Hopf algebroids and Grothendieck–Verdier duality

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Abstract

Grothendieck-Verdier duality is a powerful and ubiquitous structure on monoidal categories, which generalises the notion of rigidity. Hopf algebroids are a generalisation of Hopf algebras, to a non-commutative base ring. Just as the category of finite-dimensional modules over a Hopf algebra inherits rigidity from the category of vector spaces, we show that the category of finite-dimensional modules over a Hopf algebroid with bijective antipode inherits a Grothendieck-Verdier structure from the category of bimodules over its base algebra. We investigate the structure on both the algebraic and categorical sides of this duality.

1 Introduction

Hopf algebras are defined precisely in such a way that the category of their finite-dimensional modules inherits rigidity from the category of vector spaces. The rigid dual module structure comes from the antipode map. Hopf algebroids are generalisations of Hopf algebras, defined in the category of bimodules over some algebra. While this category is not rigid in general, it is natural to ask if the category of finite-dimensional modules over a Hopf algebroid inherits some duality structure from this underlying category. We show that this is correct, and the duality structure in question is Grothendieck-Verdier, also called non-symmetric *-autonomous, which is a generalisation of rigidity. The crucial step in this process is to determine the dualising object, for which the vector space dual of the base algebra is a natural choice. We show that the antipode map equips this space with an action of the Hopf algebroid, and that the corresponding dualising functor is an anti-equivalence, hence endowing the category of modules with a Grothendieck-Verdier structure.

1.1 Hopf algebroids

Hopf algebroids are generalisations of Hopf algebras, with a noncommutative base algebra. Recently, they have found applications in quantum gravity [18], noncommutative geometry [12], differential equations [16] and Hopf-Galois extensions [13], to name a few. While there is an accepted definition of bialgebroid, there are different definitions of Hopf algebroid, and we work with the definition from [4, Definition 4.1], which has been shown to be equivalent to the definition from [9, Section 11], by [4][Theorem 4.7]. We call this a full Hopf algebroid, or sometimes just a Hopf algebroid (with bijective antipode), which is a special case of a left Hopf algebroid, or $\times_A$-Hopf algebra, as defined in [25]. In general, the category of finite-dimensional modules over a Hopf algebroid is not rigid. If a rigid category is desired, then it is typical to consider the category of finitely generated projective modules instead.

1.2 Grothendieck-Verdier categories

Grothendieck-Verdier categories are closed monoidal categories with a duality structure that generalises rigidity [2, 5]. They are also known as non-symmetric *-autonomous, or just *-autonomous, and we will use these terms interchangably. Many of the results associated with rigid categories have analogues in the Grothendieck-Verdier setting. For example, the Eilenberg-Watts theorem [11, Lemma 3.7]. Grothendieck-Verdier categories arise in a broad range of mathematical fields [15]. For example, they have found applications in logic [20], quantum theory [8], algebra [19], conformal field theory [1] and algebraic topology [21] and finite tensor categories [22].

1.3 Connections

The connections between Hopf algebroids and Grothendieck-Verdier categories have been noticed before. In [9][Example 7.4], where Hopf algebroids are characterised by a "strong *-autonomous structure" on an opmorphism between pseudomonoids in the bicategory of algebras, bimodules and bimodule homomorphisms.

It is also known that a *-autonomous monad lifts the *-autonomous structure to its category of algebras [23, 24]. Hopf algebroids are known to be examples of Hopf monads [6], and the underlying category of bimodules is *-autonomous [10, 11]. It remains to show that Hopf algebroids are in fact *-autonomous monads. This is almost established in [14], where a sufficient condition is given for a Hopf monad to be a *-autonomous monad - namely, that the dualising object has an algebra structure. This can be established for a Hopf algebroid, with invertible antipode, as we will show. However, a different definition of *-autonomous category is used, so the result cannot be deduced directly. Further, the main result of [14] relies upon a theorem which claims that "the notions of linear distributive categories with negation and *-autonomous categories coincide", for which the references [7, 23] are provided. [7, Theorem 4.5] claims to prove this result in the symmetric case, but a large part of the proof is left "to the faith of the reader". In fact, obtaining the distributors turns out to be surprisingly subtle, see [11]. Inspired by these connections, and in light of these ambiguities, we believe it is valuable to provide an explicit proof, which leaves no room for doubt.

