Glendinning, P., Hogan, S. J., Homer, M. E., Jeffrey, M. R., & Szalai, R. (2024). Classification of Filippov type 3 singular points in planar bimodal piecewise smooth systems. *SIAM Journal on Applied Dynamical Systems*, *23*(3), 2242-2261. https://doi.org/10.1137/23M1622842

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Classification of Filippov type 3 singular points in planar bimodal piecewise smooth systems

P. Glendinning*, S. J. Hogan†, M. E. Homer†, M. R. Jeffrey†, and R. Szalai†

Abstract. We classify Filippov's [8] type 3 singular points of planar bimodal piecewise smooth systems. These singular points consist of fold or cusp tangencies of the vector fields to both sides of a switching surface. For isolated analytic type 3 singular points there are 25 topological classes, up to time reversal. For isolated general type 3 singular points there are 40 topological classes, up to time reversal.

Key words. Singularities, piecewise smooth systems, classification

MSC codes. 37C10

1. Introduction.

A piecewise smooth dynamical system [3, 4, 11] consists of a finite set of ordinary differential equations

$$\dot{x} = f^i(x), \quad x \in U_i \subset \mathbb{R}^n \tag{1.1}$$

where $i = 1 \dots k$, and the smooth vector fields f^i, defined on disjoint open regions U_i, are smoothly extendable to their closure $\overline{U}_i$, with $k, n \in \mathbb{Z}^+$, where $k, n > 1$. The regions U_i are separated by an $(n-1)$-dimensional hypersurface Σ called the *switching surface*. The union of Σ and all U_i covers the whole state space $D \subseteq \mathbb{R}^n$. Whenever the normal component of both vector fields either side of Σ points toward (or away from) Σ, the piecewise smooth system is said to exhibit *sliding* and a vector field (dynamics) must be defined *on* Σ. Various conventions have been used to define this *sliding vector field* $f^0(x)$, whose co-dimension is always greater than or equal to 1. In this paper, we use the Filippov convention [15, 8], the most widely adopted choice.

Piecewise smooth systems have wide application, including control systems [9, 18] and hybrid systems [16, 17], as well as occurring in systems with dry (Coulomb) friction [2, 5], neuronal systems [6] and the modelling of sleep-wake regulation [7]; see also [1, Ch. VIII] for numerous engineering examples. Piecewise smooth systems are not just simple extensions of smooth systems. For example, Filippov [8] shows that the usual notions of solution, separatrices, singular points, topological equivalence, stability and bifurcation all need revision.

The most important difference between smooth and piecewise smooth systems is that classical results for smooth systems concerning the uniqueness of solutions no longer hold. For any *smooth* dynamical system $\dot{x} = f(x)$ with $f(x)$ Lipschitz continuous, and initial conditions $x(t_0) = x_0$, the solution is unique. In piecewise smooth systems, the solution is not unique even if the vector fields f^i, f^0 are Lipschitz continuous. Solutions with different initial conditions in U_i may reach Σ (in forward or backward time, see Lemma 2.2) and then pass through a

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given point on Σ that exhibits sliding. As a consequence, singular points in piecewise smooth systems can be reached in finite, as well as infinite, time.

For a planar bimodal ($k = n = 2$) piecewise smooth system, with vector fields $f^{\pm}$ either side of Σ, a *regular* point is defined as one of the following:

- $x \notin \Sigma$ and $f^{\pm}(x) \neq 0$ (not an equilibrium),
- $x \in \Sigma$ and $f_N^-(x) f_N^+(x) \neq 0$, where subscript N denotes the normal direction to Σ (not a tangency),
- $x \in \Sigma$ and $f^0(x) \neq 0$, wherever $f^0(x)$ is defined (not a *pseudo-equilibrium*).

In contrast, a *singular* point is one of the following: (i) an equilibrium of at least one of $f^{\pm}$, (ii) a tangency to Σ of at least one of $f^{\pm}$ or (iii) a pseudo-equilibrium of the sliding vector field $f^0(x)$, where it is defined.

