lambda}_{k+1} = \lambda_k + \frac{t_k - 1}{t_{k+1}}(\lambda_k - \lambda_{k-1})$

$k \leftarrow k + 1$

end

Throughout the course of this thesis we will primarily use the traditional augmented Lagrangian algorithm (Algorithm 1). However, due to the improved convergence rate, the FISTA algorithm (Algorithm 2) is used instead for some calculations, which will be indicated in the text.
Converging flow of a viscoplastic fluid in a wedge or cone

Authorship: The material in this chapter is the result of original research by J. J. Taylor-West and A. J. Hogg. It was originally published in Taylor-West and Hogg, 2021 [131] which has been modified slightly for inclusion in this thesis.

3.1 Introduction

The steady flow of a viscous fluid in a planar wedge is a fundamental problem in fluid mechanics [53]. First documented by Jeffery [78] and Hamel [66], the flow is a rare example of a general analytical solution to the Navier-Stokes flow at all Reynolds numbers. ‘Jeffery-Hamel flow’ has been studied extensively (see, for example, Dean [49], Rosenhead [113], Fraenkel [59], Banks et al. [21]). For convex wedges, with half-angle $\alpha \leq 90^\circ$, converging and diverging solutions exist at vanishing Reynolds numbers for all wedges, while at larger Reynolds numbers the flow forms thin boundary layers at the walls. These layers are stable for converging flow but unstable for diverging flow, resulting, above a critical Reynolds number, in separation, and the formation of a central jet and regions of reversed flow at the walls. In essence this occurs because the weak viscous stresses are unable to balance the adverse pressure gradient of the diverging flow.

Steady converging flows of plastic materials also have a long history. Nadai [95] derived a solution for the converging flow of a perfectly rigid plastic in a wedge, with a constant
CHAPTER 3. CONVERGING FLOW OF A VISCOPLASTIC FLUID IN A WEDGE OR CONE

friction imposed on the walls (solution presented in English by Hill [72]), and Shield [126] solved the corresponding problem in an axisymmetric conical geometry. Notably, both of these flow solutions exhibit slip and divergent shear rates at the boundaries. Motivated by industrial applications such as wire drawing, polymer processing, extrusion of pastes, and the discharge of granular materials from hoppers and silos, more recent studies have examined the converging flows of other non-Newtonian fluids. For example, the slow converging flow of incompressible power law fluids has been studied by Durban [56], Brewster et al. [34], and Nagler [96], while the viscoelastic problem has been studied by Hull and Pearson [74]. The work by Brewster et al. [34] is of particular interest, as they consider a power-law fluid of small shear exponent, for which they construct similarity solutions under the assumption of purely radial flow, and demonstrate the need for a complex boundary layer structure in order to enforce no-slip at the walls.

For the slow converging flow of a viscoplastic fluid, Cristescu [44] studied the stress distribution produced by a given kinematically-feasible radial velocity profile in a cone, without attempting to solve the full system of partial differential equations required to explicitly determine the coupled velocity and stress fields. Motivated by strip drawing in the regime of low yield stress or fast drawing speed, Sandru and Camenshi [120] analysed the converging flow of a viscoplastic material when the Bingham number was relatively small (the precise definition of the Bingham number, $Bi$, for the flow analysed in this study is given in (3.13)). They imposed ‘friction’ boundary conditions in the form of an imposed ratio between shear and normal stresses at the boundaries. No-slip boundary conditions can be treated as the limiting case of this approach, in which the deviatoric normal stress vanishes and the shear stress is maximised. Although this case was not the focus of Sandru and Camenshi [120] and they stop short of explicit evaluation of the velocity fields, we demonstrate that their perturbation solution may be used to calculate the velocity fields and, in particular, emphasise that the solution implies purely radial flow is not possible. Durban [54, 55, 56] calculated flow profiles and stress fields for both a planar wedge and an axisymmetric conical die, under constant friction boundary conditions, as opposed to no-slip boundary conditions. These results are restricted by the assumption that the wall friction is low compared to the yield stress, as relevant to well lubricated, metal-forming processes at high temperatures, and the flow profiles are assumed to be purely radial and approximately uniform. Later, Alexandrov et al. [9] studied viscoplastic flow in a wedge under maximum friction boundary conditions, their definition of which reduces to the condition that the deviatoric stress consists only of shear stresses at the wall. Under this boundary condition they find that the radial
velocity must be constant at the boundary, which is inconsistent with a slipping regime in which the radial velocity increases towards the vertex of the wedge. They further consider constitutive laws with a saturation stress (where the stress remains bounded as strain rates diverge) and find that the classical assumptions of a radially independent stress state and purely radial flow are inconsistent when shear stresses become comparable to the shear yield stress. They identify the cause of these difficulties as the divergent strain rates that occur at the wall in the perfectly plastic solution. However, they do not attempt to derive velocity or stress fields for either the standard viscoplastic model, or for one with a saturation stress. Finally, Ara et al. [14] recently performed a numerical study of wedge flow of a Bingham fluid including heat transfer, in the presence of a magnetic field. Again, this study assumed a purely radial flow to derive ordinary differential equations for similarity solutions. These solutions are found to be dependent on radius through the authors’ definition of the Bingham number, and so the conclusions of this work rely on an implicit assumption that this radial dependence is sufficiently weak to be neglected in the expression of conservation of mass.

In this chapter, we present a comprehensive analysis of converging flow of viscoplastic fluid through a wedge or cone, that satisfies no-slip boundary conditions, in both the plastically \((Bi \gg 1)\) and viscously \((Bi \ll 1)\) dominated regimes. We initially carry out a full analysis for a Bingham fluid in a planar wedge, before extending the theory to a Herschel-Bulkley fluid in a wedge and a Bingham fluid in an axisymmetric conical geometry, both of which share the same analytical framework to the flow of a Bingham fluid through a wedge, but are algebraically more involved. Most importantly, in all cases we find that the viscoplasticity introduces a weak angular flow towards the centre of the wedge or cone, explaining the challenges faced by previous attempts to find purely radial solutions. Furthermore, in the plastically dominated regime, boundary layers are found at the wall, allowing the solution to satisfy no-slip. The angular extent of these boundary layers is found to decrease with Bingham number as \(Bi^{-1/2}\) (or \(Bi^{-1/(N+1)}\) for a Herschel-Bulkley fluid of flow index \(N\)) and depends also on the radial distance from the apparent apex of the wedge or cone, \(r\). Direct numerical simulations for a Bingham fluid in a planar wedge are carried out using the finite element method, verifying the analytical results in both regimes.

In the plastic regime, the converging flow field away from the boundaries is given to leading order by the solutions derived by Nadai [95] and Shield [126], with viscous effects becoming significant within relatively thin boundary layers in which the shear rates are relatively large and the shear stresses exceed the yield stress. The velocity field
CHAPTER 3. CONVERGING FLOW OF A VISCOPLASTIC FLUID IN A WEDGE OR CONE

in this regime may be determined using viscoplastic boundary layer techniques, originally
developed by Oldroyd [99] and rationalised by Balmforth et al. [20], as detailed in §2.2.
In the problems studied by Balmforth et al. [20], the boundary layer solutions are only
explicitly asymptotically matched to rigid plugs. However, in the problem considered
in this work, the velocity is varying at the outer edge of the boundary layer due to the
converging geometry, and thus our asymptotic construction is rather different. It will be
shown that the solution does share some similarities with other viscoplastic flows which
exhibit regions of nearly plastic deformation, such as flow in an eccentric annulus [142],
flow around a cylinder [137], or thin-layer gravitational flow [16]. In the weakly yielded
regions, or ‘pseudo-plugs’, present in these flows, the second invariant of the deviatoric
stress does not exceed the yield stress at leading order, but does so at higher orders.
These regions are typically delineated from fully yielded material by ‘fake yield surfaces’.
In accord with the previous results for viscoplastic boundary layers, we show that the
width of the layer scales like $\text{Bi}^{-1/2}$ for a Bingham fluid, as expected for flow bounded
by rigid walls, but that it is matched, via an intermediate asymptotic layer encompassing
a fake yield surface, to a weakly yielded, pseudo rigid plastic solution in the bulk of the
wedge. The existence of these two asymptotic layers is a consequence of the need to both
impose no-slip and to regularise the divergent shear rates found at the wall in the rigid
plastic solution of Nadai [1924].

We first define the problem and outline the solution for the converging wedge flow
of a Bingham fluid (§3.2). We then construct the asymptotic solution for the plastic
regime and compare the results to direct numerical simulations (§3.3). The asymptotic
solution for the viscous regime is derived and compared to direct simulations in §3.4. We
focus initially on the Bingham fluid and planar geometry, since this case most clearly
demonstrates the asymptotic structure with minimal algebraic complications. We then
generalise the problem to a Herschel-Bulkley fluid (§3.5), and to an axisymmetric conical
geometry (§3.6), which both follow the same asymptotic structure as the solutions in
§§3.3,3.4, but are algebraically more involved due to rheology (§3.5) and geometry (§3.6).
We summarise and conclude this chapter in §3.7. There are also four appendices in which
we provide additional algebraic details required for the derivation of the flow solutions.
3.2 Problem definition: Bingham fluid in a planar wedge

We analyse steady slow incompressible flow of a viscoplastic fluid through a planar wedge of half-angle $\alpha$ (see figure 3.1). We restrict our focus to $\alpha \leq \pi/2$, for which the geometry represents a converging flow in a wedge, as opposed to a sink flow external to a wedge when $\alpha > \pi/2$. A volume flux per unit width, $Q$ (taken to be positive for flow towards the vertex), is imposed and we assume that the wedge is sufficiently long that end effects may be ignored. The flow field is predominantly in the radial direction and is assumed symmetric about the centre-line of the wedge. Plane polar coordinates, $(r, \theta)$, are defined relative to the apparent vertex of the wedge. The key dependent variables are the velocities in the radial and angular directions, denoted by $u$ and $v$ respectively.

We initially assume the constitutive law for a Bingham fluid, before generalising to the Herschel-Bulkley model in §3.5. Thus, the constitutive law is given by

$$\begin{cases} 
(\tau_{rr}, \tau_{r\theta}) = \left( \mu + \frac{\tau_c}{\dot{\gamma}} \right) \left( \dot{\gamma}_{rr}, \dot{\gamma}_{r\theta} \right) & \text{if } \tau > \tau_c \\
\dot{\gamma} = 0 & \text{otherwise}, 
\end{cases}$$

(3.1)

where $\mu$ is the Bingham plastic viscosity and $\tau_c$ is the yield stress. The variables $(\tau_{rr}, \tau_{r\theta})$ and $(\dot{\gamma}_{rr}, \dot{\gamma}_{r\theta})$ are the components of the deviatoric stress and strain-rate tensors, respec-
tively, $\tau$ and $\dot{\gamma}$ are the corresponding second invariants given by

$$\dot{\gamma}^2 = \dot{\gamma}_{rr}^2 + \dot{\gamma}_{r\theta}^2 \quad \text{and} \quad \tau^2 = \tau_{rr}^2 + \tau_{r\theta}^2,$$

and the strain-rate tensor is defined by

$$\dot{\gamma} = (\nabla u) + (\nabla u)^T.$$

Due to the flow convergence we expect that $\dot{\gamma} > 0$, and hence that the fluid is yielding everywhere and rigid plugs do not form in the flow. As such, we need only consider the $\tau > \tau_c$ case of the constitutive law (3.1).

We have the following dimensional quantities in our problem: density, $\rho$, viscosity, $\mu$, yield stress, $\tau_c$, volume flux per unit width, $Q$, and typical radial distance from the vertex, $R$. We scale velocities by $Q/R$, strain rates by $Q/R^2$, stresses and pressures by $\mu Q/R^2$, and unless stated otherwise from henceforth all variables will be assumed dimensionless.

On the further assumption that the flow is sufficiently slow to ignore inertial effects, justified below, the non-dimensional governing equations in polar coordinates are given by

$$\frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} = 0,$$

$$\frac{\partial p}{\partial r} = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{2}{r} \tau_{rr},$$

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{rr}}{\partial \theta} + \frac{2}{r} \tau_{r\theta},$$

which express incompressibility (3.4) and the balance of momentum in the radial (3.5) and angular (3.6) directions. The deviatoric stress and strain-rate tensors are given by

$$\begin{pmatrix} \tau_{rr} \\ \tau_{r\theta} \end{pmatrix} = \left(1 + \frac{Bi}{\dot{\gamma}}\right) \begin{pmatrix} \dot{\gamma}_{rr} \\ \dot{\gamma}_{r\theta} \end{pmatrix},$$

$$\dot{\gamma}_{rr} = 2 \frac{\partial u}{\partial r}, \quad \dot{\gamma}_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r},$$

$$\dot{\gamma} = \sqrt{\left(\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 + 4 \left(\frac{\partial u}{\partial r}\right)^2}.$$

The boundary conditions are

$$(u, v) = (0, 0) \text{ at } \theta = \pm \alpha,$$  

$$v = \frac{\partial u}{\partial \theta} = 0 \text{ at } \theta = 0,$$
which express no-slip at the edges of the wedge (3.10) and symmetry about the mid-line (3.11). There is, additionally, an integral expression for the volume flux per unit width given by

\[ \int_{-\alpha}^{\alpha} ru \, d\theta = -1. \]  

The residual dimensionless parameter is given by

\[ Bi = \frac{R^2 \tau_c}{\mu Q}, \]  

which defines a Bingham number, giving the ratio of plastic to viscous stresses. Note that this Bingham number depends on the scale of the radial distance from the vertex, due to the geometry imposing no other length scale. For given rheological parameters and volume flux, the dimensional parameters define a critical dimensional radial distance, \( r_c = (\mu Q/\tau_c)^{-1/2} \), below which viscous forces dominate, and beyond which plastic forces dominate the flow. We will concern ourselves with the \( Bi \gg 1 \) (plastically-dominated) and \( Bi \ll 1 \) (viscously-dominated) regimes, which apply sufficiently far from and sufficiently close to the vertex, respectively. We note that we could have chosen to scale radial distances by \( r_c \), in which case all dimensional parameters are removed from the governing equation. However, our exposition is clearer and easier to compare with previous studies when based upon the dimensionless variables and governing equations given above. We note that, due to the quadratic dependence of \( Bi \) on \( R \), we anticipate that our asymptotic expansions will be functions of \( r^2 Bi \). The absence of inertial terms in (3.4)-(3.6) requires that the Reynolds number, \( Re = \rho Q/\mu \), is sufficiently small. In practice this requirement varies depending on the regime of interest. For the \( Bi \gg 1 \) regime it will be sufficient that \( Re = O(1) \), while for the \( Bi \ll 1 \) regime we will require \( Re \ll Bi \).

### 3.2.1 An outline of the solution

Before performing a detailed analysis of the problem, we outline the key results. In this discussion we assume that the wedge extends from a large dimensional radius, \( r \gg r_c \), to a small radius \( r \ll r_c \). Thus, in the far-field of the wedge, and away from the walls, the leading order flow is given by Nadai’s solution for converging plastic flow [95, 72],

\[ u = -A \frac{r}{r \left( c - \cos 2\psi \right)}, \quad v = 0, \quad \frac{d\psi}{d\theta} = c \sec 2\psi - 1. \]  

Here, \( A \) and \( c \) are constants determined by the flux and boundary conditions, respectively, which are given explicitly later in (3.29) and (3.30). The function \( \psi \) measures the angle between the directions of the principle stresses and the coordinate axes, and is a function
of the polar angle, such that $\tau_{r\theta} = \tau_{rr} \tan 2\psi$. Viscous shear stress dominated flow will be shown to be confined to a thin boundary layer of angular thickness $O((r^2 Bi)^{-1/2})$, allowing the solution to satisfy no-slip. The variation of the angular extent of the boundary layer with $r$ is such that the boundary layer is of a constant Cartesian thickness along the no-slip boundary. The first order corrections to the plastic flow (3.14) will be shown to also be $O((r^2 Bi)^{-1/2})$ and are driven by the flow within the boundary layer. A thin intermediate layer is also required between the bulk of the wedge and the boundary layer, for asymptotic matching of the respective solutions in these regions. One interpretation of this boundary layer structure is that the plastically dominated bulk solution, (3.14), suffers from two different inconsistencies at the wall, namely diverging shear rates and slip, which require addressing at different scales. The intermediate layer plays the role of regularising the divergent shear rates, while the boundary layer enforces the no-slip boundary condition.

In contrast, at the outflow of the wedge, the shear rate is much higher due to radial convergence, and viscous forces dominate the flow. The leading order flow is then the classical Stokes solution,

$$\begin{align*}
  u &= -\frac{\cos(2\theta) - \cos(2\alpha)}{r (\sin(2\alpha) - 2\alpha \cos(2\alpha))}, \quad v = 0, \quad p = -\frac{2 \cos 2\theta}{r^2 (\sin 2\alpha - 2\alpha \cos 2\alpha)} + \text{const},
\end{align*}$$

which defines a purely converging flow for all $\alpha \leq \pi/2$. Here, the first-order corrections due to plasticity are $O(r^2 Bi)$. Comparison of the radial velocity profiles in the plastic and viscous limits (figure 3.2), shows that the viscous profile has enhanced flow at the centre of the wedge compared to the plastic flow. A consequence of this is that we expect an angular velocity to be induced towards the centre of the wedge, away from the walls, to satisfy conservation of mass as the fluid transitions from the plastic profile at the inflow to the viscous profile at the outflow. This is indeed borne out and quantified in the analytical solution constructed below.

### 3.3 Plastic regime ($Bi \gg 1$)

In the regime of large Bingham number, $Bi \gg 1$, or equivalently at large distances from the apparent apex of the wedge, the constitutive equation (3.7) has three main regimes for different magnitudes of the strain rate, $\dot{\gamma}$: plastic stresses dominate the momentum balance when the strain rate is $O(1)$; viscous stresses dominate in a thin boundary layer where the strain rate is high; and both become significant for an intermediate magnitude of the strain rate.
3.3. PLASTIC REGIME ($Bi \gg 1$)

Figure 3.2: The radial velocity field, $u$, as a function of polar angle, $\theta$, for plastic (solid line) and viscous (dashed line) converging flow in a wedge, with $\alpha = \pi/4$ and $r = 1$ (see (3.14) and (3.15)).

3.3.1 The bulk solution

In the bulk of the wedge, away from the boundaries, we expect that $\dot{\gamma} = O(1)$. In this case the constitutive equation is dominated by the yield stress and we find that the deviatoric stresses are determined up to $O(1)$ by the plastic flow equation

$$\tau_{ij} = Bi \frac{\dot{\gamma}_{ij}}{\dot{\gamma}}, \quad (3.16)$$

which implies that, at leading order, the magnitude of the deviatoric stress is everywhere equal to the yield stress, and that the deviatoric stress aligns with the rate of strain. As for the flow in an eccentric annulus (Walton and Bittleston [142]) and gravitationally driven shallow layer flow (Balmforth and Craster [16]), while the stress does not exceed the yield stress at leading order, the material is nonetheless undergoing deformation, and so must be yielded. This region is therefore analogous to ‘pseudo-plugs’ found in other flow scenarios (Walton and Bittleston [142], Balmforth and Craster [16]).

Following the approach of Nadai [95] in the bulk of the flow we introduce a variable, $\psi$, such that

$$\begin{pmatrix} \tau_{rr} \\ \tau_{r\theta} \end{pmatrix} = Bi \begin{pmatrix} \cos 2\psi \\ \sin 2\psi \end{pmatrix}. \quad (3.17)$$

We must have $\psi(r, 0) = 0$ by symmetry, and in the analysis that follows will focus on the region $0 \leq \theta \leq \alpha$, with symmetry allowing the automatic construction of the flow in $-\alpha \leq \theta \leq 0$. In the solution of Nadai [95], (3.14), the velocity does not vanish on the wall. Hence, to satisfy the no-slip boundary condition, we expect the shear rate, $\partial u/\partial \theta$, 

to become large in a small region near the walls. This leads to a different regime where
the viscous stresses are no longer negligible in the balance of momentum. We define the
angle, \( \theta_Y \), at which this change of dynamical balance occurs by

\[
\theta_Y = \alpha - \epsilon \Phi(r),
\]

where \( \epsilon \) is a small parameter depending on \( Bi \) in a way that is yet to be determined, and
we have introduced a potential dependence on \( r \) through the function \( \Phi \). The notation
\( \theta_Y \) alludes to the fact that this is a fake yield surface, delineating weakly yielded fluid
from highly sheared, fully yielded fluid adjacent to the boundary.

The governing equation (3.17) can be rearranged to obtain

\[
\tan(2\psi) = \frac{\tau_{r\theta}}{\tau_{rr}} = \frac{\dot{\gamma}_{r\theta}}{\dot{\gamma}_{rr}}.
\]

At the wall, \( \dot{\gamma}_{rr} \) vanishes, and in the thin boundary layer, \( \dot{\gamma}_{r\theta} \) is large. Hence, both ratios
on the right-hand side of (3.19) are large in the boundary layer, and we require \( \psi \to \pi/4 \)
as \( \theta \to \theta_Y \) in order for the bulk solution to match to the boundary layer. This dependence
of \( \psi \) on \( r \) through the boundary condition at \( \theta = \pm \theta_Y(r) \), leads to apparent difficulties
in the asymptotic investigation of the boundary layer due to divergent strain rates which
obfuscate the straightforward expansion of the flow variables. To avoid this difficulty,
we use strained coordinates, introducing a coordinate transform that transforms the
fake yield surfaces, \( \theta = \pm \theta_Y(r) \), to \( \Theta = \pm 1 \). Specifically we make the transformation of
independent variables

\[
(r, \theta) = (r, \theta_Y(r)\Theta),
\]

noting that this gives partial derivatives:

\[
\frac{\partial}{\partial r} \to \frac{\partial}{\partial r} + \frac{\partial \Theta}{\partial r} \frac{\partial}{\partial \Theta}, \quad \frac{\partial}{\partial \theta} \to \frac{\partial \Theta}{\partial \theta} \frac{\partial}{\partial \Theta} = \frac{1}{\alpha - \epsilon \Phi} \frac{\partial}{\partial \Theta}.
\]

In the plastically dominated region we seek a solution that is, to leading order, radial.
The boundary layer may drive an azimuthal flow of order \( \epsilon \), so we search for a solution
of the form

\[
u = \epsilon u_1 + \ldots, \quad v = \epsilon v_1 + \ldots, \quad \psi = \psi_0 + \epsilon \psi_1 + \ldots, \quad p = p_0 + \epsilon p_1 + \ldots
\]

We have three governing equations. The curl of the momentum balance gives

\[
\left( \frac{1}{(\alpha - \epsilon \Phi)^2} \frac{\partial^2}{\partial \Theta^2} - \left( \frac{r \frac{\partial}{\partial r} + \epsilon r \Phi' \frac{\partial}{\partial \Theta}}{\alpha - \epsilon \Phi} \right)^2 - 2 \left( \frac{r \frac{\partial}{\partial r} + \epsilon r \Phi' \frac{\partial}{\partial \Theta}}{\alpha - \epsilon \Phi} \right) \right) \sin 2\psi
\]

\[
+ \frac{2}{\alpha - \epsilon \Phi} \frac{\partial}{\partial \Theta} \left( \frac{r \frac{\partial}{\partial r} + \epsilon r \Phi' \frac{\partial}{\partial \Theta}}{\alpha - \epsilon \Phi} + 1 \right) \cos 2\psi = 0;
\]

\[
(3.23)
\]
3.3. PLASTIC REGIME \((Bi \gg 1)\)

Conservation of mass gives

\[
\left( \frac{\partial}{\partial r} + \frac{\epsilon \Phi' \Theta}{\alpha - \epsilon \Phi \partial \Theta} \right) (ru) + \frac{1}{\alpha - \epsilon \Phi \partial \Theta} \frac{\partial v}{\partial \Theta} = 0; \tag{3.24}
\]

and, provided \(\epsilon \gg O(Bi^{-1})\), the rigid plastic approximation holds and the orientation of the stress tensor, parameterised through \(\psi\) (3.17), is given by

\[
2 \tan 2\psi \left( r \frac{\partial u}{\partial r} + \frac{\epsilon r \Phi' \Theta}{\alpha - \epsilon \Phi \partial \Theta} \frac{\partial u}{\partial \Theta} \right) = \frac{1}{\alpha - \epsilon \Phi \partial \Theta} \frac{\partial v}{\partial r} + \frac{\epsilon r \Phi' \Theta}{\alpha - \epsilon \Phi \partial \Theta} \frac{\partial v}{\partial \Theta} - v. \tag{3.25}
\]

The requirement that \(\epsilon \gg O(Bi^{-1})\), which will be verified later, ensures that the viscous terms do not re-enter the expansion of the constitutive equation, (3.7), up to \(O(\epsilon)\).

3.3.2 Leading order bulk flow

At \(O(1)\) the solution of (3.23)-(3.25) corresponds to Nadai’s [1924] solution up to the slight alteration of the factors of \(\alpha\) due to the scaling to unit angle. Specifically, we have

\[
\alpha \Theta = -\psi_0 + \frac{c}{\sqrt{c^2 - 1}} \arctan \left( \frac{\sqrt{c + 1}}{\sqrt{c - 1}} \tan \psi_0 \right), \tag{3.26}
\]

\[
u_0 = -\frac{A}{r (c - \cos (2\psi_0))}, \tag{3.27}
\]

\[
p_0 = 2Bi c \ln(r) + Bi c \ln(c - \cos 2\psi_0) + \text{const.} \tag{3.28}
\]

The constant, \(c\), is determined by the boundary condition \(\psi_0 (1) = \pi/4\), as required to match to a shear-stress dominated boundary layer (see discussion after equation (3.19)). This gives the implicit equation

\[
\alpha + \frac{\pi}{4} = \frac{c}{\sqrt{c^2 - 1}} \arctan \left( \frac{\sqrt{c + 1}}{\sqrt{c - 1}} \right). \tag{3.29}
\]

Finally, \(A\) is determined by the volume-flux condition. To leading order,

\[
A = \frac{2c (c^2 - 1)}{4\alpha + \pi + 2c}. \tag{3.30}
\]

3.3.3 The intermediate layer

Asymptotic analysis of this leading order solution (see Appendix 3.A) verifies that the shear rate diverges as we approach the fake yield surface, \(\theta = \theta_Y\). Thus, we will require an intermediate layer around \(\Theta = 1\), to regularise the diverging terms. We
CHAPTER 3. CONVERGING FLOW OF A VISCOPLASTIC FLUID IN A WEDGE OR CONE

proceed following Walton and Bittleston [142] and Balmforth and Craster [16], to derive a governing equation that reduces to the appropriate dynamical balances as we move out of the intermediate layer into either the bulk or boundary layer regions. We define this intermediate layer by the further coordinate transformation
\[ \delta \zeta = \theta \gamma - \theta = (\alpha - \epsilon \Phi)(1 - \Theta), \]
where \( \zeta \) is our new angular coordinate and \( \delta \) is another ordering parameter, and we re-label velocity components:
\[
\begin{pmatrix}
    u \\
v
\end{pmatrix} = \begin{pmatrix}
    U \\
\epsilon V
\end{pmatrix}, \tag{3.31}
\]
where \( U \) and \( V \) are assumed to be \( O(1) \). It is assumed, and later verified, that \( \delta \ll \epsilon \ll 1 \), so that the intermediate layer is much narrower than both the bulk region and the boundary layer. The \( \epsilon \) scaling of the azimuthal velocity is inherited from the bulk solution (3.22).

Note that
\[ \delta \zeta = \alpha - \epsilon \Phi(r) - \theta, \tag{3.32} \]
and so, as before, we have altered partial derivatives:
\[
\frac{\partial}{\partial r} \rightarrow \frac{\partial}{\partial r} - \frac{\epsilon \Phi'}{\delta} \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial \theta} \rightarrow -\frac{1}{\delta} \frac{\partial}{\partial \zeta}. \tag{3.33}
\]

Expansion of the leading order plastic solution for small \( \delta \) (see Appendix 3.A), gives
\[ u_0 = -\frac{A}{rc} - \frac{2A}{rc \sqrt{c}} \sqrt{\delta \zeta} + \ldots, \tag{3.34} \]
thus we define
\[ U = U_0(r) + \delta^{1/2} U_1(r, \zeta) + \ldots, \quad \text{with} \quad U_0(r) = -\frac{A}{cr}, \tag{3.35} \]
and the term \( U_1(r, \zeta) \) capturing the weak variation of \( U \) with \( \zeta \) over the width of the intermediate layer. In addition to the assumption that \( \delta \ll \epsilon \ll 1 \), we will now also work under the assumption that \( \epsilon^2 \ll \delta \) which will be verified below and ensures that the additional terms from the coordinate transformation do not enter the leading order balance. This assumption is also required for the second term in (3.34) to be greater than contributions to the velocity from the order \( \epsilon \) correction terms in the bulk solution.

Under this assumption, expanding the shear stress gives
\[
\tau_{r\theta} = -\frac{1}{r} \delta^{-1/2} \partial_\zeta U_1 + Bi - 2r^2 \delta Bi \left( \frac{\partial_r U_0}{\partial_\zeta U_1} \right)^2 + \ldots, \tag{3.36}
\]
where we now use short-hand notation for partial derivatives for notational clarity. The second term of (3.36) is constant and so does not contribute to the momentum balance, hence we have both viscous and plastic terms when \( \delta = Bi^{-2/3} \). The leading order radial and angular expressions of momentum balance are now given by

\[
Bi^{-1} \partial_r p = \frac{1}{r^2} \partial_\zeta \partial_\zeta U_1 + 2r \partial_\zeta \left[ \left( \frac{\partial_\zeta U_0}{\partial_\zeta U_1} \right)^2 \right],
\]

\[
Bi^{-1/3} \partial_\zeta p = 2r \partial_\zeta \left[ \frac{\partial_\zeta U_0}{\partial_\zeta U_1} \right],
\]

which, along with the fact that \( p \) is \( O(Bi) \) from the bulk solution, imply \( p = Bi P(r) = 2cBi \ln r + \text{const.} \), to leading order. Thus, the radial pressure gradient is unchanged to leading order by the intermediate layer.

The asymptotic behaviour of the shear stress in the bulk solution is given by

\[
\tau_{r\theta} = Bi \sin 2\psi_0 = Bi - 2cBi\delta\zeta + \ldots = Bi - 2cBi^{1/3}\zeta, \ldots
\]

(3.39) (see Appendix 3.A). Then, integrating (3.37) once with respect to \( \zeta \) and using (3.39) and (3.36) at \( O(Bi^{1/3}) \) to set the constant of integration, we establish a cubic equation in \( \partial_\zeta U_1 \),

\[
(\partial_\zeta U_1)^3 - r^2 \zeta \partial_r (\partial_\zeta U_1)^2 + 2r^3 (\partial_\zeta U_0)^2 = 0,
\]

(3.40) in the intermediate layer. Substituting for \( U_0(r) \) and \( P(r) \), we find the form

\[
(\partial_\zeta U_1)^3 - 2c r \zeta (\partial_\zeta U_1)^2 + \frac{2A^2}{c^2 r} = 0.
\]

(3.41)

For the angular component of the velocity, from the incompressibility equation we have

\[
\frac{\partial}{\partial r} (rU) - \frac{\epsilon r' \Phi'}{\delta} \frac{\partial U}{\partial \zeta} - \frac{\epsilon}{\delta} \frac{\partial V}{\partial \zeta} = 0,
\]

(3.42) which, on substitution of (3.35), leads to

\[
V = V_0(r) - \delta^{1/2} r \Phi' U_1(r, \zeta) + \ldots
\]

(3.43) It remains to find \( V_0 \) and \( \Phi \) by solving in the boundary layer (§3.3.4).

We deduce some matching information for the boundary layer by considering the behaviour of the cubic equation for \( \partial_\zeta U_1 \), (3.41). As \( \zeta \to -\infty \), in (3.41), the diverging positive second term can only balance with the first term, and so \( \partial_\zeta U_1 \to -\infty \) and we have

\[
\partial_\zeta U_1 \sim 2c r \zeta.
\]

(3.44)
Hence, as we move into the boundary layer, we integrate (3.44) to find the velocity profile satisfies

\[ U = U_0(r) + crBi^{-1/3} \zeta^2 + Bi^{-1/3}W(r) + \ldots, \]  

(3.45)

where \( W(r) \) is an arbitrary function of integration. Immediately, we identify the new distinguished scaling as \( Bi^{-1/3} \zeta^2 = O(1) \), or \( \zeta = O(Bi^{1/6}) \) which, when related back in terms of \( \theta \), gives \( \theta - \theta_Y = O(Bi^{-1/2}) \). Thus we have derived the width of the boundary layer as \( \epsilon = Bi^{-1/2} \), and can verify that \( \epsilon^2 = Bi^{-1} \ll \delta \ll \epsilon \), as was assumed in deriving the leading order stress balance in the intermediate layer above.

### 3.3.4 Boundary layer solution

The boundary layer solution is constructed by introducing the scaled coordinate, \( \epsilon \tilde{\phi} = \alpha - \theta \). From mass conservation we find the natural scalings:

\[
\left( \begin{array}{c}
    u \\
    v 
\end{array} \right) = \left( \begin{array}{c}
    \tilde{U} \\
    \epsilon \tilde{V} 
\end{array} \right),
\]

(3.46)

where \( \tilde{U} \) and \( \tilde{V} \) are assumed \( O(1) \). Then, expanding the shear stress gives:

\[
\tau_{r\theta} = -\frac{1}{\epsilon r} \frac{\partial \tilde{U}}{\partial \tilde{\phi}} + Bi + O(\epsilon^2 Bi),
\]

(3.47)

where the sign of the \( Bi \) term on the right-hand side arises due to \( \partial \tilde{U} / \partial \tilde{\phi} < 0 \) (since \( U \) varies from 0, at the wall, to some negative value in the bulk of the flow). Again, the second term on the right-hand side of (3.47) is constant, and hence does not contribute to the momentum balance, and the viscous first term dominates the \( O(\epsilon^2 Bi) \) terms associated with plastic shear stresses, since \( \epsilon^3 Bi = Bi^{-1/2} \ll 1 \). Hence, the plastic shear stress terms are omitted in the radial momentum balance, (3.5), and viscous stresses balance the pressure gradient. We find that, as in the intermediate layer, the pressure is constant across the extent of the boundary layer to leading order, and is thus set by the pressure in the bulk and intermediate regions. The radial momentum equation, (3.5), then becomes

\[
\frac{\partial P}{\partial r} \equiv \frac{2c}{r} = \frac{1}{r^2} \frac{\partial^2 \tilde{U}}{\partial \tilde{\phi}^2},
\]

(3.48)

which integrates to give

\[
\tilde{U} = cr \left( \tilde{\phi}^2 - 2\Phi(r)\tilde{\phi} \right).
\]

(3.49)
The constants of integration have been chosen so that $\partial \tilde{U} / \partial \tilde{\phi} = 0$ at $\tilde{\phi} = \Phi$, and $\tilde{U} = 0$ at $\tilde{\phi} = 0$.

As $\tilde{\phi} \to \Phi$ we may write

$$\tilde{U} = -cr\Phi^2 + cr(\tilde{\phi} - \Phi)^2,$$

which is consistent with the $\zeta \to -\infty$ limit, (3.43), and, from (3.35), determines $\Phi$,

$$\Phi(r) = \frac{\sqrt{A}}{cr}.$$

Thus we have found that the angular width of the boundary layer, $\epsilon \Phi$, does indeed depend on $(r^2Bi)^{-1/2}$, as anticipated. The Cartesian width of this boundary layer is $\epsilon r \Phi = Bi^{-1/2}\sqrt{A}/c$. It is therefore a constant, independent of the radial distance from the apex of the wedge. We can also now calculate the additional shear stress at the wall due to the viscoplastic boundary layer. From (3.47) we have

$$\tau_{r\theta} \bigg|_{\theta=\alpha} = Bi - \frac{1}{\epsilon r} \frac{\partial \tilde{U}}{\partial \tilde{\phi}} \bigg|_{\tilde{\phi}=0} + \ldots = Bi + \sqrt{Bi} \frac{2\sqrt{A}}{r} + \ldots,$$

demonstrating that the additional shear stress at the wall is $O(Bi^{1/2})$ and is proportional to $1/r$. Thus, the additional shear force over the extent of the upper wall, from $r = r_1$ to $r = r_2$, assuming the plastic regime applies throughout this domain, is given by

$$2\sqrt{Bi}\sqrt{A} \ln \left( \frac{r_2}{r_1} \right).$$

For the polar velocity, using incompressibility, we have

$$\frac{\partial \tilde{V}}{\partial \tilde{\phi}} = -\frac{1}{\epsilon r} \frac{\partial \tilde{V}}{\partial \theta} = -\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial r} (r\tilde{U}) = 2cr\tilde{\phi}^2 - 2\sqrt{A}\tilde{\phi}.$$

Integrating and using $\tilde{V} = 0$ at $\tilde{\phi} = 0$, yields

$$\tilde{V} = \frac{cr}{3} (2\tilde{\phi}^3 - 3\Phi\tilde{\phi}^2).$$

In particular, by matching $\tilde{V}$ to $V_0$ (see (3.43)), we find the negative angular flow,

$$V_0(r) = -\frac{cr\Phi^3}{3} = -\frac{A\sqrt{A}}{3c^2r^2}.$$

This velocity away from the boundary is transmitted through the intermediate layer to the bulk region. To determine the profile of this weak angular flow in the body of the wedge requires exploring higher orders in the bulk solution. This is done in §3.3.5.
CHAPTER 3. CONVERGING FLOW OF A VISCOPLASTIC FLUID IN A WEDGE OR CONE

It is interesting to consider the limit $\alpha \to 0$ with $r\alpha = 1$, which corresponds to channel flow with parallel boundaries, and for which $A/c^2 \to 1$. In this limit: the leading order bulk solution reduces to a uniform plug flow; the intermediate layer vanishes; the leading order solution in the uniform width, $O(Bi^{-1/2})$, boundary layer becomes a parabolic profile; and the outflow from the boundary layer, $\epsilon V_0$, vanishes. Thus the Bingham-Poiseuille flow in a channel is recovered, as it must be, in this limit.

3.3.5 Higher orders in the bulk

At $O(\epsilon)$, the governing system of equations (3.23)-(3.25) become:

\[
\left( \frac{1}{\alpha^2} \frac{\partial^2}{\partial \Theta^2} - \left( \frac{r}{\partial r} \right)^2 \right) \left( \psi_1 \cos 2\psi_0 \right) - \frac{2}{\alpha} \frac{\partial}{\partial \Theta} \left( \frac{r}{\partial r} \right) \left( \psi_1 \sin 2\psi_0 \right) = - \left( \frac{\Phi}{\alpha^2} \frac{\partial^2}{\partial \Theta^2} + \frac{\Phi \Theta}{2\alpha} \frac{\partial}{\partial \Theta} \right) \sin 2\psi_0 + \frac{\Phi \Theta}{\alpha^2} \frac{\partial^2}{\partial \Theta^2} \cos 2\psi_0,
\]

\[
\frac{\partial}{\partial r} \left( r u_1 \right) + \frac{1}{\alpha} \frac{\partial v_1}{\partial \Theta} = \frac{\Phi \Theta}{\alpha} \frac{\partial u_0}{\partial \Theta},
\]

\[
2r \tan 2\psi_0 \frac{\partial u_1}{\partial r} - \frac{1}{\alpha} \frac{\partial u_1}{\partial \Theta} - r \frac{\partial v_1}{\partial r} + v_1 - 4\psi_1 u_0 \sec^2 2\psi_0 = 2 \frac{\Phi \Theta}{\alpha} \tan 2\psi_0 \frac{\partial u_0}{\partial \Theta} + \frac{\Phi \Theta}{\alpha^2} \frac{\partial u_0}{\partial \Theta}.
\]

where the identities, $r \frac{\partial \Phi}{\partial r} = -\Phi$ and $r \frac{\partial u_0}{\partial r} = -u_0$, have been used. The right hand sides of (3.57)-(3.59) determine the $r$ dependence of $u_1, v_1$ and $\psi_1$, and so we define:

\[
u_1 = \frac{\hat{u}_1(\Theta)}{\rho^2}, \quad v_1 = \frac{\hat{w}_1(\Theta)}{\rho^2}, \quad \text{and} \quad \psi_1 = \frac{\hat{\psi}_1(\Theta)}{\rho}.
\]

Making these substitutions, along with $\Phi = \frac{\sqrt{A}}{\alpha}$, results in the coupled ODEs:

\[
\frac{1}{\alpha^2} (\hat{\psi}_1 \cos 2\psi_0)'' + \hat{\psi}_1 \cos 2\psi_0 = \frac{\sqrt{A}}{\alpha c} \left( \frac{\Theta}{\alpha} (\cos 2\psi_0)'' - \frac{1}{\alpha^2} (\sin 2\psi_0)'' - \frac{\Theta}{2} (\sin 2\psi_0)' \right),
\]

\[
-\hat{u}_1 + \frac{1}{\alpha} \hat{v}_1' = \frac{\sqrt{A} \Theta}{\alpha c} \hat{u}_0',
\]

\[
-4\hat{u}_1 \tan 2\psi_0 - \frac{1}{\alpha} \hat{u}_1' + 3\hat{v}_1 - 4\hat{\psi}_1 \hat{u}_0 \sec^2 2\psi_0 = \frac{\sqrt{A}}{\alpha c} \left( 2\Theta \tan 2\psi_0 + \frac{1}{\alpha} \right) \hat{u}_0'.
\]

There are two boundary conditions for $\hat{\psi}_1(\Theta)$:

\[
\hat{\psi}_1(0) = 0, \quad \hat{\psi}_1(1) = -\frac{A\sqrt{A}}{3c^2},
\]

\[
(3.61, 3.62, 3.63, 3.64)
\]
which respectively represent the symmetry of the flow at the mid-line and matching to the angular outflow from the boundary layer. Additionally, the symmetry conditions demand that

\[ \hat{\psi}_1(0) = 0. \]  

(3.65)

The final boundary condition arises from matching the deviatoric stresses between the intermediate layer and the bulk solution. Specifically, in the intermediate layer, using the transformed partial derivatives, (3.33), the asymptotic form of the radial velocity, (3.35), and the identity \( r\Phi' = -\Phi \), we have (in the notation of the intermediate layer)

\[ \tau_{rr} = -\frac{2rBi}{\partial \zeta U_1} \left( \frac{dU_0}{dr} - \frac{\epsilon}{\delta^{1/2}} \Phi' \partial \zeta U_1 \right) + \ldots = \frac{2U_0}{\partial \zeta U_1} Bi^{2/3} - 2\Phi Bi^{1/2} + \ldots. \]  

(3.66)

By considering the \( \zeta \to \infty \) limit of the cubic (3.41), we find that \( \partial \zeta U_1 \to -A/\left(r\sqrt{c}\zeta \right) \), and so

\[ \tau_{rr} = 2\sqrt{c}\zeta Bi^{2/3} - 2\Phi Bi^{1/2} + \ldots. \]  

(3.67)

Meanwhile, in the bulk solution, we have

\[ \tau_{rr} = Bi \cos 2\psi_0 - 2\epsilon Bi \psi_1 \sin 2\psi_0 + \ldots = 2Bi\sqrt{\alpha c (1 - \Theta)} - 2Bi^{1/2}\psi_1 + \ldots, \]  

(3.68)

where we have used the expansion of \( \cos 2\psi_0 \) and \( \sin 2\psi_0 \) about \( \Theta = 1 \), given in Appendix 3.A. Finally, using the substitution \( \delta \zeta = \theta_Y (1 - \Theta) = \alpha (1 - \Theta) + O(\epsilon \phi) \), we see that the leading order terms match, and that matching at \( O(Bi^{1/2}) \) requires

\[ \psi_1(r, \Theta = 1) = \Phi(r) = \sqrt{\frac{A}{c}} \implies \hat{\psi}_1(1) = \frac{\sqrt{A}}{c}. \]  

(3.69)

Furthermore, we may determine the behaviour of all the dependent variables in the regime \( |1 - \Theta| \ll 1 \) (see Appendix 3.B, setting \( N = 1 \) for the Bingham model).

The boundary-value problem (3.61)-(3.63) was solved using a shooting method. First we use the local form of the dependent variables, (3.161)-(3.163) in Appendix 3.B, to step a small distance, \( d \), away from the singular point before integrating to \( \Theta = 0 \) where the boundary conditions (3.64) and (3.65) determine the unknown coefficients, \( E \) and \( F \), in (3.161)-(3.163). The dependence of the solutions on \( d \) was investigated and found that profiles were essentially independent of \( d \) for \( d < 10^{-5} \). Numerical solutions are plotted for a range of values of \( \alpha \) in figure 3.3. We note that the radial velocity is enhanced by the effects of the boundary layer, and that an angular velocity away from the boundary is induced, which reaches a maximum at some interior location. Also, the magnitude of
the shear stress is promoted throughout the domain. The magnitude of these corrections decreases systematically with increasing wedge half-angle.

Physically, as we deviate from a perfectly rigid plastic, the radial flow is reduced by no-slip at the wall, over an increasingly thick boundary layer. Thus, for the same volume flux, the radial flow must be enhanced in the centre of the wedge, and conservation of mass requires an angular flow from the wall, where radial flow is small, towards the centre of the wedge, where the flow is enhanced. The dependence on half-angle arises via the decreasing boundary layer strength with increasing half-angle. This occurs through a competition between the radial pressure gradient at the edge of the boundary layer (which decreases with increasing $\alpha$, supporting a lower angular gradient of the shear stress - potentially widening the boundary layer) and the magnitude of the non-vanishing velocity at the boundary in Nadai’s [1924] plastic solution (3.14) (which decreases with $\alpha$, thus requiring a thinner boundary layer to enforce no-slip for the same radial pressure gradient). The balance between these effects is reflected in the $\sqrt{A/c}$ factor in the boundary layer width, $\epsilon\Phi$. This term turns out to be a decreasing function of $\alpha$, and so the boundary layer and the induced velocity perturbations are weaker for larger half-angles (at the same radial position, $r$). The stress behaviour is also non-trivial since the enhanced radial flow has the potential to increase both the normal radial stress, and the shear stress, but cannot increase both, since the magnitude of the deviatoric stress is everywhere equal to the yield stress at this order. However, as we have enhanced shear stress at the walls due to the viscoplastic shear layer, it is natural to expect the shear stress to be enhanced across
the domain, as is observed.

### 3.3.6 Composite solutions

To produce leading order radial and angular velocity profiles, it is useful to define a composite solution that is valid across the whole domain. We define such a solution, using the limiting behaviours of the plastic and boundary layer solutions in the vicinity of $\theta_Y$, as:

$$
\begin{align*}
    u_{\text{comp}}(r, \theta) &= \begin{cases} 
        u_0 + U + \frac{A}{rc} + \frac{2A\sqrt{\theta_Y - \theta}}{rc\sqrt{c}} & \text{for } \theta < \theta_Y, \\
        \bar{U} + U + \frac{A}{rc} - rcBi (\theta - \theta_Y)^2 & \text{for } \theta \geq \theta_Y,
    \end{cases} \\
    v_{\text{comp}}(r, \theta) &= \begin{cases} 
        \epsilon \left( v_1 + V + \frac{A\sqrt{A}}{3r^2c^2} + \frac{2A\sqrt{A}}{r^2c\sqrt{c}}\sqrt{\theta_Y - \theta} \right) & \text{for } \theta < \theta_Y, \\
        \epsilon \left( \bar{V} + V + \frac{A\sqrt{A}}{3r^2c^2} - \sqrt{\bar{A}}Bi (\theta - \theta_Y)^2 \right) & \text{for } \theta \geq \theta_Y,
    \end{cases}
\end{align*}
$$

where $\theta_Y(r)$ is the location of the fake yield surface defined by (3.18) and (3.51). These profiles (for fixed $r$) are continuous at $\theta = \theta_Y$, since the velocities here are equal to those for the intermediate layer solution at $\zeta = 0$, on both branches of the piece-wise definition.

### 3.3.7 Numerical simulations

Numerical simulations of the complete governing partial differential equations (3.4)-(3.6), were carried out using the finite element method as implemented by FEniCS [11, 84]. The domains were meshed with 60 000 triangular elements, with a higher concentration at the wall and near the outflow, where higher strain rates are expected. The fluid is yielded everywhere, so we do not expect the occurrence of plug regions and the associated singular Jacobian matrix of the discretised equations, when applying the Newton method to converge iteratively to the steady state. Nonetheless, the non-linearity in the equations causes difficulties when trying to use Newton iterations to converge to the solution from an arbitrary initial state. We found that using an augmented-Lagrangian method, as described by Saramito [123] and detailed in §2.4, resulted in slow convergence but could be used to produce an effective initial guess for the Newton method, allowing a subsequent fast convergence to a Newton residual of less than $10^{-8}$. For the following results $Bi$ has been set at 1, but the domains have a radial range of $r \in [0.01, 500]$, allowing the effective Bingham number, $r^2Bi$, to vary from $10^{-4}$ to 250 000, within a single simulation. The low value of this parameter at the outflow means the purely viscous radial solution (3.15) is a good approximation to the velocity at the outflow, and is therefore imposed as a
Chapter 3. Converging Flow of a Viscoplastic Fluid in a Wedge or Cone

Figure 3.4: a) The width of the viscoplastic boundary layer determined analytically (dashed) and numerically (symbols), at a range of radial positions and wedge angles. b) Profiles of radial velocity determined numerically for $\alpha = \pi/4$. The quantity $r(\alpha - \theta)$ defines a small Cartesian distance from the wall at $\theta = \alpha$. The theoretical position of the fake yield surface is shown by a dotted line, and the magnitude of the radial velocity is scaled up by a factor of 15000 to be visible on the same scale as $r$.