1.4 Outline

In Section 2, we introduce the definitions of bialgebroid and Hopf algebroid (with bijective antipode) and collect some useful consequences and remarks.

In Section 3, we exhibit the closed structure of the category of modules over a Hopf algebroid, and express the internal Hom functors in terms of the internal Hom functors of the base algebra. We write down the action of the Hopf algebroid on these internal Hom spaces.

In Section 4 we define a Hopf algebroid module structure on the dual of the base algebra and relate the vector space dual of a module to the internal Hom functors to this module. We state a definition of Grothendieck-Verdier category, and finally we prove that the category of finite-dimensional modules over a Hopf algebroid with bijective antipode is a Grothendieck-Verdier category, where a natural choice of dualising object is the dual of the base algebra.

Usually it will be clear from the context in which set an element takes its value. We tend to use H for the Hopf algebroid (total algebra) and A for the base algebra, M for an H-module and M^* for its dual. Then the elements are usually denoted $h, k \in H$, $a, b, c \in A$, $m \in M$ and $f \in M^*$.

2 Hopf algebroids

In this section, we define bialgebroid and Hopf algebroid. We also collect some useful consequences and remarks.

Definition 2.1 ([4, Definition 2.1]). Let k be a commutative ring and A and H be k-algebras. H is a *bialgebroid* over the *base algebra* A if it is equipped with

- two algebra maps $\alpha : A \to H$, $\beta : A^{\mathrm{op}} \to H$ with $\alpha(a)\beta(b) = \beta(b)\alpha(a)$, which endow H with the A-bimodule structure $a \cdot h \cdot b = \alpha(a)\beta(b)h$.
- a coassociative comultiplication $\Delta : H \to H \otimes_A H$, where $\otimes_A$ is the tensor product of A-bimodules, and a counit $\varepsilon : H \to A$, which are

both k-algebra maps, satisfying

$$\begin{aligned}
&\Delta(h\alpha(a)\beta(b)) = h_1\alpha(a) \otimes_A h_2\beta(b),\\
&h_1\beta(a) \otimes_A h_2 = h_1 \otimes_A h_2\alpha(a),\\
&\varepsilon(h\alpha(a)) = \varepsilon(h\beta(a)),\\
&\varepsilon(hk) = \varepsilon(h\alpha(\varepsilon(k))) = \varepsilon(h\beta(\varepsilon(k))),
\end{aligned} \tag{1}$$

where we have used Sweedler notation to write $\Delta(h) = h_1 \otimes_A h_2$.

Definition 2.2 ([4, Definition 4.1])**.** Let H be a bialgebroid with base A. Then H is a *Hopf algebroid* with base A if it is equipped with an invertible anti-algebra map $S : H \to H$, called the *antipode*, satisfying $S \circ \beta = \alpha$, and

$$\begin{aligned}
S(h_1)_1 h_2 \otimes_A S(h_1)_2 &= 1_H \otimes_A S(h),\\
(S^{-1}h_2)_1 \otimes_A (S^{-1}h_2)_2 h_1 &= S^{-1}(h) \otimes_A 1_H.
\end{aligned} \tag{2}$$

Remark. The definition of a full Hopf algebroid, or Hopf algebroid with invertible antipode has many equivalent formulations [4, Proposition 4.2], one of which is a pair of a left and right bialgebroid, such that the antipode maps between them. We choose the definition above as it will be more productive in computing the action on certain modules later.

Definition 2.3. Let H be a Hopf algebroid with base A. Let mod(H) denote the category of finite-dimensional (left) modules over H, as a k-algebra, and let bimod(A) denote the category of finite dimensional bimodules over A.

Remark.

- A bialgebroid H acts on its base algebra A by

$$h \cdot a = \varepsilon(h\alpha(a)) = \varepsilon(h\beta(a)). \tag{3}$$

- The forgetful functor from mod$(H) \to$ bimod(A) is given by

$$a \cdot m \cdot b = (\alpha(a)\beta(b)) \cdot m. \tag{4}$$

- We will assume A is finite-dimensional unless stated otherwise.