Filippov [8, p. 218] identified six different types of singular points:

1. The sliding vector field $f^0(x)$ has an equilibrium (a pseudo-equilibrium).
2. *One* of the vector fields $f^{\pm}$ is tangent to Σ.
3. *Both*[1] of the vector fields $f^{\pm}$ are tangent to Σ.
4. *One* equilibrium of $f^{\pm}$ lies on Σ.
5. *One* equilibrium of $f^{\pm}$ lies on Σ and *one* of the vector fields $f^{\pm}$ is tangent to Σ.
6. *Both* equilibria of $f^{\pm}$ lie on Σ.

In this paper, we focus on the following claim made by Filippov [8, p. 222] concerning type 3 singular points: "There exists a total of thirty-nine topological classes (in the case of analytic [vector fields], there are twenty four classes).". We have found no proof of this statement either in the book or in any of its available references[2].

In section 2, we set out the problem and introduce some results of Filippov that we need in this sequel. In section 3, using a different method, we recover the 22 classes found by Filippov for the case when the vector fields $f^{\pm}(x)$ are analytic, and give the phase plane representation for each. We show that 3 of the classes can be further sub-divided, depending on whether the type 3 singular point is reached in finite or infinite time, giving us a total of 25 topological classes; one more than claimed by Filippov. In section 4, we consider the case when the $f^{\pm}(x)$ are non-analytic and give details, for the first time, of the 15 extra classes that occur in this case (the same number as claimed by Filippov, but with no proof). Thus we prove that there are 40 topological classes of Filippov type 3 singular points in planar bimodal piecewise smooth systems.

2. Preliminaries.

Following Filippov [8, §19], we consider a bimodal piecewise smooth system in the (x, y)-plane where $f^{\pm} = (P^{\pm}, Q^{\pm})^{\top}$, where the switching surface Σ is given by the line $y = 0$, possibly after a local change of coordinates. Hence

$$\dot{x} = \begin{cases} P^+(x,y) & \text{if } y > 0 \\ P^-(x,y) & \text{if } y < 0 \end{cases}, \quad \dot{y} = \begin{cases} Q^+(x,y) & \text{if } y > 0 \\ Q^-(x,y) & \text{if } y < 0. \end{cases} \tag{2.1}$$

The functions $P^{\pm}, Q^{\pm}$ are sufficiently smooth for solutions in $y > 0$ and $y < 0$ to exist and be unique. The results in this paper are local to the singular points on Σ. If $(c, 0)$ represents

[1] This class contains the visible, visible-invisible and invisible two-folds [12, 13, 14].

[2] The claim is re-iterated in the table in [8, p. 250]. A proof is also absent from the the original Russian version of the book.

a singular point and U is an open neighbourhood of $(c, 0)$, let $U^+ = \{x \in U \mid y > 0\}$ and $U^- = \{x \in U \mid y < 0\}$. We say that system (2.1) is C^k_* on U if $P^\pm$, $Q^\pm$ are each C^k on $\overline{U}^\pm$. If $P^\pm$, $Q^\pm$ are analytic, then the system is C^ω_* on U and each of the functions $P^\pm$, $Q^\pm$ is equal to its Taylor series on $\overline{U}^\pm$.

If $Q^+(x, 0)Q^-(x, 0) > 0$ then solutions pass through Σ, with a possible discontinuous change in the tangent to the solution (*crossing*). If $Q^+(x, 0)Q^-(x, 0) \leq 0$, we have sliding and dynamics must be prescribed on Σ. Following Filippov [8], we choose the (unique) linear combination of the vector fields in U^+ and U^- such that $\dot{y} = 0$ on Σ. Hence

$$\dot{x} = f^0 \equiv P^0(x) = \frac{Q^-(x, 0)P^+(x, 0) - Q^+(x, 0)P^-(x, 0)}{Q^-(x, 0) - Q^+(x, 0)} \tag{2.2}$$

on Σ. Since $Q^+(x, 0)Q^-(x, 0) \leq 0$, the right hand side of (2.2) is well defined and belongs to C^k unless

$$Q^+(x, 0) = Q^-(x, 0) = 0 \quad \text{and} \quad P^+(x, 0) \neq P^-(x, 0)$$

since if $P^+(x, 0) = P^-(x, 0)$ then taking appropriate limits we find $P^0(x) = P^+(x, 0) = P^-(x, 0)$. We assume also that $P^0(x) = P^\pm(x, 0)$ if $Q^\pm(x, 0) = 0$, $Q^\mp(x, 0) \neq 0$ [8, pp. 217-218].