Dirichlet boundary condition here, while the large value at the inflow means the leading order asymptotic viscoplastic solution is an appropriate boundary condition here.

First, we confirm the predicted behaviour of the boundary layer thickness in the plastic regime. For a given radius, the boundary layer thickness was determined from the numerical radial flow profile by the angle at which

$$
\frac{\partial u}{\partial \theta} = \left(2Bi\frac{A^2}{c^2r}\right)^{1/3},
$$

which is the shear rate predicted at the centre of the matching region by setting $\zeta = 0$ in the equation (3.41). Figure 3.4 shows the measured boundary layer thickness alongside the predicted value, $e\Phi(r)$, with $\Phi$ given by (3.51), at a number of radial positions for three different wedge angles, confirming the analytical result. Also shown are example velocity profiles from the numerical solution for $\alpha = \pi/4$, in a small region near the upper boundary $\theta = \alpha$. We note that the boundary layer is of constant width, in a Cartesian sense, due to the $1/r$ dependence of $\Phi$, and that this is borne out by the numerical results.

Figure 3.5 provides comparisons between the numerical and composite asymptotic
3.3. PLASTIC REGIME \((Bi \gg 1)\)

Figure 3.5: The asymptotic (dashed line) and numerical (circles) radial and angular velocities as function of angle, \(\theta\), for \(\alpha = \pi/4\), \(Bi = 1\) and \(r = 100\). The numerical data points shown are only a small sub-sample of the full numerical solutions, which are of a much higher resolution. a) and b) plot the radial and angular velocities respectively, across the domain. c) and d) are close-ups of the boundary layers for a) and b) respectively. The position of the fake yield surface, \(\theta_Y\), is depicted by a red dotted line in all panels (and is almost indistinguishable from the boundary, \(\theta = \pi/4\), in the upper two panels).

Flow profiles for a choice of wedge angle \((\alpha = \pi/4)\) and radius \((r = 100)\), which is sufficiently distant from the inflow and outflow boundaries to exhibit the fully developed profile. The numerical simulations accurately reproduce the predicted velocity profiles in both the bulk and boundary layer regions. In particular, the predicted weak negative angular flow is confirmed in these simulations.

Finally, we note the small discrepancy in panel b) of figure 3.5 and explore whether this can be explained by the absence of higher order terms in the asymptotic expansions. This is confirmed by figure 3.6, which shows the difference between the leading order
For the velocity in the radial direction, the composite solution neglects terms of $O(Bi^{-1/2})$, which results in $\Delta u$ proportional to $Bi^{-1/2}r^{-2} = r^{-2}$ for fixed $Bi$. For the velocity in the angular direction we must consider the order of terms neglected above $O(Bi^{-1/2})$. From (3.45), we expect an $O(Bi^{-1/3})$ correction to the radial flow in the boundary layer. This in turn would drive a correction of $O(\epsilon Bi^{-1/3}) = Bi^{-5/6}$ in the velocity in the angular direction at the top of the boundary layer. Hence, without calculating higher orders explicitly, we expect the second order correction to the velocity to be $O(Bi^{-5/6})$ or, including radial dependence, $O(r^{-8/3}Bi^{-5/6})$, since the expansion parameter is $r^2Bi$ and the leading order flow has $r^{-1}$ dependence. Hence, we expect $\Delta v$ to scale like $r^{-8/3}$. Figure 3.6 confirms this relationship for the three wedge angles studied, providing strong evidence for the validity of our asymptotic structure. The deviation from the predicted trends at $r \approx 500$ is due to the leading order profiles being imposed here as boundary conditions in the numerical simulation, and so the difference is at the level of the implementation precision.

Figure 3.7 shows a density plot of log strain rate and radial velocity profiles from the numerical solution for the wedge of half-angle $\alpha = \pi/3$. These demonstrate how the plastically dominated regime, with a thin boundary layer structure, is valid for large values of $r$, but transitions to a different regime, without a boundary layer, towards the apparent apex of the wedge. For sufficiently small values of $r$ the viscous stresses are dominant and a new asymptotic solution can be derived. This is detailed in the next section.

### 3.4 The viscous regime

In the regime of small Bingham number, $Bi \ll 1$, or, equivalently, at small distances from the apex of the wedge, the viscoplastic problem is a regular perturbation of the viscous Stokes problem, and so we do not require a boundary layer structure. This regime has been studied previously by Sandru and Camenshi [120], so we initially outline this solution (with notation adjusted for consistency) in §3.4.1, before exploring it in greater
3.4. THE VISCOUS REGIME

Figure 3.6: The difference between the leading order asymptotic and numerical solutions for the velocity fields, evaluated as $\Delta u$ and $\Delta v$, as a function of radial position, for $\alpha = \pi/6$, $\pi/4$ and $\pi/3$. The slope markers show the predicted slopes of -2 (top) and -8/3 (bottom).

Figure 3.7: a) A density plot of log strain rate from the $\alpha = \pi/3$ numerical simulation. b) The scaled radial velocity, $ru$, as a function of angle, $\theta$, for $\alpha = \pi/3$ and $r = 1$ (dotted), 4 (dashed), and 20 (solid). The $\theta$ axis is reversed so that the no-slip boundary is located at the left of the figure.
detail by carrying out numerical integration not evaluated by Sandru and Camenshi [120], and comparing to the results of direct numerical simulations in §3.4.2.

3.4.1 Asymptotic solution

In this regime the leading order flow is given by the viscous solution (3.15), which produces an $O(Bi)$ term in the stress. Hence the solution takes the form

$$u = u_0 + Bi u_1 + \ldots, \quad v = Bi v_1 + \ldots, \quad p = p_0 + Bi p_1 + \ldots. \quad (3.75)$$

with $u_0$ and $p_0$ given by (3.15). By conservation of mass and for homogeneity in $r$ at $O(Bi)$ in the expression of conservation of momentum, we define

$$u_1 = rf'(\theta) \quad \text{and} \quad v_1 = -2rf(\theta), \quad (3.76)$$

where $f(\theta)$ is to be determined and $'$ denotes differentiation with respect to $\theta$. Note that this $r$ dependence of $u_1$ and $v_1$ was anticipated, due to the expansion parameter, in fact, being $r^2Bi$ and the leading order velocity depending upon $r^{-1}$. By eliminating pressure from the conservation of momentum equations, we find that $f(\theta)$ satisfies the differential equation

$$f'''(\theta) + 4f'(\theta) + H(\theta) = C, \quad (3.77)$$

where $C$ is a constant of integration and

$$H(\theta) = \frac{\sin 2\theta}{\sqrt{\sin^2 2\theta + (\cos 2\theta - \cos 2\alpha)^2}} \frac{2(\cos 2\theta - \cos 2\alpha)}{\sqrt{\sin^2 2\theta + (\cos 2\theta - \cos 2\alpha)^2}} \quad (3.78)$$

The governing equation (3.77) is subject to four boundary conditions:

$$f(0) = f''(0) = 0, \quad f(\alpha) = f'(\alpha) = 0, \quad (3.79)$$

which represent vertical symmetry at $\theta = 0$ and no-slip at $\theta = \alpha$, which are sufficient to evaluate $f(\theta)$ and $C$. Having determined $C$, the pressure perturbation is given by

$$p_1 = C \ln(r) + \frac{1 + 2\cos^2 2\alpha - 3\cos 2\alpha \cos 2\theta}{\sqrt{\sin^2 2\theta + (\cos 2\theta - \cos 2\alpha)^2}} + \text{const.} \quad (3.80)$$

We integrate (3.77) numerically by first finding a particular solution from an initial value problem with initial conditions given at $\theta = 0$, then determining $C$ and the coefficients of the complementary solution by the boundary conditions at $\alpha$. Specifically, we find a solution of the form

$$f(\theta) = f_p(\theta) + \frac{C}{4} \theta + D \sin 2\theta, \quad (3.81)$$
3.4. THE VISCOUS REGIME

Figure 3.8: The first order corrections to the velocities, $u$ and $v$, in the viscous regime, as functions of the scaled angle, $\theta/\alpha$, for a wedge of half-angle $\alpha = \pi/6$ (solid), $\pi/4$ (dashed), $\pi/3$ (dash-dotted), and $\pi/2$ (dotted).

where $f_p$ is the unique solution to the initial value problem:

$$f'''_p(\theta) + 4f'_p(\theta) + H(\theta) = 0,$$

subject to $f_p(0) = f'_p(0) = f''_p(0) = 0$. Then $C$ and $D$ are determined by the boundary conditions at $\theta = \alpha$:

$$C = \frac{4\left(2f_p(\alpha)\cos 2\alpha - f'_p(\alpha)\sin 2\alpha\right)}{\sin 2\alpha - 2\alpha \cos 2\alpha}$$

and

$$D = \frac{\alpha f'_p(\alpha) - f_p(\alpha)}{\sin 2\alpha - 2\alpha \cos 2\alpha}. \quad (3.83)$$

Velocity profiles for the first order corrections to the leading order viscous flow are given in figure 3.8. Note, again, the negative angular velocity, as anticipated. Also, the radial velocity is reduced at the centre of the wedge and enhanced towards the walls, as required to flatten the radial velocity profile as $r$ increases. In this regime the magnitude of the corrections increase with increasing half-angle. This can be explained by considering the magnitude of the deviatoric stress in the leading order solution,

$$\tau \equiv \sqrt{\tau_{rr}^2 + \tau_{r\theta}^2} = \frac{2\sqrt{1 + \cos^2 2\alpha - 2 \cos 2\theta \cos 2\alpha}}{r^2 (\sin 2\alpha - 2\alpha \cos 2\alpha)}, \quad (3.84)$$

which can be shown to decrease with increasing $\alpha$. Thus, the fluid is less strongly stressed by the leading order flow for larger wedge half-angles, and so the plasticity has a larger impact on the flow.
CHAPTER 3. CONVERGING FLOW OF A VISCOPLASTIC FLUID IN A WEDGE OR CONE

Figure 3.9: Features of the perturbation solution for the planar converging flow of a viscoplastic in the low Bingham number regime: a) the additional wall shear stress, \( \tau_{r\theta}^{(1)} \)\( |_{\theta=\alpha} \), and b) the scaled additional radial pressure gradient, \( r \partial p_1 / \partial r = C \), as functions of wedge half-angle, \( \alpha \).

The additional shear stress at the wall due to the viscoplasticity in this regime is given by

\[
Bi \left. \tau_{r\theta}^{(1)} \right|_{\theta=\alpha} = Bi \left( \frac{1}{r} \frac{\partial u_1}{\partial \theta} + \frac{\partial v_1}{\partial r} - \frac{v_1}{r} \right) \left|_{\theta=\alpha} \right. = Bi \left( f''(\alpha) + 1 \right),
\]

where \( f''(\alpha) \) is determined numerically. This correction to the shear stress at the wall is plotted as a function of \( \alpha \) in figure 3.9(a). An interesting feature of this profile is the minimum attained for an angle of \( \alpha \approx 32^\circ \). The behaviour of the constant \( C \) is also of particular interest, as the first order correction to the radial pressure gradient is given from (3.80) by \( Bi C/r \). The behaviour of \( C = r \partial p_1 / \partial r \) as a function of \( \alpha \) is given in figure 3.9(b), showing that \( C \) varies from 2.55 (approximately) at \( \alpha = \pi/2 \) and diverges as \( \alpha \to 0 \). In particular, we find that the perturbation to the shear stress at the wall, \( \tau_{r\theta}^{(1)} \sim 3/2 \), and \( C \sim 3/(2\alpha) \) for \( \alpha \ll 1 \) as required for consistency with the plane Poiseuille solution in the limit \( \alpha \to 0 \) with \( r\alpha = 1 \) fixed (see Appendix 3.D).

3.4.2 Numerical simulations

For numerical simulations in the viscous regime, a radial extent of \( r \in [0.1, 10] \) and a range of small values for \( Bi \) were used. Again the simulations were run in FEniCS, on a mesh of 60 000 elements. Here we imposed the purely viscous radial solution (3.15) as Dirichlet boundary conditions at both inflow and outflow, and examined the flow
Figure 3.10: The asymptotic (dashed line) and numerical (circles) radial and angular velocities as functions of angle, $\theta$, for $Bi = 0.01$ and $r = 1$. a) and b) show the radial and angular profiles, respectively, for a wedge of half-angle $\alpha = \pi/4$.

profiles at $r = 1$, which was found to be sufficiently far from these boundaries not to be affected by the approximations made in the boundary conditions. Due to the low values of the Bingham number, all simulations were solved directly by Newton iteration from the purely radial Stokes solution (3.15) converging to residuals of less than $10^{-8}$.

Figure 3.10 shows example velocity profiles for $\alpha = \pi/4$ and $Bi = 0.01$, demonstrating good agreement between asymptotic theory and numerical computation. As with the plastic regime, we measured the difference between the asymptotic profiles and the numerical solutions for $\alpha = \pi/6, \pi/4$ and $\pi/3$, this time at fixed $r = 1$ and variable $Bi$. With the absence of a boundary layer structure, there is no need for composite solutions, and so we measure the differences, $\Delta u$ and $\Delta v$, between the first order asymptotic expressions and the numerical solutions. Both differences were found to scale like $Bi^2$ (not shown), as expected having neglected second-order terms in the expansions.

3.5 Herschel-Bulkley flow

In this section we extend our examination of converging flows to Herschel-Bulkley fluids. Importantly the key features of the boundary layer structure determined for Bingham fluids when $Bi \gg 1$ (§3.3) carry over to Herschel-Bulkley fluids and here we generalise our analysis to these materials. The Herschel-Bulkley viscoplastic model is defined by
the dimensional constitutive law

$$\tau = \left(K\dot{\gamma}^{N-1} + \frac{\tau_c}{\dot{\gamma}}\right)\dot{\gamma}, \quad (3.86)$$

where $K$ is the consistency, $N$ the flow index, and $\tau_c$ the yield stress. This model extends the Bingham model, which is reproduced for $N = 1$ and $K = \mu$, by allowing for shear thickening ($N > 1$) and shear thinning ($N < 1$). For this model we have the critical distance

$$r_c = \left(\frac{KQ}{\tau_c}\right)^{\frac{1}{2N}}, \quad (3.87)$$

above which the flow is dominated by plastic stresses. For a typical radial distance, $R$, we scale velocities by $Q/R$, strain rates by $Q/R^2$, and stresses and pressures by $KQ^N/R^{2N}$ resulting in the non-dimensional constitutive equation:

$$\begin{pmatrix} \tau_{rr} \\ \tau_{r\theta} \end{pmatrix} = \left(\dot{\gamma}^{N-1} + \frac{Bi}{\dot{\gamma}}\right)\begin{pmatrix} 2\frac{\partial u}{\partial r} \\ \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{u}{r} \end{pmatrix}, \quad (3.88)$$

where the Bingham number is now given by

$$Bi = \frac{\tau_c R^{2N}}{KQ^N}. \quad (3.89)$$

The critical radial length scale is equivalent to asserting that the asymptotic analysis will break down for $r = O(Bi^{-\frac{1}{2N}})$ in non-dimensional variables.

In the $Bi \gg 1$ regime (or equivalently at large distances from the apex of the wedge), the bulk solution is unchanged at leading order since the viscous stresses are neglected here, and hence is given by (3.26)-(3.28). In the intermediate layer, using the same notation as for the Bingham case, albeit with a different small parameter, $\delta$, the radial velocity takes the form

$$U = U_0(r) + \delta^{1/2}U_1(r, \zeta), \quad (3.90)$$

and the shear stress is given by

$$\tau_{r\theta} = -\frac{1}{r^N}\delta^{-N/2}|\partial_\zeta U_1|^{N-1}\partial_\zeta U_1 + Bi - 2r^2\delta Bi \left(\frac{\partial_r U_0}{\partial_\zeta U_1}\right)^2 + \ldots \quad (3.91)$$

Hence, a balance of viscous and plastic stresses is achieved in an intermediate layer of width $\delta$, provided

$$\delta^{-N/2} = \delta Bi \implies \delta = Bi^{-\frac{N}{2N}}, \quad (3.92)$$

52
and we find that \( U_1 \) satisfies the equation

\[
|\partial_\zeta U_1|^{N+1} \partial_\zeta U_1 - 2c r^N \zeta |\partial_\zeta U_1|^2 + 2r^{N-2} A^2 \frac{c^2}{\zeta^2} = 0,
\]

(3.93)

where \( A \) and \( c \) are defined as previously for the bulk solution and given by (3.29) and (3.30). In the limit \( \zeta \to -\infty \), \( \partial_\zeta U_1 \to -\infty \) and leading order terms of (3.93) then determine

\[
\partial_\zeta U_1 = -r (-2c\zeta)^{\frac{1}{N}},
\]

(3.94)

which, integrated, reveals that the transition into the boundary layer occurs when

\[
\delta^{1/2} \frac{\zeta^{N+1}}{\hat{\zeta}} = O(1),
\]

(3.95)

or, equivalently, when

\[
|\theta - \theta_Y| = O \left( \frac{\delta^{N+1}}{\sqrt{N+1}} \right) = O \left( Bi^{-\frac{1}{N+1}} \right).
\]

(3.96)

Thus, the viscoplastic boundary layer is of thickness \( \epsilon = Bi^{-\frac{1}{N+1}} \) (cf. [107]) and, after scaling, we have the boundary layer equation (using the same notation as in the Bingham problem)

\[
\frac{\partial P}{\partial r} \equiv \frac{2c}{r} = \frac{1}{r^{N+1}} \frac{\partial}{\partial \phi} \left( |\partial_\phi \tilde{U}|^{N-1} \partial_\phi \tilde{U} \right),
\]

(3.97)

which integrates to

\[
\tilde{U} = \frac{N}{N+1} \left( 2c \right)^{\frac{1}{N}} r \left[ \left( \Phi - \hat{\phi} \right)^{\frac{N+1}{N}} - \Phi^{\frac{N+1}{N}} \right],
\]

(3.98)

and matching to leading-order at \( \hat{\phi} = \Phi \) implies that

\[
\Phi(r) = \hat{\Phi} r^{-\frac{2N}{N+1}},
\]

(3.99)

where

\[
\hat{\Phi} = \frac{1}{c} \left( \frac{(N+1) A}{N2^{\frac{N}{2}}} \right)^{\frac{N}{N+1}}.
\]

(3.100)

The additional shear stress due to the viscoplastic boundary layer is given, to leading order, by

\[
\left( \frac{1}{r^N} \left| \frac{\partial u}{\partial \theta} \right|^{N-1} \frac{\partial u}{\partial \theta} \right) \bigg|_{\theta=\alpha} = - \left( \frac{1}{r^N} \left| \partial_\phi \tilde{U} \right|^{N-1} \partial_\phi \tilde{U} \right) \bigg|_{\hat{\phi}=0} = 2Bi^{\frac{N}{N+1}} c\Phi = Bi^{\frac{N}{N+1}} \left( \frac{2(N+1) A}{Nr^2} \right)^{\frac{N}{N+1}}.
\]

(3.101)
CHAPTER 3. CONVERGING FLOW OF A VISCOPLASTIC FLUID IN A WEDGE OR CONE

Using incompressibility, we find the polar velocity at the top of the boundary layer:

\[
v(r, \Theta = 1) = \epsilon \tilde{V}(r, \tilde{\phi} = \Phi) = -er^{-\frac{3N+1}{N+1}} \frac{2N^3}{(2N + 1)(N + 1)^2} \Phi^{-\frac{2N+1}{N}} (2c)^{\frac{1}{N}}. \quad (3.102)
\]

We note that the radial dependence of both the boundary layer width and the first order corrections to the velocity field are consistent with expansions in \( r^{2N} Bi \), as anticipated. Unlike for a Bingham fluid, the Cartesian thickness of the boundary layer, \( r\epsilon \Phi(r) = \epsilon \tilde{\Phi}(1-rac{1}{N})/(1+N) \), is dependent on \( r \) - growing with \( r \) for a shear-thinning fluid and decreasing for a shear-thickening fluid. This can be rationalised by considering the magnitude of the strain rate in the bulk, which decreases with \( r \), resulting in a higher viscosity for a shear-thinning fluid, and effectively reducing the local Bingham number (the ratio of plastic and viscous stresses) relative to the global Bingham number, \( Bi \). Since the boundary layer thickness increases with decreasing Bingham number, we find a larger boundary layer thickness at larger distances from the apex for a shear-thinning fluid (and the reverse for a shear-thickening fluid).

Provided \( \epsilon \gg Bi^{-1} \), as is the case for \( N > 0 \), the first order correction to the bulk solution is found at an order before viscous stresses need to be considered. Hence we can proceed to solve for these corrections as in §3.3.5. The dependent variables in the bulk are given by the following series expansions

\[
\psi = \psi_0(\Theta) + er^{-\frac{2N}{N+1}} \hat{\psi}_1(\Theta) + \ldots, \quad (3.103)
\]

\[
u = \frac{\hat{u}_0(\Theta)}{r} + er^{-\frac{3N+1}{N+1}} \hat{u}_1(\Theta) + \ldots, \quad (3.104)
\]

\[
v = er^{-\frac{3N+1}{N+1}} \hat{v}_1(\Theta) + \ldots, \quad (3.105)
\]

and following through the analysis with the new expression for \( \Phi \) gives a similar set of ODEs to those for the Bingham case, (3.61)-(3.63). These equations, along with the asymptotic behaviour at \( \Theta = 1 \), are detailed in Appendix 3.B. They are numerically integrated using a shooting technique as in §3.3.5.

Example profiles of the first order corrections, for \( \alpha = \pi/4 \) and different values of \( N \), are given in figure 3.11. The general dependence of these profiles on \( N \) can be explained as follows. For a shear-thinning (thickening) fluid, the high shear at the walls results in a lower (higher) effective viscosity, allowing the pressure gradient to support higher (lower) shear rates and a thinner (thicker) boundary layer, resulting in a smaller (larger) adjustment to the leading order plastic velocities and stresses in the bulk of the wedge. The velocity in the radial direction depends on \( N \) in a non-trivial way. For a small flow index, for which the constitutive law is almost plastic, the profile is largely independent.
3.6 Bingham fluid in a cone

In this section we examine converging flow of a Bingham fluid through an axisymmetric cone. The results obtained for the two-dimensional flow through a wedge are generalised to this three-dimensional setting. Importantly, we find that viscoplasticity induces angular velocity away from the boundaries towards the axis of the cone and enhances boundary shear stress, and, when the material is only weakly yielded in the bulk, viscoplastic boundary layers develop. The conical geometry, however, introduces some algebraic complexity to the governing equations and asymptotic solution.

In the axisymmetric conical geometry we employ spherical coordinates, \((r, \theta, \phi)\), where \(r\) is the distance from the apparent vertex of the cone, \(\theta \in [0, \pi]\) is now the polar angle measured from the axis of symmetry of the cone, and \(\phi \in [0, 2\pi]\) is the azimuthal angle measured from an arbitrary choice of axis perpendicular to the axis of symmetry. The rigid walls are located at \(\theta = \alpha\), the half-angle of the cone, and we assume independence of \(\phi\) and no flow in the \(\phi\) direction. Thus, we define the velocity components as \(\mathbf{u} = (u, v, 0)\).

This problem is characterised by the following dimensional parameters (and for clarity of \(\Theta\), corresponding to a reasonably plastic-like velocity profile. As the index increases the profile becomes more strongly sheared and, above some value of the flow index, the magnitude of the radial flow enhancement also reduces.

Figure 3.11: First order corrections to the velocity and stress fields in the bulk as functions of the rescaled angle, \(\Theta\), for a Herschel-Bulkley fluid with flow index \(N\) (shown in legend). A solid line depicts the results for the Bingham model, \(N = 1\). All solutions are for \(\alpha = \pi/4\).
of exposition we focus only on the Bingham rheology): density, \( \rho \), viscosity, \( \mu \), yield stress, \( \tau_c \), typical radial distance from the apex, \( R \), and the volume flux, \( Q \). (We note that for flow in a cone, the volume flux, \( Q \), is dimensionally distinct from the volume flux per unit width in a wedge, \( Q/w \)). The flow variables are rendered dimensionless as follows: velocities are scaled by \( Q/R^2 \), strain rates by \( Q/R^3 \), and stresses and pressures by \( \mu Q/R^3 \). The resulting non-dimensional equations are:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( v \sin \theta \right) = 0, \quad (3.106)
\]

\[
\frac{\partial p}{\partial r} = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{3}{r} \tau_{rr} + \frac{\cot \theta}{r} \tau_{r\theta}, \quad (3.107)
\]

\[
\frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{3}{r} \tau_{r\theta} + \frac{\cot \theta}{r} (\tau_{\theta\theta} - \tau_{\phi\phi}), \quad (3.108)
\]

\[
\tau = \left( 1 + \frac{Bi}{\dot{\gamma}} \right) \dot{\gamma}, \quad (3.109)
\]

\[
\dot{\gamma}_{rr} = 2 \frac{\partial u}{\partial r}, \quad \dot{\gamma}_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \dot{\gamma}_{\theta\theta} = \frac{2}{r} \frac{\partial v}{\partial \theta} + \frac{2 u}{r}, \quad \dot{\gamma}_{\phi\phi} = \frac{2 v \cot \theta}{r} + \frac{2 u}{r}, \quad (3.110)
\]

\[
\dot{\gamma} = \sqrt{\frac{1}{2} \left( \dot{\gamma}_{rr}^2 + \dot{\gamma}_{r\theta}^2 + \dot{\gamma}_{\theta\theta}^2 + 2 \dot{\gamma}_{r\theta}^2 \right)}, \quad (3.111)
\]

representing incompressibility (3.106), conservation of momentum in the radial and polar directions (3.107) and (3.108), the Bingham constitutive law (3.109), and the definition of the non-zero components and second invariant of the strain-rate tensor (3.110)-(3.111). The Bingham number for this problem is given by

\[
Bi = \frac{R^3 \tau_c}{\mu Q}. \quad (3.112)
\]

Again, due to this dependence of the Bingham number on \( R \), we anticipate the expansions to proceed as functions of \( r^3 Bi \). Now we have the critical length scale, \( r_c = (\mu Q/\tau_c)^{1/3} \). The plastically-dominated regime, \( Bi \gg 1 \), corresponds to the dimensional distance from the apex of the cone being significantly larger than \( r_c \), and the viscously-dominated regime, \( Bi \ll 1 \), occurs for radial distances much smaller than \( r_c \). The Reynolds number is given by \( Re = \rho Q/(\mu R) \), and again for the following it will be sufficient that \( Re = O(1) \) and \( Re \ll Bi \), to neglect inertial terms in the \( Bi \gg 1 \) and \( Bi \ll 1 \) regimes, respectively. We note that, unlike the planar problem, the Reynolds number also has radial dependence and so, for given material parameters and volume flux, it will not be possible to neglect inertial terms for arbitrarily small distances from the apex of the cone.

The equations (3.106)-(3.108) are to be solved subject to boundary conditions

\[
u = v = 0 \text{ at } \theta = \alpha, \quad v = \frac{\partial u}{\partial \theta} = 0 \text{ at } \theta = 0, \quad (3.113)
\]
corresponding to no-slip and axisymmetry, respectively. We additionally have the integral expression for the volume flux,

\[ 2\pi \int_0^\alpha r^2 u \sin \theta d\theta = -1. \]  

(3.114)

The corresponding problem for a rigid-plastic was solved by Shield [126], assuming purely radial flow and a radially independent deviatoric stress state. The solution corresponds closely to Nadai’s solution for a planar wedge, and to make the correspondence clearer we adjust Shield’s notation slightly by parameterising the non-zero components of the deviatoric stress via:

\[ \tau_{rr} = \frac{2}{\sqrt{3}} Bi \cos 2\psi, \quad \tau_{\theta\theta} = \tau_{\phi\phi} = -\frac{1}{\sqrt{3}} Bi \cos 2\psi, \quad \tau_{r\theta} = Bi \sin 2\psi, \quad \psi = \psi(\theta), \]  

(3.115)

which ensures that the deviatoric stress tensor is trace-free and everywhere has constant second-invariant, \( Bi \). The equality of \( \tau_{\theta\theta} \) and \( \tau_{\phi\phi} \) is a consequence of the purely radial flow (see (3.110)).

The solution is then given by

\[ u = -\frac{A}{r^2} \exp \left(-2\sqrt{3} \int_0^\theta \tan 2\psi d\theta \right), \quad \frac{\partial p}{\partial r} = \frac{2c Bi}{r}, \]  

(3.116)

\[ \psi'(\theta) = c \sec 2\psi - \sqrt{3} - \frac{1}{2} \cot \theta \tan 2\psi, \]  

(3.117)

where \( c \) is determined by the boundary conditions

\[ \psi(0) = 0, \quad \psi(\alpha) = \frac{\pi}{4}, \]  

(3.118)

and \( A \) is determined by the flux condition (3.114).

The viscous problem of purely radial converging flow in a cone was first solved by Harrison [67], and later explored in great detail by Ackerberg [2], who found that the inclusion of inertial terms resulted in non-radial solutions occurring at the outflow of the cone. For vanishing Reynolds number, as considered by Harrison [67], the problem is more straightforward, with a purely radial solution given by

\[ u = \frac{3}{2\pi r^2} \frac{\cos^2 \alpha - \cos^2 \theta}{(1 - \cos \alpha)^2(1 + 2\cos \alpha)}, \quad p = \frac{1}{\pi r^3} \frac{1 - 3\cos^2 \theta}{(1 - \cos \alpha)^2(1 + 2\cos \alpha)} + \text{const}. \]  

(3.119)

As with the planar wedge, the viscous velocity profile (3.119) exhibits enhanced velocity in the centre of the wedge, compared to the perfectly plastic velocity profile (3.116). As the fluid flows from large to small distances from the apex, the strain rate increases due
to the converging nature of the flow, and the material evolves from plastically-dominated to viscously-dominated behaviour. Thus, superimposed upon the radial dependence of the flow due to the converging geometry, we expect an additional acceleration at the centre of the cone, and velocity reduction at the outer surface of the cone. Conservation of mass then requires a flow of the fluid in the negative polar direction, towards the centre of the cone and away from the walls, and the quantification of this velocity field is a significant outcome of this analysis.

3.6.1 The plastic regime

The plastic solution (3.116)-(3.117) does not satisfy no-slip on the boundary of the cone, and the strain rate becomes unbounded here. So, for a viscoplastic fluid in the plastic regime, \( Bi \gg 1 \) (or equivalently, sufficiently far from the apex), we construct a viscoplastic boundary layer and intermediate layer, as for the wedge solution. Since the boundary layer and intermediate region are relatively thin, the curvature in the \( \phi \) direction is negligible and the equations and solutions are identical to the planar geometry in these regions. In particular, we have the same boundary layer thickness, \( \epsilon = Bi^{-1/2} \), and intermediate region thickness, \( \delta = Bi^{-2/3} \). On the other hand, the \( r \) dependent terms, \( \Phi, U_0 \) and \( V_0 \), which are determined from matching with the bulk solution, differ.

In the bulk, we use the same strained coordinate approach, defining

\[
\theta = (\alpha - \epsilon \Phi) \Theta,
\]

and velocity expansions:

\[
u = u_0 + \epsilon u_1 + \ldots, \quad v = \epsilon v_1 + \ldots.
\]

However, we need to introduce a different parameterisation of the deviatoric stresses from that used in (3.115), which allows for a weak flow in the polar direction, and hence a non-zero normal stress difference, \( \tau_{\theta\theta} - \tau_{\phi\phi} \). One such parameterisation is:

\[
\frac{\tau_{rr}}{Bi} = \frac{2}{\sqrt{3}} \cos 2\psi \cos 2\chi, \quad \frac{\tau_{\theta\theta}}{Bi} = -\frac{1}{\sqrt{3}} \cos 2\psi \cos 2\chi + \cos 2\psi \sin 2\chi, \quad \frac{\tau_{\phi\phi}}{Bi} = -\frac{1}{\sqrt{3}} \cos 2\psi \cos 2\chi - \cos 2\psi \sin 2\chi, \quad \frac{\tau_{r\theta}}{Bi} = \sin 2\psi,
\]

which reduces to the previous parameterisation (3.115) for \( \chi = 0 \) and gives all possible symmetric, trace-free stress states satisfying \( \tau \equiv \sqrt{\tau_{ij} \tau_{ij}} / 2 = Bi \), and \( \tau_{r\phi} = \tau_{\theta\phi} = 0 \). We define asymptotic series for \( \psi \) and \( \chi \) by:

\[
\psi = \psi_0(r) + \epsilon \psi_1(r, \theta) + \ldots, \quad \chi = \epsilon \chi_1(r, \theta) + \ldots,
\]

58
3.6. BINGHAM FLUID IN A CONE

so that, at $O(1)$, $\chi = 0$ and the stress decomposition is precisely that given in (3.115), for the purely radial solution.

With the coordinate transform (3.120), the full governing equations (3.106)-(3.111) can be reduced to:

$$
\frac{1}{r^2} L_r \left( r^2 u \right) + L_\Theta v + v \cot(\alpha \Theta - \epsilon \Phi \Theta) = 0,
$$

(3.125)

$$(L_\Theta L_r + 3 L_\Theta) \tau_{rr} + \left( L_\Theta^2 - L_r^2 - 3 L_r \right) \tau_{r\theta} - L_r L_\Theta \tau_{\theta\theta} + L_\Theta \left( \cot(\alpha \Theta - \epsilon \Phi \Theta) \tau_{r\theta} \right)
$$

$$- L_r \left( \cot(\alpha \Theta - \epsilon \Phi \Theta) \left( \tau_{\theta\theta} - \tau_{\phi\phi} \right) \right) = 0,
$$

(3.126)

$$
2 \tau_{r\theta} L_r u = \tau_{rr} \left( L_\Theta u + L_r v - v \right),
$$

(3.127)

$$
2 \tau_{\theta\theta} L_r u = \tau_{rr} \left( 2 L_\Theta v + 2 u \right).
$$

(3.128)

Here (3.125) is conservation of mass, (3.126) is the curl of the momentum balances, and (3.127)-(3.128) are a statement of the proportionality between corresponding components of the deviatoric stress and strain-rate tensors. The linear operators, $L_r$ and $L_\Theta$, are given by

$$
L_r = r \frac{\partial}{\partial r} + \frac{c \Phi' \Theta}{\alpha} \frac{\partial}{\partial \Theta}, \quad L_\Theta = \frac{1}{\alpha - \epsilon \Phi} \frac{\partial}{\partial \Theta} = \left( \frac{1}{\alpha} + \frac{c \Phi}{\alpha^2} + \ldots \right) \frac{\partial}{\partial \Theta}.
$$

(3.129)

Expansion of the governing equations at $O(1)$ gives the equivalent of Shield’s solution:

$$
u_0 := \frac{\dot{u}_0}{r^2} = -\frac{A}{r^2} \exp \left( -2 \sqrt{3} \int_0^\Theta \alpha \tan 2 \psi_0 d\Theta \right), \quad \frac{\partial p_0}{\partial r} = \frac{2 c B i}{r},
$$

(3.130)

$$
\psi_0' = \alpha \left( c \sec 2 \psi_0 - \sqrt{3} - \frac{1}{2} \cot \alpha \Theta \tan 2 \psi_0 \right),
$$

(3.131)

where $c$ is determined by the boundary conditions

$$
\psi_0(0) = 0, \quad \psi_0(1) = \frac{\pi}{4},
$$

(3.132)

and $A$ is determined by the flux condition

$$
2 \pi \alpha \int_0^1 \dot{u}_0(\Theta) \sin \alpha \Theta d\Theta = 1.
$$

(3.133)

We define $\dot{U}_0 = \dot{u}_0(\Theta = 1)$ since, unlike for the planar solution, this does not have a closed form expression in terms of $A$ and $c$. We note that this solution (3.130)-(3.133) exhibits a non-vanishing radial velocity and a divergent shear rate as $\Theta \to 1$, just as for the flow through a planar wedge. We therefore must introduce both the intermediate and boundary layers in order to enforce the boundary conditions. However this matching is identical to §3.3.3 and §3.3.4 and we deduce that

$$
\Phi(r) \equiv \frac{\dot{\Phi}}{r^{3/2}} = \sqrt{-\frac{\dot{U}_0}{c r^3}}, \quad V_0(r) = -\frac{c r \Phi^3}{3} = -\frac{c \dot{\Phi}^3}{3} r^{-7/2}.
$$

(3.134)
Thus, the boundary layer width scales like \((r^3 Bi)^{-1/2}\) and the first order corrections to the velocities scale like \(r^{-2} (r^3 Bi)^{-1/2}\), as expected given the true expansion parameter \(r^3 Bi\). We may now use these expressions to calculate the effect of the boundary layer in the bulk, and, in particular, the induced angular velocity. To do so we define

\[
    u_1 = r^{-7/2} \hat{u}_1(\Theta), \quad v_1 = r^{-7/2} \hat{v}_1(\Theta), \quad \psi_1 = r^{-3/2} \hat{\psi}_1(\Theta), \quad \chi_1 = r^{-3/2} \hat{\chi}_1(\Theta),
\]

and expand the governing equations (3.125)-(3.128) at order \(\epsilon\) to obtain four ODEs for \(\hat{u}_1, \hat{v}_1, \hat{\psi}_1\) and \(\hat{\chi}_1\):

\[
    -\frac{3}{2} \hat{u}_1 + \frac{1}{\alpha} \hat{v}_1' + \hat{v}_1 \cot \alpha \Theta = \frac{3 \hat{\Phi} \Theta}{2 \alpha} \hat{u}_0',
\]

\[
    \frac{2}{\alpha^2} \left( \hat{\psi}_1 \cos 2\psi_0 \right)'' + \frac{3}{\alpha} \left( \hat{\chi}_1 \cos 2\psi_0 \right)' - \sqrt{3} \left( \hat{\psi}_1 \sin 2\psi_0 \right)' + \frac{2}{\alpha} \left( \hat{\psi}_1 \cot \alpha \Theta \cos 2\psi_0 \right)'
\]

\[
    + \left( \frac{9}{2} \hat{\psi}_1 + 6 \hat{\chi}_1 \cot \alpha \Theta \right) \cos 2\psi_0 = \frac{\hat{\Phi}}{\alpha^2} \left( \frac{3 \sqrt{3} \Theta}{2} \cos 2\psi_0'' - \sqrt{3} \left( \cos 2\psi_0 \right)' - \frac{2}{\alpha} \left( \sin 2\psi_0 \right)'' \right)
\]

\[
    - \frac{9 \alpha \Theta}{4} \left( \sin 2\psi_0 \right)' - \left( \left( \cot \alpha \Theta + \alpha \Theta \cosec^2 \alpha \Theta \right) \sin 2\psi_0 \right)' \right).
\]

\[
    -7 \sqrt{3} \hat{u}_1 \tan 2\psi_0 - \frac{2}{\alpha} \hat{u}_1' + 9 \hat{v}_1 - 8 \sqrt{3} \hat{\psi}_1 \hat{u}_0 + \frac{4}{\alpha} \hat{\psi}_1 \hat{u}_0' \tan 2\psi_0 = \frac{\hat{\Phi}}{\alpha^2} \left( 3 \sqrt{3} \alpha \Theta \tan 2\psi_0 + 2 \right) \hat{u}_0',
\]

\[
    3 \hat{u}_1 - \frac{4}{\alpha} \hat{v}_1' - 8 \sqrt{3} \hat{\chi}_1 \hat{u}_0 = - \frac{3 \hat{\Phi} \Theta}{\alpha} \hat{u}_0',
\]

with boundary conditions

\[
    \hat{\psi}_1 = \hat{v}_1 = 0 \text{ at } \Theta = 0, \quad \hat{v}_1 = -\frac{c \hat{\Phi}^3}{3} \text{ at } \Theta = 1,
\]

from symmetry at \(\Theta = 0\) and matching to the boundary layer solution at \(\Theta = 1\). As in the planar case, matching of \(\tau_{rr}\) to the intermediate layer requires

\[
    \hat{\psi}_1 = \frac{3 \sqrt{3} \hat{\Phi}}{4} \text{ at } \Theta = 1.
\]

Similarly, analysis of the normal stress difference, \(\tau_{\theta\theta} - \tau_{\phi\phi}\), in the intermediate layer gives

\[
    \tau_{\theta\theta} - \tau_{\phi\phi} \equiv \left( 1 + \frac{Bi}{\hat{\gamma}} \right) \left( \frac{2}{r} \frac{\partial v}{\partial \theta} - \frac{2v}{r} \cot \theta \right)
\]

\[
    = \left( -\frac{Bi}{r \sqrt{\delta} \frac{\partial U_1}{\partial \zeta}} + \ldots \right) \left( \frac{2 \epsilon \Phi' \partial U_1}{r \sqrt{\delta} \partial \zeta} + \ldots \right) = 3Bi^{1/2} \Phi + \ldots,
\]
3.6. BINGHAM FLUID IN A CONE

Figure 3.12: The first order corrections to the velocities, \( u \) and \( v \), and the stress orientation functions, \( \psi \) and \( \chi \), for viscoplastic flow in a cone in the plastic regime, as functions of the scaled polar coordinate \( \Theta \), for \( \alpha = \pi/6 \) (solid), \( \pi/4 \) (dashed), \( \pi/3 \) (dash-dotted), and \( \pi/2 \) (dotted).

Using (3.35), (3.43), and the identity \( r \Phi' = -3\Phi/2 \). While in the bulk solution

\[
\tau_{\theta\theta} - \tau_{\phi\phi} \equiv 2Bi \cos 2\psi \sin 2\chi = 4Bi^{1/2} \chi_1 \sqrt{2} \alpha \frac{(2c - \cot \alpha)(1 - \Theta)}{2} + \ldots ,
\]

as \( \Theta \to 1 \), using the behaviour of \( \psi_0 \) in the neighbourhood of \( \Theta = 1 \) (see Appendix 3.C). Thus, \( \hat{\chi}_1 \) diverges like \( (1 - \Theta)^{-1/2} \) as \( \Theta \to 1 \), which is consistent with the leading order behaviour of \( \hat{\chi}_1 \) from the differential equations (3.136)-(3.139) close to \( \Theta = 1 \). Although the divergence of \( \hat{\chi}_1 \) may seem problematic for the asymptotic order of our expansions, we expect the plastic regime to break down within the intermediate layer, for which \( 1 - \Theta = O(\delta) \). Here we have \( \chi = O(\epsilon \delta^{-1/2}) \ll 1 \), since \( \delta \gg \epsilon^2 \), and so \( \chi \) remains small throughout the region of validity of the plastic equations. The reason for this divergence of the normal stress difference is that the gradient of the polar velocity, \( \partial v / \partial \theta \), becomes large in the boundary layer, while the polar velocity itself remains small, thus \( \tau_{\theta\theta} \) becomes significantly larger than \( \tau_{\phi\phi} \).

The boundary value problem given by (3.136)-(3.141) was again solved by a shooting method. In this situation we have potential singularities at both ends of the domain, due
to the cot \(\alpha\Theta\) terms, so asymptotic analysis was required to step a small distance away from \(\Theta = 0\) and \(\Theta = 1\) (see equations (3.167)-(3.171) in Appendix 3.C) before numerically shooting towards the centre of the domain, and fixing unknown coefficients by enforcing continuity at some central point. Again, the solutions were found to be essentially identical for sufficiently small choices of the step sizes, and the method converges effectively for any sufficiently central choice of matching point. The profiles given in figure 3.12 were produced with a step size of \(d = 10^{-8}\) away from the singularities at both ends of the domain, and with matching at \(\Theta = 0.5\).

From the solutions to the boundary value problem, we find that the velocity profiles are qualitatively similar to those for the planar wedge (figure 3.3), with a negative polar velocity out from the boundary layer towards the centre of the cone and an enhancement of the velocity in the radial direction, both of which are increased for decreasing half-angle, due to the boundary layer thickness, \(\hat{\Phi}\), increasing for decreasing \(\alpha\). The shear stress is again enhanced throughout the domain, due to the greater shear stress at the wall, and we find that the normal stress difference becomes significant at the edge of the boundary layer, \(\Theta = 1\), via the divergence of \(\chi_1\), as discussed above.

### 3.6.2 The viscous regime

The viscous regime, \(Bi \ll 1\), can be tackled by expanding the governing equations around the leading order Stokes solution. We define the asymptotic series

\[
\begin{align*}
    u &= u_0 + Bi u_1 + \ldots, \\
    v &= Bi v_1 + \ldots, \\
    p &= p_0 + Bi p_1 + \ldots,
\end{align*}
\]

where \(u_0\) and \(p_0\) are given by (3.119). For homogeneity in \(r\) in the perturbation to the stress tensor, and using conservation of mass, we can write

\[
\begin{align*}
    u_1 &= r\hat{u}_1(\theta) = \frac{r}{\sin \theta} g'(\theta), \\
    v_1 &= r\hat{v}_1(\theta) = -\frac{3r}{\sin \theta} g(\theta),
\end{align*}
\]

where \(g\) is a function to be determined, and \(\,\prime\) represents differentiation with respect to \(\theta\). From the expression of conservation of momentum in the angular direction at \(O(Bi)\), we find that the angular pressure gradient is independent of \(r\), and so the radial pressure gradient is independent of \(\theta\), giving

\[
\begin{align*}
    r \frac{\partial p_1}{\partial r} &= \mathcal{H}(\theta) + \frac{1}{\sin \theta} g'' - \frac{\cos \theta}{\sin^2 \theta} g'' + \frac{1 + 6 \sin^2 \theta}{\sin^3 \theta} g' = C, \tag{3.146}
\end{align*}
\]

where \(C\) is an arbitrary constant, to be determined, and \(\mathcal{H}(\theta)\) is given by

\[
\mathcal{H}(\theta) = \frac{d}{d\theta} \left( \frac{\sin 2\theta}{\sqrt{\sin^2 2\theta + 3(\cos 2\theta - \cos 2\alpha)^2}} \right) + \frac{2 \cos^2 \theta + 6 \cos 2\theta - 6 \cos 2\alpha}{\sqrt{\sin^2 2\theta + 3(\cos 2\theta - \cos 2\alpha)^2}}. \tag{3.147}
\]
3.6. BINGHAM FLUID IN A CONE

Figure 3.13: The first order corrections to the velocities, $u$ and $v$, for viscoplastic flow in a cone in the viscous regime, as functions of the scaled polar coordinate $\theta/\alpha$, for $\alpha = \pi/6$ (solid), $\pi/4$ (dashed), $\pi/3$ (dash-dotted), and $\pi/2$ (dotted).

Thus we have the third-order ODE,

$$g''' - \cot \theta g'' + \left(6 + \cosec^2 \theta\right) g' = (C - H(\theta)) \sin \theta. \tag{3.148}$$

For $\hat{u}_1(0)$ to remain finite, we need $g(0) = 0$. For symmetry we require $g$ to be an even function of $\theta$, which ensures $u_1$ is even and $v_1$ is odd. Finally, we have the no-slip boundary condition at the wall, which gives

$$g(\alpha) = g'(\alpha) = 0. \tag{3.149}$$

To solve the ODE numerically we shoot from $\theta = d \ll 1$, to avoid the potential singularity at $\theta = 0$, with initial conditions

$$g(d) = \lambda d^2, \quad g'(d) = 2\lambda d, \quad g''(d) = 2\lambda, \tag{3.150}$$

and determine $\lambda$ and $C$ by enforcing the two boundary conditions at $\theta = \alpha$, (3.149). The solutions are found to be essentially independent of $d$ for $d < 10^{-5}$. Profiles of $\hat{u}_1$ and $\hat{v}_1$ are given in figure 3.13, demonstrating the anticipated negative polar velocity away from the boundary and towards the centre of the cone. As in the planar case, the first order corrections are larger for larger $\alpha$, due to the magnitude of the deviatoric stress in the leading order viscous solution being smaller at larger half-angles, resulting in a less yielded fluid and more significant plasticity effects.

The additional shear stress at the wall is given by

$$Bi \tau^{(1)}_{rr}|_{\theta=\alpha} = Bi + Bi \left( \frac{1}{r} \frac{\partial u_1}{\partial \theta} + \frac{\partial v_1}{\partial r} - \frac{v_1}{r} + 1 \right)|_{\theta=\alpha} = Bi \left( \frac{g''(\alpha)}{\sin \alpha} + 1 \right), \tag{3.151}$$
CHAPTER 3. CONVERGING FLOW OF A VISCOPLASTIC FLUID IN A
WEDGE OR CONE

Figure 3.14: Features of the perturbation solution for the conical converging flow of
a viscoplastic in the low Bingham number regime: a) the additional wall shear stress,
\( \tau^{(1)}_{r\theta} \big|_{\theta=\alpha} \), and b) the scaled additional radial pressure gradient,
\( r \frac{\partial p_1}{\partial r} = C \), as functions of cone half-angle, \( \alpha \).

which is plotted as a function of \( \alpha \) in figure 3.14(a), demonstrating a similar behaviour
to that for the planar wedge, with a minimum attained for a slightly smaller half-angle
of \( \alpha \approx 27^\circ \). The constant \( C \) is also of interest since the first order correction to the radial
pressure gradient is given by \( BiC/r \), thus the behaviour of \( C = r \frac{\partial p_1}{\partial r} \) with \( \alpha \) is given in
figure 3.14(b), showing that \( C \) varies from 4.20 (approximately) at \( \alpha = \pi/2 \) and diverges
as \( \alpha \to 0 \). In particular, we find that \( \tau^{(1)}_{r\theta} \sim 4/3 \) and \( C \sim 8/(3\alpha) \) for \( \alpha \ll 1 \), as required
for consistency with the pipe Poiseuille solution in the limit \( \alpha \to 0 \) with \( r\alpha = 1 \) fixed
(see Appendix 3.D).