3 Closed structure

In this section, we prove that the left/right internal Hom functors for Hopf algebroid modules can be expressed in terms of the internal Homs of left-/right modules over the base algebra. This is already known - see [25]. We write down the isomorphisms explicitly in order to present the module structure of the internal Hom, for later use.

Definition 3.1. Let $\mathcal{C}$ be a monoidal category. $\mathcal{C}$ is called left/right *closed* if it possesses a left/right internal Hom, which is a functor

$$[-,-]^{r/l} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{C}, \tag{5}$$

such that there are the following natural isomorphisms.

$$\mathrm{Hom}(X, [Y, Z]^r) \cong \mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(Y, [X, Z]^l). \tag{6}$$

Proposition 3.2. *The category of H-modules is right and left closed, with*

$$\begin{aligned} [M, N]^r &= \mathrm{Hom}_H(H \otimes_A M, N), \\ [M, N]^l &= \mathrm{Hom}_H(M \otimes_A H, N). \end{aligned} \tag{7}$$

Proof. Our definition of Hopf algebroid is in particular a left Hopf algebroid [4, Proposition 4.2]. Left Hopf algebroids are known to be closed, with this internal Hom functor - for the right closed case, see [25, Proposition 3.3], see also [3, Section 4.6.2]. □

Definition 3.3. We introduce the following notation which we will use throughout the paper.

$$\begin{aligned} h^1 &= S^{-1}(S(h)_2), \quad h^2 = S^{-1}(S(h)_1), \\ \tilde{h}^1 &= S(S^{-1}(h)_2), \quad \tilde{h}^2 = S(S^{-1}(h)_1). \end{aligned} \tag{8}$$

Lemma 3.4. *Let M be a left H-module. Then the module $H \otimes_A M$, with the diagonal action consisting of left multiplication on H and the action on M,*

$$k \cdot (h \otimes_A m) = k_1 h \otimes_A k_2 \cdot m, \tag{9}$$

is isomorphic to $H \otimes_{A^{\mathrm{op}}} M$, with left multiplication on H

$$k \cdot (h \otimes_{A^{\mathrm{op}}} m) = kh \otimes_{A^{\mathrm{op}}} m. \tag{10}$$

Similarly $M \otimes_A H$ and $M \otimes_{A^{\mathrm{op}}} H$, equipped with the following actions, are isomorphic.

$$\begin{aligned} k \cdot (m \otimes_A h) &= k_1 \cdot m \otimes_A k_2 h, \\ k \cdot (m \otimes_{A^{\mathrm{op}}} h) &= m \otimes_{A^{\mathrm{op}}} kh. \end{aligned} \tag{11}$$

Proof. The first isomorphism is given by

$$\begin{aligned} H \otimes_{A^{\mathrm{op}}} M &\cong H \otimes_A M \\ h \otimes_{A^{\mathrm{op}}} m &\mapsto h_1 \otimes_A h_2 \cdot m \\ h^1 \otimes_{A^{\mathrm{op}}} S(h^2) \cdot m &\leftarrow h \otimes_A m, \end{aligned} \tag{12}$$

We can see that this is an H-module homomorphism by

$$(kh)_1 \otimes_A (kh)_2 \cdot m = k_1 h_1 \otimes_A k_2 h_2 \cdot m. \tag{13}$$

Similarly, for the second isomorphism we have

$$\begin{aligned} M \otimes_{A^{\mathrm{op}}} H &\cong M \otimes_A H \\ m \otimes_{A^{\mathrm{op}}} h &\mapsto h_1 \cdot m \otimes_A h_2 \\ S^{-1}(\tilde{h}^1) \cdot m \otimes_{A^{\mathrm{op}}} \tilde{h}^1 &\leftarrow\!\shortmid m \otimes_A h, \end{aligned} \tag{14}$$

Note that these isomorphisms are essentially the same as those denoted by α and β in [4, Proposition 4.2]. □

Lemma 3.5.