By a coordinate transformation, the singular point $(c, 0)$ can be taken to be the origin $(0, 0)$. Then a type 3 singular point occurs when

$$Q^+(0, 0) = Q^-(0, 0) = 0, \quad P^+(0, 0) \neq 0, \quad P^-(0, 0) \neq 0. \tag{2.3}$$

To simplify notation we write

$$P^+P^-(x) \equiv P^+(x, 0)P^-(x, 0), \quad P^+P^- \equiv P^+P^-(0), \quad Q^+Q^-(x) \equiv Q^+(x, 0)Q^-(x, 0).$$

Also we let X^+ (resp. X^-) denote the positive (resp. negative) x-axis in U, not including the origin. Note that the choice of (2.3) excludes the possibility that the type 3 singular point is an equilibrium of either flow.

To proceed with the classification, we need a definition of equivalence. Given two systems of the form (2.1), following Filippov, we say that two points $(x, 0)$ and $(x', 0)$ are in the same *topological class* if there exist open neighbourhoods U and U' of each point and a homeomorphism $\phi : U \to U'$ such that ϕ maps trajectories of the first system in U to trajectories of the second system in U'. Then the main result of this paper is:

Theorem 2.1. *For C^ω_* isolated type 3 singular points there are 25 topological classes, up to time reversal. For general (C^1_*) isolated type 3 singular points there are 40 topological classes up to time reversal.*

Note that this is one more topological class than stated, without proof, by Filippov [8, p. 222]. Now we prove two important technical lemmas. Lemma 2.2 states that the approach to the singular point is in finite time, under certain circumstances, and otherwise there is no difference between the analytic C^ω_* classes and the general C^1_* classes.

Lemma 2.2. *Let $(0, 0)$ be an isolated type 3 singular point in C^1_* given by (2.2) and suppose that $P^+P^- > 0$. Then there is an open neighbourhood U of the origin such that any* *sliding solution in U reaches the origin in finite time (forwards time or backwards time as appropriate).*

Proof. Let U be sufficiently small so that $(0,0)$ is the only singular point in U and $\min_{(x,0)\in U}\{|P^+(x,0)|, |P^-(x,0)|\} = k > 0$. A non-trivial U can always be chosen satisfying these constraints as $(0,0)$ is isolated and $P^{\pm}(x,0)$ is continuous with both $P^{\pm}(0,0) \neq 0$. Then $Q^+Q^-(x) \neq 0$ on $X^{\pm}$, since (0,0) is isolated. So each open line segment is either sliding ($Q^+Q^-(x) < 0$) or crossing ($Q^+Q^-(x) > 0$) on $X^{\pm}$. Suppose that X^+ is a sliding region. Then $P^+P^-(x) > 0$ implies that $Q^-(x,0)P^+(x,0)$ and $Q^+(x,0)P^-(x,0)$ are non-zero and have opposite signs on X^+. Hence

$$|Q^-(x,0)P^+(x,0) - Q^+(x,0)P^-(x,0)| \geq k|Q^-(x,0) - Q^+(x,0)|$$

and so from (2.2)

$$|P_0(x)| \geq k > 0. \tag{2.4}$$

If $P_0(x) > k$ then a solution starting at $(x_0, 0)$ on X^+ reaches the origin in finite backwards time $T_b > -x_0/k$. If $P_0(x) < -k$ then the solution reaches the origin in finite forwards time $T_f < x_0/k$.

The argument is equivalent if X^- is a sliding region. ■

Remark 2.3. If $(0,0)$ is an isolated type 3 singular point, as in Lemma 2.2, but now with $P^+P^- < 0$, then, in certain circumstances, the origin can be a pseudo-equilibrium, as we shall see later.

Lemma 2.4 explains why the distinction between finite and infinite time approach is necessary.