3.7 Discussion and conclusions

Asymptotic solutions have been found for the converging flow of a viscoplastic fluid
through a planar wedge and conical geometry, in the plastically- and viscously- dominated
regimes, and verified for the case of a Bingham fluid in a wedge using direct finite element
simulations. A key feature in both regimes is that no purely radial solution is possible.
Instead, a weak angular flow is induced away from the boundaries and towards the centre
of the domains. This result can be explained as a consequence of mass conservation
as the fluid flows from a plastically dominated to viscously dominated velocity profile,
experiencing an enhanced flow velocity at the centre of the domain and reduced velocity
near the walls.
In the plastically dominated regime ($\text{Bi} \gg 1$) for a Bingham fluid, thin viscoplastic boundary layers of width $\epsilon = \text{Bi}^{-1/2}$ are required at the walls, allowing the flow to satisfy no-slip. The angular extent of these boundary layers also depends on the radial distance from the apparent apex, $r$, scaling as $r^{-1}$ in a wedge and $r^{-3/2}$ in a cone. Additionally, an intermediate asymptotic layer is required to regularise divergent strain rates close to the wall. The weak angular flow, induced by the boundary layer, is also $O(\epsilon)$ and both the boundary layer and angular flow are weaker for larger half-angles. The shear stress is enhanced compared to the rigid plastic solution, exceeding the yield stress by $O(\epsilon \text{Bi})$ at the walls.

For a Herschel-Bulkley fluid of flow index $N$, the same features are observed in the plastically dominated regime, with the distinguished scaling given instead by $\epsilon = \text{Bi}^{-1/(N+1)}$ and the radial dependence of the boundary layer width given by $r^{-2N/(N+1)}$ for flow in a wedge. We choose not to explore in detail, in this thesis, the problem of a Herschel-Bulkley fluid in an axisymmetric conical geometry in the plastically dominated regime ($\text{Bi} \gg 1$). However, we can deduce by simple scaling arguments, analogous to those in §§3.5,3.6, that the boundary layer width scales as $(r^{3N} \text{Bi})^{-1/(N+1)}$, where $N$ is the flow index. Notably, the boundary layer is of constant Cartesian thickness for the flow of a Bingham material through a wedge and for a Herschel-Bulkley fluid of flow index $N = 1/2$ through a cone, but not for the other situations.

In the viscously dominated regime ($\text{Bi} \ll 1$), the weak angular flow is $O(\text{Bi})$ and, in contrast to the plastically dominated regime, is stronger for larger half-angles. This is due to the leading order viscous flow shearing the fluid less strongly for larger half-angles, resulting in less strongly yielded fluid and a more significant effect of the yield stress. The shear stress at the wall is again enhanced compared to the purely viscous solution, with the excess shear stress being $O(\text{Bi})$.

The direct finite element simulations, carried out in FEniCS [11] using a combination of augmented-Lagrangian and Newton methods, strongly supported the validity of the asymptotic solutions derived in the case of a Bingham fluid in a planar wedge. Boundary layer widths and velocity profiles were found to be in excellent agreement with the theoretical results. The challenges posed by the converging geometry and non-linear constitutive equation leave room for a more expansive and accurate numerical study. In particular, we were unable to reproduce accurately the predictions for both the plastic and viscous regimes in the same simulations due to the vastly disparate scales involved in the different regimes.

This study has tackled the simplest and most generic scenario for converging flows
of viscoplastic materials, demonstrating both the emergence of viscoplastic boundary layers when the viscous stresses are weak, and the development of an angular flow. It would be interesting to analyse the motion in related problems, such as those driven by a body force (e.g. gravity) or situations in which there is wall slip or significant inertia. There have been several experimental investigations of extrusions and flows through dies and contractions (for example Wildman et al. [143], Jay et al. [77], Rabideau et al. [112], Luu et al. [86]), but we are unaware of any experimental observations that may be used to validate the predictions of this theory. In particular it would be interesting to detect and quantify experimentally the emergent angular flow predicted by our analysis.

3.A Asymptotic analysis of the plastic solution in a wedge

In this appendix we expand the solution for plastic flow in a wedge, (3.14), in the vicinity of the boundary, for the purposes of matching to the intermediate and boundary layer solutions derived in §§3.3.3,3.3.4.

Expansion of the equation for $\psi_0$, (3.26), for $1 - \Theta \ll 1$ gives

$$\psi_0 = \frac{\pi}{4} - \sqrt{\alpha c (1 - \Theta)} + \frac{2}{3} \alpha (1 - \Theta) + \ldots$$

(3.152)

Substituting this into the definitions of the scaled radial velocity, $\hat{u}_0$, (3.27), and the components of deviatoric stress, $\tau_{rr}$ and $\tau_{r\theta}$, (3.17), gives

$$\hat{u}_0 = -\frac{A}{c} - \frac{2A}{c} \sqrt{\frac{\alpha c}{c} (1 - \Theta)} - \frac{8A}{3c^2} (1 - \Theta) + \ldots$$

(3.153)

$$\tau_{rr} \equiv Bi \cos 2\psi_0 = 2Bi \sqrt{\frac{\alpha c}{c} (1 - \Theta)} - \frac{4}{3} Bi \alpha (1 - \Theta) + \ldots$$

(3.154)

$$\tau_{r\theta} \equiv Bi \sin 2\psi_0 = Bi - 2Bi \alpha c (1 - \Theta) + \ldots$$

(3.155)

3.B Higher orders in the bulk for a Herschel-Bulkley fluid

For a Herschel-Bulkley fluid with flow index $N$, the generalisation of equations (3.61)-(3.63), for the first order corrections to the leading order solution in the bulk of the
3.B. HIGHER ORDERS IN THE BULK FOR A HERSCHEL-BULKLEY FLUID

wedge, is given by

\[
\frac{1}{\alpha^2} \left( \hat{\psi}_1 \cos 2\psi_0 \right)'' + \frac{4N}{(N+1)^2} \hat{\psi}_1 \cos 2\psi_0 - \frac{2(1-N)}{\alpha(1+N)} \left( \hat{\psi}_1 \sin 2\psi_0 \right)' = \\
\frac{\hat{\Phi}}{\alpha^2} \left( \frac{2N}{N+1} \Theta \left( \cos 2\psi_0 \right)'' - \frac{1-N}{N+1} \left( \cos 2\psi_0 \right)' - \frac{1}{\alpha} \left( \sin 2\psi_0 \right)'' - \frac{2N}{(N+1)^2} \alpha \Theta \left( \sin 2\psi_0 \right)' \right).
\]

(3.156)

\[
-\frac{2N}{N+1} \hat{u}_1 + \frac{1}{\alpha} \hat{\psi}_1' = \frac{2N}{N+1} \frac{\hat{\Phi} \Theta}{\alpha} \hat{u}_0',
\]

(3.157)

\[
-\frac{6N+2}{N+1} \hat{u}_1 \tan 2\psi_0 - \frac{1}{\alpha} \hat{\psi}_1' + \frac{4N+2}{N+1} \hat{v}_1 - 4 \hat{\psi}_1 \hat{u}_0 \sec^2 2\psi_0 = \frac{\hat{\Phi}}{\alpha} \left( \frac{4N}{N+1} \Theta \tan 2\psi_0 + \frac{1}{\alpha} \right) \hat{u}_0',
\]

(3.158)

with boundary conditions

\[
\hat{\psi}_1(0) = \hat{\psi}_0(0) = 0, \quad \hat{\psi}_1(1) = -\frac{2N^3}{(2N+1)(N+1)^2} \hat{\Phi}^{2+\frac{\alpha}{c}} (2c)^{\frac{\alpha}{c}},
\]

(3.159)

where \( \hat{\Phi} \) is given by (3.100) and \( c \) and \( A \) are given by (3.29) and (3.30). When \( N = 1 \) we have \( \hat{\Phi} = \sqrt{A/c} \) and the equations reduce to (3.61)-(3.63), as required.

The boundary condition for \( \hat{\psi}_1 \) at \( \Theta = 1 \) can be deduced by matching the deviatoric normal stress, \( \tau_{rr} \), between the bulk and intermediate layer, as detailed for the Bingham case in §3.3.5. This gives

\[
\hat{\psi}_1(1) = \frac{2N}{N+1} \hat{\Phi}.
\]

(3.160)

Using (3.152) and (3.153), we analyse the limiting behaviour of the equations (3.156)-(3.158) for \( |1-\Theta| \ll 1 \), which yields the following local forms of all three dependent variables:

\[
\hat{\psi}_1 = \frac{2N}{N+1} \hat{\Phi} + E \sqrt{\frac{c}{\alpha}} (1-\Theta) + \ldots,
\]

(3.161)

\[
\hat{u}_1 = F + \left( \frac{6N+2}{N+1} F + \frac{2A}{c^2} \left( \frac{c \hat{\Phi}}{\alpha} - E \right) \right) \sqrt{\frac{c}{\alpha}} (1-\Theta) + \ldots
\]

(3.162)

\[
\hat{v}_1 = -\frac{2N^3 \hat{\Phi}^{2+\frac{\alpha}{c}}}{(2N+1)(N+1)^2} (2c)^{\frac{\alpha}{c}} - \frac{4N}{N+1} \frac{A \hat{\Phi}}{c} \sqrt{\frac{c}{\alpha}} (1-\Theta) + \ldots,
\]

(3.163)

where \( E \) and \( F \) are arbitrary constants.
3.C Local solutions in the plastic regime for Bingham fluid in a cone

In this appendix we analyse the equations for a Bingham fluid in a cone in the plastic regime, given in §3.6.1, to provide local forms of the dependent variables in the vicinity of the boundaries and centre of the cone.

Expansion of the differential equation for $\psi_0$, (3.131), at $\Theta = 1$ gives the asymptotic series:

$$\psi_0 = \frac{\pi}{4} - \sqrt{\frac{\alpha}{2}} (2c - \cot \alpha) (1 - \Theta) + \frac{2\sqrt{3}}{3} \alpha (1 - \Theta) + \ldots.$$  (3.164)

Matching the leading order radial velocity to the intermediate and boundary layers gives:

$$\hat{u}_0 (\Theta = 1) = -c \hat{\Phi}^2,$$  (3.165)

and expansion of the leading order solution, (3.130), gives:

$$\hat{u}_0 = -c \hat{\Phi}^2 \left( 1 + \frac{2\sqrt{6}}{\sqrt{2c - \cot \alpha}} \sqrt{\alpha (1 - \Theta)} + \frac{16\alpha}{2c - \cot \alpha} (1 - \Theta) + \ldots \right).$$  (3.166)

Then, analysis of the first order equations, (3.136)-(3.139), and application of boundary conditions, (3.140)-(3.141), reveals that solutions have the local form:

$$\hat{\psi}_1 = \frac{3\sqrt{3}}{4} \hat{\Phi} + \mathcal{E} \sqrt{1 - \Theta} + \ldots,$$  (3.167)

$$\hat{u}_1 = \mathcal{F} + \left( \frac{7\sqrt{6}\alpha}{2\sqrt{2c - \cot \alpha}} \mathcal{F} + c \hat{\Phi}^2 \left( \frac{\sqrt{6}\hat{\Phi}}{\sqrt{\alpha (2c - \cot \alpha)}} - \frac{2\sqrt{3}\mathcal{E}}{(2c - \cot \alpha)^{3/2}} \right) \right) \sqrt{1 - \Theta} + \ldots,$$  (3.168)

$$\hat{v}_1 = -\frac{c \hat{\Phi}^3}{3} \left( 1 + \frac{9\sqrt{6}}{\sqrt{2c - \cot \alpha}} \sqrt{\alpha (1 - \Theta)} + \ldots \right),$$  (3.169)

where $\mathcal{E}$ and $\mathcal{F}$ are undetermined constants. We note that $\hat{\chi}_1$ is in fact determined algebraically by (3.139), so we require no boundary conditions for $\hat{\chi}_1$. To leading order at $\Theta = 1$ we have

$$\hat{\chi}_1 = \frac{3\sqrt{2}\Phi}{8\sqrt{\alpha (2c - \cot \alpha) (1 - \Theta)}} + \ldots,$$  (3.170)

as required for matching of the normal stress difference between the intermediate layer and bulk solution, (3.142) and (3.143).

For $\Theta \ll 1$ we have the leading order local forms:

$$\hat{\psi}_0 = \frac{\alpha (c - \sqrt{3})}{2} \Theta + \ldots, \quad \hat{u}_0 = -A + \ldots, \quad \hat{\psi}_1 = \mathcal{E}\Theta + \ldots, \quad \hat{u}_1 = \mathcal{F} + \ldots,$$  (3.171)

where $\mathcal{E}$ and $\mathcal{F}$ are undetermined constants.
3.D Plane and pipe Poiseuille flow of a Bingham fluid

For small Bingham number, \( Bi \ll 1 \), we consider the plane Poiseuille flow of a Bingham fluid in a channel of width 2, occupying \(-1 \leq y \leq 1\). We assume a volume flux per unit width, \( Q = -1 \), driven by pressure gradient \( dp/dx = G \). From the volume flux condition we can write

\[
-\frac{1}{2} = \int_0^1 u dy = u_p y_c + \int_{y_c}^1 u dy = -\int_{y_c}^1 y \frac{du}{dy} dy = -\int_{y_c}^1 y (Gy - Bi) dy
\]

\[
= -\frac{G}{3} + \frac{1}{2} Bi + \ldots ,
\]

where \( u_p \) is the velocity of the plug which occupies the region \(-y_c < y < y_c\). In the above we have used integration by parts, made the substitution \( du/dy = Gy - Bi \), using the integral of the streamwise conservation of momentum equation in the yielded region, and finally used the fact that \( y_c = O(Bi) \) in neglecting terms in the final expression. Thus we find

\[
G = \frac{3}{2} + \frac{3}{2} Bi + \ldots ,
\]

(3.173)

and, by a global force balance, the shear stress at the wall is given by

\[
\tau_{xy} = G = \frac{3}{2} + \frac{3}{2} Bi + \ldots
\]

(3.174)

Similarly, for the pipe Poiseuille flow of a Bingham fluid along the axis of a cylinder of unit radius, with volume flux \(-1\) driven by a pressure gradient \( dp/dz = G \), we can write

\[
-1 = 2\pi \int_0^1 r u dr = \pi r_c^2 u_p + 2\pi \int_{r_c}^1 r u dr = -\pi \int_{r_c}^1 r^2 \left( \frac{1}{2} Gr - Bi \right) dr
\]

\[
= -\frac{\pi}{8} G + \frac{\pi}{3} Bi + \ldots ,
\]

(3.175)

and so

\[
G = \frac{8}{\pi} + \frac{8}{3} Bi + \ldots ,
\]

(3.176)

and, by a global force balance, the shear stress at the wall is given by

\[
\tau_{rz} = \frac{1}{2} G = \frac{4}{\pi} + \frac{4}{3} Bi + \ldots
\]

(3.177)
Viscoplastic corner eddies

Authorship: The material in this chapter is the result of original research by J. J. Taylor-West and A. J. Hogg. It was originally published in Taylor-West and Hogg, 2022 [132] which has been modified slightly for inclusion in this thesis.

4.1 Introduction

In a seminal paper, Moffatt [92] examined two-dimensional slow viscous flow in corners bounded by plane walls, and predicted the existence of infinite sequences of viscous, non-inertial, eddies under certain conditions. These eponymous ‘Moffatt eddies’ occur in wedges of half-angle, $\alpha$, less than a critical angle $\alpha_c \approx 73^\circ$, are driven by an arbitrary (anti-symmetric) disturbance asymptotically far from the vertex of the corner, and decay exponentially in size and intensity as the vertex is approached. In this chapter we examine corner eddies for viscoplastic fluids.

As detailed in §1, a quintessential feature of viscoplastic flows is the occurrence of unyielded regions. In food processing it is critical to avoid dead zones in corners where unyielded viscoplastic material can spoil and infect the passing product [134], thus emphasising that an understanding of unyielded and recirculating zones in viscoplastic corner flows is of great importance. Eddies occur in various examples of inertial and non-inertial flows of viscoplastic fluids, including sudden expansions and/or contractions [125, 77, 91, 1], thermal convection [80], tape casting [83], and flows through non-uniform channels [116, 118, 117]. Roustaei and Frigaard [116] compute viscoplastic flow in a wavy...
chapter in the limit of vanishing Reynolds number, and observe that, for sufficiently low Bingham number (dimensionless yield stress) and sufficiently large amplitude channel-width variations, eddies form within the expanded regions of the channel. They make the analogy with Moffatt [92] eddies, and comment that, in a sharp cornered wedge, one could theoretically observe arbitrarily many eddies, for sufficiently low Bingham number. In their numerical simulations they were only able to observe a single eddy in the parameter space studied, due to the rapid drop off of intensity with distance from the vertex analogously to the high decay rates in Moffatt’s solutions. Abbott et al. [1] analyse viscoplastic flow through an abrupt contraction, and suggest, but do not carry out, a perturbation expansion of the Moffatt solution for a right-angled corner when the yield stress is small, proposing the existence of approximately circular rotating plugs at the centre of the eddies. Finally, Chupin and Palade [40] examine the flow of viscoplastic fluids in the neighbourhood of a corner and prove that, for a concave wedge (half-angle, $\alpha < \pi/2$) the fluid must be unyielded in some neighbourhood of the vertex, the scale of which they do not determine. As noted above, the extent of this unyielded stagnant region is important for applications in which the aim is to mix or dislodge a viscoplastic fluid, as it corresponds to material undisturbed by the forcing.

In the current chapter we present a detailed numerical and analytical study of viscoplastic corner eddies, describing and rationalising the critical Bingham numbers at which new eddies form for wedges of different half-angle. We first consider an idealised case where it is assumed that the dominant solution of Moffatt [92] is fully developed at large radial distances from the vertex, and we then consider the behaviour at smaller distances, where viscoplasticity first becomes significant. We define this problem in §4.2 and describe the numerical methods in §4.3, before reporting and rationalising the results in §4.4. In §4.5, we compare this idealised case, forced by the dominant Moffatt solution, with a particular example of flow past a triangular inclusion, driven by a translating lid, to illustrate the relevance of the idealised theory to practical situations in which these eddies occur. Finally, we conclude in §4.6. There are also two appendices in which we explore the derivation of the critical Bingham number in greater detail, and demonstrate that viscoplastic eddies in rectangular channels can also be described by our work by considering the limit $\alpha \to 0$ with $r\alpha$ fixed.
4.2 Problem definition

Throughout the following, we assume slow, non-inertial, flow of a Bingham fluid, defined by the constitutive law, \( \tau = (\mu + \tau_c/\|\dot{\gamma}\|)\dot{\gamma} \) when \( \|\tau\| > \tau_c \), and \( \dot{\gamma} = 0 \) otherwise, relating the deviatoric stress tensor, \( \tau \), to the strain-rate tensor, \( \dot{\gamma} = (\nabla u) + (\nabla u)^T \), and their second invariants, \( \|\tau\| \) and \( \|\dot{\gamma}\| \), where the second invariant of a tensor, \( T \), is defined by \( \|T\| = \sqrt{T_{ij}T_{ij}/2} \). The parameters, \( \mu \) and \( \tau_c \), are the viscosity and yield stress, respectively. We consider two-dimensional motion within an infinite planar wedge of half-angle, \( \alpha \). For a viscous Newtonian fluid, the existence of Moffatt [92] eddies is derived by searching for anti-symmetric solutions for the streamfunction, \( \psi \), satisfying the biharmonic equation,

\[
\nabla^4 \psi_V = 0, \tag{4.1}
\]

and no-slip on the planar boundaries \( \theta = \pm \alpha \). In plane polar coordinates \((r, \theta)\), centered on the vertex of the wedge, making the ansatz of a separable solution, one finds a discrete set of solutions, given by the real part of

\[
\psi_V = Ar^\lambda f(\theta), \tag{4.2}
\]

where

\[
f(\theta) = \cos(\lambda \theta) \cos((\lambda - 2)\alpha) - \cos((\lambda - 2)\theta) \cos(\lambda \alpha), \tag{4.3}
\]

the eigenvalue, \( \lambda \), is a solution of

\[
\sin(2(\lambda - 1)\alpha) + (\lambda - 1) \sin(2\alpha) = 0, \tag{4.4}
\]

and \( A \) is a general (complex) constant. We will consider the dominant solution in the vicinity of the corner, given by the eigenvalue with smallest real part. For all values of \( \alpha \) below the critical value, \( \alpha_c \approx 73^\circ \), \( \lambda \) is complex \( (\lambda = \lambda_r + i\lambda_i) \) giving rise to the oscillatory behaviour interpreted as eddies. Above this critical angle, the eigenvalues are all real [50], resulting in no eddies. In this case, the dominant eigenvalue in the vicinity of the vertex corresponds to a solution in which the fluid flows towards the vertex in one half of the corner, and flows back out of the corner in the other half. In this chapter we will primarily restrict our focus to angles at which eddies form (i.e. \( \alpha < \alpha_c \)), however we briefly discuss the expected behaviour of a viscoplastic fluid for \( \alpha > \alpha_c \) in appendix 4.A.

For a given angle, \( \alpha < \alpha_c \), consecutive eddies are geometrically similar, with a length scale factor of \( S_0 \equiv e^{-\pi/\lambda_i} \) and corresponding velocity and strain-rate/vorticity factors of \( S_1 \equiv e^{-(\lambda_r-1)\pi/\lambda_i} \) and \( S_2 \equiv e^{-(\lambda_r-2)\pi/\lambda_i} \), respectively. This last factor is of particular importance when considering viscoplastic fluids, since the magnitude of the strain rate
determines the significance of the yield-stress term relative to the viscous term in the constitutive law. For all $\alpha < \alpha_c$, we have $\lambda_r > 2$, and a decaying strain rate as $r \to 0$, underpinning why fluid in the apex of the corner is unyielded. The value of the factor $S_2$ is plotted against $\alpha$ in figure 4.1 showing that it vanishes as $\alpha \to \alpha_c$, and attains a maximum of 0.0078 for $\alpha = 40^\circ$ (both given to 2 significant figures).

Since the strain rate increases with $r$, there exists a viscoplastic flow in the same domain, which asymptotically tends to this viscous solution at sufficiently large distances from the vertex. The fluid will be static and unyielded at small distances, and the eddies will be essentially unchanged at large distances. There are a few locations in the viscous solutions at which the strain rate vanishes, around which we would expect regions of unyielded fluid for a viscoplastic fluid. These include: points on the $\theta = 0$ plane near the center of each eddy; pairs of points on the upper and lower boundaries at the stagnation points between consecutive eddies; and, less intuitively, pairs of points a small distance vertically above and below the points on the $\theta = 0$ plane. However, since the ratio of strain rates between two consecutive Moffatt eddies is never greater than 0.008 for any $\alpha$ (see figure 4.1), for a given yield stress and viscosity, there will never be two consecutive eddies in which the yield stress plays a leading order role. More precisely, the material parameters define a strain-rate scale $\tau_c/\mu$, while each of the viscous Moffatt solutions has a typical strain rate. If we label the Moffatt eddies via the index $k \in \mathbb{Z}$, with $k \to -\infty$ corresponding to the tip of the corner, and define the strain-rate scale of the $k$th eddy as $\Gamma_k = U_k/L_k$, where the dividing streamline between the $k$th and $(k + 1)$th eddy passes through $(L_k, 0)$ with velocity $U_k$, then we can define a local Bingham number for each eddy via the ratio of these two strain-rate scales, $Bi_k = \tau_c/(\mu \Gamma_k)$. By the self-similarity of the Moffatt solution, we can write all $\Gamma_k$ in terms of a reference eddy, $k = 0$, via $\Gamma_k = S_2^{-k} \Gamma_0 = S_2^{-k} U_0/L_0$, where $S_2$ is the strain-rate factor defined above, and hence
$Bi_k = S_2^kB_{i0}$. Since $S_2 < 0.008$, only a single eddy can have an $O(1)$ Bingham number (with the Bingham number being a factor of over 100 smaller/larger in the eddy further from/nearer to the vertex). In other words, for the viscoplastic fluid, we expect that all but one of the eddies from the purely viscous solution will be unyielded and static, or else unchanged to leading order, with the unyielded regions around points of vanishing strain rate being negligibly small. Without loss of generality, we can choose $k = 0$ to correspond to this unique eddy, and non-dimensionalise lengths by $L_0$ and velocities by $U_0$. With this choice, in non-dimensional variables, the dividing streamline between the $0^{th}$ and $1^{st}$ eddy passes through $(r = 1, \theta = 0)$ with unit velocity in the $\theta$-direction (see figure 4.2). This fixes the constant $A$ in (4.2), and the streamfunction at large distances is given, to leading order, by

$$\psi_V = -ir^\lambda f(\theta)/\lambda f(0),$$

where $f(\theta)$ is given by (4.3) and the real part is assumed. We further non-dimensionalise stresses and pressure by the typical viscous stress, $\mu \Gamma_0 = \mu U_0/L_0$, giving the global Bingham number,

$$Bi = \frac{\tau_c}{\mu \Gamma_0} = \frac{\tau_c}{\mu U_0/L_0}. \tag{4.6}$$

In the following numerical simulations we will sometimes take a value of $Bi \ll 1$, at which new eddies open up below the one at $r = 1$. These cases are included to demonstrate the self-similarity between two consecutive generations of eddies, and to explore the critical point at which a new eddy is formed. However, we point out that when the problem is scaled as detailed above, then these Bingham numbers are technically inadmissible. Formally, due to the infinite, self-similar domain and the self-similar nature of the Moffatt solution being applied as a boundary condition, these cases should be considered as identical to rescaled problems in which the lengths are divided by $S_0$, velocities by $S_1$ and the Bingham number by $S_2$. And, after such a rescaling, they would be consistent with the scaled problem defined above, with $Bi = O(1)$ and the smallest eddy occurring just below $r = 1$.

After non-dimensionalisation, the governing equations for velocity, $\mathbf{u} = (u_r, u_\theta)$, pressure, $p$, and deviatoric stress, $\mathbf{\tau}$, are

$$\nabla \cdot \mathbf{u} = 0, \tag{4.7}$$

$$\nabla p = \nabla \cdot \mathbf{\tau}, \tag{4.8}$$

representing incompressibility and the balance of momentum. The Bingham constitutive
Figure 4.2: Schematic of viscoplastic eddies in a wedge. Black regions represent unyielded fluid, and only half of the domain is shown, with the lower half determined by anti-symmetry under vertical reflection. No eddies are present in region a), where the fluid is unyielded, and the eddies in region b) are essentially unchanged from the corresponding viscous eddies described by Moffatt [92].

The constitutive law is given in non-dimensional form by

\[ \tau = \left(1 + \frac{Bi}{\|\dot{\gamma}\|}\right)\dot{\gamma} \text{ when } \|\tau\| > Bi, \quad \dot{\gamma} = 0 \text{ otherwise.} \quad (4.9) \]

We consider anti-symmetric solutions in the upper half of the domain, \(0 \leq \theta \leq \alpha\), with boundary conditions

\[ u = 0 \text{ on } \theta = \alpha, \quad (4.10) \]
\[ u_r = 0 \text{ on } \theta = 0, \quad (4.11) \]
\[ (u_r, u_\theta) \sim \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}\right) \text{ as } r \to \infty, \quad (4.12) \]

representing no-slip, anti-symmetry and the far-field condition respectively.

### 4.3 Numerical method

We compute finite-element numerical simulations, using the augmented-Lagrangian method as described in §2.4, over a wide range of \(Bi\) and for \(\alpha = 5^\circ, 20^\circ, 45^\circ, \text{ and } 60^\circ\) (as well as \(\alpha = 0^\circ\) in Appendix 4.C). As discussed in §2.4, this algorithm circumvents the singular nature of the constitutive law at the yield surfaces via the introduction of an independent tensorial field, \(D\), representing the strain-rate tensor, and a Lagrangian multiplier, standing for the deviatoric stress tensor, which enforces the equivalence of \(D\) and \(\dot{\gamma}(u)\). In contrast to regularisation methods, in which unyielded regions
4.4 RESULTS AND KEY SCALINGS

are replaced with regions of very high viscosity, the augmented-Lagrangian method accurately represents solid regions by setting $D = 0$ for stresses below the yield stress. We implement the numerical method in FEniCS, a numerical implementation of the finite element method, [84, 11] and employ a simple adaptive mesh refinement algorithm where periodically in the augmented-Lagrangian iterations (typically every 50 iterations) we refine cells in the vicinity of the yield surface. Specifically, we refine cells for which the deviatoric stress variable lies within some tolerance of $Bi$. For the first refinement we use a tolerance $Bi/2$, and decrease the tolerance by 25% for each subsequent refinement, to encompass consistently the yield surface while somewhat limiting the number of new cells produced. We stop refining after 5 refinement steps, or once the new mesh size would be above some chosen limit - in this case 280 000 cells. In place of the far-field boundary condition, (4.12), we truncate the domain at the straight boundary $x = r \cos \theta = x_R$ and impose the viscous Moffatt [92] solution,

$$(u_r, u_\theta) = \left( \frac{1}{r} \frac{\partial \psi_V}{\partial \theta}, -\frac{\partial \psi_V}{\partial r} \right), (4.13)$$

on this boundary. When choosing the truncation position, $x_R$, we require that the strain rate is significantly larger than $Bi$ along $x = x_R$, so that the viscous solution is a good approximation to the viscoplastic solution at the truncated boundary and hence that the solution is essentially unchanged by the truncation of the domain. In particular this requires avoiding any of the points at which the strain rate vanishes in the Moffatt [92] solution. To check the impact of truncation at $x = x_R$, simulations were repeated on domains at least 50% larger and the velocity solution and inner plug depths, $d$, were found to differ by less than 1% for all solutions. The values of $x_R$ varied between 1.5 and 15 for the different values of $Bi$ and $\alpha$, with the largest domain being needed for the simulation with $\alpha = 60^\circ$ and $Bi = 2$. All simulations converged to a residual, $\| \sqrt{\left( D_{ij} - \dot{\gamma}_{ij} \right) \left( D_{ij} - \dot{\gamma}_{ij} \right)} \|_{L_2}$, of less than $10^{-5}$, with many of the smaller Bingham number simulations converging to significantly lower residuals.

4.4 Results and key scalings

A typical set of numerical solutions is given in figure 4.3 demonstrating the existence of three unyielded regions, as observed in Roustaie and Frigaard [116]: a static unyielded region in the corner of the wedge; a small static unyielded region on the boundary at the stagnation point between eddies; and a plug region in solid-body motion within the eddy. All regions decrease in size as $Bi$ decreases, until a new eddy forms at sufficiently
small $Bi$. While Abbott et al. [1] predicted an approximately circular plug rotating at the ‘centre’ of the eddy, we note that the shape and location of this plug is somewhat unintuitive, not encompassing the centre of the eddy and, as a result, being closer to a semi-circle in shape. This is due to the fact that in Moffatt eddies, the point on the wedge’s symmetry axis at which the strain rate vanishes is distinct from the point where the velocity vanishes. Furthermore, though this plug is undergoing solid-body rotation, there is no requirement that its boundary is circular, since the yield surface need not be a material surface (streamlines can exit and enter the plug as the fluid element yields and unyields). We see that the flow in the eddy in $r > 1$ is largely unchanged for these Bingham numbers, although the streamlines are slightly altered at the inner extent of the eddy for larger $Bi$. Also, once a new eddy has formed, the unyielded regions in the eddy above have become negligibly small, as anticipated. Finally, we note that, as discussed in §4.2, the solution in panel $d$) is equivalent to a rescaled problem where the first eddy has its rightmost extent at $r = 1$ and $Bi$ is multiplied by $S_2^{-1} = e^{(\lambda r - 2)\pi/\lambda} \approx 170$. This gives $Bi = 3.0$ (to 2 significant figures), and so we expect panels $a$) and $d$) to be equivalent up to scaling, as is observed.

Two key features of the problem are the extent of the stagnant unyielded plug in the corner as a function of $Bi$, and the critical Bingham numbers, $Bi_c$, at which a new eddy forms. We measure the former as the distance, $d$, of the yield surface from the vertex of the wedge, along the $\theta = 0$ plane. The stars in figure 4.4 show $d$ against $Bi$ for four values of $\alpha$, determined from the numerical simulations. The first plot shows how $d$ decreases with $Bi$ after the creation of the second eddy and before the creation of a third, while the log-log plots demonstrate the existence and location of the critical Bingham numbers at which the value of $d$ jumps due to the formation/disappearance of an eddy, and the equivalence up to scaling of consecutive eddies evidenced by the translational periodicity of the curves. In the following section we provide a heuristic argument for approximating $Bi_c$ and the values of $d$ before and after a new eddy forms.

4.4.1 The critical Bingham number

A heuristic argument to approximate the critical Bingham number, $Bi_c$, for a given half-angle, $\alpha$, is as follows. We observe that the eddy adjacent to a newly opened eddy fully contains the corresponding viscous Moffatt [92] eddy, and consider a semi-circle meeting the boundary tangentially, with diameter centered on $(x_0, 0)$, where $x_0$ is the smallest $x$-coordinate attained by this Moffatt eddy (see figure 4.5). Note that $x_0$ is known a priori, since the Moffatt solution is known analytically, but is only a heuristic
4.4. RESULTS AND KEY SCALINGS

Figure 4.3: Contours of the modulus of the strain rate, $\|\dot{\gamma}\|$ (gray-scale) and streamlines (red) for $\alpha = 20^\circ$ and a) $Bi = 3$, b) $Bi = 1$, c) $Bi = 0.25$, and d) $Bi = 0.018$. The unyielded regions are shown in black. The critical Bingham number at which a new eddy forms, $Bi_c$, lies somewhere between the value of $Bi$ for panels c) and d). Note the logarithmic scale for the strain rate.
CHAPTER 4. VISCOPLASTIC CORNER EDDIES

Figure 4.4: Extent of static plug in corner of wedge, $d$, as a function of $Bi$. Symbols show numerical results while the dotted lines show our heuristic predictions. a) $\alpha = 20^\circ$ on linear-linear scale, showing variation of $d$ with $Bi$. Log-log plots across a larger range of $Bi$ are given for b) $\alpha = 5^\circ$, c) $\alpha = 20^\circ$, d) $\alpha = 45^\circ$, and e) $\alpha = 60^\circ$, showing the jumps at critical values of $Bi$ where a new eddy forms and the self-similarity of consecutive generations of eddies. The red points $A$, $B$, $C$, and $D$ in panel c) indicate the four points derived in the heuristic approximation, as detailed in §4.4.1.
4.4. RESULTS AND KEY SCALINGS

Figure 4.5: Examples of solutions before (top row) and after (bottom row) a new eddy has formed. The dotted line shows the dividing streamline $\psi_V = 0$ from the corresponding Moffatt solutions while the red lines show the semi-circles considered in §4.4.1.

approximation to the minimum distance attained by the viscoplastic eddy. Appendix 4.B outlines a more rigorous approach to determine the region of the static corner plug that yields to rotation as the Bingham number is reduced, however for the purposes of a heuristic argument we appeal to the observation from numerical simulations that this semi-circle is a good approximation to the true yield surface, as seen in figure 4.5. The normal stresses acting on the diameter of the semi-circle exert a dimensionless torque, denoted by $2G$, around $(x_0,0)$ on the fluid contained in the semi-circle, which must be balanced by the torque due to the tangential stresses along the circumference of the semi-circle (since, in the absence of inertia, torques must balance). The dimensionless torque, $G$, is given by

$$G = \left| \int_{0}^{R} y (-p + \tau_{xx}) dy \right|,$$

(4.14)

where $R = x_0 \sin \alpha$ is the radius of the semi-circle, and the integral is calculated along $x = x_0$. While the fluid is unyielded along the circular arc, the maximum possible torque per unit length is $RB_i$. In practice the semi-circle may extend slightly beyond the yield surface (see figure 4.5) which would slightly alter the torque along the arc in this region but nonetheless the maximum torque along the circumference in the upper half of the wedge is approximately $\pi R^2 Bi/2$. At the critical Bingham number, $Bi_c$, we hence have

$$\frac{\pi R^2 Bi_c}{2} \approx G.$$  

(4.15)
The final approximation we make is to use the purely viscous solution for \( p \) and \( \tau_{xx} \), encoded by the streamfunction \( \psi \), (4.5), when evaluating \( G \) in (4.15), using (4.14). This allows us to calculate an approximation for \( Bi_c \) for any \( \alpha \), purely from the Moffatt solution given by (4.5). The left panel of figure 4.6 shows the value of \( G \) calculated using the Moffatt solution, while the stars are from the numerical simulations shown in the top row of figure 4.5. The close correspondence of these curves demonstrates the validity of the approximation. The second panel of figure 4.6 shows the predicted value of \( Bi_c \) as a function of \( \alpha \), alongside the smallest Bingham numbers of numerical simulations at which the new eddy had not yet formed (representing a numerical upper bound for \( Bi_c \)). Interestingly the predicted value of \( Bi_c \) is approximately constant over a wide range of angles, \( 15^\circ \leq \alpha \leq 45^\circ \), diverging like \( 1/\alpha \) as \( \alpha \to 0 \) and tending to 0 as \( \alpha \to \alpha_c \). The latter is anticipated since the relative intensity of consecutive eddies vanishes in this limit requiring a vanishing Bingham number to exhibit additional eddies. The divergent behaviour as \( \alpha \to 0 \) can also be understood, by instead considering \( \alpha \to 0 \) with \( r\alpha = 1 \) fixed, which is the typical way to treat this limit and which represents convergence to a uniform channel of width 2, for \( \alpha \) in radians. We previously scaled lengths by \( L_0 \) to set the eddy of interest to \( r = 1 \), so to scale this eddy instead to \( r = 1/\alpha \) requires a length scale of \( \tilde{L} = \alpha L_0 \), giving a new Bingham number \( \tilde{Bi} = \tau_c\tilde{L}/(\mu U_0) = \alpha Bi \). We anticipate that the corresponding scaled critical Bingham number, \( \tilde{Bi}_c = \alpha Bi_c \), is finite in the controlled limit, representing the Bingham number at which a new eddy forms between parallel plates, and from the heuristic calculation above, we find that

\[
\lim_{\alpha \to 0} \tilde{Bi}_c = \lim_{\alpha \to 0} \alpha Bi_c = 0.0022 \text{ (to 2 significant figures),} \tag{4.16}
\]

is the critical Bingham number for this limit. In fact this limit can be tackled directly by considering the eddy flow between parallel plates, as is demonstrated in Appendix 4.C.

Using \( x_0 \), \( R \) and \( Bi \) we can also give a heuristic approximation for the extent, \( d \), of the static plug in the corner of the wedge, measured along the \( \theta = 0 \) plane, as a function of \( Bi \). We have \( d \approx x_0 \) as \( Bi \to Bi_c \) from above, and \( d \approx x_0 - R = x_0(1 - \sin \alpha) \) as \( Bi \to Bi_c \) from below. We can then use self-similarity with the scale factors given in §4.2 to scale up/down to the values of \( d \) and \( Bi \) at the start of the eddy further from/nearer to the vertex. Specifically, this gives four points on the \( Bi - d \) curve:

\[
A = (Bi_c, x_0(1 - \sin \alpha)) \quad B = (Bi_c, x_0) \quad C = \left( e^{(\lambda_r-2)\pi/\lambda_i}Bi_c, e^{\pi/\lambda_i}x_0(1 - \sin \alpha) \right) \quad D = \left( e^{(\lambda_r-2)\pi/\lambda_i}Bi_c, e^{\pi/\lambda_i}x_0 \right),
\]

where \( A \) to \( B \) and \( C \) to \( D \) are vertical jumps occurring at the formation/disappearance of an eddy, while between \( B \) and \( C \), \( d \) is a continuous, increasing function of \( Bi \). The form
4.4. RESULTS AND KEY SCALINGS

Figure 4.6: (left) Torque, $G$, acting on the vertical radius of the semi-circle considered in §4.4.1 using the corresponding Moffatt solution (dotted) and from the viscoplastic numerical simulations shown in the top row of figure 4.5 (stars), as a function of wedge half-angle, $\alpha$. (right) The corresponding critical Bingham number, $Bi_c$, calculated from (4.15) (solid line), as a function of wedge half-angle, $\alpha$. The red dotted line shows the divergent behaviour as $\alpha \to 0$, given by $Bi_c \sim 0.0022/\alpha$, while the stars indicate the smallest Bingham numbers of numerical simulations in which the new eddy has not yet opened up (and hence represent numerical upper bounds for $Bi_c$).

of this function is shown, from numerical solutions for $\alpha = 20^\circ$, in figure 4.4(a), but for the purposes of a simple approximation, linear interpolation can be used between points $B$ and $C$. Figure 4.4 shows good agreement between this heuristic approximation and the numerical simulations, despite its simplicity.

4.4.2 Flow fields when $0 < Bi_c - Bi \ll 1$

Slightly below the yield stress at which a new eddy forms, a thin layer of yielded fluid separates the static corner plug from the rotating semi-circular plug, meaning we can employ a boundary-layer analysis similar to those detailed by Balmforth et al. [20] and Hewitt and Balmforth [69], and used earlier in §3. The purpose of this section is to utilise such a boundary layer analysis to determine the scalings of the width of the yielded layer and the rotation rate of the rotating plug, with the difference between the Bingham number and $Bi_c$. In fact there are two distinct boundary-layer scalings, with one applying between the rotating plug and the rigid boundary, and another between the static and rotating plugs. The former is asymptotically thinner than the latter; in the former, the viscous shear stresses provide the leading order contribution to the torque balance on the rotating plug, whereas in the latter, plastic stresses are non-negligible. The theory
behind these two regimes is detailed in §2.2.

Figure 4.7 shows a schematic of the boundary layer geometry. The governing small parameter is $\Delta Bi = Bi_c - Bi$, which we refer to as the Bingham number deficit, and there are a number of quantities that scale with this quantity: the boundary layer thickness between the wall and the rotating plug, $\epsilon_1$; the boundary layer thickness between the rotating plug and stagnant corner plug, $\epsilon_2$; and the rotation rate of the rotating plug, $\Omega$. Note there is also a short section of wider boundary layer after the narrow section where the boundary layer meets the adjacent eddy. The width here is also $O(\epsilon_2)$ but we will neglect this section for the clarity of the following discussion, appealing to the shortness of the region to justify this decision. The direction of rotation depends on which eddy is being considered, but we will assume clockwise rotation, as relevant to the first new eddy to form as $Bi$ is decreased as in figure 4.2. Following the construction of Balmforth et al. [20], reproduced in §2.2, we take curvilinear coordinates, $(s, n)$, and velocities, $(u_s, u_n)$, along and across the boundary layer (although we note that polar coordinates would also be an appropriate choice here), where $s = 0$ at the axis of symmetry of the wedge. The full system of equations are then (e.g. see Balmforth et al. [20])

$$\frac{\partial u_s}{\partial s} + (1 - \kappa n) \frac{\partial u_n}{\partial n} - \kappa u_n = 0,$$

$$\frac{\partial \tau_{ss}}{\partial s} + (1 - \kappa n) \frac{\partial \tau_{sn}}{\partial n} - 2\kappa \tau_{sn} = \frac{\partial p}{\partial s},$$

$$\frac{\partial \tau_{sn}}{\partial s} + (1 - \kappa n) \frac{\partial \tau_{nn}}{\partial n} + \kappa (\tau_{ss} - \tau_{nn}) = \frac{\partial p}{\partial n},$$

where $\kappa$ is the curvature of the boundary layer. For an approximately circular boundary layer we have $\kappa \approx -\frac{1}{R}$ (with the sign determined from the orientation of the coordinate axes).

The components of the strain-rate and deviatoric-stress tensors are given by

$$\dot{\gamma}_{ss} = -\dot{\gamma}_{nn} = \frac{2}{1 - \kappa n} \left( \frac{\partial u_s}{\partial s} - \kappa u_n \right), \quad \dot{\gamma}_{sn} = \frac{1}{1 - \kappa n} \left( \frac{\partial u_n}{\partial s} + \kappa u_s \right) + \frac{\partial u_s}{\partial n},$$

$$\begin{pmatrix} \tau_{ss} \\ \tau_{sn} \end{pmatrix} = \left( 1 + \frac{Bi}{\|\tau\|} \right) \begin{pmatrix} \dot{\gamma}_{ss} \\ \dot{\gamma}_{sn} \end{pmatrix} \text{ when } \|\tau\| > Bi, \text{ and } \dot{\gamma} = 0 \text{ otherwise.}$$

In each of the boundary layer regions, $j = 1$ and 2, we define scaled coordinates and velocities by

$$(s, n) = (s, \epsilon_j \eta_j), \quad (u_s, u_n) = R \Omega (U_s, \epsilon_j U_n) \text{ for } j = 1, 2,$$

where $u_s$ is scaled by the velocity of the rotating yield surface, and $u_n$ is scaled accordingly from the conservation of mass, (4.17). Retaining only potentially leading order terms we
4.4. RESULTS AND KEY SCALINGS

Figure 4.7: Schematic of boundary layer geometry shortly after a new eddy has formed. The grey regions are unyielded fluid, and the central plug is in clockwise solid body rotation around the point $O$ with rotation rate $\Omega$.

From the equations (4.23) and (4.24), we find

$$\tau_{sn} = -Bi + \frac{R\Omega}{\epsilon_j} \frac{\partial U_s}{\partial \eta_j} + 2\epsilon_j^2 Bi \left( \frac{\partial U_s}{\partial s} \right)^2 \left( \frac{\partial U_s}{\partial \eta_j} \right)^{-2} + \ldots, \quad (4.23)$$

$$\tau_{ss} = -2\epsilon_j Bi \left( \frac{\partial U_s}{\partial s} \right) \left( \frac{\partial U_s}{\partial \eta_j} \right)^{-1} + \ldots, \quad (4.24)$$

where the sign of the first term on the right hand side of (4.23) is due to the clockwise rotation of the rotating plug and we note that $R$ and $Bi$ are $O(1)$ as $\Delta Bi \to 0$. To account for the curvature term in (4.18), we write the pressure in each region as

$$p = -\frac{2Bi}{R} s + \frac{R\Omega}{\epsilon_j} P(s, \eta_j) + \ldots. \quad (4.25)$$

With this substitution we find that, to leading order, (4.18) and (4.19) are given by

$$\frac{\partial P}{\partial s} = \frac{\partial^2 U_s}{\partial \eta_j^2} + 2\epsilon_j^2 Bi \frac{\partial}{\partial \eta_j} \left( \left( \frac{\partial U_s}{\partial s} \right)^2 \left( \frac{\partial U_s}{\partial \eta_j} \right)^{-2} \right) - 2\frac{Bi\epsilon_j^3}{R\Omega} \frac{\partial^3}{\partial s^2} \left( \left( \frac{\partial U_s}{\partial s} \right) \left( \frac{\partial U_s}{\partial \eta_j} \right)^{-1} \right) + \ldots, \quad (4.26)$$

$$\frac{\partial P}{\partial \eta_j} = 2\frac{Bi\epsilon_j^3}{R\Omega} \frac{\partial}{\partial \eta_j} \left( \left( \frac{\partial U_s}{\partial s} \right) \left( \frac{\partial U_s}{\partial \eta_j} \right)^{-1} \right) + \ldots. \quad (4.27)$$

There are now two possible regimes in which viscous terms enter the momentum balance. Assuming $\epsilon_j^3/\Omega \ll 1$ we have $\partial P/\partial \eta_j = 0$ to leading order and

$$\frac{dP}{ds} = \frac{\partial^2 U_s}{\partial \eta_j^2}. \quad (4.28)$$
CHAPTER 4. VISCOPLASTIC CORNER EDDIES

giving a quadratic profile for \( U_s \). This situation applies for \( j = 1 \), where one side of the boundary layer is bounded by a rigid wall, but impossible for \( j = 2 \) since the strain rate, \( \partial U_s / \partial \eta_2 \), must vanish at two distinct points, namely at both sides of the boundary layer, where it meets unyielded fluid. Hence, in the thinner region of the boundary layer we have \( \epsilon_1^3 / \Omega \ll 1 \), with the exact scaling undetermined until later, while in the wider region we have

\[
\epsilon_2^3 Bi / (R\Omega) = 1 \implies \epsilon_2 = O(\Omega^{1/3}),
\]

and a boundary layer solution governed by the partial differential equation derived by Oldroyd [99]

\[
\frac{\partial}{\partial \eta_2} \left( \frac{\partial U_s}{\partial \eta_2} + 2 \left( \frac{\partial U_s}{\partial s} \right)^2 \left( \frac{\partial U_s}{\partial \eta_2} \right)^{-2} \right) - 4 \frac{\partial}{\partial s} \left( \frac{\partial U_s}{\partial s} \left( \frac{\partial U_s}{\partial \eta_2} \right)^{-1} \right) = F(s),
\]

for some function of integration, \( F(s) \), which vanishes for the symmetrical similarity solution given later in (4.44)-(4.45). In both sections of the boundary layer \((j = 1, 2)\) we have boundary conditions

\[
U_s = 0 \text{ at } \eta_j = \eta_j^+ \text{ and } U_s = 1, \frac{\partial U_s}{\partial \eta_j} = 0 \text{ at } \eta_j = \eta_j^-,
\]

where \( \eta_j^+ \) and \( \eta_j^- \) are the limits of the boundary layer, and in the wider section of the boundary layer, where the layer is sandwiched between regions of unyielded fluid, we have the additional boundary condition

\[
\frac{\partial U_s}{\partial \eta_2} = 0 \text{ at } \eta_2 = \eta_2^+.
\]

In the thinner region of the boundary layer, we integrate (4.28) and apply the boundary conditions to find

\[
U_s = \frac{1}{2} \frac{dP}{ds} \left( (\eta_1^+ - \eta_1)^2 - 2 (\eta_1^+ - \eta_1^-) (\eta_1^+ - \eta_1^-) \right), \quad \frac{dP}{ds} = -\frac{2}{(\eta_1^+ - \eta_1^-)^2}.
\]

Conservation of mass imposes an additional constraint

\[
\frac{d}{ds} \int_{\eta_i^-}^{\eta_i^+} U_s d\eta = -\frac{d\eta_i^-}{ds},
\]

representing the fact that divergence of the flux must be accounted for by flow through the boundaries of the boundary layer. This gives \( \eta_i^- \) in terms of \( \eta_i^+ \) as

\[
\eta_i^- = -(2\eta_i^+ + W_0)
\]
where $W_0$ is a constant of integration and is $O(1)$. Since $\eta_1^+$ is given by the fixed position of the wall, it is a known function of $s$. In the vicinity of the point at which the semi-circle meets the rigid boundary, $s \approx s_0 = R(\pi/2 - \alpha)$, we have

$$
\eta_1^+ = \frac{(s - s_0)^2}{2\epsilon_1 R}, \quad \text{and} \quad \eta_1^+ - \eta_1^- = W_0 + \frac{3(s - s_0)^2}{2\epsilon_1 R}, \quad (4.36)
$$
to leading order. Thus we interpret $W_0$ as the width of the boundary layer at $s = s_0$, in boundary layer coordinates.