$$\begin{aligned} h\beta(a) \otimes_{A^{\mathrm{op}}} m = h \otimes_{A^{\mathrm{op}}} \beta(a) \cdot m = h \otimes_{A^{\mathrm{op}}} m \cdot a, \\ m \otimes_{A^{\mathrm{op}}} \alpha(a)h = \alpha(a) \cdot m \otimes_{A^{\mathrm{op}}} h = a \cdot m \otimes_{A^{\mathrm{op}}} h. \end{aligned} \tag{15}$$

Proof. The results follow by translating the bimodule condition $m \cdot a \otimes_A n = m \otimes_A a \cdot n$ through each of the isomorphisms in Lemma 3.4. □

Proposition 3.6. *The following are isomorphic, as H-modules.*

$$\begin{aligned} \mathrm{Hom}_H(H \otimes_A M, N) \cong \mathrm{Hom}_H(H \otimes_{A^{\mathrm{op}}} M, N) \cong \mathrm{Hom}_A(M, N), \\ \mathrm{Hom}_H(M \otimes_A H, N) \cong \mathrm{Hom}_H(M \otimes_{A^{\mathrm{op}}} H, N) \cong {}_A\mathrm{Hom}(M, N), \end{aligned} \tag{16}$$

where the H-actions on $\mathrm{Hom}_A(M, N)$ and ${}_A\mathrm{Hom}(M, N)$ are given by

$$\begin{aligned} (h \cdot f)(m) = h^1 \cdot f(S(h^2) \cdot m) = S^{-1}(S(h)_2) \cdot f(S(h)_1 \cdot m), \\ (h \cdot f)(m) = \tilde{h}^2 \cdot f(S^{-1}(\tilde{h}^1) \cdot m) = S(S^{-1}(h)_1) \cdot f(S^{-1}(h)_2 \cdot m), \end{aligned} \tag{17}$$

respectively. Here, a subscript A on the hom space to the left/right denotes left/right A-module homomorphisms, respectively. We have also used the superscript notation from Definition 3.3.

Proof. For the right closed case, see [25, Theorem and Definition 3.5]. The first isomorphisms of each line follow from Lemma 3.4, and transforms the diagonal action on $H \otimes_A M$ to the action on H only. The second isomorphisms are given by

$$\begin{aligned} \mathrm{Hom}_H(H \otimes_{A^{\mathrm{op}}} M, N) &\cong \mathrm{Hom}_A(M, N), \\ f &\mapsto [m \mapsto f(1_H \otimes_{A^{\mathrm{op}}} m)], \\ [h \otimes_{A^{\mathrm{op}}} m \mapsto h \cdot g(m)] &\leftarrow\!\shortmid g. \end{aligned} \tag{18}$$

We check that both directions give the appropriate types of homomorphisms.

$$\begin{aligned} g(m) \cdot a = \beta(a) \cdot f(1_H \otimes m) = f(\beta(a) \otimes m) = g(m \cdot a), \\ k \cdot f(h \otimes m) = kh \cdot g(m) = f(kh \otimes m) = f(k \cdot (h \otimes m)). \end{aligned} \tag{19}$$

Similarly, we have

$$\begin{aligned}\mathrm{Hom}_H(M \otimes_{A^{\mathrm{op}}} H, N) &\cong {}_A\mathrm{Hom}(M, N), \\ f &\mapsto [m \mapsto f(m \otimes_{A^{\mathrm{op}}} 1_H)], \\ [m \otimes_{A^{\mathrm{op}}} h \mapsto h \cdot g(m)] &\hookleftarrow g. \end{aligned} \tag{20}$$

Again we check

$$\begin{aligned} &a \cdot g(m) = \alpha(a) \cdot f(m \otimes 1_H) = f(m \otimes \alpha(a)) = f(a \cdot m \otimes 1_H) = g(a \cdot m), \\ &k \cdot f(m \otimes h) = kh \cdot g(m) = f(m \otimes kh) = f(k \cdot (m \otimes h)). \end{aligned} \tag{21}$$

The H-actions are calculated by applying the isomorphisms to the actions on the left-hand modules. □

4 Duality structure

In this section, we prove that the internal Hom from a given module to the dual of the base algebra is isomorphic to the dual vector space of the original module, with action given by the antipode. Finally, we use this fact to prove that the category of finite-dimensional modules over a Hopf algebroid with bijective antipode is Grothendieck-Verdier.