Lemma 2.4. *Consider two piecewise smooth systems of the form (2.1), both with an isolated type 3 singular point at the origin. If one such system has a solution that approaches $(0,0)$ in finite time on X^+, and the other has a solution that approaches $(0,0)$ in infinite time on X^+, then the two systems belong to different topological classes.*

Proof. We argue by contradiction. Suppose that the two systems belong to the same topological class. One has a sliding solution $\psi_1(t)$ that reaches the origin in finite time T. Hence $\psi_1(T) = (0,0)$. The other has a sliding solution $\psi_2(t)$, conjugate to ψ_1, which reaches the origin in infinite time. Suppose that the homeomorphism ϕ takes ψ_1 to ψ_2. Hence for all t such that $\psi_1(t) \in U_1$ there exists t' such that $\psi_2(t') \in U_2$ and

$$\phi(\psi_1(t)) = \psi_2(t').$$

Set $t = T$ and note that the image of the isolated type 3 singular point of one system must be the type 3 singular point of the other system. Hence $\exists\, T' \in \mathbb{R}$ such that

$$\phi(\psi_1(T)) = \phi((0,0)) = (0,0) = \psi_2(T').$$

But by definition $\psi_2(t) \neq (0,0)$ for all $t \in \mathbb{R}$, giving the required contradiction. ■

In section 3, we consider the case of analytic functions $P^{\pm}, Q^{\pm}$ in (2.1), expanding them as Taylor series to give a classification based on the signs of leading order coefficients, and whether the leading order terms are odd or even powers. We also consider whether the type 3 singular point can reached in finite or infinite time (or both). We compare our approach with that of Filippov. We find a total of 25 *analytic classes* (in contrast to Filippov's unproved assertion that there are 24 such classes [8, p. 222]).

3. $P^{\pm}, Q^{\pm}$ analytic. In this section, we show that there are 25 different different classes of type 3 singular points when $P^{\pm}, Q^{\pm}$ are analytic.

By rescaling time, we can set $P^{+}(0,0) = 1$. Since system (2.1) is now in C^{ω}_{*}, we represent the functions by Taylor series

(3.1)
$$\begin{aligned} P^{+}(x,y) &= 1 + O(|x|,|y|) \\ Q^{+}(x,y) &= b_0 x^n + O(|x|^{n+1},|y|) \\ P^{-}(x,y) &= c_0 + O(|x|,|y|) \\ Q^{-}(x,y) &= d_0 x^m + O(|x|^{m+1},|y|) \end{aligned}$$

where, to avoid degeneracy, we assume

(3.2)
$$c_0 \neq 1, \quad b_0 \neq d_0$$

and where

(3.3)
$$b_0 \neq 0, \quad c_0 \neq 0, \quad d_0 \neq 0, \quad 1 \leq n, m < \infty.$$

The first four conditions in (3.3) follow from (2.3). The last condition stems from the fact that if *all* derivatives of $Q^{\pm}(0,0)$ with respect to x were to vanish then $Q^{\pm}(x,y) = yg^{\pm}(x,y)$ with $g^{\pm}$ analytic. So $Q^{\pm}(x,0) = 0$ for all $x \in U^{+}$. But that would contradict the assumption that the origin is an isolated singular point. So there must be a finite value of n, m such that the Taylor series coefficient of $x^{n,m}$ does not vanish. Note that n determines the nature of the tangency in U^{+}, and m in U^{-}.

Informally, ignoring higher order terms, from (2.1) and (3.1) solutions in U^{+} satisfy

$$\frac{dy}{dx} \approx b_0 x^n$$

and hence, for some constant y_0, we have

$$y \approx y_0 + \frac{b_0}{(n+1)} x^{n+1}.$$

If n is odd, solutions lie on generalized parabolae, or *folds*. Folds can be either *visible* ($b_0 > 0$) or *invisible* ($b_0 < 0$). If n is even, solutions lie on generalized cubics, or *cusps*. Cusps can be either *increasing* ($b_0 > 0$) or *decreasing* ($b_0 < 0$). Similar conclusions hold in U^{-}, based on whether m is odd or even and the sign of d_0/c_0. This informal argument is made rigorous by determining the derivatives of $y(x)$ on integral curves through the origin, see Appendix A. The setting up of a conjugacy between systems in the same topological class is now straightforward.