Since the pressure is chosen to vanish at $\theta = 0$ by anti-symmetry of the solution, the total pressure change along the boundary layers is $O(1)$, given by the pressure at the point where the boundary layer meets the adjacent eddy- and here the solution remains unchanged to leading order by small changes in Bingham number, $\Delta Bi$. The contribution to the pressure gradient along the boundary layer, $\partial p/\partial s$, due to the curvature of the boundary layer is $-2Bi/R = O(1)$. This is the leading order contribution in the wider sections of the boundary layer, and contributes a total pressure drop of $-\pi Bi$ over the length of the boundary layer. In general this does not match the pressure where the boundary layer meets the fully yielded adjacent eddy (for example see the second panel in figure 4.8) and thus we require an additional $O(1)$ pressure jump along the thinner section of the boundary layer. This is only possible if the dominant contribution to the pressure gradient, $\partial p/\partial s$, in the thinner section of the boundary layer is due to the second term in (4.25). Substituting for $\eta_1^+$ we find

$$
\frac{dP}{ds} = -\frac{2}{\left( W_0 + \frac{3(s - s_0)^2}{2\epsilon_1 R} \right)^2}, \quad (4.37)
$$
which is $O(1)$ over the region where $s - s_0 = O(\sqrt{\epsilon_1})$, and decays outside this region. Hence the total pressure drop over the thinner region is $O(\sqrt{\epsilon_1}R\Omega/\epsilon_1^2)$. In fact, we can integrate (4.37) analytically to obtain the leading order pressure drop over the thinner region as

$$
\frac{R\Omega}{\epsilon_1^2} \int_{-\infty}^{\infty} \frac{dP}{ds} ds = \sqrt{\frac{2}{3}} \frac{\pi R^{3/2}\Omega}{(\epsilon_1 W_0)^{3/2}}, \quad (4.38)
$$
In either case, we conclude

$$
\epsilon_1 = O(\Omega^{2/3}) = O(\epsilon_2^2). \quad (4.39)
$$

Finally, we consider the additional torque along the circular arc, above that provided by the yield stress, given by

$$
- \int_{0}^{\frac{\pi}{2}} R(\tau_{sn} + Bi) ds = G(Bi) - \frac{\pi R^2 Bi}{2}, \quad (4.40)
$$
CHAPTER 4. VISCOPLASTIC CORNER EDDIES

where \( G(Bi) \) again represents the torque acting on the upper half of the diameter of the semi-circle due to normal stresses in the yielded flow to the right of the plug (see (4.14)), and is substituted for \(- \int_0^{R \pi / 2} R \tau_{sn} ds\) by torque balance. Since this diameter lies primarily in the adjacent eddy, in which the solution varies smoothly with \( Bi \), we can use a Taylor series to write

\[
G(Bi) = G(Bi_c) + O(Bi_c - Bi) = \frac{\pi R^2 Bi_c}{2} + O(\Delta Bi),
\]

and thus

\[
- \int_0^{R \pi / 2} R(\tau_{sn} + Bi) ds = O(\Delta Bi).
\]

Substituting for \( \tau_{sn} \) from (4.23), and using (4.29), the contributions to this additional torque from region 2 is \( O(\Omega / \epsilon_2^2) = O(\Omega^2 / 3) \). In region 1, similarly to the pressure gradient, the viscous shear stress is dominated by a region of length \( O(\sqrt{\epsilon_1}) \) where \( \partial U_s / \partial \eta_1 = O(1) \). And hence, using (4.23) and (4.39), the additional torque from region 1 is \( O(\sqrt{\epsilon_1} \Omega / \epsilon_1) = O(\Omega^2 / 3) \), as for region 2. Thus \( \Delta Bi \sim \Omega^2 / 3 \), and we obtain the scalings

\[
\Omega \sim \Delta Bi^{3/2}, \quad \epsilon_2 \sim \Delta Bi^{1/2}, \quad \text{and} \quad \epsilon_1 \sim \Delta Bi.
\]

Figure 4.8 provides evidence for the validity of this boundary-layer theory, from a sequence of numerical simulations for \( \alpha = 20^\circ \). We measure \( \Omega \) as the rotation rate at the point on the wedge’s symmetry-axis at which the strain rate vanishes in the corresponding Moffatt [92] solution. This point is always inside the central rotating plug in the viscoplastic solutions. The left panel shows how \( \Omega \sim \Delta Bi^{3/2} \) when \( Bi \rightarrow Bi_c \) from below, and rises up to the rotation rate, \( \Omega_V \), from the viscous, Moffatt solution, as \( Bi \) decreases. The right panel shows how pressure varies along the boundary layer, for three values of the Bingham number deficit, \( \Delta Bi \), verifying that the pressure gradient approaches the constant \(-2Bi_c / R\) in the first, wider, part of the boundary layer, before becoming large in a short section of the layer, resulting in an \( O(1) \) overall pressure change, essentially independent of \( \Delta Bi \).

In region 2 we can solve (4.30) by means of a similarity solution detailed by Balmforth et al. [20] and find

\[
U_s = \frac{1}{2} + \frac{1}{4} \zeta \left( \zeta^2 - 3 \right), \quad \zeta = \frac{\eta_2 - \eta_2^m}{Y(s)} \tag{4.44}
\]

where \( \eta_2^m = (\eta_2^+ + \eta_2^-) / 2 \) is the mid-line of the boundary layer and \( Y(s) = (\eta_2^+ - \eta_2^-) / 2 \) is its half-width, given implicitly by

\[
Y_E^{3/2} \left( \tan^{-1} \sqrt{\frac{Y/Y_E}{1 - Y/Y_E}} - \sqrt{\frac{Y}{Y_E}} \left( 1 - \frac{Y}{Y_E} \right) \right) = \frac{\sqrt{3}}{2} (\tilde{s}_0 - s). \tag{4.45}
\]
4.4. RESULTS AND KEY SCALINGS

Figure 4.8: (left) Rotation rate $\Omega$ as a function of $Bi$ for $\alpha = 20^\circ$, measured in numerical simulations (stars) and the scaling relationship $\Omega^{(2/3)} \sim Bi_c - Bi$. The horizontal dashed line shows $\Omega_V^{(2/3)}$ where $\Omega_V$ is the rotation rate at the point where strain rate vanishes in the viscous solution. (right) Pressure, $p$, as a function of the coordinate along the boundary-layer, $s$, for three values of the Bingham number deficit, $\Delta Bi$, indicated in the legend. The black dotted line shows the constant gradient $-2Bi_c/R$ predicted in the thicker section of the boundary layer for small $\Delta Bi$, and the vertical grey dashed line marks the narrowest point of the boundary layer.

Here $s = \tilde{s}_0$ is the apparent origin of the similarity solution, at which location $Y(\tilde{s}_0) = 0$, and $Y_E = \left(\sqrt{3}\tilde{s}_0/\pi\right)^{2/3}$, is the maximum half-width of the boundary layer. Asymptotically, to leading order this section of the boundary layer must end at $s = s_0 = R(\pi/2 - \alpha)$ with vanishing width, and hence to leading order we must have $\tilde{s}_0 = s_0$. The complete leading order boundary-layer solution would then be obtained by fixing the constants $W_0$ and $\Omega_0$, where $\Omega \sim \Omega_0\Delta Bi^{3/2}$, by enforcing both the pressure drop over region 1 and the torque over the entire boundary layer. This calculation requires additional detailed analysis of the fully yielded flow within the eddy adjacent to the boundary layers and is not attempted here. Instead, the boundary-layer structure has revealed the asymptotic scalings of $\epsilon_1$, $\epsilon_2$ and $\Omega$ upon the Bingham number deficit, $\Delta Bi$.

The predictions of boundary-layer shape can also be compared with numerics by directly measuring the empirical rotation rate, $\Omega$, radius of the rotating plug, $R$, and widths of the boundary layer at $s = s_0$ and $s = 0$, $\epsilon_1 W_0$ and $2\epsilon_2 Y_E$ respectively, for a given $\alpha$ and $Bi$. Equations (4.36) and (4.45) can then be used to predict the boundary layer width along the rest of the boundary layer. An example of such a comparison, for $\alpha = 20^\circ$ and $Bi = 0.02$, is given in figure 4.9 showing good agreement within the limited regions each approximation applies.
Figure 4.9: (left) Boundary layer width as a function of streamwise coordinate, $s$, from the numerical solution for $\alpha = 20^\circ$ and $\hat{B}i = 0.02$ (black stars) and the asymptotic solutions (4.36) (cyan dotted) and (4.45) (red dashed). (right) A contour plot of log strain rate from the same numerical simulation (black regions represent unyielded plugs) and the predicted boundaries of the shear-layer from these asymptotic solutions (colours correspond to left panel).

4.5 Comparison with flow past triangular inclusion

In the numerical solutions discussed so far, the velocity from an appropriate Moffatt [92] solution has been imposed as a boundary condition far from the vertex of the wedge on the right-hand side of the domain, with the intention of studying the idealised problem in which Moffatt eddies occupy an infinite wedge. It is also of interest to verify that these solutions are relevant to situations in which the eddies are driven by a more readily realised flow configuration. We thus simulate numerically a lid-driven problem in which a rigid wall translates past a triangular inclusion (see figure 4.10) under no imposed pressure gradient. This problem can be non-dimensionalised by the half-height (in the orientation shown) of the triangular inclusion, $L$, and the velocity of the translating wall, $U$. If we non-dimensionalise stresses by $\mu U/L$, this leaves the non-dimensional yield stress, $\hat{\tau}_c L/\mu U$, the wedge half-angle, $\alpha$, and the dimensionless distance of the translating wall from the top of the triangular inclusion, $\epsilon$, as free parameters. We have denoted the Bingham number for this flow problem by $\hat{B}i$ to distinguish it from the Bingham number used in the idealised problem, $B_i$, given by (4.6). The dimensionless inflow length, $L_{in}/L$, is taken sufficiently large that the solution in the wedge is independent of it, which we verify by doubling $L_{in}/L$ and comparing the solutions. We seek an anti-symmetric solution, and so need only consider the inflow and top half of the wedge, as indicated in figure 4.10. For the purposes of demonstrating the applicability of the idealised solutions
4.5. COMPARISON WITH FLOW PAST TRIANGULAR INCLUSION

Figure 4.10: Domain (grey), streamlines (red), and unyielded zones (black) for a lid-driven disturbance in a wedge. The motion is driven by a translating boundary, moving with velocity $U$. In this example $\alpha = 20^\circ$ and $\hat{Bi} = 0.006$.

to more general flows, we choose not to explore the parameter space fully and instead set $\alpha = 20^\circ$ and $\varepsilon = 0.1$, while varying the Bingham number, $\hat{Bi}$. These problems are solved using the same numerical methods as described in §4.3. The boundary conditions imposed are no-slip on the three rigid walls, anti-symmetry and $p = 0$ on the $x$-axis, and a linear vertical flow profile at the inflow boundary.

For this particular flow configuration, we find that eddies analogous to those described by Moffatt [92] do indeed form in the wedge. As discussed by Roustaei and Frigaard [116], very small Bingham numbers are required to observe multiple eddies. We find that we require $\hat{Bi} \approx O(10^{-3}), O(10^{-5}), O(10^{-7})$, to observe two, three and four eddies, respectively. Note that this is consistent with the Moffatt [92] solution for $\alpha = 20^\circ$, for which the strain rate decreases by a factor of $169.6 = O(10^2)$ between consecutive eddies.

Using a purely viscous, $\hat{Bi} = 0$, solution to the same problem, we can measure the $x$-coordinate and velocity at the dividing streamline between the first and second eddy, and rescale in the manner described in §4.2. In particular we find that, to two significant figures, the dividing streamline passes through $x = 1.0$, with velocity $\|u\| = 2.0 \times 10^{-3}$. Thus, to compare results from the lid-driven problem and the idealised problem of §§4.2-4.4, the Bingham number for the lid-driven problem should be approximately a factor of $1.0 / (2.0 \times 10^{-3}) = 500$ smaller than for the idealised problem. Figure 4.11 shows comparisons between the lid-driven and idealised problem at two pairs of roughly equivalent Bingham numbers, demonstrating that the unyielded zones correspond very closely. This provides evidence that the idealised problem of §§4.2-4.4 is indeed relevant to more specific flow configurations, and hence we may use the estimates of the occurrence
CHAPTER 4. VISCOPLASTIC CORNER EDDIES

Figure 4.11: Comparison of unyielded regions for the idealised problem (top row) and the lid-driven problem (bottom row) with $\alpha = 20^\circ$. The black regions represent unyielded fluid while grey regions represent yielded fluid. The Bingham numbers were chosen to be equivalent after scaling for the velocity and length scales of the first eddy in the lid-driven problem. Only a portion of the lid-driven domain is shown.

and length scales of unyielded regions and eddies, developed in §4.4, in more general flow scenarios.

4.6 Conclusion

In this chapter, we have studied corner eddies in viscoplastic fluids. Such corner eddies occur in a range of flow configurations including flows through abrupt contractions and past triangular inclusions, and are of particular relevance to food processing applications in which it is crucial to avoid stagnant unyielded regions of fluid. The idealised problem consists of Bingham fluid occupying an infinite wedge of half-angle $\alpha$, driven by the corresponding dominant viscous eddy [92] at large distances from the vertex. In the presence of a yield stress, the fluid remains static and unyielded in the corner of the wedge, but forms eddies away from this stagnant region. Further unyielded regions exist on the boundary in between two eddies, where the fluid is static, and at the centre of the eddies, where it rotates in solid-body motion. The size of these unyielded regions is of concern to applications in which one aims to stir or dislodge viscoplastic material in a sharp corner, where unyielded regions correspond to undisturbed fluid. Direct numerical simulations were carried out at five values of $\alpha$ and a large range of Bingham numbers, $Bi$, to quantify the extent of the inner stagnant region and the critical Bingham numbers at which new
4.A. VISCOPLASTIC FLOW IN CORNERS WITH $\alpha > \alpha_c$

Eddies form, decreasing the extent of this stagnant fluid. This occurs when the torque exerted on a section of the unyielded fluid by the stresses in the adjacent eddy exceeds the torque that can be provided by the yield stress in the unyielded fluid, providing a heuristic method to calculate approximations for these critical Bingham numbers using only the well established Moffatt [92] solution. The results of this heuristic approximation are compared with the results of numerical solutions, showing good agreement. We further study the behaviour of the smallest eddy at Bingham numbers just below the critical value, $Bi = Bi_c - \Delta Bi$, for which a boundary-layer method can be employed in a thin layer between the stagnant and rotating plugs. The dimensionless rotation rate scales like $\Delta Bi^{3/2}$, and the dimensionless width of the thinnest region of the boundary layer, which occurs between the rotating plug and the boundary, scales like $\Delta Bi$. We demonstrate that the results and insights from this idealised problem are relevant to more readily realised flows in which eddies form, by comparing solutions for $\alpha = 20^\circ$ with a problem in which eddies are driven by a translating lid over the top of the wedge. We also explore the $\alpha \to 0$ limit, demonstrating how this can be used to predict the number and dimension of viscoplastic eddies forming between parallel plates (Appendix 4.C).

Future work could include calculating the dimensions of the other stagnant regions, located on the rigid boundary at the stagnation points between consecutive eddies (see figure 4.3 and §6.6.1), and exploring the impact of non-negligible inertial forces, slip and shear thinning on the occurrence and character of viscoplastic corner eddies. Regarding the last of these, Meyer and Creyts [90] have provided the solution for Moffatt eddies of a shear-thinning power-law fluid, and applied their results to the formation of such eddies in the ice of subglacial valleys. As such, this solution could be taken as the far-field condition in determining the solution for a Herschel-Bulkley fluid, following the methodology of this chapter; thereby combining the effects of shear thinning and a non-zero yield stress.

4.A Viscoplastic flow in corners with $\alpha > \alpha_c$

As discussed in §4.2, when $\alpha > \alpha_c \approx 73^\circ$, the eigenvalue, $\lambda$, becomes real, and the viscous Newtonian solution no longer exhibits eddies. In this case the dominant antisymmetric solution instead exhibits flow towards the vertex in one half of the domain (e.g. $\theta > 0$) and away from the vertex in the other. If $\alpha < 90^\circ$, the geometry still corresponds to flow within a concave corner and we have $\lambda > 2$. Thus, the strain rate scales like $r^{\lambda-2}$ and for a Bingham fluid we would anticipate a single stagnant corner plug in the vicinity of the vertex. Where this corner is embedded in some global flow and the global dimensionless
yield stress is small, $Bi \ll 1$, the plug would occur within the radius at which the strain rate and yield stress are of the same magnitude, and thus would have a dimension that scales roughly as $r \sim Bi^{1/(\lambda-2)}$. A schematic of the expected viscoplastic flow pattern in this case is shown in figure 4.12.

When $\alpha = 90^\circ$ the geometry corresponds to a straight wall, the eigenvalue is $\lambda = 2$, and the solution reduces to parallel flow past a wall (with Cartesian streamfunction $\psi = Ay^2$). Since the strain rate is then independent of position, a solution with the exact same streamfunction exists for a Bingham fluid which is yielded throughout the domain.

Finally, when $\alpha > 90^\circ$, the problem now represents flow around the outside of a concave corner. In this case, the Newtonian solution has $\lambda < 2$ and we can no longer define the viscoplastic problem in the same way as before, since the strain rate decays at large distances from the vertex and it would not be valid to impose the Newtonian solution at $r \rightarrow \infty$. The solution thus depends on the exact choice of boundary condition at infinity, for example it may be possible to construct a plastically dominated solution at large distances from the vertex, which is matched to the Newtonian solution as $r \rightarrow 0$.

### 4.B Torque induced yielding of fluid in a wedge

For the purposes of this appendix we take Cartesian coordinates $(x, y) = (r \cos \theta, r \sin \theta)$. We consider a region of the static unyielded corner plug, symmetrical in the $\theta = 0$ axis, with upper-half given by $ABC$ in figure 4.13 (and bottom half by symmetry). This region will yield along its boundary when the torque exerted on $BC$ exceeds the torque that can be exerted by unyielded fluid on $AB$. Since no net force acts on the region, we can take this torque balance around any origin, $O$, and, since the torque on any closed loop also vanishes, we can consider the torque exerted on the straight line $BC'$ rather than
4.B. TORQUE INDUCED YIELDING OF FLUID IN A WEDGE

The maximum magnitude of the deviatoric stress in the unyielded fluid is \( B_i \), thus the fluid yields along \( AB \) when

\[
\left| \int_{C'}^B (-pr \times \hat{a} + r \times \tau \cdot \hat{a}) \, dy \right| \geq \left| \int_{A}^{B} -pr \times n \pm Bi r \times \hat{\tau} \cdot n \, ds \right|,
\]  

(4.46)

where \( \hat{\tau} \) is a unit tensor oriented with the deviatoric stress. Note that the pressure term on the right-hand side can be made arbitrarily large, unless \( r \times n = 0 \) everywhere for some choice of origin, \( O \), in which case the maximum torque that can be supplied by the unyielded fluid occurs when the deviatoric stress is purely shear-stress. This is simply a statement of the physically intuitive result that such a region can only yield to rotation if its boundary is a circular arc. Now taking \( O \) as the centre of this circular arc, the fluid will yield along \( AB \) when

\[
0 \leq \frac{\left| \int_{C'}^B (-pr \times \hat{a} + r \times \tau \cdot \hat{a}) \, dy \right|}{\int_{A}^{B} R \, ds} - Bi \equiv S(AB, Bi) .
\]

(4.47)

In principle then, starting from an unyielded corner plug, we can decrease \( Bi \), evaluating

\[
\mathcal{T}(Bi) = \max_{AB} S(AB, Bi) ,
\]

(4.48)

where the maximum is taken over all possible circular arcs, \( AB \), fitting inside the unyielded plug, with the centre of the circle lying on the symmetry axis of the wedge (see for example the left panel of figure 4.14). Since the plug is initially unyielded, \( \mathcal{T} \) is initially negative. If \( \mathcal{T}(Bi) \) becomes 0, then a new eddy yields along the circular arc, \( AB \), for which \( S(AB, Bi) = 0 \), and the Bingham number at which this occurs is, by definition, \( Bi_c \). The space of all possible circular arcs, \( AB \), can be parameterised by the intersection with the yield surface and the centre of rotation via \( Y \) and \( \delta \) as shown in figure 4.14. We
Figure 4.14: Contours of the components of stress, $\sigma_{xx}$ and $\sigma_{xy}$, in the eddy adjacent to the static corner plug from the numerical simulation for $\alpha = 20^\circ$ and $Bi = 0.022$. The red dashed line shows a streamline in the eddy, while the white circular arc shows an example of a potential yield surface in the static plug and indicates the parametrisation of these arcs via $Y$ and $\delta$.

can then write the first term of $S$ (which is the term that varies with $AB$) as

$$\frac{\int_0^Y y \sigma_{xx} \, dy - \delta Y \int_0^Y \sigma_{xy} \, dy}{(\pi/2 + \arctan \delta) (1 + \delta^2) Y^2} = \frac{N(Y) + \delta S(Y)}{(\pi/2 + \arctan \delta) (1 + \delta^2)},$$

where

$$N(Y) = \int_0^Y \dot{y} \sigma_{xx} (x_Y(Y), Y \dot{y}) \, d\dot{y}, \quad S(Y) = \int_0^1 -\sigma_{xy} (x_Y(Y), Y \dot{y}) \, d\dot{y},$$

by making the substitution $y = Y \dot{y}$ in the integrals in (4.49), and $x = x_Y(y)$ describes the yield surface of the static corner plug. Contours of $\sigma_{xx}$ and $\sigma_{xy}$ are given in figure 4.14 for one example of $\alpha = 20^\circ$ and $Bi = 0.022$ demonstrating that $S$ does not depend strongly on $Y$ but $N$ increases with $Y$. Thus we anticipate $\partial S/\partial Y > 0$ and hence that the maximum of $S$ is attained on the boundary of the region

$$\{Y, \delta : Y \sqrt{1 + \delta^2} \leq (x_Y(Y) - \delta Y) \sin \alpha\},$$

which represents the condition that the radius of the circular arc is at most the perpendicular distance from the centre to the boundary of the wedge. The conclusion that we expect the maximum of $S$ to be attained on the boundary of the region (4.51) corresponds to the conclusion that the circular arc along which the static plug yields at $Bi = Bi_c$ meets the boundary of the wedge tangentially, as is observed in the numerical simulations (e.g. see figure 4.5).

It is not possible to proceed analytically to determine exactly the circular arc $AB$ that maximises $S$ for general $\alpha$ and $Bi$, as neither the stress field in the yielded region, nor the plug geometry, are known analytically. Thus, for the purposes of a simple approximation,
4.C. FLOW WITHIN A PARALLEL-SIDED CHANNEL: $\alpha \to 0$ LIMIT

rather than attempting to use this approach to calculate critical Bingham numbers, we instead consider the yield surface along which the static plug yields at $Bi = Bi_c$ to be semi-circular, with centre of rotation set by the inner-most horizontal extent of the adjacent Moffatt [92] eddy. This approximation is supported by the numerical simulations and is detailed in §4.4.1.

4.C Flow within a parallel-sided channel: $\alpha \to 0$ limit

As demonstrated by Moffatt [92], the limit $\alpha \to 0$ with $r\alpha = O(1)$. Specifically if we make the coordinate transformation $r = 1/\alpha + \tilde{x}$ and $\theta = \alpha \tilde{y}$, the limit $\alpha \to 0$ represents eddy flow in a gap of width 2 between parallel boundaries. With the additional substitution $\lambda = k/\alpha$ we find the Cartesian streamfunction given by

$$
\psi_c = \hat{A} e^{k \hat{x}} (\sin k \tilde{y} - \tilde{y} \cos k \sin k \tilde{y}),
$$

(4.52)

where $k = k_r + ik_i \approx 2.11 + 1.13i$ and the real part of expressions is assumed for all physical quantities. If we choose to fix a dividing streamline with unit velocity through the origin, this sets $\hat{A} = -i/(k_i \sin k)$. Following the heuristic derivation of $Bi_c$ given in §4.4.1, we consider a semi-circle of unit radius with centre $(\tilde{x}_0, 0)$, where $\tilde{x}_0 = \exp(-\pi/k_i)$. The normal stress acting on the radius of the semi-circle in the viscous solution (4.52) is given by

$$
-p + \tau_{\tilde{x}\tilde{y}} = 2 \hat{A} e^{-k \tilde{x}_0} (k \sin k \tilde{y} + k \tilde{y} \cos k \sin k \tilde{y} + 2 \cos k \sin k \tilde{y}),
$$

(4.53)

and hence the torque is given by

$$
G = 2 \hat{A} e^{-k \tilde{x}_0} \int_0^1 k \tilde{y} \sin k \tilde{y} \sin k \tilde{y} + k \tilde{y}^2 \cos k \cos k \tilde{y} + 2 \tilde{y} \cos k \sin k \tilde{y} \tilde{y} d\tilde{y} = -2 \hat{A} e^{-k \pi/k_i} \sin^2 k = \frac{2i}{k_i} e^{-k \pi/k_i} \sin k.
$$

(4.54)

(4.55)

The approximation to the critical Bingham number is then given by

$$
Bi_c \approx 2 |G| / \pi = \left| \Re \left( \frac{4i}{k_i \pi} e^{-k \pi/k_i} \sin k \right) \right| = 0.0022 \text{ (to 2 significant figures)},
$$

(4.56)

as is found via the numerical limit for $\alpha Bi_c$ as $\alpha \to 0$ (see §4.4.1).

This theory allows us to make some general conclusions about flow configurations in which eddies may form between parallel walls, such as flow over the top of a rectangular inclusion, or the flow configuration described by Moffatt [92], in which fluid
between parallel plates is disturbed by a rotating cylinder. In general there will be
a region close to the disturbance where the solution depends strongly on the specific
form of the driving, but if the inclusion is sufficiently long, and the yield stress suffi-
ciently low, then the theory above will become relevant for predicting the number of
eddies and extent of disturbed fluid in the cavity. We consider a non-dimensionalisation
in which the distance between the parallel plates is scaled to \( \frac{2}{k} \), as above, and note
that the dimension of each Moffatt eddy in the direction parallel to the plates is
\( \frac{\pi}{k} \approx 2.78 \). Neglecting the region close to the disturbance, for a rectangular cavity
of length \( L \) we would therefore anticipate at most approximately \( \frac{L}{2.78} \) eddies oc-
cupying the full width of the cavity, before end-effects become significant, including
potential eddies in the corners of the rectangle. For a yield-stress fluid, the number
of these eddies actually present will depend on the Bingham number, \( \tilde{Bi} \), much in the
same way as described for the wedge in §4.4. An initial viscous (\( \tilde{Bi} = 0 \)) calculation
can be carried out to determine the velocity, \( \tilde{U} \), at the first dividing streamline, and
then the set of critical Bingham numbers at which new eddies form, is given approxi-
mately by
\[
\{ \tilde{Bi}_c = 0.0022 \times \tilde{U} \times \exp(Nk_i,\pi/k_i) = 0.0022 \times 353^N \times \tilde{U} : N \in \mathbb{Z}, N \leq N_0 \}.
\]
Here the upper limit, \( N_0 \), corresponds to the Moffatt eddy that occurs closest to the
disturbance for the particular flow configuration, and as \( N \) ranges from \( N_0 \) to \( -\infty \) we
obtain the full infinite sequence of eddies in the cavity (which for a finite rectangular
cavity would also be truncated due to end effects as discussed above).

Figure 4.15 shows the result of numerical simulations for the example of fluid disturbed
by a rotating cylinder. In these simulations the boundary of the cylinder has unit velocity.
\( \tilde{U} \) was found to be 0.14 to 2 significant figures and \( N_0 \) was found to be 1. Thus, with
\( N = 1, 0 \), we find the first two critical Bingham numbers as approximately 0.1 and 0.0003.
The numerical simulations show that new eddies, with large, roughly semi-circular rotating
plugs, have formed at these critical Bingham numbers, and that the fully developed
eddy in the third panel has approximately the dimensions of a viscous Moffatt eddy [92]
between parallel plates (i.e. of length approximately 2.78 times the half-height), as
anticipated.
Figure 4.15: Viscoplastic eddies between parallel plates, driven by a rotating cylinder located at the origin (outline in blue). Plots show strain rate on a logarithmic scale (gray-scale) and streamlines (red dashed lines) for \( \tilde{Bi} = 0 \) (top), \( \tilde{Bi} = 0.001 \) (middle), and \( \tilde{Bi} = 0.0003 \) (bottom).
Chapter 5

Compression of a viscoplastic fluid between hinged plates

Authorship: The material in this chapter is the result of original research by J. J. Taylor-West and A. J. Hogg. It has been published in Taylor-West and Hogg, 2023 [133] which has been modified slightly for inclusion in this thesis.

5.1 Introduction

In this chapter, we investigate the incompressible flow of a Herschel-Bulkley viscoplastic fluid between two rigid, semi-infinite plates, hinged at the origin and rotating towards one another with angular velocity, \( \Omega \) (see figure 5.1), thus extending the classical problem of viscous Newtonian fluid flow in this configuration (see, for example Moffatt [92]). The previous chapters (§§3,4) on viscoplastic fluids in converging and recirculating corner flows have demonstrated how the existence of a yield stress changes the structure of the Newtonian solutions significantly, leading to the occurrence of rigid unyielded regions of fluid, or “plugs”, and the development of viscoplastic boundary layers when the dimensionless yield stress is large. In both of these previous studies, the magnitude of the strain rate varies with the distance from the vertex, resulting in viscously and yield-stress dominated behaviour in different regions of the wedge. In contrast, it will be shown that the flow driven by hinged plates is self-similar, with the dimensionless strain rate and deviatoric stresses varying only with the polar angle. This flow configuration has...
CHAPTER 5. COMPRESSION OF A VISCOPLASTIC FLUID BETWEEN HINGED PLATES

application to coating [46], lubrication (for example in the biomechanics of synovial joints, Hou et al. [73]) and extrusion flows of viscoplastic fluid [4]. In sensory evaluation of foods, Chen [38] suggests that squeeze flow in a wedge more accurately models the flow between the tongue and roof of mouth than the flow between parallel plates, which has been used as a model to predict oral sensory response (for example Demartine and Cussler [51], Elejalde and Kokini [57]). In addition, albeit with different boundary conditions, the flow in a closing wedge could be used to understand the local flow near a moving contact line in a drop of yield-stress fluid (see for example Jalaal et al. [76]). Beyond its direct relevance to applications, the flow configuration under consideration in this study also offers a rare example of a quasi-analytical solution for non-Newtonian, non-parallel flow. As such it is a useful problem for bench-marking computational codes and for determining the validity of constitutive models beyond simple-shear flows in which they are typically defined and experimentally determined.

Figure 5.1: Schematic of problem geometry. Only half of the geometry is shown, with the other half given by symmetry in $\theta = 0$. The shaded region indicates unyielded material.

The flow between rotating hinged plates has been studied for a number of different non-Newtonian constitutive laws. Phan-Thien [103] studied the case of a viscoelastic fluid, showing that exact similarity solutions exist for a general viscoelastic constitutive law, and analysing in detail the time evolution for the specific examples of the Oldroyd-B and Phan-Thien and Tanner (PTT) models, with a prescribed exponential closing rate of the hinged plates. They showed that the velocity fields do not deviate significantly from the Newtonian solution, but that viscoelasticity can have a significant impact on the evolution of the stresses in the wedge, with the Oldroyd-B model, in particular, predicting unbounded extensional stresses above a critical Weissenberg number (the
5.1. INTRODUCTION

ratio of typical elastic and viscous stresses), while stresses in the PTT model remain bounded but can become oscillatory. Phan-Thien and Zheng [104] further explored the existence of this critical Weissenberg number, by focussing on the steady-state solution at a given hinge angle (of $\pi/4$) and showing that, for an Oldroyd-B fluid, the critical Weissenberg number corresponds to a limiting point above which the steady-state solution ceases to exist. Chen [38] studied the problem for a power-law fluid, using an assumed approximate form for the radial velocity field in the cases of slip and no-slip at the walls, with the aim of providing more easily calculated solutions than the exact similarity solutions derived by Phan-Thien [103]. Wilson [144] investigated the flow of a biviscosity fluid in a closing wedge of half angle less than $\pi/4$. Under this regularised rheological model (see §2.4.1), the fluid is assumed to be Newtonian with relatively high viscosity up to an imposed transitional shear stress (or equivalently a transitional strain rate) and then exhibits a Bingham-like constitutive law for high shear stresses and strain rates; by construction the constitutive law is continuous. In the limit where the ratio of the viscosities above and below the transitional strain rate vanishes, the Bingham law is retrieved. Often with this model, a ‘yield surface’ is defined as a location at which the flow attains the transitional strain rate and changes its rheological model from Newtonian to Bingham-like. Evidently this surface does not demark the boundary of an unyielded rigid plastic region, since deformation is still permitted, albeit with a potentially high viscosity. Wilson [144] determined the existence of the ‘yield surface’ and its dependence upon the dimensionless transitional shear stress and the ratio of the viscosities. They showed that the material close to the symmetry line of the wedge could be ‘unyielded’ (i.e. its strain rate falls below the transitional value), but that this region vanished when the viscosity ratio became sufficiently small for any fixed dimensionless transitional shear stress, in which case the entire fluid was ‘yielded’. As we demonstrate below (see §5.3 and §5.4), for wedge angles less than $\pi/4$, the fluid is indeed yielded throughout, and is plastically dominated and only weakly yielded when the dimensionless yield stress is large. It is also possible to explore the dynamical behaviour under other regularised rheologies within the wedge. For example, Al Khatib [5] considered the problem for a regularised version of the Herschel-Bulkley constitutive law, deriving the governing similarity equations for the time evolution of the flow under a prescribed exponential closing rate, utilising the Papanastasiou regularisation (see §2.4.1). They found that the radial velocity distribution was very close to the Newtonian solution for the range of parameters explored, and showed how the pressure load on the plates varied with time and depended on the shear thinning and yield stress of the fluid. They also considered the
existence of unyielded regions in the wedge and found that no such region exists, however
their analysis is for just a single choice of the hinge angle and constitutive parameters
and, moreover, the regularisation of the constitutive law precludes the occurrence of any
true plugs (as for the bi-viscous model).

Compression between rotating hinged plates has also been widely studied for a plastic
material, under a range of different constitutive laws [7, 6, 10, 8]. A distinguished feature
of the rigid plastic problem is the existence of unyielded regions in rigid body rotation
adjacent to the rotating plates, for sufficiently large wedge angles. Of particular relevance
to the current chapter are the studies for a Bingham viscoplastic [6], and for a viscoplastic
with a saturation stress which the magnitude of the deviatoric stress approaches as the
strain rate tends to infinity [8]. In the former, Alexandrov and Jeng [6] show that the
deviatoric stresses are functions of polar angle only, and hence derive governing ordinary
differential equations (ODEs) for the stress and velocity fields in the domain. Their
results do not exhibit plug formation or boundary layers in part due to the focus on
relatively small yield stresses and angles between the plates. In the latter, Alexandrov
and Miszuris [8] show that a rigid zone can occur for wedge angles above some critical
value but, in the absence of a specific constitutive law, do not calculate this angle or
evaluate how it depends on the non-dimensional yield stress. The existence of typical
viscoplastic boundary layers in this case is precluded by the saturation stress which
results in plastic behaviour when the shear rate is large. Herein, we revisit the solution
of Alexandrov and Jeng [6] for the case of a Bingham fluid, which we generalise to the
Herschel-Bulkley constitutive law, demonstrating that the self-similar solution does in
fact include the existence of unyielded regions for sufficiently large angles between
the plates, and elucidating the boundary-layer structure that emerges in the regime of large
non-dimensional yield stress.

We assume the constitutive law of a Herschel-Bulkley fluid, relating the deviatoric
stress tensor, $\tau$, to the strain-rate tensor, $\dot{\gamma} = \nabla u + \nabla u^T$, via

$$
\tau = \left(K\dot{\gamma}^{N-1} + \frac{\tau_Y}{\dot{\gamma}}\right)\dot{\gamma} \quad \text{when } \tau > \tau_Y, \quad \dot{\gamma} = 0 \text{ otherwise},
$$

where $K$ is the consistency, $N$ is the flow index, $\tau_Y$ is the yield stress and $\tau \equiv (\tau_{ij}\tau_{ij}/2)^{1/2}$
and $\dot{\gamma} \equiv (\dot{\gamma}_{ij}\dot{\gamma}_{ij}/2)^{1/2}$ are the second invariants of the deviatoric stress and strain-rate
tensors, respectively. This constitutive law reduces to the Bingham model for $N = 1$ and
$K = \mu$, the viscosity. We adopt polar coordinates, $(r, \theta)$, centered on the hinge and with
$\theta$ measured from the line of symmetry between the two plates (see figure 5.1). We denote
the components of the velocity as $u = (u, v)$. Assuming $\rho \Omega^2 r^2 / K \ll 1$ (with density,
\( \rho \) in the region of interest, we can neglect inertia and search for a quasi-static solution. In this case, the system of equations is given by

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{v} \frac{\partial v}{\partial \theta} = 0, \tag{5.2}
\]

\[
\frac{\partial p}{\partial r} = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{2}{r} \tau_{rr}, \tag{5.3}
\]

\[
\frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{\partial \tau_{r\theta}}{\partial r} - \frac{1}{r} \frac{\partial \tau_{rr}}{\partial \theta} + \frac{2}{r} \tau_{r\theta}, \tag{5.4}
\]

representing incompressibility, and the balance of momentum in the radial and azimuthal directions, respectively. The boundary conditions arise from symmetry at \( \theta = 0 \) and no-slip at the rigid wall, and are given by

\[
v = \frac{\partial u}{\partial \theta} = 0 \text{ at } \theta = 0, \text{ and } (u, v) = (0, -\Omega r) \text{ at } \theta = \alpha, \tag{5.5}
\]

when the rigid plates are at \( \theta = \pm \alpha \).

We note that, in the absence of a length scale in the problem, the only velocity scale is \( \Omega r \) and so we write

\[
(u, v) = \frac{\Omega r}{2} (f'(\theta), -2f(\theta)), \tag{5.6}
\]

where \( f(\theta) \) is a function to be determined, and then incompressibility is automatically satisfied; the streamfunction is therefore given by \( \Psi = \Omega r^2 f(\theta)/2. \) We further scale strain rates by \( \Omega/2, \) and pressure and stresses by the viscous stress scale \( K(\Omega/2)^N. \) Having done so, the governing equations for the dimensionless variables are unchanged but the constitutive law becomes

\[
\tau = \left( \frac{\dot{\gamma}^{N-1}}{\dot{\gamma}} + \frac{Bi}{\dot{\gamma}} \right) \dot{\gamma} \quad \text{when } \tau > Bi, \quad \dot{\gamma} = 0 \text{ otherwise,} \tag{5.7}
\]

where the dimensionless parameter, \( Bi = 2^N \tau_Y / (K\Omega^N), \) is the Bingham number, representing the ratio of the yield stress to a typical viscous stress. Further, the boundary conditions become \( f = f'' = 0 \) at \( \theta = 0, \) and \( f = 1, f' = 0 \) at \( \theta = \alpha, \) representing assumed symmetry at \( \theta = 0 \) and no slip at \( \theta = \alpha. \)

As shown by Alexandrov and Jeng \[6\], under the self-similar ansatz (5.6), the governing equations reduce to ODEs. By careful analysis of these governing differential equations we will show that a rigid zone adjacent to the boundary, in which the fluid is in solid body rotation, does exist for angles above a critical angle, \( \alpha \geq \alpha_c \) (with \( \alpha_c \geq \pi/4 \)), and demonstrate how this critical angle depends on the Bingham number, in particular reducing asymptotically to \( \pi/4 \) in the plastic limit \( Bi \to \infty. \) We will also show that
viscoplastic boundary layers occur when $Bi \gg 1$, demonstrating the dependence of the width of these layers on the Bingham number.

The theory of viscoplastic boundary layers, in which shear becomes concentrated in a thin layer when the yield stress is large, is detailed in §2.2. For the current problem we will show that boundary layers of this kind form at the rigid boundaries when the Bingham number is large and the wedge half-angle is below $\pi/4$ ($\alpha_c - \alpha = O(1)$). Conversely, for $\alpha$ beyond this regime the flow can satisfy the velocity boundary conditions without requiring a region of high strain rate, and so the strain rates remain $O(1)$, and the fluid remains at the yield stress to leading order throughout the wedge. In accord with previous studies, the dimensionless width of the boundary layers scales with the Bingham number via $Bi^{-1/(N+1)}$, reducing to the anticipated $Bi^{-1/2}$ scaling for Bingham fluids. We note that Wilson’s [1993] analysis of the biviscosity model in wedges with $\alpha < \pi/4$ in the regime of relatively high transitional shear stress also features boundary layers attached to the wedge boundaries. They scale in accord with the Bingham case of $N = 1$ with the dimensionless width being proportional to $Bi^{-1/2}$. It will be shown in §5.4.1 that our approach circumvents some of the difficulties of Wilson’s [1993] analysis by a different choice of independent variable (see §5.2) and explicitly matches between a plastically-dominated region in the bulk and viscously-deforming region adjacent to the wedge boundaries. This allows extension to both Herschel-Bulkley rheology and to wider wedge angles ($\alpha > \pi/4$) for which rigid plug regions exist adjacent to the wedge boundaries.

We first derive the similarity equations, extending the methodology of Alexandrov and Jeng [6] in §5.2, then detail the numerical integration of these ODEs in §5.3 and set out the boundary layer analysis in §5.4. In §5.5 we carry out full numerical simulations of the problem, to show how the similarity solution is embedded in the full simulations, before briefly concluding in §5.6. There is also one appendix in which we detail how the results reduce to the Newtonian solution when $Bi < 1$ with $N = 1$, and determine the asymptotic dependence of a thin unyielded region near the plates when $\alpha = \pi/2$ in this regime.

5.2 Similarity equations

Given the assumed form of the velocity (5.6), the strain-rate components and magnitude of the strain rate are given in dimensionless form by

$$
\dot{\gamma}_{rr} = 2f'(\theta), \quad \dot{\gamma}_{r\theta} = f''(\theta), \quad \dot{\gamma} = \sqrt{f''^2 + 4f'^2},
$$

(5.8)
5.2. SIMILARITY EQUATIONS

where a prime denotes differentiation with respect to $\theta$. Importantly these are independent of radial distance from the vertex and this underpins the solution that we develop in what follows. An immediate consequence is that the components of the stress tensor are also functions of only the polar angle and we can write

$$
\begin{pmatrix}
\tau_{rr}(\theta) \\
\tau_{r\theta}(\theta)
\end{pmatrix}
= k(\theta) \begin{pmatrix}
\cos 2\psi \\
-\sin 2\psi
\end{pmatrix},
$$

(5.9)

where $\psi = \psi(\theta)$ is a variable representing the orientation of the deviatoric stress tensor and the magnitude of the deviatoric stress, $k(\theta)$, is given by

$$
k(\theta) = \left( f''^2 + 4f'^2 \right)^{N/2} + Bi,
$$

(5.10)

wherever the fluid is yielded. By symmetry, $\psi = 0$ at $\theta = 0$ and, since $u$ vanishes on the wall, $\tau_{rr} = 0$ and $\psi = \pi/4$ at $\theta = \alpha$. Furthermore, if there is a rigid region for $\theta \geq \alpha_c$ then $\psi = \pi/4$ and $f'' = 0$ at $\theta = \alpha_c$ (since the strain rate must vanish at the unyielded plug). The azimuthal pressure gradient is found to be independent of $r$ and thus the pressure takes the form

$$
p = 2ABi \log r + g(\theta),
$$

(5.11)

where $A$ is a constant. As such, the balance of radial and angular momentum reduce to

$$
2BiA = -\frac{dk}{d\theta} \sin 2\psi + 2k \cos 2\psi \left( 1 - \frac{d\psi}{d\theta} \right),
$$

(5.12)

$$
\frac{dg}{d\theta} = -\frac{dk}{d\theta} \cos 2\psi - 2k \sin 2\psi \left( 1 - \frac{d\psi}{d\theta} \right).
$$

(5.13)

The constitutive law implies

$$
\frac{\dot{\gamma}_{r\theta}}{\dot{\gamma}_{rr}} = \frac{\tau_{r\theta}}{\tau_{rr}} \implies \frac{f''}{f'} = -2 \tan 2\psi,
$$

(5.14)

and hence, from (5.10), $k = (2f' \sec 2\psi)^N + Bi$. Following Alexandrov and Jeng [6] we define $F = df/d\theta$, and change independent variable to $\psi$. Using (5.12) and (5.14), and substituting for $k$, we arrive at the system of ODEs

$$
\frac{d\theta}{d\psi} = \frac{(N + (1 - N) \cos^2 2\psi)(2f' \sec 2\psi)^N + Bi \cos^2 2\psi}{(N + (1 - N) \cos^2 2\psi)(2f' \sec 2\psi)^N + Bi \cos^2 2\psi - ABi \cos 2\psi},
$$

(5.15)

$$
\frac{dF}{d\psi} = -2F \tan 2\psi \times \frac{d\theta}{d\psi},
$$

(5.16)

$$
\frac{df}{d\psi} = F \times \frac{d\theta}{d\psi},
$$

(5.17)
which are identical to the equations of Alexandrov and Jeng [6] when \( N = 1 \). For hinge half-angles below the critical value, \( \alpha \leq \alpha_c \), the boundary conditions are

\[
(a, b) \ f = \theta = 0 \text{ at } \psi = 0, \text{ and } (c) \ f = 1, (d) \ \theta = \alpha \text{ at } \psi = \pi/4. \tag{5.18}
\]

This represents a third-order system of equations with an eigenvalue, \( A \), and four boundary conditions. Note that we no longer require the boundary conditions \( F' = 0 \) at \( \theta = 0 \) and \( F = 0 \) at \( \theta = \alpha \), since these are implied by \( -2 \tan 2\psi = F' / F \) at \( \psi = 0 \) and \( \pi/4 \), respectively. Alternatively, if \( \alpha > \alpha_c \) there is a rigid plug occupying \( \alpha_c \leq \theta \leq \alpha \). At \( \psi = \pi/4 \), instead of (5.18d), we impose \( \theta = \alpha_c \), with the additional condition \( dF/d\psi = 0 \), which enables the critical angle, \( \alpha_c \), also to be calculated as part of the solution.

A particular solution, for which \( \alpha = \pi/4, A = 0, \psi = \theta \) and \( f = \sin 2\theta \) has been noted in previous work [e.g. 144, 6]. In fact, this solution exists for any generalised Newtonian fluid for which the constitutive law is given by \( \tau = \mu(\dot{\gamma})\dot{\gamma} \) (and hence in current notation \( k = k(\dot{\gamma}) \)), since the strain rate is spatially constant for this solution, and so any strain-rate dependence in the rheology is irrelevant to the solution. For this special case, since \( A = 0 \), the pressure is also independent of radial distance. We furthermore note that, since the governing equations (5.2)-(5.4) are time-reversible due to the omission of inertial terms, the resulting self-similar solutions can also be used for the case in which the wedge is being opened slowly. However, for some viscoplastic materials it may not be true that the fluid maintains adhesion to the plates as the wedge is expanded, and so we have chosen to focus on the case of compression between the two plates.