Proposition 4.1. *The vector space $A^* = \mathrm{Hom}_k(A, k)$ can be equipped with the following module structures, for $a, b, c \in A$, $h \in H$, $f \in A^*$.*

- *A-bimodule with action*

$$(c \cdot f \cdot a)(b) = f(abc), \tag{22}$$

- *H-module with action*

$$(h \cdot f)(a) = f(S(h) \cdot a) = f(S^{-1}(h) \cdot a). \tag{23}$$

Proof. In [17, Section 2.6.8], two different right bialgebroid structures are presented, which, together with the left bialgebroid and antipode, produce isomorphic full Hopf algebroids. By [4, Proposition 4.3], these differ by an isomorphism which is trivial on the algebra H. We can deduce that there is an isomorphism $\phi : A \to A$ such that

$$S\alpha = \beta\phi, \quad \phi\varepsilon S^{-1} = \varepsilon S \implies S^2\beta = \beta\phi, \quad \varepsilon S^2 = \phi\varepsilon. \tag{24}$$

This isomorphism yields an isomorphism of modules, between A with the action of h, and A with the action of $S^2(h)$.

$$\begin{aligned}\phi(h \cdot a) &= \phi(\varepsilon(h\beta(a))) = \varepsilon(S^2(h\beta(a))) = \varepsilon(S^2(h)S^2(\beta(a))) \\ &= \varepsilon(S^2(h)\beta(\phi(a))) = h \cdot \phi(a). \end{aligned} \tag{25}$$

□

In proving the following two lemmas, we make use of the counit condition

$$\alpha(\varepsilon(h_1))h_2 = \varepsilon(h_1) \cdot h_2 = h = h_1 \cdot \varepsilon(h_2) = \beta(\varepsilon(h_2))h_1 = h. \tag{26}$$

Lemma 4.2. *The following are isomorphic, as vector spaces*

$$\begin{aligned} M^* = \mathrm{Hom}_k(M, k) &\cong \mathrm{Hom}_A(M, A^*) \\ f &\mapsto [\phi_f(m) : a \mapsto f(m \cdot a)], \end{aligned} \tag{27}$$

Further, inducing the first action from Proposition 3.6, with $N = A^*$, yields the following action of H on M^*.

$$(h \cdot f)(m) = f(S(h) \cdot m). \tag{28}$$

Proof. We first check that the forward direction yields an A-module homomorphism.

$$\phi_f(m \cdot b)(a) = f(m \cdot ba) = \phi_f(m)(ba) = (\phi_f(m) \cdot b)(a). \tag{29}$$

For the reverse direction, one can take $\phi \mapsto [m \mapsto \phi(m)(1) \in k]$. The action is defined by

$$\begin{aligned} (h \cdot f)(m) &= (h^1 \cdot \phi_f(S(h^2) \cdot m))(1) \\ &= S^{-1}(S(h)_2) \cdot \phi_f(S(h)_1 \cdot m)(1) = \phi_f(S(h)_1 \cdot m)(S(h)_2 \cdot 1) \\ &= f((S(h)_1 \cdot m)) \cdot (S(h)_2 \cdot 1)) \\ &= f((S(h)_1 \cdot m) \cdot \varepsilon(S(h)_2)) = f(\beta(\varepsilon(S(h)_2))S(h)_1 \cdot m) \\ &= f(S(h) \cdot m), \end{aligned} \tag{30}$$

where we have used the superscript notation from Definition 3.3. □

Lemma 4.3. *The following are isomorphic, as vector spaces*

$$\begin{aligned} M^* = \mathrm{Hom}_k(M, k) &\cong {}_A\mathrm{Hom}(M, A^*) \\ f &\mapsto [\phi_f(m) : a \mapsto f(a \cdot m)]. \end{aligned} \tag{31}$$

Further, inducing the second action from Proposition 3.6, with $N = A^*$, yields the following action of H on M^*.

$$(h \cdot f)(m) = f(S(h) \cdot m). \tag{32}$$

Proof. Similarly to the first case, we have

$$\phi_f(b \cdot m)(a) = f(ab \cdot m) = \phi_f(m)(ab) = (b \cdot \phi_f(m))(a). \tag{33}$$