We map the integral curves to the integral curves in U^+ and U^- separately using the graphs in Appendix A, and sliding motion is dealt with by mapping Σ to itself.

Key quantities from section 2 can be calculated explicitly. On Σ, we find

$$P^+P^-(x) = c_0 + O(|x|), \tag{3.4}$$

so

$$P^+P^- = c_0, \tag{3.5}$$

and

$$Q^+Q^-(x) = b_0 d_0 x^{n+m} + O(|x|^{n+m+1}). \tag{3.6}$$

Given the Taylor expansions (3.1) above, the sliding flow $P^0(x)$ in (2.2) is given by

$$P^0(x) = \frac{d_0 x^m - b_0 c_0 x^n + \dots}{d_0 x^m - b_0 x^n + \dots}. \tag{3.7}$$

In particular, if $m < n$ then

$$P^0(x) = 1 + O(|x|), \tag{3.8}$$

whilst if $m > n$ then

$$P^0(x) = c_0 + O(|x|), \tag{3.9}$$

and if $m = n$ then

$$P^0(x) = \frac{d_0 - b_0 c_0}{d_0 - b_0} + O(|x|). \tag{3.10}$$

In the last case, if $d_0 - b_0 c_0 \neq 0$, then the constant term dominates locally and the approach to the origin is in finite (forwards or backwards) time. But if $d_0 - b_0 c_0 = 0$, since the origin is an isolated singular point, $\exists\, p \in \mathbb{N}$ and $A \neq 0$ such that the sliding flow becomes

$$P^0(x) = Ax^p + O(|x|^{p+1}), \quad (m = n,\ d_0 - b_0 c_0 = 0), \tag{3.11}$$

and the local gradients of the flows $f^{\pm}$ are equal. Since $P^0(0) = 0$, we have a pseudo-equilibrium and the approach to the singular point is now in *infinite* time. If p is odd, the approach is in forwards time if $A < 0$ (a *stable pseudo-node*) and in backwards time if $A > 0$ (a *unstable pseudo-node*). If p is even, the approach is in forwards time from one side and backwards time from the other (a *pseudo-saddle-node*).

When $c_0 > 0$ on Σ, Lemma 2.2 with (3.5) implies that any sliding solution has to arrive at the singular point in finite time. So a full classification has to take account of those cases with $c_0 < 0$ and sliding, where $d_0 - b_0 c_0 = 0$, when the singular point becomes a pseudo-equilibrium.

The various combinations of visible and invisible folds, increasing and decreasing cusps on either side of Σ (i.e., whether m, n are even or odd, and the signs of b_0, d_0/c_0) lead to 8 candidate configurations of type 3 singular points, in the absence of any direction of time, originally shown in [8, Figures 64-71] and reproduced here in Figures 3.1 to 3.8. We consider each of these figures in turn, following the order used by Filippov. In all our phase plane diagrams, x is the horizontal axis, y is the vertical axis, with arrows denoting the direction of time increasing. Sliding along $\Sigma : y = 0$, if it occurs, is denoted by a thick line and a single arrow.

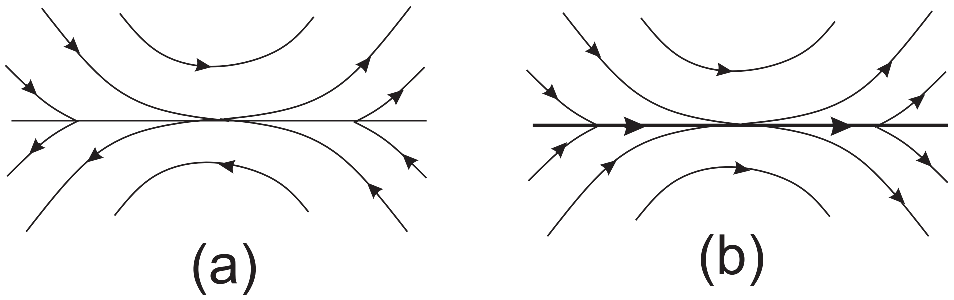

Figure 3.1: Visible two-fold [8, Figure 64]; n, m both odd, $b_0 > 0$, $d_0/c_0 < 0$. (a) crossing ($b_0 d_0 > 0$, $c_0 < 0$), (b) sliding in finite time ($b_0 d_0 < 0$, $c_0 > 0$).