### 5.3 Numerical integration

We integrate the governing equations numerically using a shooting method. First we note that the governing ODEs (5.15)-(5.17) have a potential singular point at \( \psi = \pi/4 \), due to the \( \sec 2\psi \) terms, which occurs at \( \theta = \alpha \) (or \( \theta = \alpha_c \) if a rigid zone occurs). It is therefore helpful to expand the dependent variables in terms of \( \delta = \pi/4 - \psi \ll 1 \). When \( \alpha < \alpha_c \) we have \( F = 0 \) and \( dF/d\psi \neq 0 \) at \( \delta = 0 \), so we can write

\[
F = D_0\delta + D_1\delta^2 + \ldots, \tag{5.19}
\]

with \( D_0 \neq 0 \). Using (5.19), substituting into (5.15)-(5.17) and equating powers of \( \delta \) gives \( \theta = \alpha - \delta + \ldots, f = 1 - D_0\delta^2/2 + \ldots, \) and \( D_1 = 2ABiD_0^{-N}/N \). Using these local forms of the dependent variables we can solve the ODEs numerically in the case \( \alpha < \alpha_c \) (so no
Figure 5.2: Strain-rate (colour-plot) and streamlines (black) from numerical integration as described in §5.3, with $\alpha = 60^\circ$, $Bi = 1000$, and $N = 1$. The solid blue region shows the unyielded plug.

rigid region occurs) by making a guess for $A$ and $D_0$, integrating from $\psi = \pi/4 - \delta$ to $\psi = 0$ and iterating to satisfy the boundary conditions $\theta(0) = f(0) = 0$.

We can determine $\alpha_c$ by imposing $D_0 = 0$, since $dF/d\psi = 0$ at the yield surface ($\psi = \pi/4$). In this case, analysis of (5.15)-(5.17) gives a different form of the local expansions, with

$$F = \left( \frac{2(N+1)ABi}{N} \right)^{1/N} \delta^{1+1/N} + \ldots, \quad (5.20)$$

$$\theta = \alpha_c - \left( 1 + \frac{1}{N} \right) \delta + \ldots, \quad (5.21)$$

$$f = 1 - \frac{N+1}{2N+1} \left( \frac{2(N+1)ABi}{N} \right)^{1/N} \delta^{2+1/N} + \ldots. \quad (5.22)$$

Then, the constants $A$ and $\alpha_c$ are determined by integrating from $\psi = \pi/4 - \delta$ and requiring $\theta(0) = f(0) = 0$. The complete solution for $\alpha > \alpha_c$ is given by the solution for $\alpha = \alpha_c$ in the region $\theta \leq \alpha_c$ and a rigid plug attached to the rotating boundary ($f = 1, F = 0$) in the region $\alpha_c \leq \theta \leq \alpha$.

Using the approach detailed above we can integrate the equations numerically for all $\alpha$, $Bi$ and $N$. Figure 5.2 shows streamlines and a colour-plot of the magnitude of the strain rate for $\alpha = 60^\circ$, $Bi = 1000$ and $N = 1$, indicating the unyielded region adjacent to the wall. Figure 5.3(a,b) shows the velocity profiles for $Bi = 1/\sqrt{3}$ (corresponding to the value used by Alexandrov and Jeng [6]) with $N = 1$ (solid) and $N = 0.5$ (dotted) at a selection of wedge angles, $\alpha$, up to and including the critical value, $\alpha_c$, at which the plug forms (for $N=1$, $\alpha_c = 86.4^\circ$ and for $N = 0.5$, $\alpha_c = 99.2^\circ$). We note that the value
of $\alpha_c$ exceeds the largest $\alpha$ computed by Alexandrov and Jeng [6] for $Bi = 1/\sqrt{3}$ and $N = 1$, contributing to their conclusion that no rigid zones occur. Figure 5.3(c,d) shows the velocity profiles for $Bi = 10^4$ (and the same values of $N$) indicating that $\alpha_c$ is close to (but exceeds) $45^\circ$ for $Bi \gg 1$ and boundary layers occur for $\alpha < 45^\circ$. These boundary layers are most readily observed in figure 5.3(c) as the narrow angular range over which the radial velocity is adjusted to satisfy no-slip. They are also present in figure 5.3(d) since the angular velocity must have vanishing angular gradient at the boundary; however this transition is more difficult to observe in these figures. These behaviours are explored in the following section where we analyse the equations in the plastic regime $Bi \gg 1$.

The inclusion of shear thinning (flow index $N < 1$) has a minor impact on the velocity profiles in most cases, with the effect being most significant at small half-angles, $\alpha$, since the shear rate is largest for these hinge angles. The strain rate is greatest at the rigid boundary in this case, and hence the effect of shear thinning is to reduce the effective viscosity at the boundary relative to the centre of the wedge, resulting in an increased strain rate at the boundary and a reduced radial velocity at the centre.
5.4 Viscoplastic boundary layers: \( Bi \gg 1 \)

The dependence on the Bingham number of the critical angle, \( \alpha_c \) (now returning to radians), and the value of the constant \( A \) at this critical angle, denoted \( A_c \), are shown in Figure 5.4 for \( N = 1 \) and \( N = 0.5 \). We see that \( \alpha_c \to \pi/4 \) (as noted above) and \( A_c \to 0 \) as \( Bi \to \infty \), while \( \alpha_c \to \pi/2 \) and \( A_c \to \infty \) in the Newtonian limit, \( Bi \to 0 \) with \( N = 1 \), which is a consequence of the choice to scale pressure by \( Bi \) in (5.11). The former is analysed in the following section, while the latter is analysed in Appendix 5.A. When Bingham numbers are order unity (\( Bi = O(1) \)), shear thinning increases the critical angle above which a plug first forms. The physical mechanism for this is that, for a shear-thinning fluid, the shear rate decays more rapidly as the plug is approached (see (5.20)) because the lower strain rate near the plug results in a higher effective viscosity and a further hinderence of shear there. This region of low strain rate means the radial velocity tends to zero more slowly as the plug is approached from the bulk of the wedge, and so the true plug occurs at a larger angle. Roughly speaking, we can think of the plugged region for the Bingham case (\( N = 1 \)) being replaced by a smaller plugged region plus a region in which the strain rate is very low and the effective viscosity very high, but in which the fluid is nonetheless yielded. In contrast, shear thinning results in a slightly smaller value of \( \alpha_c \) when the Bingham number is large (see figure 5.4(a) inset).

The reduction in strain rate near the plug due to shear thinning becomes less significant when \( Bi \gg 1 \) since \( \alpha_c \sim \pi/4 \) and the solution in the bulk of the wedge approaches the uniform strain rate solution of \( \alpha = \pi/4 \). Since the dimensionless value of this uniform strain rate is \( \dot{\gamma} = 4 > 1 \), the effect of shear thinning is to reduce the stress in the bulk of the wedge, \( \theta < \pi/4 \), and hence we anticipate that the fluid is yielded over a smaller region. Consequently, in this regime, \( \alpha_c \) is smaller for the shear thinning case (although this effect is quite slight as shown in figure 5.4 and expounded using asymptotics in §5.4).

5.4 Viscoplastic boundary layers: \( Bi \gg 1 \)

The numerical results have demonstrated that when \( Bi \gg 1 \), regions emerge with high velocity gradient adjacent to the hinge boundary when \( \alpha < \alpha_c \) (figure 5.3). Also the critical angle, \( \alpha_c \), is a function of \( Bi \), which asymptotes to \( \pi/4 \) as \( Bi \to \infty \). In this section we elucidate both of these phenomena mathematically, by introducing matched asymptotic expansions between the interior flow away from the boundary or the plug, and a relatively thin region within which the velocity and stress fields adjust to the conditions at the boundary or the plug.

We first examine the leading order solutions in the ‘bulk’ \( (\pi/4 - \psi = O(1)) \), which
we term the ‘outer’ region. We introduce regular series expansions for the dependent
variables and eigenvalue, \((\theta, F, f, A) = (\theta_0, F_0, f_0, A_0) + o(1)\). Then to leading order
\[
\frac{d\theta_0}{d\psi} = \frac{\cos 2\psi}{\cos 2\psi - A_0}, \quad \frac{dF_0}{d\psi} = -\frac{2F_0 \sin 2\psi}{\cos 2\psi - A_0} \quad \text{and} \quad \frac{df_0}{d\psi} = \frac{F_0 \cos 2\psi}{\cos 2\psi - A_0}.
\]
Following Nadai [95], we may integrate these equations subject to the boundary conditions
\(f_0(0) = 0\) and \(\theta_0(0) = 0\) to find that
\[
F_0 = c_1 (\cos 2\psi - A_0), \quad f_0 = \frac{c_1}{2} \sin 2\psi, \quad \theta_0 = \psi + \frac{A_0}{1 - A_0^2} \tanh^{-1} \left[ \left( \frac{1 + A_0}{1 - A_0} \right)^{1/2} \tan \psi \right] \equiv G(\psi, A_0).
\]
The constant \(c_1\) and the eigenvalue \(A_0\) are yet to be determined; their values will follow
as part of the matching process, as shown below. When \(\psi = \pi/4 - \delta\ (\delta \ll 1)\), we find
the leading order expressions
\[
F_0 = -c_1 A_0 + \ldots, \quad f_0 = \frac{c_1}{2} + \ldots \quad \text{and} \quad \theta_0 = G(\pi/4, A_0) + \ldots.
\]
Immediately we can see the need for a boundary layer because these leading order
expressions cannot simultaneously satisfy the boundary conditions at \(\psi = \pi/4\), namely

Figure 5.4: (a) \(\alpha_c\) and (b) \(A_c\) as functions of \(Bi\) from numerical integration for \(N = 1\)
(solid blue) and \(N = 0.5\) (solid red). The corresponding asymptotic predictions for \(Bi \gg 1\)
are given by dotted lines. The inset shows a close up of the region \(10^5 < Bi < 10^6\) with
the numerically determined values shown as stars. The black dashed line in (a) shows
the asymptote \(\alpha_c = \pi/4\).
\( F(\pi/4) = 0, \ f(\pi/4) = 1 \) and \( \theta(\pi/4) = \alpha \). The outer solutions were derived on the basis that \( F^N \ll ABi (\cos 2 \psi)^{N+1} \), and thus the size of the boundary layer is determined by assessing when this regime becomes invalid. We further note that the matching condition, (5.26), requires that \( F \sim A \) and thus, when \( A \sim O(1) \), \( F^N \sim A Bi (\cos 2 \psi)^{N+1} \) when \( \delta^{N+1} Bi \sim 1 \).

If the eigenvalue, \( A \), is smaller than order unity, then the outer solution takes a different form. In particular, when \( A \) and \( \delta \) are of the same order we will also need to include the neglected \( Bi \cos^2 2 \psi \) terms (see (5.15)) in the boundary layer equations. In this case, as the boundary layer is approached we have \( F \sim A \) and thus, when \( A \) is \( O(1) \), \( F^N \sim A Bi (\cos 2 \psi)^{N+1} \) when \( \delta^{N+1} Bi \sim 1 \). This regime therefore occurs when \( A = O(Bi^{-1/2}) \), so we write \( A = a Bi^{-1/2} + \ldots \) with \( a = O(1) \) and expand the governing equations up to \( O(Bi^{-1/2}) \) to find the outer solutions,

\[
F = c_1 \cos 2 \psi + \frac{1}{Bi^{1/2}} (c_2 \cos 2 \psi - c_1 a) + \ldots, \tag{5.27}
\]
\[
f = \frac{c_1}{2} \sin 2 \psi + \frac{c_2}{2Bi^{1/2}} \sin 2 \psi + \ldots, \tag{5.28}
\]
\[
\theta = \psi + \frac{a}{Bi^{1/2}} \tanh^{-1} (\tan \psi) + \ldots, \tag{5.29}
\]

where \( c_2 \) is a constant. When \( \pi/4 - \psi = \delta = O(Bi^{-1/2}) \), expanding (5.27)-(5.29) up to \( O(Bi^{-1/2}) \) gives

\[
F = c_1 \left( 2 \delta - \frac{a}{Bi^{1/2}} \right) + \ldots, \quad f = \frac{c_1}{2} + \frac{c_2}{2Bi^{1/2}} \ldots \tag{5.30}
\]

and \( \theta = \frac{\pi}{4} + \left( -\delta + \frac{a}{2Bi^{1/2}} \log \left( \frac{1}{\delta} \right) \right) + \ldots \tag{5.31} \)

The requirement to include more than just the leading order term in the outer solution in this case is highlighted by the breaking of order in \( F \), (5.27), within the boundary layer, with contributions from leading and first order terms from (5.27)-(5.29) contributing to the dominant term in the local expansion (5.30).

In the following subsections we complete the asymptotic matching by deriving the inner solutions in the two different regimes.

### 5.4.1 Below the critical angle: \( 0 < \pi/4 - \alpha = O(1) \)

When the eigenvalue is of order unity \( (A = A_0 + \ldots) \), we define a rescaled independent variable within the boundary layer given by \( \eta = (\pi/4 - \psi) Bi^{1/(N+1)} \), as anticipated by the analysis above. We define the inner dependent variables as

\[
\phi_i = (\theta - \alpha) Bi^{1/(N+1)}, \quad F_i = F, \quad f_i = (f - 1) Bi^{1/(N+1)}. \tag{5.32}
\]
Then to leading order the governing equations are given by

\[
\frac{dF_i}{d\eta} = \frac{NF_i^{N+1}}{N F_i^N \eta - 2 A_0 \eta^{2+N}}, \quad \frac{d\phi_i}{d\eta} = -\eta \frac{dF_i}{F_i d\eta} \quad \text{and} \quad \frac{df_i}{d\eta} = -\eta \frac{dF_i}{d\eta}.
\] (5.33)

Provided \(\theta_i\) and \(f_i\) remain order unity as \(\eta \to \infty\), which will be verified later, matching to the outer field (5.26) determines \(c_1 = 2\) and \(A_0\) is given implicitly by

\[
\alpha = G\left(\frac{\pi}{4}, A_0\right) = \frac{\pi}{4} + \frac{A_0}{(1 - A_0^2)^{1/2}} \tanh^{-1}\left[\left(\frac{1 + A_0}{1 - A_0}\right)^{1/2}\right].
\] (5.34)

This relation implies \(A_0\) is of order unity and negative when 0 < \(\pi/4 - \alpha = O(1)\).

Integrating (5.33a), with \(F_i(0) = 0\) and \(F_i \to -2A_0\) as \(\eta \to \infty\) (as required by matching to (5.26)), we find an implicit relation for \(F_i\),

\[
NF_i^{N+1} - 2(N + 1) A_0 F_i \eta^{N+1} = 4A_0^2 (N + 1) \eta^{N+1}.
\] (5.35)

Next, integrating (5.33b,c), with \(\phi_i = f_i = 0\) when \(F_i = 0\), we find

\[
\phi_i = (-2A_0)^{\frac{N-1}{N+1}} \left(\frac{N + 1}{N}\right)^{\frac{N}{N+1}} \left(1 + \frac{F_i}{2A_0}\right)^{\frac{N}{N+1}} - 1,
\] (5.36)

\[
f_i = (-2A_0)^{\frac{2N}{N+1}} \frac{N + 1}{2N + 1} \left(\frac{N + 1}{N}\right)^{\frac{N}{N+1}} \times \left(1 - \frac{NF_i}{2(N + 1) A_0}\right) \left(1 + \frac{F_i}{2A_0}\right)^{\frac{N}{N+1}} - 1,
\] (5.37)

and verify that both tend to a constant as \(\eta \to \infty\) and \(F_i \to -2A_0\).

The composite solutions for \(F\) and \(\theta\), denoted by \(C\{F\}\) and \(C\{\theta\}\) respectively, are then formed using the outer, (5.23), and inner, (5.35)-(5.36), solutions, to give

\[
C\{F\} = 2 \cos 2\psi + F_i, \quad C\{\theta\} = G(\psi, A_0) + Bi^{-1/(N+1)} \phi_i.
\] (5.38)

These are compared with a numerically integrated solution for \(\alpha = \pi/6, Bi = 10^4\), and \(N = 1\) and 0.5 in panels a) and c) of figure 5.5, respectively, showing excellent agreement.

### 5.4.2 Near the critical angle: \(\pi/4 - \alpha = O(Bi^{-1/2})\)

When the half angle of the hinge is close to \(\pi/4\) (and hence, as we will show, close to \(\alpha_c\)), the structure of the solution changes. The radial velocity, encoded through \(F\), undergoes a less extreme change across the boundary layer since when \(A = O(Bi^{-1/2})\), the matching condition (5.30) requires that \(F = O(Bi^{-1/2})\). Note that in this case the term ‘boundary
5.4. VISCOPLASTIC BOUNDARY LAYERS: $Bi \gg 1$

Figure 5.5: The numerically computed solution, $F \equiv 2u/(r\Omega)$, as a function of the polar angle, $\theta$, (solid line) and the asymptotic composite, $C\{F\}$ as a function of $C\{\theta\}$, (dotted) for $Bi = 10^4$, $\alpha = \pi/6$ (a,c) and $\alpha = \alpha_c$ (b,d), and $N = 1$ (a,b) and $N = 0.5$ (c,d). The curves are plotted parametrically via the independent variable, $\psi$, as $(\theta(\psi), F(\psi))$.

layer’ is used in the asymptotic sense but does not constitute a region where the velocity gradient is large (rather a region where the gradient of the strain rate is large) so that the existence of a boundary layer is visibly non-obvious for $\alpha = \alpha_c$ in figure 5.3(c), but is clearer in figure 5.2 where the strain rate exhibits a sharp gradient in a region adjacent to the unyielded plug.

In this regime we define the rescaled independent variable via $\eta = (\pi/4 - \psi) Bi^{1/2}$ and it is convenient to write the ‘inner’ dependent variables as

$$\phi_c = (\theta - \pi/4) Bi^{1/2}, \quad F_c = FBi^{1/2}, \quad f_c = (f - 1) Bi^{1/2}, \quad (5.39)$$

while the eigenvalue $A = Bi^{-1/2}a + \ldots$. At $O(1)$ we find

$$\frac{dF_c}{d\eta} = \frac{F_c(N(F_c/\eta)^N + 4\eta^2)}{N(F_c/\eta)^N \eta + 4\eta^3 - 2a\eta^2}, \quad \frac{d\phi_c}{d\eta} = -\frac{\eta}{F_c} \frac{dF_c}{d\eta}, \quad \text{and} \quad \frac{df_c}{d\eta} = 0, \quad (5.40)$$

subject to $F_c(0) = f_c(0) = 0$ and $\phi_c(0) = (\alpha - \pi/4)Bi^{1/2}$. Thus, we find $f_c = 0$ is constant, and, matching with the outer solution, $(5.30)$, requires $c_1 = 2$ (as in §5.4.1) and $c_2 = 0$.

On substituting $a^2V = N(F_c/\eta)^N$ into the first equation in $(5.40a)$, we have

$$\frac{dV}{d\eta} = \frac{2aNV}{a^2V - 2a\eta + 4\eta^2}, \quad (5.41)$$
and integrating yields the implicit solution

\[ \eta = \frac{a \sqrt{V} \left( c_3 Y_{1+1} \left( \frac{2}{N} \sqrt{V} \right) + J_{1+1} \left( \frac{2}{N} \sqrt{V} \right) \right)}{2 \left( c_3 Y_1 \left( \frac{2}{N} \sqrt{V} \right) + J_1 \left( \frac{2}{N} \sqrt{V} \right) \right)}, \tag{5.42} \]

where \( c_3 \) is a constant and \( J_i \) and \( Y_i \) denote Bessel functions of order \( i \) of the first and second kind, respectively. This expression automatically satisfies the boundary condition \( F_c(0) = 0 \), since \( V(0) \) is finite and \( F_c = \eta \left( a^2 V/N \right)^{1/N} \), and the constant \( c_3 \) is related to the rescaled eigenvalue \( a \) through matching to the far-field.

The matching is most readily explained as follows. We suppose that \( V(0) = V_0 \) and this determines the constant \( c_3 \) in terms of \( V_0 \) by demanding that the numerator of (5.42) vanishes. The denominator of (5.42) then vanishes at various values of \( V \) and we select the values \( V_-^\infty < V < V_+^\infty \) (\( V_-^\infty < V_0 < V_+^\infty \)), such that the denominator is non-vanishing in the range \( V_-^\infty < V < V_+^\infty \). From (5.42), we deduce that \( \eta \to \infty \) as \( V \to V_+^\infty \) if \( a > 0 \) and as \( V \to V_-^\infty \) if \( a < 0 \).

Next, using (5.41), we deduce the far-field form of \( V(\eta) \) as

\[ V = V_\infty - \frac{NV_\infty a}{2\eta} + \ldots \text{ and so } F = Bi^{-1/2} \left( \frac{a^2 V_\infty}{N} \right)^{1/N} \left( \eta - \frac{a}{2} + \ldots \right). \tag{5.43} \]

Matching with the far-field (5.30) then determines two values for \( a \) depending on its sign, namely \( a^+ = 2^N \sqrt{N/V_\infty} \) and \( a^- = -2^N \sqrt{N/V_\infty} \).

The final step in the analysis is to integrate (5.40b), which gives

\[ \phi_c(\eta) - \phi_c(0) = -\eta - \int_0^\eta \hat{\eta} \frac{dV}{NV} d\hat{\eta} \tag{5.44} \]

\[ = -\eta - a \frac{\log \eta}{2} - \int_0^1 \hat{\eta} \frac{dV}{NV} d\hat{\eta} - \int_1^\eta \left( \frac{\hat{\eta}}{NV} d\hat{\eta} - \frac{a}{2\hat{\eta}} \right) d\hat{\eta}. \tag{5.45} \]

Matching to the outer solution (5.31) requires that

\[ \phi_c \to -\eta - a \frac{\log \eta + \log(Bi^{-1/2})}{2} \text{ as } \eta \to \infty. \tag{5.46} \]

Then, denoting \( V(1) = V_1 \) we determine that

\[ \alpha - \pi/4 = Bi^{-1/2} \left( \int_{V_0}^{V_1} \eta \frac{dV}{NV} + \int_{V_1}^{V_\infty} 2a\eta - a^2 V + \frac{a}{4} \log Bi \right). \tag{5.47} \]

This final expression relates the half-angle of the wedge to properties of the inner solution, all of which are determined as functions of \( V_0 \). In other words this equation completely determines the matched asymptotic expressions.
5.4 VISCOPLASTIC BOUNDARY LAYERS: \( Bi \gg 1 \)

<table>
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<th>( V_1 )</th>
<th>( a )</th>
<th>( I )</th>
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<td>0.4683</td>
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</tbody>
</table>

Table 5.1: Table of constants in the asymptotic prediction for the critical angle, \((5.49)\), for different values of the flow index, \( N \). The magnitude of the radial pressure gradient when \( \alpha \geq \alpha_c \) is given asymptotically by \( 2aBi^{1/2}/r \) when \( Bi \gg 1 \).

An important consequence of this analysis is that we may determine asymptotically the critical angle at which the rigid plug adjacent to the boundary first forms. As in §5.2 this demands the additional condition that \( dF_c/d\eta = 0 \) at \( \psi = \pi/4 \), which in terms of the expansion developed here requires that \( V_0 = 0 \) and hence \( c_3 = 0 \). Then, \( V_\infty = \left( N j_{1/2}^{1/2}/2 \right)^2 \) and \( a = 2^{N+1}/(\sqrt{N}j_{N+1}) \), where \( j_{N+1} \) is the first positive root of the Bessel function \( J_{1/2} \), and \( V_1 \) is given by the first positive solution of

\[
\frac{a\sqrt{V_1}J_{N+1}^{1/2}\left(\frac{2}{N}\sqrt{V_1}\right)}{2J_{1/2}^{1/2}\left(\frac{2}{N}\sqrt{V_1}\right)} = 1.
\]

The asymptotic behaviour of \( \alpha_c \) with \( Bi \) is then given by

\[
\alpha_c = \frac{\pi}{4} + aBi^{-1/2}\left(\frac{1}{4}\log Bi + I\right) + \ldots,
\]

where \( I \) is determined from the integrals

\[
I = \int_0^{V_1} \frac{J_{N+1}^{1/2}\left(\frac{2}{N}\sqrt{V}\right)}{2N\sqrt{V}} dV + \int_{V_1}^{V_\infty} \frac{1}{2N\sqrt{V}} - \frac{J_{1/2}^{1/2}\left(\frac{2}{N}\sqrt{V}\right)}{2N\sqrt{V}J_{N+1}^{1/2}\left(\frac{2}{N}\sqrt{V}\right)} dV.
\]

A table of values of \( V_\infty, V_1, a \) and \( I \) for various values of \( N \), are given in table 5.1, and figure 5.4 shows a comparison of the asymptotic and numerical results for \( \alpha_c \) and \( A_c = aBi^{-1/2} + \ldots \), as a function of \( Bi \) for \( N = 1 \) and \( N = 0.5 \). Both \( \alpha_c \) and \( A_c \) are very accurately captured by their asymptotic form in the regime \( Bi \gg 1 \).

The composite solutions for \( F \) and \( \theta \) are formed from the inner solutions determined above, and the outer solutions \((5.28)\) and \((5.29)\), to obtain

\[
\mathcal{C}\{F\} = 2\cos 2\psi + Bi^{-1/2}F_c - 4\left(\frac{\pi}{4} - \psi\right),
\]

\[
\mathcal{C}\{\theta\} = \frac{\pi}{4} + Bi^{-1/2}\left( a\tanh^{-1}(\tan\psi) + \phi_c - \frac{a}{2}\log\left(\frac{1}{\pi/4 - \psi}\right) \right).
\]

These are compared with a numerically integrated solution for \( \alpha = \alpha_c, Bi = 10^4 \), and \( N = 1 \) and \( 0.5 \) in panels \( b) \) and \( d) \) of figure 5.5, again showing excellent agreement.
5.5 Comparison to full numerical simulations

The similarity solutions derived in this chapter are planar and apply for hinged plates of semi-infinite extent. We therefore anticipate that these solutions emerge sufficiently close to the vertex of finite hinged plates and for sufficiently long plates in the third dimension, but in general would be perturbed by out-of-plane flow and the outer radial boundary condition. To explore the impact of the outer boundary condition and the potential embedding of the similarity solution derived above within a more general flow, full 2D numerical simulations were carried out for the compression of a Bingham fluid \((N = 1)\) between hinged plates, using a triangular domain with a “vertical” stress-free boundary at \(x = r \cos \theta = 1\). As for the analytical solutions, only the upper half of the domain is considered, with solutions given in the bottom half by (anti-)symmetry. This problem models the extrusion of a finite quantity of viscoplastic fluid by squeezing from between hinged plates. While we do not evolve the problem forwards in time by reducing the hinge angle and evolving the stress-free surface, the solution provides the instantaneous stresses and velocities (including of the free surface) at the moment at which the squeezing begins.

The solution of the governing equations (5.2)-(5.4) and (5.7) was carried out using the FISTA algorithm proposed by Treskatis et al. [138] and given in §2.4, which has also been used by Muravleva [94] and Pourzahedi et al. [108]. The FISTA algorithm is an accelerated version of the widely used augmented-Lagrangian algorithm for viscoplastic flows (e.g. see [123]), accurately resolving unyielded regions of the flow and circumventing the singular nature of the constitutive law at the yield surfaces via the introduction of an additional tensorial field, \(D\), representing the strain-rate tensor but decoupled from the velocity field, \(u\), along with a Lagrange multiplier, standing for the deviatoric stress tensor, which enforces the equivalence of \(D\) and \(\dot{\gamma}(u)\) at convergence. The algorithm is given explicitly by Algorithm 2 in §2.4, and was carried out using the finite element method as implemented by FEniCS [11, 84], using Taylor-Hood elements for the velocity and pressure, and discontinuous piecewise linear elements for the strain-rate and deviatoric stress tensors. The initial (triangular) meshes used for the simulations have greater resolution at the vertex of the wedge and, in the case of \(\alpha < \alpha_c\), for which we anticipate a boundary layer adjacent to the rigid boundary, also along the rigid boundary. When an unyielded region occurs, a simple mesh refinement method was used to increase the resolution at the yield surface. Specifically, every 1000 iterations of the FISTA algorithm, any cells that have vertices lying in both yielded and unyielded regions (according to the
magnitude of the deviatoric stress for the current iteration) were divided into four smaller cells. This refinement step was carried out 5 times or until the number of cells became larger than 150,000. The convergence of the algorithm was tracked by the residual, \( R \),

\[
R = \| \sqrt{(D_{ij} - \dot{\gamma}_{ij})(D_{ij} - \dot{\gamma}_{ij})} \|_{L^2} / \| \sqrt{\dot{\gamma}_{ij}\dot{\gamma}_{ij}} \|_{L^2},
\]

which measures the discrepancy between the additional tensor field, \( D \), and the strain rate tensor \( \dot{\gamma}(u) \). The given solutions converged to a residual of less than \( 10^{-5} \).

Figure 5.6 shows solutions for \( Bi = 1000 \) and \( \alpha = 60^\circ \) and \( 30^\circ \). For \( \alpha = 60^\circ \) the unyielded zone is observed adjacent to the wall as predicted by the similarity solution (for example compare panel a) with figure 5.1(b)), and the radial velocity profiles agree well with the similarity solution but deviate with increasing distance from the vertex, as anticipated. Similarly, for \( \alpha = 30^\circ \) the viscoplastic boundary layer, in which the strain rate becomes large, is visible adjacent to the rigid boundary (note the logarithmic colour-scale for the strain rate in this case) and again the radial velocity profiles agree well with the similarity solution close to the vertex of the wedge. In fact, in both cases, the deviation between the numerical simulation (on the domain bounded at \( x = 1 \)) and the similarity solution is quite small even up to \( r = 0.8 \).

### 5.6 Discussion and conclusions

We have solved for the viscoplastic flow of a Herschel-Bulkley fluid between hinged plates, and have demonstrated that the flow is self-similar with the dimensionless deviatoric stresses being functions only of the polar angle, the Bingham number, and the flow index, \( N \). We have also shown that plugs and boundary layers form. The former occur for half-angles, \( \alpha \), greater than a critical value, \( \alpha_c \), which depends on the Bingham number and the flow index, and which decreases to \( \pi/4 \) as \( Bi \to \infty \). A complicated boundary layer structure, dependent on the value of \( \alpha \), occurs in the plastic regime, \( Bi \gg 1 \). Classical “viscoplastic boundary layers”, in which the strain rate becomes large to enforce the no slip boundary condition, occur when \( 0 < \pi/4 - \alpha = O(1) \) and have an angular width which scales like \( Bi^{-1/(N+1)} \). This structure is modified somewhat as the half angle approaches \( \pi/4 \), in which case the boundary layer features logarithmic corrections to a \( Bi^{-1/2} \) dependence and only adjusts the strain rate, not the velocity, over the thin layer. In the Newtonian regime, \( Bi \ll 1 \) with \( N = 1 \), the solution reduces to the classical solution described by Moffatt [92] to leading order. In this regime, unyielded regions only exist when the wedge angle is within \( O(Bi) \) of \( \pi/2 \).
The similarity solutions derived in this chapter are planar and apply for an infinite wedge. We therefore anticipate that, in general, these solutions would be perturbed by out-of-plane flow and the outer radial boundary condition. To briefly explore the impact of the outer boundary condition, 2D numerical simulations were carried out with a stress-free boundary at \( r \cos \theta = 1 \). In this case the similarity solution is observed close to the vertex of the wedge but becomes altered at larger radial distances from the vertex, as anticipated. In addition to further elucidating the impact of these non-planar and finite-wedge effects, future work could involve the solution of the governing equations with constitutive laws that allow for elastic or thixotropic effects or with the inclusion of non-negligible inertial stresses. Furthermore, many examples of viscoplastic materials have been shown to exhibit wall slip [22, 105, 41, 45], as opposed to the no-slip boundary condition applied at the rigid boundaries in this work. Thus, further work could also...
consider the impact of such wall slip on the flow of a viscoplastic fluid between hinged plates.

5.A The Newtonian regime: $Bi \ll 1, N = 1$.

In the Newtonian regime, $Bi \ll 1$ and $N = 1$, it is easiest to carry out the analysis without the change of independent variable from $\theta$ to $\psi$. Writing $p = 2ABi \log r + g(\theta)$, and making the additional substitution $\theta = \alpha \Theta$, the conservation of momentum in the radial direction is expressed as

$$2\alpha^2 Bi A = F'' + 4\alpha^2 F + 4\alpha^2 Bi \frac{(F'' + 4\alpha^2 F) F^2}{(F'' + 4\alpha^2 F)^{3/2}},$$

where $F(\Theta) = f'(\Theta)/\alpha$ (5.54)

and primes now represent differentiation with respect to $\Theta$. We expand the dependent variables and the eigenvalue via

$$F = F_0 + Bi F_1 + \ldots, \quad f = f_0 + Bi f_1 + \ldots, \quad \text{and} \quad A = A_{-1} Bi^{-1} + A_0 + \ldots.$$

(5.55)

The case of no unyielded region is then given by solving (5.54) with boundary conditions $f(0) = F'(0) = 0$, $f(1) = 1$ and $F(1) = 0$, with leading order solution

$$F_0 = \frac{2 (\cos 2\alpha \Theta - \cos 2\alpha)}{\sin 2\alpha - 2\alpha \cos 2\alpha}, \quad f_0 = \frac{\sin 2\alpha \Theta - 2\alpha \Theta \cos 2\alpha}{\sin 2\alpha - 2\alpha \cos 2\alpha},$$

(5.56)

and

$$A_{-1} = \frac{-4 \cos 2\alpha}{\sin 2\alpha - 2\alpha \cos 2\alpha},$$

(5.57)

which recovers the Newtonian solution given by Moffatt [92].

To consider the case in which an unyielded region occurs for $\theta \geq \alpha_c$, we substitute $\alpha_c = \alpha_0 + Bi \alpha_1 + \ldots$ for $\alpha$ in (5.54), and solve with the additional boundary condition $F'(1) = 0$, which is sufficient to determine $\alpha_c$. At leading order this gives

$$F_0 = \frac{2}{\pi} (1 + \cos \pi \Theta), \quad f_0 = \frac{1}{\pi} (\pi \Theta + \sin \pi \Theta), \quad A_{-1} = \frac{4}{\pi} \text{ and } \alpha_0 = \frac{\pi}{2},$$

(5.58)

while at $O(Bi)$ we find

$$F_1 = \frac{1}{6\pi} (4 + 4 \cos \pi \Theta + \pi \Theta \sin \pi \Theta) - \frac{1}{3} \cos \frac{\pi \Theta}{2},$$

$$f_1 = \frac{1}{12\pi} (3\pi \Theta + 4 \sin \pi \Theta - \pi \Theta \cos \pi \Theta) - \frac{1}{3} \sin \frac{\pi \Theta}{2},$$

(5.59)
Figure 5.7: a) $F$ as a function of $\theta$ for $\alpha = \pi/2$ and $Bi = 10^{-3}$, determined by asymptotic predictions (black/red dotted) and numerical integration (blue solid). The asymptotic solution in black retains only the leading order terms in $F$ and $\alpha_c$, while the solution in red retains terms up to $O(Bi)$. The inset shows the thin unyielded region ($F = F' = 0$) near the plate, predicted by the first order asymptotic solution. b) $A_c$ and c) $\alpha_c$ as functions of $Bi$ from the asymptotic predictions (5.61) (red), and numerical integration (blue).

and $A_0 = 4/(3\pi)$, $\alpha_1 = -\pi/24$. Thus, the asymptotic solutions for $A_c$ and $\alpha_c$ in this regime are given by

$$A_c \sim \frac{4}{\pi} Bi^{-1} + \frac{4}{3\pi} + \ldots, \quad \text{and} \quad \alpha_c \sim \frac{\pi}{2} - \frac{\pi}{24} Bi + \ldots \quad (5.61)$$

Figure 5.7(a) shows the asymptotic solution, $F = F(\theta/\alpha_c)$, compared with a numerically determined solution for $\alpha = \pi/2$ and $Bi = 10^{-3}$. This figure confirms that the asymptotic solution accurately reproduces the numerical computation and that two terms are required in the asymptotic solution ($F = F_0 + BiF_1$, $\alpha_c = \alpha_0 + Bi\alpha_1$) to capture the required behaviour close to the plates. Panels (b) and (c) compare the asymptotic predictions for $A_c$ and $\alpha_c$ with numerical results, strongly supporting the validity of the asymptotic analysis. Indeed the asymptotic predictions remain accurate up to relatively large values of $Bi$. 

122
6.1 Introduction

Stagnation points occur in many flow configurations of interest, such as flow through a T-junction in a pipe, flow around a body moving through fluid, and squeeze flow between two plates. In many applications of such flows the flowing material is a viscoplastic fluid, a particular class of non-Newtonian fluid which acts as a rigid plastic or flows as a viscous fluid, depending on whether the stress is less than or exceeds a critical yield stress, respectively. In particular, this behaviour is common for slurries and suspensions, and the viscoplastic model has wide ranging applications in geophysics and industry [12, 19, 61]. For a viscoplastic fluid, the vanishing strain rate at a stagnation point results in the stress falling below the yield stress, and the existence of regions of stagnant, unyielded fluid. These stagnant regions of unyielded viscoplastic material can have significant implications in the food industry where stagnant material may spoil and contaminate the product, and more generally may impact efficiency when transporting viscoplastic fluids. In this chapter, we investigate the geometry of these plug regions for planar flows.

The geometry of unyielded zones around stagnation points has been discussed for a number of specific flow configurations including flow around a sphere [27, 31], flow around a cylinder [3, 137, 146, 69], flow around non-circular cylinders [97, 37], axisymmetric
and planar squeeze flow [128, 88, 93], flow through a T-junction [75], and flow around an inclined plate [100]. In this existing literature, the authors are typically primarily interested in other characteristics of the flow, such as drag coefficients or pressure drops, and the description of the stagnant zones is often a secondary result from direct numerical simulations, with limited definite conclusions being made for the general case. Nonetheless, some results are known. For a given flow configuration, the plug size increases with non-dimensional yield stress (given by the Bingham number, the ratio of the yield stress to a typical viscous stress in the flow). For planar flows, the stagnant zones are approximately triangular in shape, with a vertex at the point where the flow diverges. Tokpavi et al. [137] and Hewitt and Balmforth [69] both note that the angle subtended at this vertex must approach a right-angle in the limit of infinite Bingham number since, in this limit, one can apply the slipline theory of plane plasticity (see §2.1.2) for which yield surfaces must follow mutually-orthogonal sliplines. Similarly, Chaparian and Frigaard [37] provide an empirical rule for the geometry of the unyielded region around a settling particle in viscoplastic fluid, concluding from a large number of numerical simulations that the angle at the vertex of the stagnation-point plug is at most a right-angle, while Nirmalkar et al. [97] study the flow around a square cylinder and observe that the angle at the vertex of the stagnation-point plug is at least a right-angle, measuring an obtuse angle for any finite Bingham number. One key result of the current chapter is an analytical argument that this angle must, in-fact, always be a right-angle, independent of Bingham number or flow configuration. Further, we study the geometry of plugs occurring in a generic example of a stagnation-point flow, that of a viscoplastic fluid converging against an infinite straight boundary which provides a good approximation to the flow local to a stagnation point in a general flow configuration.

In §6.2 we define the problem and discuss how it can be viewed as a local solution in the neighbourhood of a more complex flow configuration. In §6.3 we construct an asymptotic solution valid sufficiently far from the plug which is used as the boundary condition for direct numerical simulations. In §6.4 we discuss these numerical simulations, before detailing and rationalising the key features of the plug geometry in §6.5. Finally, in §6.6 we discuss the embedding of this theory in a global flow, considering the specific flow configurations of flow around a cylinder and recirculating flow in a closed wedge, and offer conclusions in §6.7.
6.2 Problem definition

To investigate the flow of a viscoplastic fluid in the vicinity of a stagnation point we consider the prototypical example of a stagnating flow, namely a flow in the half-plane exhibiting a single stagnation point on the infinite planar boundary (see figure 6.1). For a Newtonian fluid, this configuration gives a good approximation to the neighbourhood of a stagnation point embedded in a more general flow configuration, provided the boundary is smooth and thus can be locally approximated as a straight line. For a viscoplastic fluid the situation is more complicated since the vanishing strain rates at the stagnation point result in a region of unyielded fluid. The length scale of this stagnant zone, or plug, is determined by the yield stress and viscosity of the material, and the typical velocity away from the stagnation point. In particular, a larger yield stress will result in a larger plug, while a faster flowing fluid will result in a smaller plug. The suitability of this idealised problem as an approximation to the flow in the neighbourhood of a stagnation point in a more general flow configuration then also relies on this plug length scale being significantly smaller than the geometrical length scales involved in the global flow configuration, such as would be the case for a sufficiently low yield stress.

In Appendix 6.A we more rigorously explore how the solution of this idealised problem relates to a local solution in the neighbourhood of a stagnation point in a global flow, provided the Bingham number (which measures the ratio of the yield stress to typical viscous stresses in the flow, defined formally in §6.6) is small.

![Figure 6.1: A diagram of the flow in the neighbourhood of a stagnation point](image)

We consider viscoplastic fluid occupying the upper half-plane and assume the Bingham model for the fluid, with constitutive law

$$
\dot{\tau}_{ij} = \begin{cases} 
\left( \mu + \frac{\tau_c}{\dot{\gamma}} \right) \dot{\gamma}_{ij} & \text{for } \tau > \tau_c, \\
\dot{\gamma} = 0 & \text{otherwise},
\end{cases}
$$

(6.1)
relating the deviatoric stresses, $\tau_{ij}$, to the strain rates, $\dot{\gamma}_{ij}$. Here $\mu$ is the viscosity, $\tau_c$ is the yield stress, and $\tau$ and $\dot{\gamma}$ are the second invariants of the deviatoric stress and strain-rate tensors respectively, given (for example) by

$$\tau = \sqrt{\frac{1}{2} \tau_{ij} \tau_{ij}}. \quad (6.2)$$

We neglect inertia (which is appropriate in the neighbourhood of a stagnation point, where velocities are low), and we impose no-slip and no penetration on the straight boundary.

Finally, we seek a solution that, at large distances (where strain rates are high and viscous stresses dominate), is asymptotic to the viscous stagnation flow (c.f. [25, pg. 226]) with streamfunction,

$$\tilde{\psi} = k \tilde{r}^3 \sin^2 \theta \sin(\theta_0 - \theta), \quad (6.3)$$

$$\tilde{u} = \frac{1}{\tilde{r}} \frac{\partial \tilde{\psi}}{\partial \theta}, \quad \tilde{v} = -\frac{\partial \tilde{\psi}}{\partial \tilde{r}}, \quad (6.4)$$

where $(\tilde{r}, \theta)$ are dimensional polar coordinates defined around the stagnation point, $(\tilde{u}, \tilde{v})$ are dimensional velocities in the radial and polar directions, and $k$ and $\theta_0$ are constants that would be determined from matching to a global flow. $\theta_0$ represents the angle between the $\psi = 0$ streamline and the wall (see figure 6.1), so any symmetrical flow configurations (e.g. flow through a symmetrical T-junction or around a symmetrical particle) would automatically imply $\theta_0 = \pi/2$.

### 6.2.1 Non-dimensionalisation and governing equations

This problem has no explicit length scale, so we introduce the viscoplastic length scale $L_V = \mu U / \tau_c$, where the velocity-scale, $U$, is given by $kL_V^2$ (and hence, eliminating $U$, we have $L_V = \tau_c / (\mu k)$). We non-dimensionalise lengths by $L_V$ and velocities by $U$, and remove the tildes for non-dimensional quantities, so that the non-dimensional streamfunction is given, far from the stagnation point, by

$$\psi = r^3 \sin^2 \theta \sin(\theta_0 - \theta) + \ldots. \quad (6.5)$$

We further non-dimensionalise pressure and stresses by $\mu U / L_V = \tau_c$. With this non-dimensionalisation, the Bingham constitutive law becomes

$$\tau_{ij} = \left( 1 + \frac{1}{\dot{\gamma}_i} \right) \dot{\gamma}_{ij}, \quad (6.6)$$
and the non-dimensionalised problem has only one residual parameter, \( \theta_0 \). From (6.5) the strain rate, \( \dot{\gamma} \) of the viscous solution is proportional to \( r \), and thus, as anticipated, viscous stresses dominate at large \( r \) and the viscous streamfunction (6.5) is an appropriate leading order term for an asymptotic series in \( r \), valid far from the stagnation point. Close to the stagnation point, viscous stresses decay and plasticity becomes important, resulting in a static rigid plug forming here.

To summarise, the full system of dimensionless equations we wish to solve is given by

\[
\begin{align*}
\frac{1}{r} \frac{\partial \psi}{\partial \theta} &= u, \quad \frac{\partial \psi}{\partial r} = v, \quad (6.7) \\
\dot{\gamma}_{rr} &= 2 \frac{\partial u}{\partial r}, \quad \dot{\gamma}_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \dot{\gamma} = \sqrt{\dot{\gamma}_{rr}^2 + \dot{\gamma}_{r\theta}^2}, \quad (6.8) \\
\tau_{ij} &= \left(1 + \frac{1}{\dot{\gamma}}\right) \dot{\gamma}_{ij} \text{ when } \tau > 1, \quad \dot{\gamma}_{ij} = 0 \text{ otherwise}, \quad (6.9) \\
\frac{\partial p}{\partial r} &= \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta}, \quad \frac{\partial \tau_{r\theta}}{\partial r} = \frac{1}{r} \frac{\partial \tau_{rr}}{\partial \theta} + \frac{2}{r} \tau_{r\theta}, \quad (6.10) \\
\frac{1}{r} \frac{\partial p}{\partial \theta} &= \frac{\partial \tau_{r\theta}}{\partial r} - \frac{\partial \tau_{rr}}{\partial \theta} + \frac{2}{r} \tau_{r\theta}, \quad (6.11)
\end{align*}
\]

subject to the no-slip boundary conditions

\[
\begin{align*}
u = v = 0 & \text{ at } \theta = 0, \pi. \quad (6.12)
\end{align*}
\]

Eliminating \( p \) gives

\[
2 \frac{\partial^2 \tau_{rr}}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial^2 \tau_{r\theta}}{\partial \theta^2} - \frac{\partial}{\partial r} \left( r \frac{\partial \tau_{r\theta}}{\partial r} \right) + \frac{2}{r} \frac{\partial \tau_{rr}}{\partial \theta} - 2 \frac{\partial \tau_{r\theta}}{\partial r} = 0. \quad (6.13)
\]

### 6.3 Asymptotic solution far from the stagnation point

Far from the stagnation point we look for an asymptotic series in \( r \) for the streamfunction, of the form

\[
\psi = r^3 f_0(\theta) + r^2 f_1(r, \theta) + \ldots \quad (6.14)
\]

where

\[
f_0(\theta) = \sin^2 \theta \sin(\theta_0 - \theta), \quad (6.15)
\]

and we allow a radial dependence of the unknown function, \( f_1 \), in order to be able to identify logarithmic terms in the expansion. We note that all solutions will depend on \( \theta_0 \) as a parameter throughout the following.
Equation (6.14) gives
\[ \tau_{rr} = 4r f_0' + 2 \partial \theta f_1 + 2r \partial_r \partial \theta f_1 + \frac{4f_0'}{\sqrt{16f_0'^2 + (f_0'' - 3f_0)^2}} + \ldots, \]  
(6.16)
\[ \tau_{r \theta} = r \left( f_0'' - 3f_0 \right) + \partial^2 \theta f_1 - 2r \partial_r f_1 - (r \partial_r)^2 f_1 + \frac{f_0'' - 3f_0}{\sqrt{16f_0'^2 + (f_0'' - 3f_0)^2}}, \]  
(6.17)
where \( \partial \) denotes differentiation with respect to \( \theta \). At \( O(1) \) in (6.13) we have
\[ 4 \partial^2 \theta (r \partial_r + 1)^2 f_1 + \left( \partial^2_\theta - 2r \partial_r - (r \partial_r)^2 \right)^2 f_1 + H'(\theta) = 0, \]  
(6.18)
where
\[ H(\theta) = \frac{d}{d\theta} \left( \frac{f_0'' - 3f_0}{\sqrt{16f_0'^2 + (f_0'' - 3f_0)^2}} \right) + \frac{8f_0'}{\sqrt{16f_0'^2 + (f_0'' - 3f_0)^2}}. \]  
(6.19)
The form of this PDE suggests the ansatz
\[ f_1(r, \theta) = \ln(r) F(\theta) + G(\theta). \]  
(6.20)
In this case (6.19) becomes the pair of ODEs:
\[ F''' + 4F'' = 0, \]  
(6.21)
\[ G''' + 4G'' + 4F'' + H' = 0, \]  
(6.22)
with
\[ F(0) = F'(0) = G(0) = G'(0) = F(\pi) = F'(\pi) = G(\pi) = G'(\pi) = 0. \]  
(6.23)
Integrating (6.21) and applying the boundary conditions gives
\[ F(\theta) = A (1 - \cos 2\theta), \]  
(6.24)
where \( A \) is a constant not determined by the boundary conditions for \( F \), and may depend on the parameter \( \theta_0 \). Substituting and integrating (6.22) then gives
\[ G'''(\theta) + 4G'(\theta) + 8A \sin 2\theta + H(\theta) = C. \]  
(6.25)
The solution is given by
\[ G = G_p + A \theta \sin 2\theta + \frac{C}{8} (2\theta - \sin 2\theta) + D (1 - \cos 2\theta), \]  
(6.26)
where \( G_p(\theta) \) is the solution to the initial value problem
\[ G''' + 4G' + H = 0, \quad G_p(0) = G_p'(0) = G_p''(0) = 0, \]  
(6.27)
6.3. ASYMPTOTIC SOLUTION FAR FROM THE STAGNATION POINT

Figure 6.2: $G_p(\theta)$ as a function of $\theta$, for values of $\theta_0$ given in the legend.

and $A$, $C$ and $D$ are constants of integration, dependent on $\theta_0$, to be determined by the boundary conditions at $\theta = \pi$. Plots of $G_p(\theta)$ are given for a selection of $\theta_0$ values is given in figure 6.2.

The constants $A$ and $C$ are given by

$$A = -\frac{1}{2\pi} G_p'(\pi), \quad C = -\frac{4}{\pi} G_p(\pi),$$

and plots of $A(\theta_0)$ and $C(\theta_0)$ are given in figure 6.3.