Again, similarly, we have

$$\begin{aligned} (h \cdot f)(m) &= (\tilde{h}^2 \cdot \phi_f(S^{-1}(\tilde{h}^1) \cdot m))(1) = S(S^{-1}(h)_1) \cdot \phi_f(S^{-1}(h)_2 \cdot m)(1) \\ &= f((S^{-1}(h)_1 \cdot 1)) \cdot (S^{-1}(h)_2 \cdot m)) \\ &= f(\varepsilon(S^{-1}(h)_1) \cdot (S^{-1}(h)_2 \cdot m)) = f(\alpha(\varepsilon(S^{-1}(h)_1))S^{-1}(h)_2 \cdot m) \\ &= f(S^{-1}(h) \cdot m) = f(S(h) \cdot m), \end{aligned} \tag{34}$$

where we have used the superscript notation from Definition 3.3, as well as the equivalence of the actions of $S(h)$ and $S^{-1}(h)$ from (23). □

Definition 4.4. [2, 5] A *Grothendieck-Verdier* category is a pair $(\mathcal{C}, K)$, where $\mathcal{C}$ is a monoidal category and $K \in \mathcal{C}$, such that there is a natural isomorphism

$$\mathrm{Hom}(X \otimes Y, K) \cong \mathrm{Hom}(X, D(Y)), \tag{35}$$

where the contravariant functor D is an anti-equivalence. K is called a dualising object and D a dualising functor.

Remark. A closed monoidal category $\mathcal{C}$ is Grothendieck-Verdier if there exists an object $K \in \mathcal{C}$, such that the internal Hom functor $D = [-, K]^r$ is an anti-equivalence.

We are now in a position to prove the main result.

Theorem 4.5. *Let H be a Hopf algebroid with finite-dimensional base k-algebra A and an invertible antipode S. Then the category of finite-dimensional H-modules is a Grothendieck-Verdier category. A natural choice of dualising object is given by the vector space dual of the base algebra A^*, with the following action of H.*

$$(h \cdot f)(a) = f(S(h) \cdot a), \qquad h \in H,\ f \in A^*,\ a \in A. \tag{36}$$

Proof. The dualising functor is given by

$$D = [-, A^*]^r \simeq \mathrm{Hom}_H(H \otimes_A (-), A^*) \simeq \mathrm{Hom}_A(-, A^*) \simeq \mathrm{Hom}_k(-, k), \tag{37}$$

with the H action on $D(M)$, for some H-module M, is given by $(h \cdot f)(m) = f(S(h) \cdot m)$. The inverse of the dualising functor is the left internal Hom to the same module. That is,

$$D^{-1} = [-, A^*]^l \simeq \mathrm{Hom}_H((-) \otimes_A H, A^*) \simeq {}_A\mathrm{Hom}(-, A^*) \simeq \mathrm{Hom}_k(-, k), \tag{38}$$

As we are considering only finite-dimensional vector spaces, we have

$$D(D^{-1}(M)) \cong D^{-1}(D(M)) \cong \mathrm{Hom}_k(\mathrm{Hom}_k(M, k), k) \cong M. \tag{39}$$

The action on $\phi_m \in M^{**}$, defined by $\phi_m(f) = f(m)$ is given by

$$\begin{aligned}(h \cdot \phi_m)(f) &= \phi_m(S(h) \cdot f) = (S(h) \cdot f)(m) = f(S^2(h) \cdot m) \\ &= f(h \cdot m) = \phi_{h \cdot m}(f).\end{aligned} \tag{40}$$

Therefore the action on M is the original one, and D is an anti-equivalence. □

Remark. Hopf algebroids also satisfy the following relation.

$$S(S^{-1}(h)_2 \otimes S^{-1}(h)_1) = S^{-1}(S(h)_2 \otimes S(h)_1). \tag{41}$$

This defines a second coproduct, which parallels the following definition of the second tensor product in a Grothendieck-Verdier category, given by

$$X \odot Y = D(D^{-1}Y \otimes D^{-1}X) \cong D^{-1}(DY \otimes DX). \tag{42}$$

In a rigid category these two tensor products are equivalent, just as one can see that for a Hopf algebra, the terms in the first line above coincide with the usual comultiplication.

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