3.1. Visible two-fold, Figure 3.1 [8, Figure 64] . We take

$$n, m \text{ both odd}, \quad b_0 > 0, \quad d_0/c_0 < 0. \tag{3.12}$$

Since $n + m$ is even, (3.6) implies that either there is crossing ($b_0 d_0 > 0$) or sliding ($b_0 d_0 < 0$) locally on Σ.

3.1.1. crossing. If $b_0 d_0 > 0$, then $d_0 > 0$, $c_0 < 0$ from (3.12) and the flow is completely determined (Figure 3.1a). There is no sliding.

3.1.2. sliding. If $b_0 d_0 > 0$ we have $d_0 < 0$, $c_0 > 0$ and so by Lemma 2.2 the sliding motion on Σ reaches the origin in finite time, forwards in X^- and backwards in X^+ (Figure 3.1b). There is no pseudo-equilibrium.

Hence there are **two classes of the visible two-fold** for analytic functions.

3.2. Visible/invisible two-fold, Figure 3.2 [8, Figure 65]. We take

$$n, m \text{ both odd}, \quad b_0 > 0, \quad d_0/c_0 > 0. \tag{3.13}$$

Since $n + m$ is even, (3.6) implies that either there is crossing ($b_0 d_0 > 0$) or sliding ($b_0 d_0 < 0$) locally on Σ.

3.2.1. crossing. If $b_0 d_0 > 0$, then $d_0 > 0$, $c_0 > 0$ from (3.13) and the flow is completely determined (Figure 3.2a). There is no sliding.

3.2.2. sliding. If $b_0 d_0 < 0$ then $d_0 < 0$, $c_0 < 0$. Hence Lemma 2.2 does not hold. So we must explore the possibility that the origin can, for certain parameter values, be a pseudo-equilibrium and hence approached in *infinite* time. There are three possibilities to consider:

(a) $m < n$: $P^0(x) = 1 + O(|x|)$ from (3.8). Since $P^0(x) > 0$ locally, solutions on Σ approach the singular point in finite time (forwards in X^-, backwards in X^+). See Figure 3.2b.

(b) $m > n$: $P^0(x) = c_0 + O(|x|)$ from (3.9). In particular U may be chosen so that $P^0(x) \leq -k < 0$ and so solutions on Σ approach the singular point in finite time (backwards in X^-, forwards in X^+). See Figure 3.2c.