$D$ is, in fact, not determined by the boundary conditions. This term is a solution to the viscous flow problem, representing a horizontal shear flow

$$\psi = 2Dr^2 \sin^2 \theta = 2Dy^2 \Rightarrow u_x = 4Dy, \quad u_y = 0.$$  

(6.29)

One way of interpreting this undetermined constant, $D$, is to consider a horizontal translation $x = \hat{x} - x_0$, $y = \hat{y}$. Under this transformation, the leading order solution can be written as

$$\psi_0 = r^3 \sin^2 \hat{\theta} \sin(\theta_0 - \hat{\theta}) = r^3 \sin^2 \hat{\theta} (\cos \hat{\theta} \sin \theta_0 - \sin \hat{\theta} \cos \theta_0)$$

$$= x\hat{y}^2 \sin \theta_0 - y^2 \cos \theta_0 = \hat{x}\hat{y}^2 \sin \theta_0 - \hat{y}^2 \cos \theta_0 - x_0\hat{y}^2 \sin \theta_0$$

$$= \hat{r}^3 \sin^2 \hat{\theta} \sin \left( \theta_0 - \hat{\theta} \right) - \frac{1}{2} x_0 \sin \theta_0 \hat{r}^2 \left( 1 - \cos 2\hat{\theta} \right).$$

(6.30)
For $x_0 = O(1) \ll r$, we have $\dot{r} = r + x_0 \cos(\theta) = r + O(1)$, $\dot{\theta} = \theta - (x_0 \sin \theta)/r = \theta + O(1/r)$. Substituting into the expression for the streamfunction, and dropping the hats, we get

$$
\psi = r^3 \sin^2 \theta \sin(\theta_0 - \theta) + A r^2 \ln r \left(1 - \cos 2\theta\right) + r^2 \left(G_p + A \theta \sin 2\theta + \frac{C}{8} (2\theta - \sin 2\theta) + \left(D - \frac{x_0}{2} \sin \theta_0\right) (1 - \cos 2\theta)\right) + O(r \ln r),
$$

which is equivalent to the untransformed solution, except with $D \to D - (x_0 \sin \theta_0)/2$. Thus, the choice of $D$ can be interpreted as a choice of origin, as opposed to giving a genuinely distinct family of solutions. To justify the natural choice of origin, consider the streamline given by $\psi = 0$ for a general $D$. This is given by

$$
r \sin^2 \theta \sin(\theta_0 - \theta) + A \ln r \left(1 - \cos 2\theta\right) + G(\theta) = 0.
$$

Since $r$ is large, this implies $\theta = \theta_0 + \delta \theta$, where $\delta \theta \ll 1$. Expanding gives, at leading order,

$$
-r \delta \theta \sin^2 \theta_0 + 2 A \ln r \sin^2 \theta_0 + G(\theta_0) = 0,
$$

$$
\delta \theta = 2 A \frac{\ln r}{r} + \frac{1}{r \sin^2 \theta_0} G(\theta_0) + \ldots.
$$

The Cartesian distance between the $\psi = 0$ streamline and $\theta = \theta_0$ is given by

$$
r \delta \theta = 2 A \ln r + G(\theta_0) \csc^2 \theta_0 + \ldots.
$$
6.4. NUMERICAL SIMULATIONS

The second term on the right hand side of (6.37) is independent of $r$ and hence represents a translation relative to $\theta = \theta_0$, thus choosing $G(\theta_0) = 0$ is a natural (though non unique) way of fixing the origin for the general problem.

Finally, this gives the asymptotic solution valid far from the stagnation point as

$$\psi = r^3 \sin^2(\theta) \sin(\theta_0 - \theta) + Ar^2 \ln r (1 - \cos 2\theta) + r^2 \left(G_p(\theta) + A\theta \sin 2\theta + \frac{C}{8} (2\theta - \sin 2\theta) + D (1 - \cos 2\theta)\right) + \ldots, \tag{6.38}$$

with $G_p$ the unique solution of the initial value problem, (6.27), $A$ and $C$ given by (6.28), and $D$ given by

$$D = -\frac{8G_p(\theta_0) + 8A\theta_0 \sin 2\theta_0 + C (2\theta_0 - \sin 2\theta_0)}{8 (1 - \cos 2\theta_0)}. \tag{6.39}$$

We can now use this asymptotic solution, as the boundary condition for numerical simulations in the simplified geometry of figure 6.1 and infer conclusions about the shape of the stagnant zone for more general flow configurations. These numerical simulations are discussed in the following section.

6.4 Numerical Simulations

We carried out direct numerical simulations of the idealised stagnation-flow problem using an augmented-Lagrangian method (as described in §2.4), implemented in FEniCS [11] on a rectangular domain, $\{(x, y) : -5 \leq x \leq 5, 0 \leq y \leq 5\}$. A selection of problems were also tested on domains 3, 6, and 10 times larger than this, verifying that the solutions were essentially independent of domain size and the smaller mesh could be used. The asymptotic solution, (6.38), was imposed as a velocity boundary condition on $x = \pm 5$ and $y = 5$, and zero velocity was imposed on $y = 0$. Starting from a mesh of 10 000 triangular cells, a simple adaptive method was used to refine the mesh to resolve the yield surface accurately. Namely, every 50 iterations in the augmented-Lagrangian method we check for cells where the magnitude of the deviatoric stress is within some tolerance of the dimensionless yield stress, 1, and split these cells into four smaller cells. After a small number of these refinements, the yield surface is very well resolved by the mesh, after which the augmented-Lagrangian method is continued until the magnitude of the strain-rate increment falls below $10^{-6}$.

The resulting plug shapes for four values of $\theta_0$ are shown in figure 6.4. We note several properties of these plugs. Firstly, these simulations are consistent with the yield
CHAPTER 6. STAGNATION POINT FLOW OF A VISCOPLASTIC FLUID

Figure 6.4: Stagnation point plugs (blue) and streamlines (black) from numerical simulations with a) $\theta_0 = 90^o$, b) $\theta_0 = 60^o$, c) $\theta_0 = 45^o$, d) $\theta_0 = 30^o$. The red dotted lines indicate an angle of $\theta_0/2$ from the horizontal, which is found to be a good approximation for the slope of the upper-left yield surface at the vertex of the plug.

surface meeting the boundary tangentially and the vertex of the plug being a right-angle. Although not conclusive from the simulations, §6.5.1 gives a general argument that both these results must be the case for a stagnation-point plug in any flow configuration and Bingham number. Secondly, as the stagnation angle, $\theta_0$, decreases, the plug becomes increasingly asymmetric and its aspect ratio (height to width) decreases. In §6.5.3 we rationalise these trends by approximating the upper surfaces of the plug as arcs of circles. Finally, we note that the plug size also increases with decreasing $\theta_0$. This can largely be explained as a result of our choice of non-dimensionalisation, as discussed in §6.5.4.
6.5 Plug geometry

In the following sections we rationalise properties of the stagnation point plug geometry. Let $\phi$ be the angle between the yield surface and the $x$-axis at a point on the yield surface, and $\{t, n\}$ be a basis formed by the tangent and normal vectors at that point. Let $(x_V, y_V)$ be the coordinates of the vertex of the plug, and let $x_L$ and $x_R$ be the $x$-coordinates of the intersection of the yield surface with the plug on the left and right, respectively. Figure 6.5 provides a schematic of these definitions.

6.5.1 Vertex angles

By definition, $||\tau|| = 1$ (or $Bi$ in the non-dimensional global problem) at the yield surface. We also have the component $\tau_u = 0$ since $u \cdot t = 0$ from no-slip on the plug. By considering the direction of shear we can write, in the tangent-normal basis,

$$
\tau = \begin{cases} 
- \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & x < x_V \\
0 & 1, & x > x_V 
\end{cases} \quad (6.40)
$$

Rotating by $\phi$ gives the deviatoric stress tensor in the Cartesian basis as

$$
\tau = \begin{cases} 
- \begin{pmatrix} -\sin 2\phi & \cos 2\phi \\ \cos 2\phi & \sin 2\phi \end{pmatrix}, & x < x_V \\
- \begin{pmatrix} -\sin 2\phi & \cos 2\phi \\ \cos 2\phi & \sin 2\phi \end{pmatrix}, & x > x_V 
\end{cases} \quad (6.41)
$$

When $x \to x_R$, we have $\tau_{xx} \to 0$ and $\tau_{xy} \to 1$ due to no slip on the wall. Thus $\phi \to 0$ and the plug joins the wall tangentially. The same can be deduced for $x \to x_L$ (for which
τ_{xy} \to -1). Now consider the limit \( x \to x_V \), in which the stress should be continuous. If the left and right limits of \( \phi \) are denoted \( \phi_L \) and \( \phi_R \), respectively, then we have

\[
\sin 2\phi_L = -\sin 2\phi_R, \\
\cos 2\phi_L = -\cos 2\phi_R,
\]

which implies \( 2\phi_L \equiv \pi + 2\phi_R \) (mod \( 2\pi \)) and so \( \phi_L - \phi_R = \pi/2 \). Thus the angle at the vertex must be \( \pi/2 \), as is observed in the numerical simulations. This argument would apply at the vertex of any planar stagnation-point plug regardless of Bingham number or boundary geometry, and thus generalises and clarifies the observations of Tokpavi et al. [137], Hewitt and Balmforth [69], and Chaparian and Frigaard [37] that this angle attains \( \pi/2 \) in certain limits and configurations—by demonstrating that it is, in fact, always \( \pi/2 \). We note that this differs from stagnation points in viscoplastic flow down a thin slot, for which the vertex of the plug forms a sharp cusp, as demonstrated by Hewitt et al. [70].

### 6.5.2 Determining \( \phi_L \)

We define polar coordinates around the vertex of the plug. This has a sub-leading effect on the streamfunction at large \( r \). Thus the leading order is given still by

\[
\psi = r^3 \sin^2 \theta \sin(\theta_0 - \theta) + \ldots.
\]

The dividing streamline, on which \( \psi = 0 \), is given asymptotically by \( \theta = \theta_0 \). On this streamline, the deviatoric stress is given to leading order by

\[
\begin{pmatrix}
\tau_{rr}
\tau_{r\theta}
\end{pmatrix}
= -4r \sin \theta_0 \begin{pmatrix}
\sin \theta_0 \\
\cos \theta_0
\end{pmatrix} + \ldots,
\]

which, converted to the Cartesian basis, gives

\[
\begin{pmatrix}
\tau_{xx}
\tau_{xy}
\end{pmatrix}
= 4r \sin \theta_0 \begin{pmatrix}
\sin \theta_0 \\
-\cos \theta_0
\end{pmatrix} + \ldots.
\]

Thus, the orientation of the deviatoric stress tensor is independent of \( r \) and given by \( (\sin \theta_0, -\cos \theta_0) \). The dividing streamline, \( \psi = 0 \), must hit the vertex of the plug. Thus, if we assume the orientation of the deviatoric stress remains unchanged along this streamline, we have that the stress at the vertex is given by \( (\tau_{xx}, \tau_{xy}) = (\sin \theta_0, -\cos \theta_0) \). Earlier we saw that the stress here is given by \( (\tau_{xx}, \tau_{xy}) = (\sin 2\phi_L, -\cos 2\phi_L) = (-\sin 2\phi_R, \cos 2\phi_R) \),
6.5. PLUG GEOMETRY

Figure 6.6: The inclination of the vertex of the stagnation point plug, $\phi_L$, as a function of stagnation angle, $\theta_0$, from numerical simulations (blue dots) and the heuristic approximation $\phi_L = \theta_0/2$ (dotted).

which would imply $\phi_L = \theta_0/2$. For $\theta_0 = 90^\circ$ the problem is symmetrical and hence we must have $\phi_L = 45^\circ$, as predicted by $\phi_L = \theta_0/2$. Figure 6.4 also shows the numerically determined plug geometries for $\theta_0 = 60^\circ, 45^\circ,$ and $30^\circ$, with an overlaid slope of angle $\theta_0/2$, indicating the effectiveness of this approximation to the slope of the top of the plug at the vertex. Furthermore, figure 6.6 shows the values of $\phi_L$ determined from the numerical simulations, showing good agreement with the value predicted by $\phi_L = \theta_0/2$.

For the purposes of producing this figure, $\phi_L$ was determined by measuring the direction of the fluid velocity (assumed to be parallel to the yield surface of the stagnation plug) in the vicinity of the vertex of the plug. This direction becomes poorly determined very close to the plug, since the velocity vanishes here, but at larger distances from the plug the velocity direction could deviate from being parallel to the yield surface. Thus, we measure the angle of the velocity at a small but finite distance from the vertex of the plug, specifically choosing to measure it at the point a distance 0.005 from the vertex of the plug, in the direction perpendicular to the upper-left slope of the plug.

This argument for the angle $\phi_L$ is heuristic in nature, and a more rigorous matching of the stress orientation between the far-field and the neighbourhood of the plug vertex is required to rigorously determine $\phi_L$. Notably, from (6.36) we see that the dividing streamline is expected to steepen as it approaches the plug, due to the logarithmic terms in the asymptotic solution, and thus we may expect $\phi_L$ to be slightly larger that $\theta_0/2$, which is consistent with figure 6.6.
6.5.3 Relative dimensions of the plug

From the numerical results it appears we can roughly approximate the upper surfaces of the plug as arcs of circles. With $\phi_L$ determined, this allows us to make approximate conclusions about the relative dimensions of the plugs, rationalising the trends seen in figure 6.4. If the two circles have radii $r_L$ and $r_R$ on the left and right respectively (see figure 6.7), then we find the ratio of radii

$$\frac{r_R}{r_L} = \frac{1 - \cos \phi_L}{1 - \sin \phi_L}, \quad (6.47)$$

the ratio of right and left widths

$$\frac{x_R - x_V}{x_V - x_L} = \frac{1 - \cos \phi_L}{1 - \sin \phi_L} \cot \phi_L, \quad (6.48)$$

and the aspect-ratio of height to total-width

$$\frac{y_V}{x_R - x_L} = \frac{(1 - \cos \phi_L)(1 - \sin \phi_L)}{\sin \phi_L + \cos \phi_L - 1}, \quad (6.49)$$

where $y_V$ is the $y$-coordinate of the plug vertex.

Figure 6.8 compares the ratios (6.48) and (6.49) obtained from numerical simulations with those predicted by the right-hand sides using the approximation $\phi_L = \theta_0/2$. The general behaviour of the ratios with $\theta_0$ is captured reasonably well by these approximations, although they systematically underestimate both ratios. This is somewhat explained by the observation in §6.5.2, that $\phi_L$ is actually slightly larger than $\theta_0/2$, since both (6.48) and (6.49) are increasing functions of $\phi_L$. 
6.5. PLUG GEOMETRY

Figure 6.8: The geometrical ratios of the stagnation-point plugs as functions of $\theta_0$. The left panel shows a measure of symmetry, namely the ratio of widths to the right and left of the vertex, $(x_R - x_V)/(x_V - x_L)$, and the right panel shows the aspect ratio of height to width, $y_V/(x_R - x_L)$. The blue dots are determined from numerical simulations while the dotted lines indicate the predictions from the expressions (6.48) and (6.49) using the approximation $\phi_L = \theta_0/2$.

6.5.4 Plug width

From figure 6.4 we can see that the overall size of the plug grows with decreasing $\theta_0$. One explanation for this is that, due to our choice of non-dimensionalisation, the leading order far-field strain rate at the walls is given by

$$\dot{\gamma} = 2r \sin \theta_0,$$

(6.50)

while the strain rate along the leading-order dividing streamline, $\theta = \theta_0$, is given by

$$\dot{\gamma} = 4r \sin \theta_0.$$

(6.51)

An alternative non-dimensionalisation, specifically $L_V = \tau_c/\mu k \sin \theta_0$ and $U = kL_V^2 \sin \theta_0$, could be chosen to remove the factor of $\sin \theta_0$ from these expressions, making the far-field strain rates more comparable for different stagnation angles. With a comparable far-field strain rate, we might expect the plugs to be approximately the same size under this new non-dimensionalisation, and hence, in our original non-dimensionalisation,
CHAPTER 6. STAGNATION POINT FLOW OF A VISCOPLASTIC FLUID

Figure 6.9: The plug width, $x_R - x_L$, and area, $\int_{x_L}^{x_R} y_Y(x) \, dx$ (where $y = y_Y(x)$ is the equation of the yield surface), as functions of stagnation angle, $\theta_0$. Blue dots are determined from numerical simulations, while the dotted lines show the curves proportional to $\csc(\theta_0)$ (left) and $\csc^2(\theta_0)$ (right) fitted to pass through the numerical data at $\theta_0 = \pi/2$.

the width and area of the plugs to be proportional to $1/\sin \theta_0$ and $1/\sin^2 \theta_0$. Figure 6.9 compares the numerically determined plug widths and areas with this prediction, showing a reasonable agreement, particularly for the plug area. For the global problem, with the non-dimensionalisation discussed in §6.6 and Appendix 6.A, this approximation gives

\[
\text{Plug area} \approx 0.037Bi^2 \csc^2(\theta_0). \tag{6.52}
\]

6.6 Embedding in global flow

We consider a global flow in which a stagnation point is expected to occur, such as flow in a T-junction, or around a blunt object. The geometry of this global flow imposes a geometrical length scale, $L_G$. If the typical velocity of the fluid is $U_0$, then we can define a Bingham number $Bi = \tau_c L_G/(\mu U_0)$. The conclusion that the solutions calculated in §6.4 apply locally to a stagnation point in this global flow is not immediate, since the analysis in §6.3 assumes large distances from the stagnation point, while applying
the viscous stagnation point flow (6.3), and neglecting curvature of the boundary, both assume small distances from the stagnation point. When $Bi$ is asymptotically small, we can justify these apparently contrasting assumptions by working at an intermediate length scale, which is large compared to the viscoplastic length scale, $L_{V}$ (defined in §6.2), but small compared to the global length scale, $L_{G}$. The details of this argument are given in Appendix 6.A, however this embedding of the local theory in a global flow problem is perhaps best demonstrated via application to some specific flow configurations. We consider the examples of recirculating flow in a wedge and flow around a cylinder in the following sections.

6.6.1 Corner Eddies

As shown in §4, stagnation point plugs occur on the boundary of a wedge in which eddies are being driven by a disturbance far from the vertex of the wedge. We can use the viscous Moffatt solution [92] to determine the angle between the dividing streamline and the boundary at the stagnation points in the viscous limit $Bi \to 0$. The streamfunction for the viscous solution is given (see (4.5)) by

$$\Psi_{V} = Ar^{\lambda}f(\theta),$$

(6.53)

where

$$f(\theta) = \cos(\lambda \theta) \cos((\lambda - 2)\alpha) - \cos((\lambda - 2)\theta) \cos(\lambda \alpha),$$

(6.54)

the eigenvalue, $\lambda = \lambda_{r} + i\lambda_{i}$, is the solution of

$$\sin(2(\lambda - 1)\alpha) + (\lambda - 1)\sin(2\alpha) = 0,$$

(6.55)

$A$ is an arbitrary complex constant, and the real part is assumed. Using the non-dimensionalisation given in §4 we have

$$A = -\frac{i}{\lambda_{i}f(0)}$$

(6.56)

(see (4.5)).

The dividing streamline between two eddies is given by $\Psi_{V} = 0 \implies \Re(Ar^{\lambda}f(\theta)) = 0$, where $\Re$ denotes the real part. The stagnation point on the rigid boundary is found where

$$\dot{\gamma} = 0 \implies \frac{1}{r} \frac{\partial u}{\partial \theta} = 0,$$

(6.57)
since $v, \partial v/\partial r,$ and $\partial u/\partial r$ all vanish on the boundary, due to the boundary conditions $u = v = 0$ along $\theta = \alpha$. The stagnation point is therefore found at the point $(r_c, \alpha)$, satisfying

$$
\frac{1}{r} \frac{\partial u}{\partial \theta} = 0 \quad \Rightarrow \quad \Re \left( -\frac{i r_c^\lambda}{\lambda f(0)} f''(\alpha) \right) = \Im \left( \frac{r_c^\lambda}{\lambda f(0)} f''(\alpha) \right) = 0,
$$

$$
\Rightarrow \quad r_c = \exp \left( \frac{1}{\lambda} \left[ n\pi - \arg \left( \frac{f''(\alpha)}{f(0)} \right) \right] \right),
$$

where $\Im$ denotes the imaginary part, and $n$ is an arbitrary integer with $n = 0$ giving the stagnation point associated to the dividing streamline passing through $r = 1$.

The angle at which this streamline meets the boundary, $\theta = \alpha$, is thus given by

$$
\theta_0 = \arctan \left( -\frac{\lambda_i}{\Im \left( \lim_{\theta \to \alpha} \frac{f''(\theta)}{f'(\theta)} \right)} \right).
$$

Since $f(\alpha) = f'(\alpha) = 0$, we use l'Hôpital's rule to obtain

$$
\theta_0 = \arctan \left( -\frac{\lambda_i}{\Im \left( \frac{f''(\alpha)}{f'(\alpha)} \right)} \right).
$$

The stagnation angle, $\theta_0$, is plotted against wedge half-angle, $\alpha$, in figure 6.10.

We assume that the streamfunction in the vicinity of the stagnation point takes the form

$$
\Psi = K \rho^3 \sin(\theta_0 - \phi) \sin^2 \phi + \ldots,
$$

where $(\rho, \phi)$ are polar coordinates defined around the stagnation point such that $\phi$ is measured anticlockwise with $\phi = 0$ pointing along the boundary of the wedge towards the vertex, $r = 0$. Then, we consider how the streamfunction varies in the direction perpendicular to the boundary. For the local form, this corresponds to setting $\phi = \pi/2$ and taking derivatives with respect to $\rho$, while for the global form, this corresponds to
6.6. EMBEDDING IN GLOBAL FLOW

6.6.1 Evaluation of the Gradient

Evaluating the gradient in the negative $\theta$ direction at the point $(r_c, \alpha)$. These must be equivalent and, in particular, taking the third derivative we find

$$6K \cos \theta_0 = - \left( \frac{1}{r^3} \frac{\partial^3 \Psi_V}{\partial \theta^3} \right)_{(r_c, \alpha)} = \Im \left[ \frac{r_c^{\lambda-3} f''''(\alpha)}{\lambda f'(0)} \right], \quad (6.66)$$

which determines $K$ in terms of $\alpha$.

Figure 6.11 shows a comparison between stagnation plugs from global numerical simulations of viscoplastic corner eddies, with $Bi = 1$ (as defined in §4), and those determined by the method described in §§6.3-6.4 for the idealised problem, with the appropriate choice of $\theta_0$ determined from (6.64) and scaled by $Bi/K$ with $K$ given by (6.66). For $\alpha = 60^\circ$ we find $\theta_0 \approx 45.4^\circ$ and $K \approx 3.33$, while for $\alpha = 20^\circ$ we have $\theta_0 \approx 57.6^\circ$ and $K \approx 46.4$. The agreement between the geometries of the plugs demonstrates the validity of the local theory embedded in the global problem, and greater resolution is achievable for the idealised problem, since the flow in the remainder of the domain does not need to be calculated.

6.6.2 Flow around a cylinder

For uniform flow of velocity $U$, past a cylinder of radius $a$, in a viscoplastic fluid, stagnation points exist on the boundary of the cylinder upstream and downstream, resulting in unyielded material at these points. The symmetry of the problem implies that the stagnation point flow is perpendicular to the surface of the cylinder (i.e. $\theta_0 = 90^\circ$), so we can compare the stagnation-point plugs with the geometry found in the first panel of
figure 6.4. To non-dimensionalise the problem we use $a$ for lengths, $U$ for velocities, and $\mu U/a$ for stresses. In the general theory of Appendix 6.A we require the local expansion of the stream function for this non-dimensionalised global problem, in the vicinity of the stagnation point, when the Bingham number is small. Hewitt and Balmforth [69] give an expansion for this stream function in small $Bi = \tau_c a / (\mu U)$, with the leading order term given by

$$\Psi = -\sin \theta \left( \frac{2r \log r - r + r^{-1}}{2 \log Bi^{-1}} \right).$$  (6.67)

Here, $(r, \theta)$ are polar coordinates measured from the centre of the cylinder, and with the background flow in the direction $\theta = \pi$. The logarithmic dependence on $Bi$ arises by matching to the unyielded solution in the far field, and this significance of the yield stress in the leading order Newtonian solution near the cylinder is analogous to the role of inertia in the removal of the classical Stokes paradox for Newtonian flow around a cylinder [110]. Notably, the asymptotic expansion is expected to continue above this order with further powers of $(\log Bi^{-1})^{-1}$ which results in a very slow decay in $Bi$ and relatively poor accuracy of the leading order solution for finite values of $Bi$. This is demonstrated in [69] by the discrepancy between the force exerted on the cylinder in the direction of the far-field flow predicted by the leading order solution (6.67) and calculated from direct numerical simulations. For example, at $Bi = 1/64$ this force was found to be $F_x = 6.09$ from the numerical simulations (numerical value provided from private correspondence), while the leading order solution gives a prediction of
\[ F_x = 4\pi / \log Bi^{-1} = 4\pi / \log 64 = 3.02 \] (both to 3 significant figures), which differs from the numerical solution by approximately a factor of 2. Nonetheless, in an asymptotic sense, the leading order solution (6.67) governs the behaviour of the solution in the vicinity of the stagnation point for small Bingham numbers.

To expand around the stagnation point, we define a secondary set of polar coordinates, \((\rho, \phi)\), around the point \((r, \theta) = (1, 0)\) with \(\rho \ll 1\). We then have

\[ r = \sqrt{(1 + \rho \cos \phi)^2 + (\rho \sin \phi)^2}, \quad \sin \theta = \frac{\rho \sin \phi}{r}. \]

(6.68)

Substituting into (6.67) and expanding in \(\rho\) we find

\[ \Psi \sim -\frac{1}{\log Bi^{-1}} \rho^3 \cos^2 \phi \sin \phi + \ldots, \]  

(6.69)

which, after a rotation of \(\pi/2\), corresponds to a local stagnation point flow with \(\theta_0 = \pi/2\) (as anticipated) and coefficient \(K = \left(\log Bi^{-1}\right)^{-1}\). The theory given in Appendix 6.A then implies that the solution found in §6.4 for \(\theta_0 = \pi/2\) should be scaled by a factor of \(Bi/K = Bi \log Bi^{-1}\) to compare with the stagnation plug in the non-dimensionalised global problem.

To make this comparison we compute the flow around a cylinder using the same algorithm as described in §6.4. For uniform flow of a viscoplastic fluid around a cylinder, the fluid yields in an envelope around the cylinder but is unyielded in the far-field (e.g. see Tokpavi et al. [137], Hewitt and Balmforth [69]). The domain for the simulation (much larger than the region shown in the first panel of figure 6.12) was therefore chosen sufficiently large to enclose the entire yielded zone, while symmetry was exploited to restrict the domain to the first quadrant.

Figure 6.12 shows such a comparison for \(Bi = 1/32\), demonstrating good agreement between the shapes of the stagnation-point plug, but that the scaling is an overestimate by about a factor of 2. This discrepancy is consistent with the discrepancy in the force exerted on the cylinder found by Hewitt and Balmforth [69] as discussed above, and so we can attribute it to the slow decay of the logarithmic terms in the asymptotic series for the “outer” streamfunction (away from the stagnation point), for \(Bi \ll 1\). As such, we anticipate that the accuracy of the predicted scaling would increase as the Bingham number is reduced, however it becomes difficult to accurately resolve the stagnation point plug in the numerical simulations for significantly smaller Bingham numbers. The comparison also highlights the absence of curvature from the leading order considerations at low Bingham number.
For larger Bingham numbers, the length scale of the plug becomes comparable to the length scale of the geometry and the local theory developed above does not apply. The plug in this regime has been well studied by Tokpavi et al. [137] who showed that the plug grows and the sides straighten as the Bingham number is increased, eventually tending to a triangular cap with straight sides meeting the cylinder tangentially, as predicted by plasticity theory for $Bi \rightarrow \infty$.

### 6.7 Conclusions

In this chapter we have considered stagnation points in slow viscoplastic flows. We calculated numerical simulations of the prototypical example of flow against a straight boundary, using an asymptotic solution for the far-field boundary condition, and show that this problem can be considered as the solution local to a stagnation point in a more general flow configuration. The dependence of the plug geometry on the stagnation angle, $\theta_0$, between the dividing streamline and the boundary, was explored, showing that the plug becomes less symmetrical, and of smaller aspect ratio (height perpendicular to the boundary to width along the boundary), as $\theta_0$ decreases. We further show that the angle at the vertex of a stagnation-point plug is always a right-angle, and the plug always meets the boundary tangentially, clarifying observations in the literature regarding the
angle subtended by the vertex in numerical simulations. Finally, we give two examples of stagnation-point flows in specific flow configurations, demonstrating the effectiveness of the local theory when the plug is small relative to the length scale of the global flow.

6.A Details of embedding

For the global problem, we non-dimensionalise lengths by $L_G$, velocities by $U_0$, and pressures and stresses by $\mu U_0/L_G$. We define polar coordinates $(R, \theta)$ around the stagnation point, with $\theta = 0, \pi$ being tangent to the boundary, and velocities $(U, V)$ in the radial and polar directions. Then, the Bingham constitutive law is given by

$$
\begin{pmatrix}
\tau_{RR} \\
\tau_{R\theta}
\end{pmatrix} = \left( 1 + \frac{Bi}{\dot{\gamma}} \right) \begin{pmatrix}
\dot{\gamma}_{RR} \\
\dot{\gamma}_{R\theta}
\end{pmatrix},
$$

where $Bi = \tau_c L_G/(\mu U_0)$ is the Bingham number for the global problem. If this Bingham number is small then the yield stress only becomes significant in the neighbourhood of the stagnation point where the strain rate vanishes. In particular, comparing to §6.2 we have the dimensional radial coordinate $\tilde{r} = L_V r = L_G R$ and so

$$R = \frac{L_V}{L_G} r = \frac{\tau_c/(\mu k)}{L_G} r = \frac{U_0}{k L_G^2} Bi r. \quad (6.71)$$

The dimensionless constant $U_0/(k L_G^2)$ is typically $O(1)$ (although the example of flow around a cylinder gives a counter-example to this, as discussed in §6.6.2), since $k L_G^2$ gives the scale of the stagnation-point velocity at distances from the stagnation point on the order of the global length scale, which generally matches to the typical velocity $U_0$ of the global flow. Thus, $R = O(Bi)$ when $r = O(1)$. Due to this separation of scales, we can define an intermediate radial coordinate, $\eta$, by $R = Bi^{\alpha} \eta$ with $0 < \alpha < 1$, so that, for $\eta = O(1)$ we have $R \ll 1$ and $r \gg 1$. For the global problem we have non-dimensional streamfunction, $\Psi = \Psi(R, \theta)$ which, in the intermediate region, we can expand in powers of $R \ll 1$. Since the strain rate vanishes as $R \to 0$, to leading order the streamfunction takes the form

$$\Psi = R^3 F_0(\theta) + \ldots = Bi^{3\alpha} \eta^3 F_0(\theta) + \ldots \quad (6.72)$$

Provided the boundary is smooth, it is straight to leading order in $R \ll 1$, and so we can apply no slip and no penetration boundary conditions at $\theta = 0, \pi$. Substituting into the conservation of momentum at leading order, we find $F_0$ is again given by the viscous solution (c.f. (6.5))

$$F_0 = K \sin^2 \theta \sin(\theta_0 - \theta) = K f_0(\theta; \theta_0), \quad (6.73)$$

145
for some value of the constants $K$ and $\theta_0$. Thus, we can indeed match to an asymptotic solution of the form derived in §6.2. The leading order dimensional streamfunction in the intermediate region is given by

$$\tilde{\psi} = U_0 L G \Psi = U_0 L G K R^3 f_0(\theta), \quad (6.74)$$

from (6.73). While, in the notation of §6.2, from (6.5), we have

$$\tilde{\psi} = k r^3 f_0(\theta) = k L_G^3 R^3 f_0(\theta). \quad (6.75)$$

For equivalence, we require $K = k L^2 G / U_0$ and the relationship between the local and global length scales is given by

$$R = \frac{1}{K} B i r. \quad (6.76)$$

Note that, where $K = O(1)$, a consistent and convenient choice of $U_0$ (a velocity scale for the global flow) is such that $U_0 = k L^2 G$, which gives $R = B i r$ and $K = 1$. We finally need to check that moderate curvature of the boundary at the stagnation point in the global flow configuration does not enter the asymptotic solution until higher orders than those considered in §6.3.

In general, any $O(1)$ curvature, $\kappa$ (non-dimensionalised by $1/L_G$), would enter our asymptotic problem after leading order, via the boundary conditions being applied at

$$\theta = 0 + \kappa R + O(R^2) = 0 + B i^3 \kappa \eta + O(\text{Bi}^{2\alpha}), \quad (6.77)$$

$$\theta = \pi - \kappa R + O(R^2) = \pi - B i^3 \kappa \eta + O(\text{Bi}^{2\alpha}). \quad (6.78)$$

For the no-slip and no penetration boundary conditions, this gives

$$\left. \frac{\partial \Psi}{\partial R} \right|_{\theta=0} = -B i^3 \kappa \eta \left. \frac{\partial^2 \Psi}{\partial R \partial \theta} \right|_{\theta=0} + \ldots, \quad (6.79)$$

$$\left. \frac{\partial \Psi}{\partial \theta} \right|_{\theta=0} = -B i^3 \kappa \eta \left. \frac{\partial^2 \Psi}{\partial \theta^2} \right|_{\theta=0} + \ldots, \quad (6.80)$$

$$\left. \frac{\partial \Psi}{\partial R} \right|_{\theta=\pi} = B i^3 \kappa \eta \left. \frac{\partial^2 \Psi}{\partial R \partial \theta} \right|_{\theta=\pi} + \ldots, \quad (6.81)$$

$$\left. \frac{\partial \Psi}{\partial \theta} \right|_{\theta=\pi} = B i^3 \kappa \eta \left. \frac{\partial^2 \Psi}{\partial \theta^2} \right|_{\theta=\pi} + \ldots. \quad (6.82)$$

$\Psi = O(\text{Bi}^{3\alpha})$ to leading order, and so the right-hand sides of (6.79)-(6.82) are zero up to $O(\text{Bi}^{4\alpha})$. Meanwhile, substituting $r = B r^{\alpha-1} \eta$ into the asymptotic solution (6.14), we have

$$\Psi = \frac{1}{U_0 L G} \tilde{\psi} = B i^3 \tilde{\psi}(r, \theta) = B i^3 \left( B i^{3\alpha-3} \eta^3 f_0 + B i^{2\alpha-2} \eta^2 f_1(r, \theta) + \ldots \right), \quad (6.83)$$
which includes terms up to $O(Bi^{2\alpha+1} \ln(1/Bi))$ (from the logarithmic term in $f_1$). Hence, provided we choose our intermediate region with $\alpha > 1/2$ (i.e. sufficiently close to the plug) and the Bingham number is small, then an $O(1)$ curvature of the boundary in the global flow can be neglected in the asymptotic solution derived in §6.3, up to the orders considered.

Since the solution in the inner $r$-region, calculated in §6.4, depends only on $\theta_0$, which is fixed for a given flow configuration, and the inner coordinate is given by $r = KR/Bi$, with $K$ typically $O(1)$, we can immediately deduce the expected result that, for $Bi \ll 1$, the length scale of the plug in a general flow configuration is $O(Bi)$ compared to the length scale of the global flow. Note, further, that the free choice of the constant $D$ in the function $G$, (6.26), then corresponds to an $O(Bi)$ translation of the origin in the outer $R$ coordinate, which is consistent with the fact that the origin could be chosen anywhere within the $O(Bi)$ plug that surrounds the stagnation point.
Scraping of a thin layer of viscoplastic fluid

Authorship: The material in this chapter is the result of original research by J. J. Taylor-West and A. J. Hogg.

7.1 Introduction

The following chapter concerns the scraping of a thin layer of viscoplastic fluid from a plane surface. This flow configuration and rheology is relevant to a number of environmental and industrial processes. For example, the removal of excess plaster from a wall or of a layer of mud from a road following a mudslide may be carried out by the translation of a vertical scraper, where both of these materials are known to exhibit viscoplastic rheology [13, 89, 121]. A similar flow configuration is also relevant to certain blade coating and screen printing processes where the desired result is a uniform residual layer of fluid behind the scraper [114, 145, 87, 71]. Finally, in geophysics, similar flow configurations have been used as a model for the formation of fold-and-thrust belts and accretionary wedges at converging tectonic plates, whereby a layer of relatively soft sediment is scraped off a descending plate by the more rigid overlying plate during plate subduction. Emerman and Turcotte [58] used a Newtonian rheology and shallow layer theory, identifying a late time similarity solution in which the shape of the wedge is quasi-static, although they do not explicitly determine the time dependence of the height or length of the mound in this regime. Similarly, Perazzo and Gratton [102] considered the uplift due to the convergence of two shallow layers of Newtonian fluid, identifying
an early time similarity solution in which the height and width of the wedges grow like \( t^{1/2} \) and a late time similarity solution in which the height grows like \( t^{1/4} \) and the length like \( t^{3/4} \) (corresponding to the shape found by Emerman and Turcotte [58]). Ball et al. [15] considered an extension to the viscous wedge theory in which the underlying plate is modelled as an elastic beam. Stockmal [129] and Davis et al. [47] studied the problem for a plastic Coulomb rheology by using shallow layer theory and slipline theory respectively, under the assumption that the material is at failure throughout the wedge. Fully two-dimensional finite-element simulations accounting for the free surface have also been carried out for Coulomb plastic [130] and viscoelastic-plastic [127, 119] rheologies. These simulations benefit from being able to resolve heterogeneous deformation and complex surface features that are otherwise averaged out by the shallow layer approximation, in particular they predict deformation via a sequence of folds or thrusts along localised shear bands. On the other hand, shallow-layer theory allows the development of simplified results which can predict average mound shapes and identify different scaling regimes for the time evolution, without the need for intensive computation.

The scraping of a viscoplastic fluid was investigated by Lister and Hinton [81], who studied the steady state problem in which fluid entering the mound is balanced by flux under the scraper or around a finite width scraper under the approximations of shallow-layer theory. They solved for the steady state shape of a quasi-rigid (yield-stress dominated) mound in front of an infinite scraper and a long finite scraper, with the latter held perpendicular or obliquely to the direction of travel. They also showed how the steady state is approached at late times for a Newtonian fluid and infinite scraper, by assuming that the flux under the scraper is proportional to the free-surface height upstream from the scraper. Maillard et al. [87] carried out experiments for the scraping of a layer of Carbopol gel with a sharp scraper. They observed an instability resulting in undulations of the free surface and a residual layer of roughly uniform height was left behind the scraper. They also found that the height of the mound was proportional to the square root of its length (after correcting for some initial transient), which they explain theoretically by considering a force balance between the yield stress and hydrostatic pressure, under the assumption that the mound was predominantly unyielded. This assumption was supported by particle image velocimetry (PIV) measurements that indicated that the mound was approximately rigid, being pushed along by the scraper, and separated from the base by a uniform sheared layer. They do not attempt to compare the experimental free-surface profiles with predictions from shallow-layer (or other) theory.

In this chapter, we provide predictions from shallow-layer theory for the shape and
time evolution of the free surface of a layer of Bingham or Herschel-Bulkley fluid being scraped from a plane surface by a vertical rigid scraper. We provide similarity solutions at early and late times, and show how these predictions are effected by leakage under the scraper, which we model explicitly via a Couette-Poiseuille flow in a narrow rectangular gap under the scraper. The predictions for the free-surface profiles are compared against preliminary experiments using a commercial hair gel and show reasonable agreement once a slip boundary condition is included in the model, as motivated by experimental observations of slip.

7.2 Problem definition

We consider the scraping of a layer of viscoplastic fluid, of depth $h_{\infty}$, from a horizontal plane surface by a scraper travelling at a constant velocity, $U$, parallel to the surface. We assume a planar flow, and that the scraper has infinite width in the out-of-plane direction. Figure 7.1 shows a schematic of the flow geometry, in the frame of reference moving with the scraper. The free surface, $z = h(x,t)$, has height $h_0(t) = h(0,t)$ at the point where it meets the scraper on the up-stream side, and height $h_{\infty}$ as $x \to \infty$. We first consider a Bingham fluid, with density, $\rho$, yield stress, $\tau_c$, and viscosity $\mu$, before generalising to a Herschel-Bulkley fluid in §7.7. The acceleration due to gravity is $g$.

Under the lubrication approximation, as detailed in §2.3, the flow is in the horizontal direction to leading order, and takes the form of an apparently unyielded plug riding on top of a yielded layer. At higher orders this apparent plug is typically weakly yielded, allowing the plug velocity to vary with $x$. In the absence of surface tension, the governing equation, obtained by integrating the mass-conservation equation over the height of the
layer, is
\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\rho g Y^2 (3h - Y)}{6\mu} \frac{\partial h}{\partial x} + Uh \right),
\]  
\tag{7.1}
\]
where \( Y(x, t) \) is the location of the apparent yield surface,
\[
Y = \max \left( 0, h - \frac{\tau_c}{\rho g |\partial h/\partial x|} \right).
\tag{7.2}
\]

We non-dimensionalise vertical lengths by \( h_\infty \), velocities by \( U \), horizontal lengths by \( L = \rho gh_\infty^3/(\mu U) \), a length-scale over which a typical hydrostatic pressure gradient balances vertical gradients in viscous stresses, and time by \( T = L/U \). After non-dimensionalising (and relabelling the variables without hats) we obtain the governing equation
\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{6} Y^2 (3h - Y) \frac{\partial h}{\partial x} + h \right),
\]  
\tag{7.3}
\]
where
\[
Y = \max \left( 0, h - \frac{Bi}{|\partial h/\partial x|} \right),
\tag{7.4}
\]
and the residual non-dimensional parameter is the Bingham number,
\[
Bi = \frac{\tau_c h_\infty}{\mu U},
\tag{7.5}
\]
which measures the magnitude of the yield stress relative to viscous stresses developed in the flowing layer. The addition of the term, \( h \), on the right hand side of (7.3) (compared to (2.65) in §2.3), arises due to the change of frame of reference to that of the translating scraper.

Due to the lubrication approximation, we are unable to enforce no-slip or penetration at the scraper due to the absence of vertical structure in the leading order solution. Instead boundary conditions are obtained by imposing a unit dimensionless flux in the negative \( x \)-direction far upstream of the scraper, and a dimensionless leakage flux at the upstream edge of the scraper, \( x = 0 \). In the case that there is no gap below the scraper, \( h_\alpha = 0 \), this outgoing flux is 0, otherwise there is a leakage flux, \( Q_\alpha \). Lister and Hinton [81] explore the steady state that occurs when this leakage flux approaches unity due to a large hydrostatic pressure at the upstream side of the gap. In this case, the residual unyielded layer left behind the scraper is the same height as the incoming layer, and no additional volume is being removed from the layer. In practice this steady state may take a large time to be reached, and, at least at early times, we might instead expect the layer behind the scraper to be approximately uniform and comparable to the gap height \( h_\alpha \).
(particularly for a high value of the yield stress). This has been illustrated in experiments. For example, Maillard et al. [87] carried out experiments for a similar configuration using Carbopol, and observed an approximately uniform residual layer height, given by \( h \approx 1.1 h_a \), for the range of motion they were limited to, corresponding to a constant leakage flux. In §7.3 we propose a simple model for the leakage flux as a function of the free-surface height in front and behind the scraper. Motivated by this model, in §7.5, we then explore the evolution of the free surface in the case of a constant leakage flux, for which similarity solutions can be deduced at early and late times, before examining the full time-dependent leakage flux in §7.6 to show how this alters the dynamics.

### 7.3 Leakage flux model

As noted by Lister and Hinton [81] a large mound height at the scraper results in a high hydrostatic pressure at the gap between the scraper and surface, which should drive additional leakage flux through this gap. One model for the leakage flux when the gap is small, i.e. \( h_a \ll L_a \) in figure 7.1, is to assume a Couette-Poiseuille flow in the gap. In dimensional variables, assuming hydrostatic pressure behind the scraper, the horizontal pressure gradient in the thin gap is given approximately by

\[
\frac{\partial p}{\partial x} = G = \frac{\rho g (h_0 - h_b)}{L_a},
\]

(7.6)

where \( h_0 \) and \( h_b \) are the (dimensional) free-surface heights in front of and behind the scraper, respectively.

Depending on the size of the pressure gradient, we have three different flow regimes in the gap: for sufficiently small \( G \) (in a non-dimensional sense, given explicitly below) the fluid is fully yielded in the gap due to the translating bottom boundary (in the frame of the scraper); for intermediate \( G \) the fluid is yielded between the scraper and an unyielded plug is attached to the translating bottom boundary; and for large \( G \) the fluid is yielded in regions adjacent to the scraper and the bottom boundary, while a central plug translates down the gap at a speed, \( U_p \), which exceeds the bottom boundary speed, \( U \) (see figure 7.2). The dimensional flux (in the \( x \)-direction) is given by

\[
Q = \begin{cases} 
-\frac{U h_a}{2} - \frac{G h_a^3}{12 \mu} & \text{for } 0 < G < \frac{2 \mu U}{h_a}, \\
-U h_a + \frac{U_p}{3} \sqrt{\frac{2 \mu U}{G}} & \text{for } \frac{2 \mu U}{h_a} < G < \frac{2 \tau c}{h_a} + \frac{\mu U}{h_a} \left(1 + \sqrt{1 + \frac{4 \tau c \mu}{\mu U}}\right), \\
-U_p h_a + \frac{U_p}{3} \sqrt{\frac{2 \mu U}{G}} + \frac{U_p - U}{3} \sqrt{\frac{2 \mu (U_p - U)}{G}} & \text{for } G > \frac{2 \tau c}{h_a} + \frac{\mu U}{h_a} \left(1 + \sqrt{1 + \frac{4 \tau c \mu}{\mu U}}\right),
\end{cases}
\]

(7.7)
where $U_p$ is the solution to

$$\sqrt{U_p - U} + \sqrt{U_p} = \sqrt{\frac{G}{2\mu}} \left(h_\alpha - \frac{2\tau_c}{G}\right). \tag{7.8}$$

After substituting for $G$ and non-dimensionalising, we have

$$Q_\alpha = \begin{cases} \frac{\hat{h}_0}{2} + \frac{\hat{h}_b \hat{G}}{12} & \text{for } 0 < \hat{G} \leq \hat{G}_{c1}, \\ \hat{h}_\alpha - \frac{1}{3} \sqrt{\frac{2}{G}} & \text{for } \hat{G}_{c1} < \hat{G} \leq \hat{G}_{c2}, \\ \hat{U}_p \hat{h}_\alpha - \frac{\hat{U}_p}{3} \sqrt{\frac{2\hat{U}_p}{G}} - \frac{\hat{U}_{p-1}}{3} \sqrt{\frac{2(\hat{U}_{p-1})}{G}} & \text{for } \hat{G} > \hat{G}_{c2}, \end{cases} \tag{7.9}$$

where $Q_\alpha$ is the dimensionless leakage flux and $\hat{G} = (\hat{h}_0 - \hat{h}_b) / \hat{L}_\alpha$, is the dimensionless pressure gradient. Here, $\hat{h}_0$ and $\hat{h}_b$ have both been non-dimensionalised by $h_\infty$, $\hat{L}_\alpha$ has been non-dimensionalised by $L$, and $\hat{U}_p$ is the non-dimensional plug velocity, satisfying

$$\sqrt{\hat{U}_p - 1} + \sqrt{\hat{U}_p} = \sqrt{\frac{G}{2}} \hat{h}_\alpha - \sqrt{\frac{2}{G}} Bi. \tag{7.10}$$

This plug velocity can be written in closed form as

$$\hat{U}_p = \frac{1}{4} \left( R + \frac{1}{R} \right)^2, \tag{7.11}$$

where $R$ is given by the right-hand side of (7.10). The critical values of the non-dimensional pressure gradient are given by

$$\hat{G}_{c1} = \frac{2}{\hat{h}_\alpha^2}, \quad \hat{G}_{c2} = \frac{1}{\hat{h}_\alpha^2} + \frac{2Bi}{\hat{h}_\alpha} + \frac{1}{\hat{h}_\alpha^2} \sqrt{1 + 4Bi\hat{h}_\alpha}. \tag{7.12}$$

When the gap under the scraper is small relative to the thickness of the fluid layer, we have $\hat{h}_\alpha \ll 1$, and so $\hat{G} \gg 1$ in the final region of the piecewise function (7.9). We
can then expand in \( \dot{G} \) to find that
\[
\dot{U}_p \sim \frac{1}{4} \left( R^2 + 2 + \ldots \right) = \frac{1}{8} \left( \dot{h}_a^2 \dot{G} - 4\dot{h}_a Bi + \frac{4Bi^2}{\dot{G}} + 4 + \ldots \right),
\]
(7.13)
\[
Q_\alpha \sim \frac{1}{12} \left( \dot{h}_a^3 \dot{G} - 3\dot{h}_a^2 Bi + \frac{4Bi^3}{\dot{G}^2} + 6\dot{h}_a + \ldots \right),
\]
(7.14)
where we have expanded up to first order and included any terms that could be significant when \( Bi \gg 1 \). When \( Bi \leq O(1) \) the result reduces to,
\[
Q_\alpha \sim \frac{\dot{h}_a^3 \dot{G}}{12} + \frac{\dot{h}_a}{2} + \ldots
\]
(7.15)
Thus, since \( \dot{h}_0 \gg 1 \) in this regime, we have \( \dot{G} \sim \dot{h}_0 / \dot{L}_\alpha \) and the leakage flux will approach 1 when the height of the fluid in front of the scraper, \( \dot{h}_0 \), approaches a critical value, \( \dot{H}_0 \), satisfying
\[
1 = \frac{1}{12} \left( \frac{\dot{h}_a^3 \dot{H}_0}{\dot{L}_\alpha} - 3\dot{h}_a^2 Bi + \frac{4Bi^3 \dot{L}_\alpha^3}{\dot{H}_0^3} + 6\dot{h}_a + \ldots \right).
\]
(7.16)
Hence, the system approaches a steady state in which there is no net volume flux into the heap of fluid, as described by Lister and Hinton [81], and, in particular, \( \dot{H}_0 \) corresponds to the final, steady state value of the the maximum mound height, \( \dot{h}_0 \). Note that, at this stage, it is not clear whether the steady state is reached in finite time or approached asymptotically; this is determined later, in §7.6.1, where we show that the approach to steady state is exponential.

Figure 7.3 shows plots of \( Q_\alpha \) against \( \dot{G} \) for \( \dot{h}_\alpha = 0.1 \) and three different values of \( Bi \). Both show how, since \( \dot{h}_\alpha \) is small, the gradient \( Q'_\alpha(\dot{G}) \), which is bounded above by \( \dot{h}_\alpha^3 / 12 \), is very small, requiring significant variation in \( \dot{G} \) (and hence \( \dot{h}_0 \)) to see variation in \( Q_\alpha \). In the case of a large Bingham number, there is also a wide range of \( \dot{G} \) over which the leakage flux varies even more slowly. This is because the difference between the two critical pressure gradients in (7.12) is \( O(Bi / \dot{h}_\alpha) \) so the middle regime occurs over a large range of pressure gradients when \( Bi \gg 1 \). Throughout much of this regime the dimensionless leakage flux is approximately given by the gap size, \( Q_\alpha \approx \dot{h}_\alpha \) (see figure 7.3c), which can be understood as the sheared region between the unyielded plug and the underside of the scraper (see figure 7.2) becoming a thin boundary layer when \( Bi \gg 1 \), and hence the flow under the scraper being given to leading order by a plug flow of height \( \dot{h}_\alpha \) and unit velocity (for which the leakage flux is \( \dot{h}_\alpha \)). This is consistent with the intuition that, for a large yield stress fluid, such a scraper would typically “cut” the
fluid down to a layer of height dictated by the gap height. Regardless of the magnitude of \( Bi \), the steady state takes a very long time to be reached, only occurring after a sufficiently high mound (\( \geq O(10^4) \)) has built up in front of the scraper. Given the slow variation in \( Q_\alpha \), an appropriate simplification, which is valid over \( O(1) \) time scales, is to take the leakage flux as constant. The evolution of the free surface in this case is discussed in §7.5. Over very large time scales, however, the varying flux becomes relevant, in particular implying a steady state is approached. Numerical simulations are used in §7.6 to explore the evolution of the free surface in this case. First we summarise the full system in §7.4.

### 7.4 The full system

To summarise, given an initial free-surface profile \( h(x, 0) \), the Bingham number, \( Bi \), and the non-dimensional height, \( \hat{h}_\alpha \), and length, \( \hat{L}_\alpha \), of the gap under the scraper, we wish to
determine the non-dimensional free-surface elevation, \( h(x, t) \), and yield-surface height, \( Y(x, t) \), for times \( t > 0 \), in front \( (x > 0) \) and behind \( (x < -\hat{L}_a) \) the scraper. The yield surface height at any given moment is determined directly from the free-surface profile, via (7.4), and the evolution of \( h \) is governed by (7.3), with boundary conditions,

\[
\begin{aligned}
&h = 1 \quad \text{for } x \to \pm \infty, \\
&\frac{1}{6} Y^2 (3h - Y) \frac{\partial h}{\partial x} + h = Q_\alpha \quad \text{for } x = -\hat{L}_a \text{ and } x = 0,
\end{aligned}
\]

where \( Q_\alpha \) is either chosen constant, or else \( Q_\alpha = Q_\alpha(\hat{G}) \) as given by (7.9) with

\[
\hat{G} = \frac{\hat{h}_0 - \hat{h}_b}{L_a},
\]

where \( \hat{h}_0(t) = h(0, t) \) and \( \hat{h}_b(t) = h(-\hat{L}_a, t) \) are the dimensionless free-surface heights immediately upstream and downstream of the scraper, respectively.

### 7.5 Constant leakage flux

In the case of constant leakage flux, \( Q_\alpha \), we can find leading order similarity solutions for early and late time profiles in the region upstream of the scraper \( (x > 0) \). We seek solutions of the form

\[
\begin{aligned}
h &= 1 + t^b \mathcal{H}(\xi), \\
Y &= t^c \mathcal{Y}(\xi),
\end{aligned}
\]

where

\[
\xi \equiv \frac{x}{L(t)} = \frac{x}{x_N t^a},
\]

\( L(t) = x_N t^a \) is the length of the deformed region, \( 0 \leq \xi \leq 1 \), and \( a, b \) and \( c \) are all non-negative. Substitution into (7.3) and (7.4), and assuming the fluid is not completely unyielded at any point in the deformed region, since deformation is taking place here, gives

\[
t^{b-1} (b \mathcal{H} - a \xi \mathcal{H}') = \frac{1}{x_N} \left( t^{2c+b-2a} \frac{1}{6x_N} \gamma^2 \left( 3 + 3t^b \mathcal{H} - t^c \mathcal{Y} \right) \mathcal{H}' + t^{-a} \mathcal{H} \right)',
\]

\[
t^c \mathcal{Y} = 1 + t^b \mathcal{H} + \frac{Bix_N t^{a-b}}{\mathcal{H}'},
\]

where \( ' \) represents differentiation with respect to \( \xi \). The boundary conditions are given by

\[
\mathcal{H}(1) = \mathcal{Y}(1) = 0,
\]

\[
t^{2c+b-2a} \frac{1}{6x_N} \gamma^2 \left( 3 + 3t^b \mathcal{H} - t^c \mathcal{Y} \right) \mathcal{H}' + t^{-a} \mathcal{H} = Q_\alpha - 1 \text{ at } \xi = 0,
\]

157
which represent the conditions that the fluid is unyielded and of height 1 at the nose, (7.23), and the flux is $-Q_\alpha$ at the wall, (7.24). Note that unit flux at the nose, where the deformed free surface meets the incoming unyielded layer, is automatically satisfied by (7.23), and that one of the flux conditions can be replaced by the global mass conservation condition

$$t^{a+b} x_N \int_0^1 \mathcal{H} d\xi = (1 - Q_\alpha)t,$$

which gives $a+b=1$ and $\int_0^1 \mathcal{H} d\xi = (1 - Q_\alpha)/x_N$.

### 7.5.1 Early time

For $t \ll 1$, if the left-hand side of (7.21) dominates, then

$$\frac{\mathcal{H}}{\xi^{b/a}} = \text{const.} = 0,$$

which does not satisfy global mass conservation. Since $b - 1 < b - a$, we must therefore have the balance $b - 1 = 2c + b - 2a$ (i.e. $2c = 2a - 1$). Then, for balance in (7.22), we have $a - b = c = 0$ and so $a = b = 1/2$. The governing equation then becomes the ordinary differential equation (ODE)

$$\mathcal{H} - \xi \mathcal{H}' = \frac{1}{3x_N^2} \left( \mathcal{Y}^2 \left( 3 - \mathcal{Y} \right) \mathcal{H}' \right)', \quad \mathcal{Y} = 1 + \frac{Bi x_N}{\mathcal{H}'} ,$$

with boundary conditions

$$\mathcal{H}(1) = \mathcal{Y}(1) = 0 \quad \text{and} \quad \int_0^1 \mathcal{H} d\xi = \frac{1 - Q_\alpha}{x_N} .$$

This ODE is solved as described in Appendix 7.A, with results for a selection of $Bi$ and $Q_\alpha$ shown in figure 7.4. The free surface is slightly convex, and becomes steeper and more triangular as the Bingham number is increased. The height of the apparent yield surface decreases with $Bi$, as one would expect, and the height and width of the mound and yielded region both decrease with increasing leakage flux $Q_\alpha$. The behaviour at large Bingham number can be rationalised by considering (7.27b), which requires that $\mathcal{H}'/x_N = O(Bi)$ while (7.28) requires $\mathcal{H} x_N = O(1)$, which together imply that $x_N = O(Bi^{-1/2})$. To achieve a balance in (7.27a), we then require that $\mathcal{Y} = O(Bi^{-1/2})$ and (7.27b) reduces to

$$1 + \frac{Bi x_N}{\mathcal{H}'} = 0,$$

with solution

$$\mathcal{H} = Bi x_N \left( 1 - \xi \right), \quad x_N = \sqrt{\frac{2(1 - Q_\alpha)}{Bi}} .$$
7.5. CONSTANT LEAKAGE FLUX

Figure 7.4: Early time solutions, $\mathcal{H}$ and $\mathcal{Y}$, for $Bi = 0.5$ (left column), $Bi = 5$ (middle column), and $Bi = 100$ (right column) and various $Q_\alpha$ (values shown in legend). Profiles are plotted against $x_N \xi$ to indicate the differing values of $x_N$, and both $x$ and $y$-axes are shared where not indicated. The black dotted lines in the top right panel shows the yield-stress dominated solution (7.30).

This yield-stress dominated solution gives a triangular free surface height with $O(Bi)$ slope, which is compared with the solution of (7.27) for the case of $Bi = 100$, showing good agreement.

Early time solutions were also calculated numerically from the full equations (7.3) and (7.4). The numerical method described in Appendix 7.B was used to integrate from $t = 0$ to $t = 10^{-2}$, and the results compared with the similarity solutions, showing excellent agreement up to $t = 10^{-4}$, where the full solutions begin to deviate slightly from the leading order similarity solution (see figure 7.5). The similarity solution was derived on the basis that the divergence of the flux was dominated by variations in the free-surface gradient (i.e. “slumping”), neglecting the advective term in the equation due to variations

159
in the free-surface height. This required that \( \partial^2_x h \gg \partial_x h \) which implies \( L \sim t^{1/2} \ll 1 \). Thus, the similarity solution typically becomes invalid once \( t \) becomes order unity, when the length of the mound is no longer small compared to the initial layer height. When the Bingham number is large, however, the similarity solution actually becomes invalid at earlier times, since the yield stress inhibits slumping. Specifically, when \( Bi \gg 1 \) we have shown that \( h \sim \mathcal{H} t^{1/2} \sim (Bi t)^{1/2} \), while \( L \sim x_N t^{1/2} \sim (t/Bi)^{1/2} \) and \( Y \sim \mathcal{Y} \sim Bi^{-1/2} \).

Hence, the advection term can be neglected in comparison to the slumping term when \( \partial_x (Y^2 \partial_x h) \gg \partial_x h \), which implies \( Y^2/L \gg 1 \) and \( (tBi)^{1/2} \ll 1 \). Thus, in this regime we anticipate the similarity solution to become invalid when \( t = O(1/Bi) \).

If the fluid layer initially also has \( h = 1 \) behind the scraper, then the early time deformation behind the scraper is a reflection of the free surface in front of the scraper. In particular, if we define

\[
\mathcal{H}_b(\xi) = -\mathcal{H}(-\xi), \quad \mathcal{Y}_b(\xi) = \mathcal{Y}(-\xi),
\]

then \( \mathcal{H}_b \) and \( \mathcal{Y}_b \) satisfy (7.27) in the region \(-1 \leq \xi \leq 0\), representing the deformed region behind the scraper, and \( \mathcal{H}_b \) satisfies

\[
\int_{-1}^{0} \mathcal{H}_b d\xi = -(1 - Q_\alpha)/x_N,
\]

as required behind the scraper by mass conservation.

### 7.5.2 Late time

At late time, \( t \gg 1 \), we have \( t^{b-a} \gg t^{b-1} \) and so the leading order balance is between the two terms on the right-hand side of (7.21), thus representing a quasi-static balance between slumping and advection. This requires \( 2c + b = 2a = b - a \implies c + b = 1/2 \). In (7.22) we cannot have \( c > b \) since the yield surface cannot grow faster than the free surface. If \( c = b \) then \( a = 3/4, b = c = 1/4, t^{a-b} \gg t^b \gg 1 \) and no balance is possible in (7.22). Thus, we must have \( c < b \) and a balance between the final two terms in (7.22), representing a quasi-rigid solution in which the material is unyielded to leading order. This gives \( a - b = b \) and so \( a = 2/3, b = 1/3 \) and \( c = 1/6 \). \( \mathcal{H} \) and \( x_N \) are then determined by

\[
\mathcal{H} + \frac{Bix_N}{\mathcal{H}} = 0, \quad \mathcal{H}(1) = 0, \quad \int_0^1 \mathcal{H} d\xi = (1 - Q_\alpha)/x_N,
\]

with solution

\[
\mathcal{H} = 3^{1/3}Bi^{1/3} (1 - Q_\alpha)^{1/3} \sqrt{1 - \xi}, \quad x_N = \frac{3^{2/3} (1 - Q_\alpha)^{2/3}}{2Bi^{1/3}}.
\]
The yield surface is determined from

\[ \left( \frac{1}{2x_N} \mathcal{Y}^2 \mathcal{H}' + \mathcal{H} \right)' = 0, \] (7.35)

which can be integrated with boundary conditions \( \mathcal{Y}(1) = 0, \mathcal{H}(1) = 0 \), to obtain

\[ \frac{1}{2x_N} \mathcal{Y}^2 \mathcal{H}' + 1 = 0, \] (7.36)

with solution

\[ \mathcal{Y} = \frac{24^{1/6} (1 - Q_\alpha)^{1/6}}{Bi^{1/3}} (1 - \xi)^{1/4}. \] (7.37)

Note that the shape of this solution corresponds to the quasi-rigid bow-wave discussed by Lister and Hinton [81]. The similarity solution above provides the explicit time-dependence of the quasi-rigid bow-wave, in the case of a constant leakage-flux. Late-time numerical simulations are compared with these similarity solutions in figure 7.6 showing approach to the similarity solution as \( t \to \infty \). The large times required to approach
the similarity solution can be rationalised by the fact that the term omitted in (7.33) is only an order $Bi^{2/3}t^{1/6}$ smaller than the leading order terms. This also implies that the leading order approximation is better for larger Bingham numbers, as is observed in the figure 7.6.

Technically, this asymptotic solution is not well-ordered in the vicinity of the nose, $\xi = 1$, due to the divergence of the free-surface gradient. When $1 - \xi = O(t^{-2/3})$ (i.e. $x_N t^{2/3} - x = O(1)$), we have $h$ and $Y$ both $O(1)$, and we can no longer neglect $Y$ in the yield condition (7.4). If we define a boundary layer coordinate via $\chi = x_N t^{2/3} - x = x_N t^{2/3}(1 - \xi)$, then the leading order equations for the inner solutions, $h_i$, $Y_i$, are given by

$$Y_i = h_i - \frac{Bi}{\partial h_i / \partial \chi}, \quad h_i - \frac{1}{6}Y_i^2 (3h_i - Y_i) \frac{\partial h_i}{\partial \chi} = 1.$$  

These equations can be integrated from $\chi = 0$ where $h_i = 1$ and $Y_i = 0$. To match to the outer solution as $\chi \to \infty$ we derive the far field behaviour, $h \sim \sqrt{2Bi\chi}$ and $Y \sim (8\chi/Bi)^{1/4}$, from (7.38). The outer solution is given by the late time similarity solution above, but with the constant of integration set by matching to the inner solution, rather than imposing the boundary condition at the nose, $\xi = 1$. In practice the result is the same since, in the outer, we have

$$h_o = H(\xi)t^{1/3} + \ldots = \sqrt{2Bi x_N (1 - \xi) t^{2/3}} + \ldots = \sqrt{2Bi\chi} + \ldots$$

which matches automatically to the inner solution (and similarly the outer solution for $Y$ matches automatically to the inner solution, $Y_i$). The final step is to determine $x_N$ via the volume condition. To leading order the volume of the mound is given by the volume of the outer solution, with a sub-leading contribution coming from the boundary layer solution. Thus, if we impose the volume condition only on the outer solution, we recover the integral condition for the late time similarity solution, $\int_0^1 \mathcal{H} d\xi = 1 - Q_\alpha$. Then $x_N$ is given by (7.34) and the leading order solution in the outer is precisely the late time similarity solution (7.34). This similarity solution has the advantage that it has a simple, closed-form expression, and since the boundary layer has a sub-leading impact on the size and shape of the mound at late times, we mostly opt to use only the similarity solution, without the boundary layer, when plotting the late time solution.

Alternatively, we can construct a composite solution via

$$h = H(\xi)t^{1/3} + h_i(\chi) - \sqrt{2Bi\chi} = h_i(\chi), \quad Y = \mathcal{Y}(\xi)t^{1/6} + Y_i(\chi) - \left(\frac{8\chi}{Bi}\right)^{1/4} = Y_i(\chi).$$

Thus the composite solution simply corresponds to integrating (7.38), all of the way back to the scraper, $\chi = \chi_\infty \equiv x_N t^{2/3}$. Previously to determine $x_N$, we imposed the volume
Figure 7.6: Example profiles of free-surface height (top row) and yield-surface height (bottom row) from numerical simulations at $t = 10^2, 10^4, 10^6$ and $10^8$ (colored dotted lines, often coincident with the corresponding dashed lines), the similarity solutions given by (7.34) and (7.37) (black solid lines), and the composite solution given by (7.38) and (7.40) (colored dashed lines). The solutions are scaled according to the similarity solution (7.34)-(7.37).

c condition on the outer similarity solution. However, when using this composite solution, one might choose to instead impose the volume constraint on the composite solution, corresponding to $\int_0^{x_{\infty}} h_i \, dx = (1 - Q_\alpha) t$. This no longer provides a similarity solution since the time dependence comes in through this integral condition, but, as $Y$ is included in (7.38) throughout the integration, the solution is expected to be more accurate than the similarity solution, particularly for small $Bi$. This is demonstrated in figure 7.6, which compares the results of the composite solution, found by integration of (7.38), against the full numerical solutions and the late-time similarity solution.
7.6 Variable leakage flux

As discussed in §7.3, the assumption of constant leakage flux does not apply over very large time scales, due to increasing hydrostatic pressure gradients driving increasing flux under the scraper as the height of the mound increases in front of the scraper. Following the model of leakage flux given in §7.3, if we assume \( \hat{h}_0 \ll 1 \) and \( \hat{L}_0 = O(1) \), then at early times (when \( \hat{h}_0, \hat{h}_b \approx 1 \)) we lie in the first regime and the constant term, \( \hat{h}_a/2 \), dominates the varying term, \( \hat{h}^3 \hat{G}/12 \). Thus we can assume an initially constant leakage flux of magnitude \( \hat{h}_a/2 \) to provide an initial free-surface profile via the early-time similarity solution detailed in §7.5.1. We then integrate numerically from this initial condition, using the method described in Appendix 7.B, with leakage flux given by (7.9), to explore the role of variable leakage flux. As discussed in Appendix 7.B, behind the scraper the free surface first decreases before reaching the singular point where \( \hat{h}_b = Q_\alpha \).

After this point, provided the leakage flux does not vary sufficiently rapidly for the layer to become yielded again, the scraper simply leaves an unyielded residual layer behind it, such that \( \hat{h}_b = Q_\alpha \) at the back of the scraper at all times. The layer is advected to the left with unit velocity, and so, within this unyielded layer, we have \( h(x, t) = Q_\alpha(t - x) \). As discussed in §7.3, \( Q_\alpha \) varies very slowly, and thus the free-surface height \( \partial h/\partial x \) will be small in this region- supporting the assumption that the layer does not ever become yielded again. Furthermore, since the rate of increase of the free surface height in front of the scraper, \( \hat{h}_0 \), decreases with time, the rate of increase of \( Q_\alpha \), and the slope of the free surface behind the scraper decrease with time, further preventing additional yielding in the layer. Figure 7.7 shows snapshots of the dimensionless free surface layer height, \( h(x) \), from a numerical simulation with the parameters: \( \hat{h}_a = 0.2 \), \( \hat{L}_a = 0.5 \) and \( Bi = 5 \). This demonstrates the initial development of the mound upstream and depression downstream of the scraper at early times (figure 7.7a), the stage in which the residual layer immediately behind the scraper is unyielded and we have \( \hat{h}_b = Q_\alpha \) (figure 7.7b), and the increase of the leakage flux to \( Q_\alpha = 1 \) over a very large timescale and the resulting steady state mound upstream of the scraper (figure 7.7c). In this final panel, the vertical scale is logarithmic and the horizontal scale is piece-wise linear, but differs between the upstream and downstream regions, so that both can be viewed on the same plot.

Figure 7.8 shows the height, \( \hat{h}_0 \), length, \( L \), and leakage flux, \( Q_\alpha \), as functions of time, from the same numerical solution as shown in figure 7.7. The behaviour of the solutions is typical for solutions with small \( \hat{h}_a \). Namely, a very large time is required before the steady state is reached, and both early and late time scalings are observed as detailed
7.6. VARIABLE LEAKAGE FLUX

Figure 7.7: Profiles of layer height, $h(x)$, at $t = 0.01$, 1, and $10^8$, from a numerical solution for the variable leakage flux model described in §7.6 with $\hat{h}_\alpha = 0.2$, $\hat{L}_\alpha = 0.5$, and $Bi = 5$. The outline of the scraper is shown in black. Panels (a) and (b) show all regions of the layer on a linear scale while panel (c) shows only the upstream mound and the unyielded residual layer behind the scraper at late times, using a logarithmic scale for the vertical axis. The horizontal scale of panel (c) is non-uniform, with the different scales upstream and downstream of the scraper indicated on the axis, and the scale linear between the indicated points.

for the constant leakage flux case in §7.5. We can understand the approach to steady state ($Q_\alpha \sim 1$) and intermediate times (namely $t \gg 1$ with $Q_\alpha \ll 1$) by assuming a quasi-steady shape of the mound in-front of the scraper, as suggested for the viscous problem by Lister and Hinton [81]. We detail this approach in the next section.

7.6.1 Intermediate-time and approach to steady state

Motivated by the numerical simulations and the late-time similarity solutions for constant leakage-flux, we assume a quasi-steady solution upstream of the scraper for sufficiently large times (we expect $h \sim t^a$, $L \sim t^b$, with $0 < a, b < 1$, in which case time derivatives are smaller than horizontal gradients for large $t$). In this case the governing equation for
Figure 7.8: Numerical solution for the variable leakage flux model described in §7.6 with \( \hat{h}_a = 0.2, \hat{L}_a = 0.5, \) and \( Bi = 5. \) (left) Free-surface height upstream of the scraper relative to the initial layer thickness, \( \hat{h}_0 - 1, \) as a function of time, \( t, \) on a log-log scale. (middle) Length of disturbance upstream of scraper, \( L, \) as a function of time, \( t, \) on a log-log scale. (right) Leakage flux, \( Q_\alpha, \) as a function of time, \( t, \) on a semi-log scale and detail on a linear scale (inset). Dotted red lines show the transition between the different regimes in (7.9).

The shape, assuming \( Y \neq 0 \) across the extent of the mound, is given by

\[
\frac{1}{6} Y^2 (3h - Y) \frac{\partial h}{\partial x} + h = 1, \quad \text{and} \quad Y = h + \frac{Bi}{\partial h/\partial x}. \tag{7.41a,b}
\]

Note this is the same equation as (7.38), without the change of horizontal coordinate, thus this approximation can be interpreted as using the leading order composite solution from the late time asymptotic expansion, to determine the shape of the mound. We can then find the volume of the mound, \( V, \) as a function of the free-surface height at the scraper, \( \hat{h}_0, \) by numerical integration of (7.41a,b) from the nose of the mound, where \( h = 1 \) and \( Y = 0, \) up until reaching \( h = \hat{h}_0, \) where we evaluate the volume \( V(\hat{h}_0). \)

Additionally, at late times the free-surface height upstream of the scraper satisfies \( \hat{h}_0 \gg 1, \) while downstream we have \( \hat{h}_b \leq 1. \) Hence, the hydrostatic pressure gradient under the scraper, and by implication the leakage flux, is dominated by the variation in the upstream free-surface height, \( \hat{h}_0. \) Hence, we can approximate the leakage flux (7.9)
7.6. VARIABLE LEAKAGE FLUX

by

\[ Q_\alpha = Q_\alpha \left( \hat{G} = (\hat{h}_0 - \hat{h}_b)/\hat{L}_\alpha \right) \approx Q_\alpha \left( \hat{h}_0/\hat{L}_\alpha \right). \]  

which can be interpreted as a function only of \( \hat{h}_0 \), since \( \hat{L}_\alpha \) is a fixed parameter of the problem.

We can then define an ODE for \( \hat{h}_0 \), by noting that

\[ \frac{d}{dt} V(\hat{h}_0) = V'(\hat{h}_0) \frac{d\hat{h}_0}{dt} = 1 - Q_\alpha, \]

where \( ' \) represents differentiation with respect to \( \hat{h}_0 \). Integrating gives

\[ t(\hat{h}_0) \approx \int_1^{\hat{h}_0} \frac{V'(h)}{1 - Q_\alpha} dh = \frac{V(\hat{h}_0)}{1 - Q_\alpha} - \int_1^{\hat{h}_0} \frac{V(h)}{(1 - Q_\alpha)^2} Q'_\alpha dh, \]

where integration by parts is used to express the integral in terms of \( Q'_\alpha(\hat{h}_0) \), which can be derived analytically from (7.9). This gives an implicit relation for the free-surface height, \( \hat{h}_0 \), as a function of \( t \). In deriving this relation two approximations were made, namely that the mound is quasi-static and that the leakage flux is independent of the height behind the scraper. Both of these approximations require large time, \( t \gg 1 \), for which we assume the height and the length of the scraper are both large, \( h, L \gg 1 \). Along with the volume constraint, \( hL \sim t \), this implies \( h \sim t/L \ll t \) and \( L \sim t/h \ll t \). The quasi-static assumption requires that the non-dimensional adjustment time, \( t \), is much larger than the advection time, which is given by the time taken for the bottom boundary to travel the length of the mound, \( L \). Thus, this approximation applies when \( t \gg L \), which we have shown is true for \( t \gg 1 \). The free-surface height upstream of the scraper scales like the height of the mound, while the free-surface height behind the scraper is bounded above by 1, and so the leakage flux is dominated by the upstream height, \( \hat{h}_0 \), when \( h \gg 1 \) which is again valid provided \( t \gg 1 \).

7.6.1.1 Yield-stress dominated quasi-static regime

We can deduce some analytical results by considering viscous or yield-stress dominated regimes. As noted in §7.5.2, at sufficiently late times the yield stress always dominates, since the horizontal scale grows faster than the vertical scale, and hence the surface slope becomes asymptotically small. Since very large times are reached before approaching steady-state, we therefore predict that the yield-stress dominated solution will apply at some stage of the solution for all \( Bi \geq O(1) \) (the exact range of \( Bi \) for which yield-stress
dominated behaviour occurs before reaching the steady state depends on the geometry of the gap under the scraper, and is given explicitly later). In this regime we have

\[ Y = h + \frac{Bi}{\partial h/\partial x} = 0 \implies h = \sqrt{2Bi(L-x)}, \] (7.45)

and

\[ V(\hat{h}_0) = \frac{\hat{h}_0^3}{3Bi}. \] (7.46)

When \( Q_\alpha \ll 1 \) and \( Q'_\alpha \ll 1 \), which is a reasonable approximation at intermediate times when \( \hat{h}_\alpha \ll 1 \), we can evaluate (7.44) to obtain

\[ \hat{h}_0 \sim (3Bi t)^{1/3}, \] (7.47)

which recovers the 1/3 power law for \( \hat{h}_0 \) and the similarity solution (7.34), with \( Q_\alpha = 0 \). As we approach the steady-state, we can no longer assume \( Q_\alpha \ll 1 \), but we can instead use the large \( \hat{G} \) approximation, (7.14). Assuming \( Bi \ll 1/\hat{h}_\alpha^2 \), to leading order (7.14) gives

\[ Q_\alpha \sim \frac{\hat{h}_\alpha^2 \hat{G}}{12} \sim \frac{\hat{h}_\alpha^3 \hat{h}_0}{12\hat{L}_\alpha} \sim \frac{\hat{h}_0}{\hat{H}_0}, \] (7.48)

where \( \hat{H}_0 \approx 12\hat{L}_\alpha/\hat{h}_\alpha^3 \) is the value of \( \hat{h}_0 \) which drives a unit leakage flux, resulting in the steady state. Substituting \( Q_\alpha = \hat{h}_0/\hat{H}_0 \) and (7.46), we can integrate (7.44) to find

\[ \frac{Bi t}{\hat{H}_0^3} = \log \left( \frac{1}{1 - \hat{h}_0/\hat{H}_0} \right) - \frac{\hat{h}_0}{\hat{H}_0} - \frac{1}{2} \left( \frac{\hat{h}_0}{\hat{H}_0} \right)^2, \] (7.49)

which again reduces to \( \hat{h}_0 = (3Bi t)^{1/3} \) when \( \hat{h}_0/\hat{H}_0 \) is small, and additionally implies that the steady state is approached exponentially as

\[ \hat{h}_0 \sim \hat{H}_0 \left( 1 - \exp \left\{ -\frac{3}{2} - \frac{Bi}{\hat{H}_0^2 t} \right\} \right). \] (7.50)

The assumption that the bow-wave is quasi-rigid at intermediate times, as used for this yield-stress dominated regime, may not apply if \( Bi \) is very small. Specifically, the quasi-rigid approximation is valid provided \( Y \ll h \) which implies \( Bi t^{1/4} \gg 1 \) (since \( h \sim (Bi t)^{1/3} \) and \( Y \sim (t/Bi^2)^{1/6} \)). When \( Bi t^{1/4} \ll 1 \), we instead have a visously dominated bow-wave.

### 7.6.1.2 Viscously dominated quasi-static regime

When the quasi-static mound is viscously dominated, we find

\[ Y \approx h \implies \frac{1}{3} \hat{h}^2 \frac{\partial h}{\partial x} + h = 1, \] (7.51)
which can be solved to find an implicit equation for the bow-wave shape, as given by Lister and Hinton [81]:

\[ x = X(\hat{h}_0) - X(h), \text{ where } X(h) = \frac{1}{3} \left( \frac{1}{3} h^3 + \frac{1}{2} h^2 + h + \log(h - 1) \right). \]  (7.52)

Since \( h, \hat{h}_0 \gg 1 \), this can be approximated by

\[ h = (9(L - x))^{1/3} = (\hat{h}_0^3 - 9x)^{1/3}, \]  (7.53)

and so

\[ V(\hat{h}_0) = \frac{1}{12} \hat{h}_0^4. \]  (7.54)

Repeating the approach above, while the leakage flux is negligible we have

\[ \hat{h}_0 \sim (12t)^{1/4}, \]  (7.55)

as found by Lister and Hinton [81], and at later times, the approach to steady-state is again exponential, with

\[ \hat{h}_0 \sim H_0 \left( 1 - \exp \left\{ -\frac{11}{6} - \frac{3}{H_0^2} t \right\} \right). \]  (7.56)

This regime applies as long as \( h \gg \text{Bi} \left| \frac{\partial h}{\partial x} \right| \), which implies \( \text{Bi}^{-1/4} 

\section{7.6.1.3 Transitions between viscously and yield-stress dominated quasi-static regimes}

For a given Bingham number and gap geometry, we may see only the yield-stress dominated or viscously dominated regimes before reaching steady state, or there might be a transition from viscously to yield-stress dominated behaviour. To determine which occurs, we first note that the predicted transition from viscously dominated to yield stress dominated behaviour occurs after a time \( t = T_i \sim 1/\text{Bi}^4 \), but the quasi-static regime only occurs for \( t \gg 1 \), thus the quasi-static regime will only exhibit viscously dominated behaviour if \( \text{Bi} \ll 1 \). Next, we identify an approximation to the free-surface height at which the steady state is reached, which we saw before is given by \( \hat{H}_0 \approx 12\hat{L}_\alpha/\hat{h}_0^3 \), provided \( \text{Bi} \ll 1/\hat{h}_0^2 \). If we have \( \text{Bi} \geq O(1) \) then the behaviour is yield stress dominated throughout the evolution to the steady state, and hence the time scale on which the steady state is reached is given by \( T_y \approx \hat{H}_0^3/(3\text{Bi}) \approx 576\hat{L}_\alpha^3/(\hat{h}_0^3\text{Bi}) \). If, on the other hand, we assume that the behaviour is viscously dominated throughout the evolution to the steady state, then this free-surface height is reached on a time scale given by
CHAPTER 7. SCRAPING OF A THIN LAYER OF VISCOPLASTIC FLUID

\( T_v \approx \dot{h}_0^4/12 \approx 1728 \dot{h}_a^4/\dot{h}_a^{12} \), thus there will be no transition to yield-stress dominated behaviour if the time at which the transition occurs is equal to or larger than the time at which steady state is reached, \( T_t \geq T_v \), which occurs for \( Bi \lesssim \dot{h}_a^3/(12^{3/4} \dot{L}_a) \). Hence, the quasi-static regime is yield-stress dominated throughout the evolution to the steady state if \( Bi \geq O(1) \), is viscously dominated throughout the evolution to the steady state if \( Bi \lesssim \dot{h}_a^3/(12^{3/4} \dot{L}_a) \), and will exhibit a transition from viscously to yield-stress dominated behaviour if \( \dot{h}_a^3/(12^{3/4} \dot{L}_a) \ll Bi \ll 1 \).

Figure 7.9 shows how \( \dot{h}_0 \) evolves as a function of time, comparing the full numerics with the quasi-steady prediction given by (7.44) for the specific case of \( \dot{h}_a = 0.1 \) and \( \dot{L}_a = 0.5 \), showing excellent agreement in the regime of validity, \( t \gg 1 \). The gradient indicators show the appropriate power law for the yield-stress dominated (1/3) and viscously dominated (1/4) regimes, and the predicted time scale on which the steady state is reached are indicated by vertical lines in panels (a) and (c). Panel (b) shows the case of \( Bi = 0.01 \), which lies in the regime in which a transition between viscously dominated and yield-stress dominated behaviour is predicted before the steady state is reached. The numerics show that such a transition does indeed take place, and occurs around the predicted time \( t = T_t \sim 1/Bi^4 \).

7.7 Scraping of Herschel-Bulkley fluid

The above model can be generalised to the scraping of a Herschel-Bulkley fluid. For a Herschel-Bulkley fluid with consistency, \( K \), and shear index, \( N \), the non-dimensionalisation, evolution equation, and leakage flux are all altered slightly. The horizontal length-scale is now given by \( L = \rho g h_\infty^{2+N}/(K U^N) \) and the Bingham number by

\[
Bi = \frac{\tau_c h_\infty^N}{K U^N}. \tag{7.57}
\]

The evolution equation is altered to (see §2.3)

\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\sigma N}{(N + 1)(2N + 1)} Y^{1+1/N} ((2N + 1) h - NY) \left| \frac{\partial h}{\partial x} \right|^{1/N} + h \right), \tag{7.58}
\]

where \( \sigma = \text{sgn} (\partial h/\partial x) \) and the pseudo yield surface is still given by

\[
Y = \max \left( 0, h - \frac{Bi}{\left| \partial h/\partial x \right|} \right). \tag{7.59}
\]

The details of the leakage flux model for a Herschel-Bulkley fluid are given in Appendix 7.C, but again this is modelled by a generalised-Couette flow under the scraper and consists of three regimes depending on the location of plugged regions in the flow.
Figure 7.9: Numerical solutions (solid blue) for dimensionless maximum free-surface height, \( \hat{h}_0 \), as a function of time, \( t \), on a log-log scale. Parameters are \( \hat{h}_0 = 0.1, \hat{L}_0 = 0.5 \), and \( Bi = 10 \) (a), \( Bi = 0.01 \) (b) and \( Bi = 0.001 \) (c). The red dashed line shows the quasi-static prediction given by (7.44), and the gray dashed lines give the predicted time scales for the approach to steady state in the yield-stress dominated regime, \( T_y \) (panel a), the transition between viscously and yield-stress dominated behaviour, \( T_t \) (panel b), and the approach to steady state in the viscously dominated regime, \( T_v \) (panel c).

While the full analysis in the sections above could in theory be carried out again for the Herschel-Bulkley model, we instead choose to pull out the key predicted scalings at early and late times. Once again, at early times \( t \ll 1 \), we have a small perturbation to the initial uniform layer, \( h = 1 + \tilde{h} \). If the length of the mound is given by \( L \) then the volume condition implies
\[
\tilde{h}L \sim t, \tag{7.60}
\]
and the yield condition implies
\[
\frac{Bi}{\tilde{h}/L} \sim 1. \tag{7.61}
\]
Thus, we again have an early time regime in which
\[
h - 1 \sim (Bi t)^{1/2} \quad \text{and} \quad L \sim (t/Bi)^{1/2}. \tag{7.62}
\]
At the intermediate times for which \( t \gg 1 \) but the leakage flux is still small, \( Q_\alpha \ll 1 \), we have the possibility of viscously dominated and yield-stress dominated regimes. For the
CHAPTER 7. SCRAPING OF A THIN LAYER OF VISCOPLASTIC FLUID

former we have $Y \sim h$ and the balance between slumping and advection terms in (7.58) requires

$$h^{2+2/N}/L^{1/N} \sim h,$$  \hfill (7.63)

which, combined with the volume condition, $hL \sim t$, implies

$$h \sim t^{1/(N+3)} \text{ and } L \sim t^{(N+2)/(N+3)}.$$  \hfill (7.64)

On the other hand, if the behaviour is yield-stress dominated, then the yield condition implies

$$h \sim \frac{Bi}{h/L},$$  \hfill (7.65)

and the predicted scaling is the same as for the Bingham fluid, namely

$$h \sim (Bi/t)^{1/3} \text{ and } L \sim (t^2/Bi)^{1/3}.$$  \hfill (7.66)

This is to be expected since any shear-thinning (or thickening) should not enter the leading order solution when the yield stress dominates the viscous stresses. As with the Bingham case, the viscously dominated behaviour can only be observed for $Bi \ll 1$ and only before a transition to yield-stress dominated behaviour occurs due to the surface slope becoming sufficiently shallow at $Bi t^{N/(N+3)} \sim 1$ (unless the steady state is reached on a shorter timescale than the timescale implied by this condition).

The predictions for the maximum free-surface height, $\hat{h}_0$, given in (7.62), (7.64) and (7.66) are compared against full numerical simulations for $N = 0.5$ and $Bi = 10$ and $Bi = 0.001$ in figure 7.10, supporting the validity of the predicted scalings.

7.8 Comparison to experiments

To test the validity of our shallow-layer theory for scraping of a layer of viscoplastic fluid, we have conducted preliminary experiments with a model yield-stress fluid. The methodology and results of these experiments are detailed in the following sections.

7.8.1 Methodology

7.8.1.1 Experimental configuration

The configuration of the experiments is shown in figure 7.11. The channel (clear acrylic) was mounted on the table of a Bridgeport milling machine, while the scraper was attached to the milling machine head. The machine has a motorized bed, allowing for
Figure 7.10: Numerical solutions (solid blue) for dimensionless maximum free-surface height, $\hat{h}_0$, as a function of time, $t$, on a log-log scale for a Herschel-Bulkley fluid. Parameters are $\hat{h}_\alpha = 0.1$, $\hat{L}_\alpha = 0.5$, $N = 0.5$, and $\hat{B}i = 10$ (a) and $\hat{B}i = 0.001$ (c). The slope indicators show the predicted scalings at early times, $t \ll 1$, and intermediate times, $t \gg 1$ with $Q_\alpha \ll 1$, for the yield-stress and viscously dominated behaviours (in (a) and (b) respectively).

linear translation of the table at a range of speeds (1 – 20 in/min) while the head (and hence scraper) remains stationary. The bed can also be raised vertically, allowing for control over the size of the gap under the scraper. To prepare the initial uniform layer thickness, a hand-held scraper was used to remove fluid above a fixed height. Still images were taken at fixed time intervals using a Canon EOS 250D camera with a timer-controlled automatic shutter trigger. The camera was positioned on a tripod in front of the scraper, which is stationary during the experiments, at a distance of approximately 70 cm from the tank. Finally, for one of the experiments, a dial indicator was mounted at the back of the scraper (see figure 7.13) to check for any deflection of the scraper. This dial indicated a maximum deflection of less than 0.002 inches ($\lesssim 0.05$ mm) and so the scraper is well represented by a static vertical boundary as assumed in the shallow-layer theory.
CHAPTER 7. SCRAPING OF A THIN LAYER OF VISCOPLASTIC FLUID

Figure 7.11: Schematic of experiment configuration. The cuboidal scraper is fixed to the stationary milling machine head, while the tank containing a layer of hair gel is fixed to the bed of the machine, which can translate at uniform velocity, \( U \). The scraper fits snugly in the channel, with the gaps on either side being approximately 0.1 mm.

7.8.1.2 Materials

For our model yield stress fluid we used a commercially available hair gel (Enliven Hair Gel Extreme, hold 4). Tests were carried out in a Kinexus ultra+ rotational rheometer (Malvern Instruments, Worcestershire, United Kingdom) using parallel plates of diameter 40 mm and gap 1 mm to measure the steady-state flow curve of the material. Shear-rate controlled tests were carried out for shear rates stepped up and down between \( 10^{-4} \) s\(^{-1} \) and \( 30 \) s\(^{-1} \) and amplitude controlled oscillatory measurements were taken for strains between \( 10^{-4} \) and 5, at a fixed frequency of 1 Hz. To mitigate the effects of slip on the measurements, P400 grit sandpaper was glued to the surfaces of the parallel plates. Figure 7.12(a) shows results for two separate up- and down-stepped shear-rate controlled tests, separated by a 100 s rest period, demonstrating that the rheology of the material is well fitted by a Herschel-Bulkley constitutive rule, with \( \tau_c = 70 \) Pa, \( K = 80 \) Pa \( \cdot \) s\(^N\), and \( N = 0.25 \). Apart from at very low strain rates, the results show excellent agreement both between the up and down stepped tests, and between the two tests separated by the period of rest, indicating that the material does not exhibit any significant thixotropy.

The large yield stress of the hair gel poses a challenge for measuring an accurate volume, and hence density, of the material. The density was found to be close to that of water, namely in the range \( 1000 \pm 50 \) kg/m\(^3\), and hence we take the value of 1000 kg/m\(^3\) in the absence of more precise volume measurements. Figure 7.12(b) shows the storage modulus, \( G' \), and the loss modulus, \( G'' \), as a function of strain. At low strains the material behaves like a linear elastic solid, as evidenced by approximately constant values of both \( G' \) and \( G'' \). At higher strains, the storage modulus decreases dramatically, and eventually falls.
Figure 7.12: Hair gel rheometry. a) Steady state flow curve for the commercial hair gel used in the experiments. Data (symbols) is shown for two up and down, shear-rate stepped tests, separated by a 100 s rest period. The solid black line indicates the flow curve for a Herschel-Bulkley constitutive law with $\tau_c = 70$ Pa, $K = 80$ Pa $\cdot$ s$^N$, and $N = 0.25$. b) Storage modulus, $G'$ (blue stars), and loss modulus, $G''$ (red circles), as functions of strain. The dashed blue and red lines indicate the values 813 Pa and 52.5 Pa, respectively, obtained by averaging $G'$ and $G''$ over low strains ($\gamma < 10^{-2}$). The vertical grey line indicates the critical strain, $\gamma = 0.12$, at which the stress attains the yield stress, $G'\gamma = \tau_c = 70$ Pa, as obtained from the Herschel-Bulkley fit.

below the loss modulus, indicating a transition to a fluid regime. The beginning of this transition roughly corresponds to the critical strain, $\gamma = 0.12$, where the stress is equal to the yield stress, $\tau_c = 70$ Pa. The average value of the storage modulus, $G'$, over the linear elastic regime (averaged over $\gamma < 10^{-2}$) is 813 Pa, while at the critical strain (i.e. at yielding) it takes the value of 578 Pa. We will discuss the relative significance of elastic effects in §7.8.3.

To reduce the number of bubbles present in the sample, each bottle of hair gel was spun in a lathe at 2000 rpm, resulting in the bubbles converging to the centre of the bottle. Inevitably some bubbles were reintroduced to the fluid in the depositing and scraping of the layer. These were seen as a convenience for visualising the flow within the layer, and represent a sufficiently small volume fraction that we do not anticipate they had a significant impact on the rheology or dynamics of the flow.
Figure 7.13: Typical images from a scraping experiment (Test I). The dial indicator behind the scraper is used to test for any deflection of the scraper.
7.8. COMPARISON TO EXPERIMENTS

7.8.1.3 Image processing and data extraction

Figure 7.13 shows a selection of three typical images obtained from an experiment (Test I shown). We note that the mound shape is qualitatively consistent with those reported by Maillard et al. [87]. To compare with the predictions of shallow-layer theory, we wish to extract the free-surface profiles from these raw images. To do so, we first crop the image to a 3000×1000 pixel region containing the hair-gel layer. We then perform a sequence of transformations to adjust for perspective and lens distortion effects. The guiding principles in these transformations are that the base of the layer should be horizontal (corresponding to the \( x \)-axis), the bolts visible along the bottom of the tank should be equispaced in the horizontal direction, the scraper is a known width of 1.8 cm, and the upper surface should be approximately horizontal at a known height, prior to the initiation of the experiment. We thus define the following quantities, in pixels:

\[
(\mathbf{X}_0, \mathbf{Y}_0) : \text{coordinates of the point on the base, directly below the upstream edge}\n\]
\[
(\mathbf{X}_0s, \mathbf{Y}_0s) : \text{coordinates of the point on the free surface, directly upstream of the}\n\]
\[
(\mathbf{X}_1, \mathbf{Y}_1) : \text{coordinates of the point on the base at the far right of the cropped}\n\]
\[
(\mathbf{X}_{1s}, \mathbf{Y}_{1s}) : \text{coordinates of the point on the free surface at the far right of the}\n\]
\[
(\mathbf{X}_A, \mathbf{X}_B, \mathbf{X}_C) : \text{horizontal coordinates of the centres of three consecutive bolts};\n\]
\[
\mathbf{W} : \text{width of the scraper, measured at the bottom of the scraper}.\n\]

Note that the vertical coordinate is measured downwards from the top left of the image, as is the convention for images. Given these quantities and the coordinates of the image, \((\mathbf{X}, \mathbf{Y})\), we first make the transformation

\[
\hat{\mathbf{X}} = \frac{1.8}{\mathbf{W}} \times (X - \mathbf{X}_0) \times (1 + \alpha_x(X - \mathbf{X}_0)), \quad (7.67)\n\]
\[
\hat{\mathbf{Y}} = -\frac{\mathbf{h}_\infty}{\mathbf{Y}_0 - \mathbf{Y}_{s0}} \times (Y - \mathbf{Y}_0 - \beta_x(X - \mathbf{X}_0)) \times (1 + \alpha_y(X - \mathbf{X}_0)), \quad (7.68)\n\]

where \(\alpha_x\) and \(\alpha_y\) account for variations of the horizontal and vertical scales with horizontal
position due to perspective, and are given by

\[ \alpha_x = \frac{(X_A - X_0) + (X_C - X_0) - 2(X_B - X_0)}{2(X_B - X_0)^2 - (X_A - X_0)^2 - (X_C - X_0)^2}, \]  
\[ \alpha_y = \frac{(Y_0 - Y_{0s}) - (Y_1 - Y_{1s})}{(Y_1 - Y_{1s})(X_1 - X_0)}, \]

and \( \beta_x \) accounts for a small (and hence “linearised” rotation), and is given by

\[ \beta_x = \frac{Y_1 - Y_0}{X_1 - X_0}. \]

After carrying out this transformation, we find that there remains some slight unwanted curvature in the image, perhaps due to lens distortion (see figure 7.14). The effect is minor, but can appear significant when the vertical scale is exaggerated. We correct for this by making a further transformation which translates coordinates vertically according to a parabola fitted through the ends and midpoint of the base of the fluid layer, as shown by the red dashed line in figure 7.14. Specifically, we define the midpoint of the layer, \( \hat{X}_m = \hat{X}_1/2 \), where \( \hat{X}_1 \) is the transformation of \( X_1 \). Given the \( \hat{Y} \)-coordinate of the bottom of the layer at this point, \( \hat{Y}_m \), we make the further transformation,

\[ x = \hat{X}, \quad y = \hat{Y} - \hat{Y}_m \left( 1 - \left( \frac{\hat{X} - \hat{X}_m}{\hat{X}_m} \right)^2 \right), \]

which ensures that the ends and midpoint of the bottom of the layer all lie along the straight line \( y = 0 \). The final result of these transformations is shown in the third panel of figure 7.14.