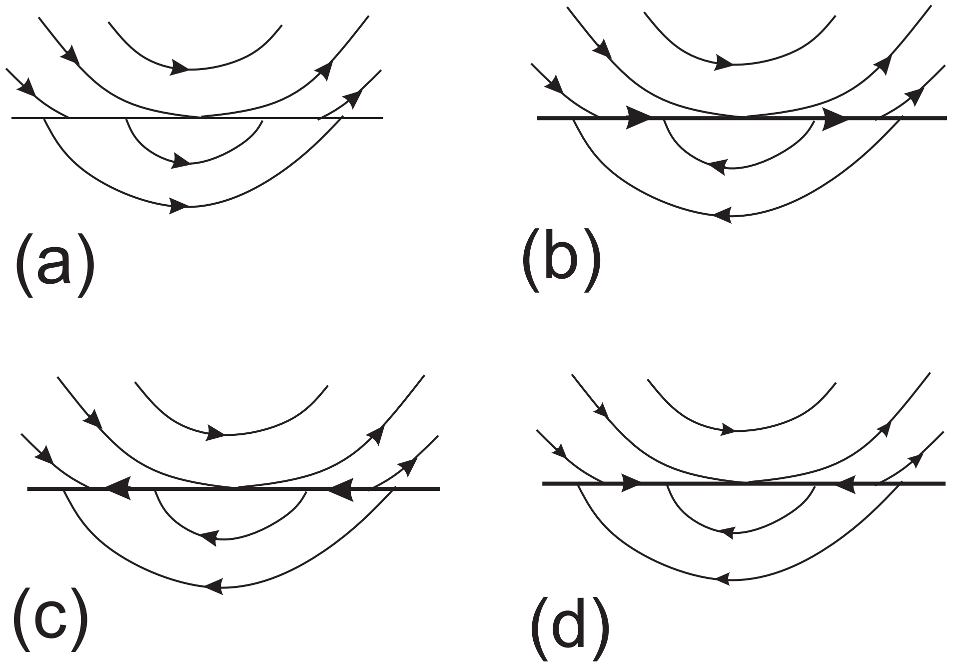

Figure 3.2: Visible/invisible two-fold [8, Figure 65]; n, m both odd, $b_0 > 0$, $d_0/c_0 > 0$. (a) crossing ($b_0 d_0 > 0$, $c_0 > 0$), (b) sliding ($b_0 d_0 < 0$, $c_0 < 0$) in finite time with $m < n$ or $m = n, (d_0 - b_0 c_0)/(d_0 - b_0) > 0$ *or* in infinite time with $m = n, d_0 - b_0 c_0 = 0, p$ even, $A > 0$, (c) sliding ($b_0 d_0 < 0$, $a_0 c_0 < 0$) in finite time with $m > n$ or $m = n, (d_0 - b_0 c_0)/(d_0 - b_0) < 0$ *or* in infinite time with $m = n, d_0 - b_0 c_0 = 0, p$ even, $A < 0$, (d) sliding ($b_0 d_0 < 0$, $c_0 < 0$) in infinite time with $m = n, d_0 - b_0 c_0 = 0, p$ odd, A either sign.

(c) $m = n$: $P^0(x) = (d_0 - b_0 c_0)/(d_0 - b_0) + O(|x|)$ from (3.10). Since $b_0 > 0, d_0 < 0$ the denominator is negative and cannot vanish.

(i) If $d_0 - b_0 c_0 \neq 0$, then $P^0(x) \neq 0$ can take both signs and has the same sign in both $X^{\pm}$. The approach to the origin is in finite time, as in the previous classes $m < n$ and $m > n$. See Figure 3.2b, Figure 3.2c.

(ii) If[3] $d_0 - b_0 c_0 = 0$, then $P^0(x) = Ax^p + O(|x|^{p+1})$ for some $p \geq 1$ from (3.11). The origin is now a pseudequilibrium and the approach is in infinite time (either forwards or backwards).

If p is even, the approach is in the same direction in $X^{\pm}$, depending on the sign of A; see Figure 3.2b, Figure 3.2c.

If p is odd, the infinite time approach is in opposite directions in $X^{\pm}$. There appear to be two classes, but they are related by the symmetry $(x, y, t, A) \to (-x, y, -t, -A)$ and shown in Figure 3.2d.

So, for analytic functions, both Figure 3.2b and Figure 3.2c are divided into two different classes: finite and infinite approach to the origin. Figure 3.2d is only possible with infinite

[3]Since $d_0 < 0, b_0 c_0 < 0$, the numerator in $P^0(x)$ can vanish.

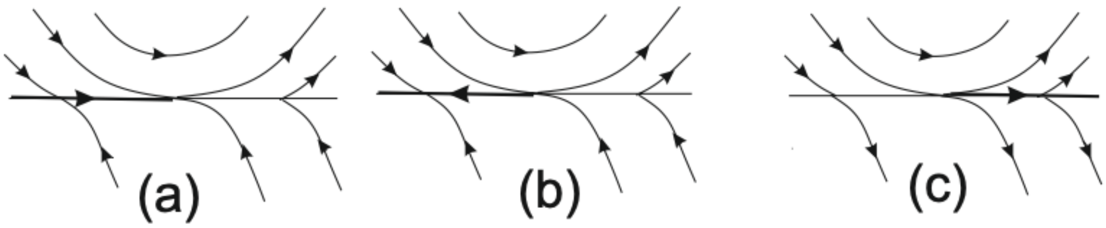

Figure 3.3: Visible fold-cusp [8, Figure 66]; n odd, m even, $b_0 > 0$, $d_0/c_0 < 0$. (a) sliding in finite time in $x \leq 0$ ($b_0 d_0 > 0, c_0 < 0, m < n$), (b) as (a) with $m > n$, (c) sliding in finite time in $x \geq 0$ ($b_0 d_0 < 0$, $c_0 > 0$).

time approach to the origin.