As in figures 7.13 and 7.14, the images typically show the upper surface of the fluid layer, however the region where the fluid lies against the front wall of the tank is a more saturated blue colour. We take advantage of this difference to extract the free surface from the transformed images by converting to HSV (Hue Saturation Value) format and extracting a contour from the saturation channel at a value of 0.8. Before doing so, we first put the HSV image through five iterations of the denoising function \texttt{medfilt2} provided by the MATLAB Image Processing toolbox [136, 135], to remove some noise. In figure 7.15 we show the resulting contour overlain on the transformed image for two images from Test I (corresponding to the first two images shown in figure 7.13). This figure demonstrates that the process is reasonably effective in identifying the free surface of the fluid layer at the front of the tank, however it is also susceptible to noise and can result in the contour having disconnected sections. To select a single connected contour for the free surface, we start from the contour point on the free surface at the far right of
7.8. COMPARISON TO EXPERIMENTS

Figure 7.14: Comparison of images before and after scaling and transformation as described in §7.8.1.3. Note the exaggerated vertical scale. The red crosses in the first panel indicate the key landmarks used to transform the image, and the dashed line in the second panel indicates the parabola fitted to the bottom of the layer, which we use to adjust for the remaining curvature after the first transformation.

the image and iteratively select the next contour point in the list which has a smaller $x$-coordinate. When there is no such contour point left in the list, we are at the far left of the image and the selected points form a single contour line which we define to be the free surface for the given experiment image.

7.8.2 Accounting for slip

Wall slip is a feature commonly reported for yield stress fluids [22, 105, 41, 45] and was identified in our experiments through two observations. Firstly, the thickness of the residual layer behind the scraper (in the absence of an instability we detail in §7.8.4) was typically uniform and equal to the gap height (see figure 7.16), which is consistent with the fluid being unyielded under the scraper, translating with the bottom boundary and slipping against the underside of the scraper. Secondly, bubbles which are close to the bottom of the tank can be seen to move relative to the tank while not moving significantly relative to one another (see figure 7.16), indicating slip against the bottom of the tank. We thus choose to alter the theory detailed in §7.2-7.7 to introduce the effect
CHAPTER 7. SCRAPING OF A THIN LAYER OF VISCOPLASTIC FLUID

Figure 7.15: Example of extraction of the free surface by contouring according to saturation value. The panels show two snapshots from Test I.

Figure 7.16: Evidence of wall slip in the scraping experiments. Two images from Test IV, taken 50 seconds apart. The residual layer behind the scraper has a uniform thickness set by the gap height, suggesting that the fluid is slipping against the underside of the scraper. The red circles indicate two bubbles that are close to the bottom of the tank but have moved relative to the tank (the red arrows indicate the distance of the bubbles to the bolts, which are moving with the tank), indicative of slip at the base.
of slip.

Among experimental studies of wall slip of viscoplastic fluids, of particular relevance to our experiments are the studies of Piau [105] and Daneshi et al. [45], who characterise the slip of Carbopol solutions (a key ingredient in commercial hair gels) against glass and plexiglass surfaces, for different concentrations and yield stresses of solution. In both cases the data is well characterised by a slip velocity law of the form,

\[ U_s = \beta \max(\tau - \tau_s, 0)^m, \]

(7.73)

where \( U_s \) is the slip velocity of the fluid relative to the boundary, \( \tau \) is the shear stress at the boundary, \( \tau_s \) is a slip yield stress, and \( \beta \) and \( m \) are parameters which, along with \( \tau_s \), can depend on the concentration of the Carbopol solution. Daneshi et al. [45] found the exponent, \( m \), to be \( 1.04 \pm 0.06 \), and hence well represented by a linear relationship, \( m = 1 \). Piau [105] also found that the exponent was close to one for smooth surfaces such as plexiglass, but could be greater than 2 for rougher surfaces (machined chromium). Given this evidence, and for simplicity, we choose a linear relationship, \( m = 1 \), for our model. For the slip yield stress, \( \tau_s \), both studies found a strong dependence on the yield stress of the fluid, with larger yield stresses resulting in larger slip yield stresses. However, the specific relationship varied significantly between the studies. Daneshi et al. [45] found that the ratio \( \tau_s/\tau_c \) varied between 0.009 and 0.24. We choose to use a value at the lower end of this range, \( \tau_s = 0.01 \tau_c \), for two reasons. Firstly, Piau [105] reports a value of \( \tau_s = 0.07 \) Pa for a sample with \( \tau_c = 75 \) Pa, which is close to the yield stress of our fluid (while the samples used by Daneshi et al. [45] reach a maximum yield stress of 32 Pa). Secondly, we observe slip occurring at relatively shallow free surface slopes (and hence low basal shear stresses). The constant of proportionality, \( \beta \), also varies significantly with concentration and between studies. Combining the results of Daneshi et al. [45] and Piau [105], we find a range of between \( 2.2 \times 10^{-6} \) m s\(^{-1}\)Pa\(^{-1}\) and \( 6.0 \times 10^{-1} \) m s\(^{-1}\)Pa\(^{-1}\), with the upper limit corresponding to low concentrations (\( \leq 0.1\% \)) of Carbopol, reported by Daneshi et al. [45], and the lower limit corresponding to high concentrations, reported by Piau [105]. We choose to use the value \( \beta = 1.6 \times 10^{-5} \) m s\(^{-1}\)Pa\(^{-1}\), which approximately corresponds to the value reported by Piau [105] for the 75 Pa yield stress sample.

Under the lubrication model, the magnitude of the basal shear stress, \( \tau \), is given by

\[ \tau = \rho gh \left| \frac{\partial h}{\partial x} \right|. \]

(7.74)
Thus, to introduce slip into our model, we replace the bottom no-slip boundary condition, 
\( u = -U \), with
\[
\begin{align*}
  u &= -U + \beta \max \left( \rho g h \left| \frac{\partial h}{\partial x} \right| - 0.01 \tau_c, 0 \right). 
\end{align*}
\]
(7.75)

After non-dimensionalisation, and dropping hats from the new variables, this can be written as
\[
\begin{align*}
  u &= -1 + U_s = -1 + L_s \max \left( h \left| \frac{\partial h}{\partial x} \right| - 0.01 Bi, 0 \right) \text{ at } z = 0, 
\end{align*}
\]
(7.76)

where \( U_s \) is the dimensionless slip velocity and \( L_s \) is the dimensionless slip length, given by
\[
\begin{align*}
  L_s &= \frac{\beta \mu}{h_\infty} = \frac{\beta K (U/h_\infty)^{N-1}}{h_\infty}. 
\end{align*}
\]
(7.77)

\( \mu = K (U/h_\infty)^{N-1} \) is the typical viscosity for the Herschel-Bulkley model, and we use the terminology, “slip length”, to reflect the fact that \( \beta \mu \) is a length, which becomes non-dimensionalised by \( h_\infty \) in the definition of \( L_s \). With this alteration to the basal boundary condition, the evolution equation for \( h \), (7.58), gains a term, becoming
\[
\begin{align*}
  \frac{\partial h}{\partial t} &= \frac{\partial}{\partial x} \left( \frac{\sigma N}{(N+1)(2N+1)} Y^{1+1/N} ((2N+1)h - NY) \left| \frac{\partial h}{\partial x} \right|^{1/N} + h - U_s h \right), 
\end{align*}
\]
(7.78)

with \( \sigma \) and \( Y \) defined as before. The effect of slip on the shape of the predicted free-surface profile and on the evolution of the height of the mound is shown in figure 7.17 for the particular case of \( Bi = 1, N = 1 \), and no leakage flux, \( \hat{h}_\alpha = 0 \). In figure 7.17(a) we see that the inclusion of slip reduces the gradient of the free surface at the nose (although the gradient discontinuity remains, due to the non-smoothness of the maximum function in (7.76)). The horizontal extent is thus increased and the height decreased. Figure 7.17(b) indicates that the excess height of the mound follows the same trend with time, but is reduced as the slip length is increased. Another feature we observe is that the yield surface, \( Y \), can vanish over the entirety of the mound for sufficiently large slip lengths. This may seem contradictory since it suggests the mound is deforming despite being unyielded everywhere, however this can be rationalised as in [16] by showing that the fluid remains above the yield stress when higher orders are included in the asymptotic expansion. Writing
\[
\begin{align*}
  u &= U_s(x) - 1 + \epsilon v_1 + \ldots, \quad v = \epsilon v_1 + \ldots, 
\end{align*}
\]
(7.79)
we find that the deviatoric stresses (using the Bingham model for demonstration) are given by

\[
\tau_{xx} = \tau_{xx}^{(0)} + O(\epsilon) = 2 \left( 1 + \frac{Bi}{\dot{\gamma}_0} \right) U'_s + O(\epsilon), \quad (7.80)
\]

\[
\tau_{xz} = \tau_{xz}^{(0)} + O(\epsilon) = \left( 1 + \frac{Bi}{\dot{\gamma}_0} \right) \frac{\partial u_1}{\partial z} + O(\epsilon), \quad (7.81)
\]

where

\[
\dot{\gamma}_0 = \sqrt{(2U'_s)^2 + \left( \frac{\partial u_1}{\partial z} \right)^2} \geq 0, \quad (7.82)
\]

and thus the deviatoric stress lies above the yield stress,

\[
\tau_{xx}^2 + \tau_{xz}^2 = (Bi + \dot{\gamma}_0)^2 + O(\epsilon) > Bi^2. \quad (7.83)
\]

Next, we verify that the shallow-layer equation we derived above remains valid when \( Y = 0 \). To leading order in the expression of conservation of momentum in the vertical direction we have

\[
\frac{\partial}{\partial z} \left( -p + \tau_{xx}^{(0)} \right) = 1 \quad \Rightarrow \quad -p + \tau_{xx}^{(0)} = z - h, \quad (7.84)
\]

then, from the conservation of horizontal momentum, we have

\[
\frac{\partial \tau_{xx}^{(0)}}{\partial z} = -\frac{\partial}{\partial x} \left( -p + \tau_{xx}^{(0)} \right) = \frac{\partial h}{\partial x} \quad \Rightarrow \quad \tau_{xx}^{(0)} = -\frac{\partial h}{\partial x} (h - z). \quad (7.85)
\]

Thus the magnitude of the basal shear stress remains equal to \( h |\partial h/\partial x| \), and the slip velocity is given as above. The leading order horizontal flux is then

\[
q = \int_0^h u \, dz = h (U_s - 1) + \ldots, \quad (7.86)
\]

and conservation of mass gives

\[
\frac{\partial h}{\partial t} = -\frac{\partial q}{\partial x} = \frac{\partial}{\partial x} \left( h - U_s h \right), \quad (7.87)
\]

as given by (7.78) with \( Y = 0 \).

When \( \dot{h}_\alpha \neq 0 \) we should also include slip in the model of leakage flux under the scraper. This can be done, in principle, by including further regimes in the analysis of Appendix 7.C, dependent on whether or not, and in which direction, the fluid is slipping at the top and bottom of the thin gap. However, this results in a large number of possible regimes and a rather complicated model for the flux under the scraper. Rather than
Figure 7.17: a) Free-surface profiles, \( h \) (solid), and yield-surface profiles, \( Y \) (dashed), from numerical solutions at \( t = 20 \) with \( Bi = 1 \), \( N = 1 \), \( \hat{h}_\alpha = 0 \), and a selection of dimensionless slip-lengths, \( L_s \) (see legend). The yield surface for \( L_s = 1 \) and \( L_s = 1.5 \) are both at \( Y = 0 \). b) Increase in height immediately upstream of the scraper, \( h_0 - 1 \), as a function of time, \( t \), for the same parameters shown in a).

Taking this approach, we instead appeal to the experimental observation that the residual layer is typically uniform and of thickness given by the gap height (see figure 7.16). Since the dimensionless thickness of the residual layer sufficiently far downstream of the scraper is equal to the dimensionless leakage flux, this is consistent with a particular regime in which the fluid is unyielded in the gap, and slipping only on the upper surface (giving a dimensionless flux equal to the gap height). Thus, when computing shallow-layer solutions to compare to the experiments, we use the constant value \( Q_\alpha = \hat{h}_\alpha \) for the dimensionless leakage flux.

### 7.8.3 Dimensionless parameters

To summarise, for the material parameters of the fluid, we take \( \tau_c = 70 \) Pa, \( K = 80 \) Pa·s\(^N\), \( N = 0.25 \), and \( \rho = 1000 \) kg/m\(^3\). The storage modulus was measured to be 813 Pa in the linear-elastic regime and 578 Pa at yielding. For the model of slip, we take \( \tau_s = 0.7 \) Pa and \( \beta = 1.6 \times 10^{-5} \) ms\(^{-1}\)Pa\(^{-1}\). The initial layer thickness was set as 1 cm (Tests I and II) and 2 cm (Tests III-V). The scraper velocity was varied between approximately 0.4 mm/s and 8.5 mm/s, and the gap height between approximately 0.6 mm and 6 mm.
After non-dimensionalising according to §7.7 we obtain the dimensionless parameters shown in table 7.1, for the five experiment runs. Also shown are the typical length and time scales, \( L \) and \( T \).

As detailed in §2.3, the significance of inertia in the governing equations is measured by the modified Reynolds number, \( \epsilon Re \), where

\[
Re = \frac{\rho U^2}{K (U/h_\infty)^N}.
\]

For the parameters of our experiments, this quantity varies between a minimum of \( 9.1 \times 10^{-7} \) for Tests IV and V, and a maximum of \( 7.3 \times 10^{-4} \) for Test I, indicating that our experiments are conducted in a regime for which inertia can be safely neglected.

To determine the significance of elasticity in the experiments, we wish to compare an elastic relaxation time, \( t_e \), to the typical timescale of the flow, \( T \), defining a Deborah number, \( De = t_e / T \). When the Deborah number is small, elastic stresses relax quickly, relative to the timescale of the experiment, and can be neglected. There are multiple ways of defining such an elastic timescale [30, p.234] and the correct choice is not widely agreed. One choice which has been used for power-law and Herschel-Bulkley fluids [30, 85, 140] is given by \((K/G')^{1/N}\) where \( G' \) is either measured in the linear-elastic regime [85] or near yielding [140]. For our material properties this gives an elastic timescale in the range \( 9.4 \times 10^{-5} \text{ s} < t_e < 3.7 \times 10^{-4} \text{ s} \), which is significantly smaller than the timescales of our experiments (see table 7.1), resulting in Deborah numbers smaller than \( 2.4 \times 10^{-4} \). Another way to define an elastic timescale is \( t_e = \mu / G' \) where \( \mu = K (U/h_\infty)^{N-1} \) is a typical viscosity of the fluid [30, p. 351]. In this case, \( t_e \) depends on the experimental parameters and, using the value of \( G' \) at yielding, the Deborah number is found to vary between \( 8 \times 10^{-3} \) (for Tests IV and V) and 0.1 (for Test I). Thus we anticipate that our experiments are carried out in a regime in which elastic effects are largely negligible.

Finally, due to the presence of a free surface in our flow, surface tension effects could also be present in the experiments. To determine the significance of these effects we consider the capillary length, \( l_c = (\sigma_t/\rho g)^{1/2} \), where \( \sigma_t \) is the surface tension of the fluid. Below this lengthscale we anticipate surface tension effects to be significant, while on significantly larger length scales they will be negligible. Alternatively we can define the Bond number, \( Bo = \rho g L^3 / \sigma_t = L^2 / l_c^2 \), which measures the relative sizes of gravitational and capillary stresses. Like the elastic timescale, surface tension is difficult to define and measure for a yield-stress fluid [32, 65, 79, 140]. These previous studies have typically reported a value for Carbopol slightly below the surface tension of water (0.072 Nm\(^{-1}\)), in the range \( \approx 0.05 - 0.07 \text{ Nm}^{-1} \). However, our commercial hair gel contains a number
of other ingredients such as Polyvinylpyrrolidone (PVP) and Triethanolamine, which could act to alter the surface tension significantly. Both of these ingredients reduce the surface tension of water when in solution [35, 98], thus we report the typical capillary length and corresponding Bond numbers for a value of $\sigma_t = 0.07 \text{ Nm}^{-1}$, noting that these correspond to a generous upper limit for the significance of surface tension effects in our experiments. This gives a capillary length of $l_c \approx 2.7 \text{ mm}$ and thus a Bond number that varies between 23 (Test I) and 2300 (Tests IV and V), indicating that surface tension is not a significant factor in our experiments.

### 7.8.4 Results and discussion

Figure 7.18 shows free-surface profiles at a selection of four times from Tests I-IV, compared against the corresponding shallow-layer predictions. These show reasonable agreement, particularly given that we have not introduced any fitted parameters and the aspect ratio, $\epsilon$, is not especially small in any of the examples, ranging between 0.16 and 0.78 (see table 7.1). Excluding a region close to the scraper, which we discuss later, the weakest agreement is for Test I, which corresponds to the experiment with the largest aspect ratio, $\epsilon = 0.78$, smallest Bond number, $Bo = 23$, and largest Deborah number, $De = 0.1$. Hence this experiment could be being more effected by non-shallow, elastic or surface tension effects. Nonetheless there are some significant discrepancies which we discuss further below.

One notable discrepancy is that the areas under the curves do not agree between experiments and theory, typically being larger in the experiments than the theoretical profiles. This could be due to discrepancies between the leakage flux used in the model and occurring in the experiment, but we anticipate this effect to be minor, since we are able to observe the uniform layer behind the scraper in the experiments (except for

<table>
<thead>
<tr>
<th>Test</th>
<th>$Bi$</th>
<th>$\hat{h}_{\alpha}$</th>
<th>$L_s$</th>
<th>$\epsilon$</th>
<th>$L$ (cm)</th>
<th>$T$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test I</td>
<td>0.91</td>
<td>0.07</td>
<td>0.15</td>
<td>0.78</td>
<td>1.3</td>
<td>1.5</td>
</tr>
<tr>
<td>Test II</td>
<td>1.93</td>
<td>0.30</td>
<td>1.4</td>
<td>0.37</td>
<td>2.7</td>
<td>63.9</td>
</tr>
<tr>
<td>Test III</td>
<td>1.82</td>
<td>0.03</td>
<td>0.58</td>
<td>0.20</td>
<td>10.2</td>
<td>96.6</td>
</tr>
<tr>
<td>Test IV</td>
<td>2.29</td>
<td>0.30</td>
<td>1.2</td>
<td>0.16</td>
<td>12.9</td>
<td>303.8</td>
</tr>
<tr>
<td>Test V</td>
<td>2.29</td>
<td>0.03</td>
<td>1.2</td>
<td>0.16</td>
<td>12.9</td>
<td>303.8</td>
</tr>
</tbody>
</table>

Table 7.1: Table of dimensionless parameters and typical length and time scales for the five scraping experiments. The dimensionless parameters are the Bingham number, $Bi$, the dimensionless gap height, $\hat{h}_{\alpha}$, the dimensionless slip length, $L_s$, and the aspect ratio, $\epsilon = h_\infty / L$. 

CHAPTER 7. SCRAPING OF A THIN LAYER OF VISCOPLASTIC FLUID
7.8. COMPARISON TO EXPERIMENTS

Test II which features an instability behind the scraper as discussed below). Rather, we believe this disagreement arises primarily due to non-planarity of the experiments and the image processing method used to distinguish the free surface. Firstly, as can be seen in the second panel of figure 7.13, the free surface may dip at the centre of the tank, resulting in the free surface at the tank sides being larger than the average value across the tank, causing the area under the extracted experimental free surface to be larger than expected through conservation of mass. Secondly, even where the free surface does not dip at the centre of the tank, as the mound grows, regions of the upper surface visible in the images can become darker and eventually reach the saturation threshold required to be identified with the free surface in the image processing method. Again this would result in an over-estimation of the free-surface height, and hence an exaggeration of the area under the curve.

The discrepancy between the theoretical and experimental profiles is generally greatest in a region close to the scraper. This is to be anticipated since, in the shallow-layer theory, vertical velocities are assumed small relative to the horizontal velocities, whereas they must become of the same magnitude at the scraper where there is a stagnation flow against the no-penetration boundary. The horizontal extent of the region where the vertical velocity becomes non-negligible is $O(\epsilon)$ in the scaled shallow-layer theory, or $O(h_\infty)$ in dimensional coordinates. We hence indicate the horizontal positions corresponding to $x = 2h_\infty$ by dashed lines in figure 7.18, demonstrating that much of the discrepancy between experiments and theory can be attributed to this $O(h_\infty)$ region from the scraper, where shallow-layer theory fails.

We note two further interesting observations from the experiments. Firstly, the free surface upstream of the scraper undergoes a buckling/wrinkling instability also observed by Maillard et al. [87]. These can be seen in the extracted surface profiles in figure 7.18 but perhaps are more clearly seen in the raw images in figures 7.13 and 7.16. This instability cannot be described through shallow-layer theory, since the wavelength of the resulting wrinkles is on the order of the height of the layer and perhaps arises through elastic or plastic buckling of an unyielded layer of fluid at the free surface. Indeed similar surface features are seen in fully two-dimensional numerical simulations of a slumping slender vertical block of viscoplastic fluid under gravity with a small but finite Reynolds number [82], suggesting the instability can plausibly be understood as an effect of non-shallowness, without the need for additional physics such as surface tension or elasticity. A second instability can be observed in the layer downstream of the scraper for sufficiently slow speeds and thin gaps under the scraper, resulting in a periodic ripple
CHAPTER 7. SCRAPING OF A THIN LAYER OF VISCOPlastic FLUID

Figure 7.18: Comparison of surface profiles from experiments (red) and shallow-layer theory (blue). The panels show profiles at a selection of times from four different experiments, for which the dimensionless parameters are given in table 7.1. The profiles shown correspond to the following dimensionless times (increasing from bottom to top as indicated in panel b)): $t = 1.4, 4.1, 9.4, 20$ (Test I); $t = 0.55, 2.5, 5.1, 10$ (Test II); $t = 0.041, 0.21, 0.50, 0.99$ (Test III); $t = 0.045, 0.11, 0.24, 0.60$ (Test IV). The vertical dashed line indicates $x = 2h_\infty$, to the left of which we anticipate the shallow-layer theory to fail.
7.9. CONCLUSIONS

Figure 7.19: The instability observed behind the scraper for slow speeds and narrow gaps. The images are from Tests II (left) and V (right), with dimensional gap sizes, $h_\alpha = 3$ mm and 0.6 mm, respectively, and speed, $U \approx 0.4$ mm/s for both. The red lines and text indicate a typical scale for the wavelength of the instability.

pattern of a reasonably well defined wavelength behind the scraper (see figure 7.19). The pattern formed is reminiscent of the washboard instability where an inclined plate towed over a layer of viscoplastic fluid can become unstable to vertical oscillations, resulting in a periodic pattern behind the scraper [71], however this mechanism is not applicable to our instability, since the scraper is rigid and not free to lift vertically or deflect horizontally. It appears the wavelength of the pattern is set by the gap size, with the two panels in figure 7.19 corresponding to a factor of 5 difference in gap size, and exhibiting a similar scaling between the resulting wavelengths. We do not believe the upstream and downstream instabilities are connected since the wavelength of the pattern downstream does not appear to be related to the wavelength upstream, and because the pattern upstream of the scraper is approximately stationary in the frame of the scraper, thereby not producing any periodic behaviour at the leading edge. We are not currently able to provide a definite mechanism for this instability, which would warrant further experiments to detail the dependence on scraper velocity and gap size more fully, as well as an exploration of the further effects of material properties notably including yield stress and slip. An observed pinching-off of a slightly reduced layer height behind the scraper, and the eventual de-wetting of the base between the ripples suggests that surface tension could be important in the process.

7.9 Conclusions

We have used lubrication theory to predict the evolution of the free surface of a layer of viscoplastic fluid when scraped by a infinitely wide scraper with a small gap underneath. At early times the depression of the free surface behind the scraper is a reflection of the elevation ahead of the scraper, which grows as $t^{1/2}$ in height and length. When a constant
leakage flux is assumed, the mound upstream of the scraper eventually approaches a quasi-rigid similarity solution, which grows vertically as $t^{1/3}$ and horizontally as $t^{2/3}$, and a layer of constant thickness is left behind the scraper after the initial transient adjustment of the free surface. If we instead allow the leakage flux to vary according to the hydrostatic pressure difference between the upstream and downstream edges of the scraper, then a steady state is eventually reached, though only after very large times when the gap under the scraper is small. On the approach to this steady state there are three regimes, the initial early-time adjustment, followed by a quasi-steady intermediate-time regime, before an exponential approach to the steady state. Behind the scraper there is again an initial adjustment of the free surface, after which the residual layer is unyielded and varies very slowly in height.

To test the predictions of the shallow-layer theory, we carried out preliminary experiments using a commercial hair gel. These indicated a need to include slip into the model, after which the free-surface profiles agreed reasonably well with the theory, except for in a region close to the scraper which is not well described by shallow-layer theory due to significant vertical velocities. We observed a previously reported buckling instability of the free surface as well as a novel instability associated with the thin gap under the scraper. The results of these preliminary experiments support additional experimental investigation, in particular to better understand the instability downstream of the scraper, and to further explore the effects of material properties including yield stress and slip. Future theoretical work on the problem could involve extending the time-dependent theory given in this chapter, to a scraper of finite length, using the three-dimensional shallow-layer equations, (2.66)-(2.67) (given in §2.3), to determine the transient dynamics of the mound as it evolves to the steady-state solution given by Lister and Hinton [81].

7.A Early time ODE

We solve the ODE (7.27) using a shooting method. We first write it as the system

$$\frac{d}{d\xi} (H, H', Y) = \left( H', \frac{3x_N^2 (H - \xi H')}{2Y(3 - 3Y + Y^2)}, -\frac{3x_N^3 Bi (H - \xi H')}{2Y(3 - 3Y + Y^2) H'} \right),$$  

(7.89)

for an initial guess of the unknown $x_N$. This system is singular at the point $\xi = 1$ so we integrate from $\xi = 1 - \delta$, with $\delta \ll 1$, down to $\xi = 0$ and use a Newton solver to solve for $x_N$, by enforcing global mass conservation $\int_0^1 H d\xi = 1 - Q_\alpha$. The form of the
7.B. NUMERICAL SCHEME FOR INTEGRATING (7.3)-(7.4)

Independent variables at $\xi = 1 - \delta$ is given by

$$H \sim Bix_N\delta, \quad H' \sim -Bix_N, \quad Y \sim x_N\sqrt{\delta}, \quad (7.90)$$

and the solutions are found to be essentially independent of $\delta \ll 1$ by comparing solutions for $\delta = 10^{-6}, 10^{-8}$ and $10^{-10}$. A value of $\delta = 10^{-8}$ is taken for all calculations shown.

7.B Numerical scheme for integrating (7.3)-(7.4)

For efficient resolution of the evolving mound of fluid, we choose to restrict the computational domain to the width of the mound, and scale the $x$-coordinate by the length of the mound

$$\zeta = \frac{x}{L(t)}, \quad (7.91)$$

This introduces an advection term to the equation, which becomes

$$h_t = \frac{1}{L} \left( \frac{1}{6L} Y^2 (3h - Y) h_\zeta + h \right) + \frac{\zeta L}{L} h_\zeta \equiv \frac{1}{L} q_\zeta + r, \quad (7.92)$$

$$Y = \max \left( 0, h - \frac{L Bi}{|h_\zeta|} \right), \quad (7.93)$$

where subscripts represent partial differentiation, dot represents differentiation with respect to time, and $q$ and $r$ are the flux and rescaling terms. We solve this equation numerically using the numerical scheme proposed by Balmforth et al. [17]. We construct a (in general) non-uniform spatial grid,

$$\zeta = \{ \zeta^i, i = 0, \ldots, N \}, \quad \zeta^0 = 0 \quad \text{and} \quad \zeta^N = 1.$$}

We then define

$$\zeta^{(i+1)/2} = \frac{\zeta^{(i+1)} + \zeta^{(i)}}{2}, \quad h^{(i+1)/2} = \frac{h^{(i+1)} + h^{(i)}}{2}, \quad h^{(i+1)/2}_\zeta = \frac{h^{(i+1)} - h^{(i)}}{\zeta^{(i+1)} - \zeta^{(i)}}, \quad (7.94)$$

Discretisation of (7.92) provides ordinary differential equations (ODEs) for $h^{(i)}$, $i = 1, \ldots, N - 1$, given by

$$\dot{h}^{(i)} = \frac{2}{L} q^{(i+1)/2} - q^{(i-1)/2} + \frac{\zeta^{(i+1)} - \zeta^{(i)}}{\zeta^{(i+1)} - \zeta^{(i-1)}} r^{(i+1)/2} + \frac{\zeta^{(i)} - \zeta^{(i-1)}}{\zeta^{(i+1)} - \zeta^{(i-1)}} r^{(i-1)/2}, \quad (7.95)$$

where $q^{(i+1)/2}$ and $r^{(i+1)/2}$ are the flux and rescaling terms in (7.92) evaluated at $\zeta^{(i+1)/2}$. We have $\dot{h}^{(N)} = 0$ (with initial condition satisfying $h^{(N)} = 1$), and the flux boundary conditions provide ODEs for $h^{(0)}$ and $L$, via

$$\dot{h}^{(0)} = \frac{q^{(1/2)} - Q_\alpha}{L \zeta^{(1/2)}} + r^{(1/2)}, \quad \dot{L} = 2 \frac{q^{(N-1/2)} - 1}{1 - h^{(N-1)}}, \quad (7.96)$$

191
where \( Q_\alpha \) is the leakage flux. These coupled ODEs are solved using a backward differentiation formula, as implemented in SciPy’s `solve_ivp` function [141].

When integrating from \( t = 0 \) with an initially uniform layer of fluid, the approach above fails since \( L(0) = 0 \). Instead, we carry out the method above on a fixed horizontal grid (i.e. \( L = 1, \frac{dL}{dt} = 0 \)) and a domain size chosen sufficiently large to encompass the disturbed region throughout the time period under consideration. It is then possible to use the resulting solution as an initial profile to commence further time-stepping using the method detailed above.

It is also possible to solve for the free-surface height behind the scraper, using an equivalent scheme to above. The length of the disturbed region behind the scraper is denoted \( L_b \) and is solved for in the same manner as \( L \) in front of the scraper. The leakage flux, \( Q_\alpha \) is taken either constant or as a function of the height immediately in-front (\( \hat{h}_0 \)) and behind (\( \hat{h}_b \)) the scraper due to hydrostatic pressure driving additional flux through the gap as discussed in §7.6. Behind the scraper there is an additional complication that occurs when the fluid behind the scraper drops to a level at which it is unyielded. The condition immediately behind the scraper is given by

\[
\frac{1}{6}Y^2(3h - Y) \frac{\partial h}{\partial x} + h = Q_\alpha, \quad (7.97)
\]

thus, as long as \( \hat{h}_b \neq Q_\alpha \), we require \( Y \neq 0 \) and the layer to be yielded, but when \( \hat{h}_b \) equals \( Q_\alpha \) the layer must be unyielded. Thus \( \hat{h}_b = Q_\alpha \) constitutes a singular point of the equations which permits discontinuous derivatives, presenting numerical difficulty. Rather than attempting to integrate through this singular point, the integration is interrupted when \( \hat{h}_b \) first equals \( Q_\alpha \). Provided the layer does not become re-yielded immediately behind the scraper, after this time there are three distinct regions to the free surface: the region behind the scraper that deformed in the initial stages of the problem; an unyielded layer behind the scraper that, in the frame of the scraper, is purely advected to the left and satisfies \( \hat{h}_b = Q_\alpha \) at the scraper boundary; and the growing mound in front of the scraper. The free surface in each of these regions is solved for using a numerical scheme as above.

### 7.C Leakage flux for a Herschel-Bulkley fluid

The leakage flux of a Herschel-Bulkley fluid through the thin gap under the scraper, can again be modelled using a generalised-Couette flow driven by the translating bottom boundary and the dimensionless pressure gradient, \( \hat{G} \). We consider the flow profile under
the scraper which, in general, can have two yielded regions and one plugged region (see figure 7.20). In each of the yielded regions the horizontal pressure gradient is balanced by the vertical gradient in shear stress and one can deduce that the horizontal velocity, $u$, in these regions takes the form

$$\hat{U}_p - u = \frac{N}{N + 1} \hat{G}^\frac{1}{N} z \frac{N+1}{N},$$

(7.98)

where $\hat{U}_p$ is the non-dimensional plug velocity, and $z$ is a positive coordinate measuring the vertical distance from the plug in each of the regions. Thus the velocity profiles above and below the plug are reflections of one another, save for the lower profile being truncated sooner due to the translating bottom boundary and the static top boundary.

We can invert to find the non-dimensional heights, $\hat{h}_1$ and $\hat{h}_2$, via

$$\hat{h}_1 = \left( \frac{N + 1}{N} \left( \hat{U}_p - 1 \right) \right)^{\frac{N}{N+1}} \hat{G}^{-\frac{1}{N+1}}, \quad \hat{h}_2 = \left( \frac{N + 1}{N} \hat{U}_p \right)^{\frac{N}{N+1}} \hat{G}^{-\frac{1}{N+1}},$$

(7.99)

The dimensionless flux under the scraper is given by

$$Q_\alpha = \hat{h}_\alpha \hat{U}_p - \frac{N^2 \hat{G}^{\frac{1}{N}}}{(N + 1)(2N + 1)} \left( \hat{h}_1^{\frac{2N+1}{N+1}} + \hat{h}_2^{\frac{2N+1}{N+1}} \right),$$

(7.100)

$$= \hat{h}_\alpha \hat{U}_p - \frac{N}{2N + 1} \left( \frac{N + 1}{N} \right)^{\frac{N}{N+1}} \hat{G}^{-\frac{1}{N+1}} \left( \hat{U}_p^{\frac{2N+1}{N+1}} + \left( \hat{U}_p - 1 \right)^{\frac{2N+1}{N+1}} \right).$$

(7.101)

Finally, force balance on the plug requires that

$$2Bi = \left( \hat{h}_\alpha - \hat{h}_1 - \hat{h}_2 \right) \hat{G},$$

(7.102)

$$\Rightarrow \hat{U}_p^{\frac{N}{N+1}} + \left( \hat{U}_p - 1 \right)^{\frac{N}{N+1}} = \left( \frac{N}{N + 1} \right)^{\frac{N}{N+1}} \hat{G}^{\frac{1}{N+1}} \left( \hat{h}_\alpha - \frac{2Bi}{\hat{G}} \right).$$

(7.103)
We can thus set
\[ \hat{G}_{c2} h_\alpha - \left( \frac{N + 1}{N} \right)^{\frac{N}{N+1}} \hat{G}_{c2}^{\frac{N}{N+1}} = 2Bi. \] (7.104)

Thus, for \( \hat{G} > \hat{G}_{c2} \), (7.103) defines \( \hat{U}_p \) implicitly in terms of \( \hat{G} \) and (7.101) then gives the leakage flux under the scraper. For pressure gradients below this critical value, and provided \( \hat{h}_2 < \hat{h}_\alpha \) there is a plug attached to the bottom boundary with velocity \( \hat{U}_p = 1 \). We can thus set \( \hat{U}_p = 1 \), \( \hat{h}_1 = 0 \) and
\[ \hat{h}_2 = \left( \frac{N + 1}{N} \right)^{\frac{N}{N+1}} \hat{G}^{\frac{1}{N+1}}, \] (7.105)
and the leakage flux is given still by (7.101), with \( \hat{U}_p = 1 \). The final regime, for sufficiently low \( \hat{G} \), occurs when \( \hat{h}_2 \geq \hat{h}_\alpha \) and so
\[ \hat{G} \leq \hat{G}_{c1} = \left( \frac{N + 1}{N} \right)^N \hat{h}_\alpha^{-(N+1)}. \] (7.106)

The leakage flux then takes a different form. We now have \( \hat{h}_2 \geq \hat{h}_\alpha \) and from (7.98) we find
\[ \hat{U}_p = \frac{N}{N + 1} \hat{G}^{\frac{1}{N+1}} \hat{h}_2^{\frac{N+1}{N+1}}, \quad \hat{U}_p - 1 = \frac{N}{N + 1} \hat{G}^{\frac{1}{N+1}} (\hat{h}_2 - \hat{h}_\alpha)^{\frac{N+1}{N}}, \] (7.107)
from which we can eliminate \( \hat{h}_2 \) to find
\[ \hat{U}_p^{\frac{N}{N+1}} - (\hat{U}_p - 1)^{\frac{N}{N+1}} = \left( \frac{N}{N + 1} \right)^{\frac{N}{N+1}} \hat{G}^{\frac{1}{N+1}} \hat{h}_\alpha, \] (7.108)
and the leakage flux is given by
\[ Q_\alpha = \hat{h}_\alpha \hat{U}_p - \frac{N}{2N + 1} \left( \frac{N + 1}{N} \right)^{\frac{N}{N+1}} \hat{G}^{\frac{1}{N+1}} \left( \hat{U}_p^{\frac{2N+1}{N+1}} - (\hat{U}_p - 1)^{\frac{2N+1}{N+1}} \right). \] (7.109)

To summarise the three regimes, we can define
\[ f_\pm(G, V; B) \equiv V^{\frac{N}{N+1}} \pm (V - 1)^{\frac{N}{N+1}} - \left( \frac{N}{N + 1} \right)^{\frac{N}{N+1}} \hat{G}^{\frac{1}{N+1}} (\hat{h}_\alpha - \frac{2B}{G}), \] (7.110)
\[ q_\pm(G, V) \equiv \hat{h}_\alpha V - \frac{N}{2N + 1} \left( \frac{N + 1}{N} \right)^{\frac{N}{N+1}} \hat{G}^{\frac{1}{N+1}} \left( V^{\frac{2N+1}{N+1}} \pm (V - 1)^{\frac{2N+1}{N+1}} \right), \] (7.111)
and then the leakage flux is given by
\[ Q_\alpha(\hat{G}) = \begin{cases} q_-(\hat{G}, \hat{U}_p) & \text{with } \hat{U}_p \text{ satisfying } f_-(\hat{G}, \hat{U}_p; 0) = 0 \quad \text{for } \hat{G} < \hat{G}_{c1}, \\ q_+(\hat{G}, 1) & \text{for } \hat{G}_{c1} \leq \hat{G} < \hat{G}_{c2}, \\ q_+(\hat{G}, \hat{U}_p) & \text{with } \hat{U}_p \text{ satisfying } f_+(\hat{G}, \hat{U}_p; Bi) = 0 \quad \text{for } \hat{G}_{c2} \leq \hat{G}. \end{cases} \] (7.112)

194
Figure 7.21: Dimensionless leakage flux, $Q_\alpha$, as a function of the dimensionless pressure gradient under the scraper, $\hat{G}$, for $\hat{h}_\alpha = 0.1$, $N = 0.5$ and $Bi = 1$ (a) and $Bi = 100$ (b). The transitions between different flow regimes are marked by vertical dotted lines.

Figure 7.21 shows the leakage flux calculated by this model for shear index $N = 0.5$. Compared with the Bingham case (see figure 7.3) the steady state, $Q_\alpha = 1$, is reached for significantly lower (though still relatively large) dimensionless pressure gradients, $\hat{G}$. This is to be expected since the high shear rate in the thin gap under the scraper will result in a lower effective viscosity for the fluid and, thus, in a greater leakage flux for the same imposed pressure gradient.
In this thesis we have considered a range of problems relating to the flow of a viscoplastic fluid, modelled with a Bingham or Herschel-Bulkley constitutive law. Some of the problems have previously been solved for the flow of a Newtonian fluid, and exhibit similarity solutions in this case. The inclusion of a yield stress in these problems has a non-trivial effect, resulting in rigid unyielded regions and/or viscoplastic boundary layers which separate rigid, or almost rigid, regions from their neighbours or the domain boundaries. We have elucidated these features through the use of asymptotic and numerical methods. Furthermore, we have used shallow-layer theory and laboratory experiments to study the scraping of a thin layer of viscoplastic fluid.

In §3 we studied the converging flow in a wedge or cone, demonstrating that, as the fluid converges from the slower to faster flowing regions it develops from an almost unyielded plastic flow with viscoplastic boundary layers, to a strongly yielded flow given by the Newtonian solution to leading order. As it does so, the flow becomes more strongly focussed at the centre of the geometry and an azimuthal flow develops as a consequence of conservation of mass. We find that the thickness of the viscoplastic boundary layer is spatially constant for the case of a Bingham fluid in a planar wedge, or for a Herschel-Bulkley fluid with shear index $N = 0.5$ in a cone, and otherwise depends on the radial distance from the vertex of the geometry, $r$. More specifically, we find that the boundary layer width (in a Cartesian sense) scales as $(r^{N-1} Bi)^{1/(1+N)}$ and $(r^{2N-1} Bi)^{1/(1+N)}$ for a Herschel-Bulkley fluid in a wedge and cone respectively. In this work the fluid was assumed to yield everywhere, as one might expect where the fluid is being pushed through
CHAPTER 8. CONCLUSIONS

the geometry (for example by some form of plunger), however the behaviour could be quite different in a gravity driven flow. Notably, if the outflow of the converging geometry is of infinitesimal width, then no finite value of gravity would succeed in yielding the fluid to drive a flow through the geometry. If instead the outflow is of finite size, then we might anticipate that the outflow would have a non-trivial effect on the flow, with the possibility of static unyielded regions that remain stuck to the boundaries, while the fluid in the centre of the domain drains out of the bottom of the geometry. The size and shape of these unyielded regions would likely depend on the magnitude of the yield stress and the size of the outflow, and determining this dependence would be an interesting problem for future study.

In §4 we considered the disturbance of viscoplastic fluid occupying a corner, by some forcing far from the vertex. Under certain conditions it is possible for eddies to form, in the absence of inertia, analogously to the viscous eddies described by Moffatt [92]. Whereas Moffatt’s eddies extend infinitely far into the corner via a diminishing infinite series, for a viscoplastic fluid the yield stress ultimately inhibits the development of eddies at small distances from the corner and a plugged region forms at the tip. As the Bingham number is reduced (e.g. by the yield stress being decreased or the strength of the imposed disturbance being increased), the eddy adjacent to this plug develops from an almost unyielded plug in solid body rotation, to a fully yielded shearing eddy. This eddy imposes stresses on the unyielded plug and at some critical Bingham number, \( Bi_c \), the torque exerted by these stresses exceeds the torque that can be provided by the yield stress in the unyielded plug. The fluid in the plug therefore yields along a circular arc, forming a new eddy. A heuristic torque-balance argument was given to estimate the value of this critical Bingham number and the dependence of the depth of the unyielded plug on the Bingham number, and both were shown to give a good approximation to the values determined by numerical simulations. The prediction of these quantities is a key result of this chapter.

At Bingham numbers just below the critical value \( (Bi = Bi_c - \Delta Bi) \) the smallest eddy is almost entirely unyielded and a viscoplastic boundary layer method can be used to describe the flow. In particular we find that the dimensionless rotation rate of the plug scales like \( \Delta Bi^{3/2} \) and the width of the thickest and thinnest points of the boundary layer scale like \( \Delta Bi^{1/2} \) and \( \Delta Bi \), respectively. In this chapter we primarily considered the case of an infinite wedge, for which we show there is a discrete self-similarity of the solution as a function of the Bingham number. Namely, the solution for a given \( Bi \) can be scaled to match the solution for \( S_2 Bi \), via \( r \rightarrow S_0 r, \ u \rightarrow S_1 u, \) and \( (\tau, p) \rightarrow S_2(\tau, p) \), where the scaling factors \( S_i \) are given in terms of the eigenvalue of the viscous problem solved by
Moffatt [92]. However, we also show that the conclusions can be relevant to more readily realised flow configurations by numerical simulation of the flow in a triangular corner driven by a translating lid, which compares favourably with the theoretical results for the infinite wedge. Future work could consider the influence of shear-thinning and other rheological effects, such as elasticity and thixotropy, on shape and scaling of viscoplastic eddies in a corner.

In §5 we considered the compression of viscoplastic fluid between rotating hinged plates. At sufficiently large angles between the plates, unyielded regions occur attached to the plates and rotate in solid-body motion with them. We calculated the critical angle, \( \alpha_c \), above which these plugs exists, as a function of the Bingham number and flow index for a Herschel-Bulkley fluid. In particular, we detailed the asymptotic behaviours in the regime of large Bingham number. The similarity solutions given in this chapter were for a planar flow between semi-infinite plates, and so we anticipate that, for flow in a three-dimensional, finite hinge, this solution would be embedded in the flow sufficiently close to the hinge and sufficiently far from the boundaries in the third dimension. In general, however, this solution would be perturbed by out-of-plane flow and the outer radial boundary conditions. To explore the impact of the outer boundary conditions, we computed full numerical simulations of the planar flow between finite rotating plates with a no-stress boundary at the opening of the hinge, showing how the similarity solution, including the unyielded regions and viscoplastic boundary layers, was embedded in the full flow solution. In addition to further understanding how the similarity solution is perturbed by the outer boundary condition and out-of-plane flow, future work could study the impact of other rheological effects such as thixotropy or elasticity on the existence and evolution of unyielded regions and viscoplastic boundary layers in this flow configuration.

In the study of recirculating flow of a viscoplastic fluid in a corner, §4, we saw that stagnant unyielded plugs exist at the stagnation points where separating streamlines between adjacent eddies meet the rigid boundary. Similar stagnation-point plugs are also found in the planar flow of a viscoplastic fluid through a t-junction, and in the flow of a viscoplastic fluid past an object such as a cylinder. In §6 we studied these stagnation-point plugs, deriving an asymptotic expansion at large distances from the stagnation point and using this asymptotic solution as a boundary condition for numerical simulations of the stagnation point flow of a Bingham fluid against a rigid wall. In particular, these numerical simulations allow us to calculate the resulting plug geometry. We proved that the vertex of the plug always subtends an angle of \( 90^\circ \) and that the plug meets the rigid
wall tangentially. We further showed how the solutions can be embedded in more general flow configurations, when the Bingham number is sufficiently small, using the examples of recirculating flow in a corner and flow around a cylinder to demonstrate the generality of the computed stagnation-point plugs.

Finally, in §7 we studied the scraping of a thin layer of viscoplastic fluid from a horizontal surface by a translating vertical scraper, using the shallow-layer approximation to model the evolution of the free surface in front of and behind the scraper, and modelling the flow under the scraper by a generalised Couette flow. When there is no gap underneath the scraper, or if the leakage flux under the scraper is modelled as a constant value (less than one in dimensionless variables), then the mound grows indefinitely. The early behaviour of the solution in this case can be described to leading order by a similarity solution in which the height and length of the mound both grow with time via $t^{1/2}$. At late times, on the other hand, the length of the mound grows faster than the height and the gradient of the free surface decreases with time. As a result, the late time behaviour of the solution is given by a quasi-rigid solution in which the fluid in the mound is unyielded to leading order and the surface takes a characteristic square-root profile. In this regime the height of the mound grows like $t^{1/3}$, while the length grows like $t^{2/3}$. When the leakage flux under the scraper is coupled to the free surface height in front and behind the scraper via the resulting pressure drop, the solution eventually reaches a steady state in which the leakage flux under the scraper balances the incoming flux from the constant thickness layer in front of the scraper. However, when the gap under the scraper is thin, we showed that the leakage flux varies only very slowly, and that the steady state is reached at very large times. The early time similarity solution from the constant leakage flux case remains valid in this regime since the leakage flux was found initially to be constant to leading order. At intermediate times, once the evolution of the mound has become quasi-static but before the steady state has been reached, the problem can be reduced to an ordinary differential equation (ODE) for the free surface height immediately in front of the scraper, $\hat{h}_0$, as a function of time. When the leakage flux is negligible, this ODE reproduces the quasi-rigid scaling in which $\hat{h}_0 \sim t^{1/3}$ when the Bingham number is of order unity ($Bi = O(1)$), and a viscously dominated solution in which $\hat{h}_0 \sim t^{1/4}$ when the Bingham number is sufficiently small. We also identified the range of Bingham numbers for which a transition occurs between viscously and yield-stress dominated behaviour. When the leakage flux approaches unity, the ODE implies an exponential approach to the steady state. Finally, we carried out preliminary experiments using a commercial hair gel to test the validity of the predictions of shallow layer theory. These
experiments showed evidence of wall slip, and agreed reasonably well with the predictions of shallow-layer theory once a slip boundary condition was included in the model. There were also two instabilities observed in the experiments. One of these involved the buckling or folding of the free surface of the mound upstream of the scraper, as reported by Maillard et al. [87], and the second resulted in the residual layer downstream of the scraper breaking into a periodic pattern of ripples. These are interesting features not captured by the shallow-layer theory presented in this chapter, and could be the basis for future theoretical and experimental work on the problem. Future theoretical work could also study the time-dependent evolution of the mound in front of a scraper of finite length, to understand how the system evolves to the steady-state solutions of Lister and Hinton [81].

This thesis has explored the role that the yield stress plays in flows of idealised viscoplastic models, in particular detailing the development of viscoplastic boundary layers and unyielded regions in these flows. Viscoplasticity remains a rich vein for future study. The asymptotic theory of viscoplastic boundary layers and the use of slip-line theory for large Bingham number flows has predominantly been developed for application to planar flows. In this thesis we carry out one example of a non-planar boundary layer solution in the case of an axisymmetric converging flow of a viscoplastic fluid through a cone, but a general theory for viscoplastic boundary layers and unyielded regions in three-dimensional flows remains to be developed. Numerical methods for the accurate solution of viscoplastic flows continue to be computationally expensive, and there are limited results for fully three-dimensional flows, so any theoretical results for such problems would serve as useful benchmarks for computational studies. Beyond the study of idealised viscoplastic models, experimental studies indicate that many viscoplastic fluids exhibit thixotropic and elastic behaviours in addition to the existence of a yield stress. Constitutive laws have been proposed to account for some of these features (for example [122], [48]) and understanding how these properties affect the existence, size and shape of unyielded regions and boundary layers in viscoplastic flows would be of great significance. Finally, while the viscoplastic model is commonly used for application to environmental and geophysical flows, often these systems have a rheology that evolves spatially and temporally (for example through solidification, melting or entrainment of eroded sediment). Therefore, developing accurate models and an improved understanding of how rheological evolution controls the time-dependent dynamics of a flow is a significant area for future research.
Bibliography


215