Hence there are **six classes of the visible/invisible two-fold** for analytic functions.

3.3. Visible fold-cusp, Figure 3.3 [8, Figure 66]. We take

$$n \text{ odd}, \quad m \text{ even}, \quad b_0 > 0, \quad d_0/c_0 < 0. \tag{3.14}$$

Since $m+n$ is odd, there is always sliding[4]: in $x \leq 0$ if $b_0 d_0 > 0$ and in $x \geq 0$ if $b_0 d_0 < 0$. The condition $b_0 > 0$ appears arbitrary. However, the symmetry $(x, y, t) \to (-x, y, -t)$ changes the sign of the $\dot{y}$ equation in the upper flow and the other constants are unchanged. Thus this choice can be made without loss of generality.

3.3.1. sliding. If $b_0 d_0 > 0$, then $d_0 > 0, c_0 < 0$. The sliding flow $P^0(x)$ is given by (3.8) if $m < n$ or (3.9) if $m > n$, both with finite time approach to the origin, in forward and backward time respectively. These classes are shown in Figure 3.3a and Figure 3.3b. There can be no possibility of infinite time approach since $m \neq n$ by assumption.

If $b_0 d_0 < 0$, then $d_0 < 0, c_0 > 0$. Then $P^+ P^- > 0$, hence Lemma 2.2 holds and the sliding region exists in $x \geq 0$ with finite approach to the origin in backwards time (Figure 3.3c).

Hence there are **three classes of the visible fold-cusp** for analytic functions.

3.4. Invisible two-fold, Figures 3.4 and 3.5 [8, Figures 67-68]. We take

$$n, m \text{ both odd}, \quad b_0 < 0, \quad d_0/c_0 > 0. \tag{3.15}$$

Since $n+m$ is even, (3.6) implies that either there is crossing ($b_0 d_0 > 0$) or sliding ($b_0 d_0 < 0$) locallly on Σ.

3.4.1. crossing. If $b_0 d_0 > 0$, then $d_0 < 0$ and $c_0 < 0$ from (3.15). Successive intersections spiral either out or in (equivalent under x and t reversal) to give the sewed focus (Figure 3.4a), or the solutions lie on closed curves to give the sewed centre (Figure 3.5a). For the sewed focus, it can be shown that the approach to singular point is in infinite time [8, pp. 234-238].

[4]Sliding occurs when $Q^+Q^-(x) \approx b_0 d_0 x^{n+m} < 0$ from (3.6)

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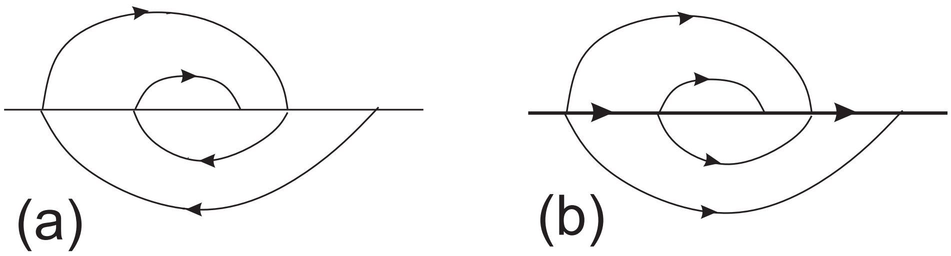

Figure 3.4: Invisible two-fold [8, Figure 67]; n, m both odd, $b_0 < 0$, $d_0/c_0 > 0$. (a) sewed focus crossing ($b_0 d_0 > 0$, $c_0 < 0$), (b) focus-like sliding in finite time ($b_0 d_0 < 0$, $c_0 > 0$).

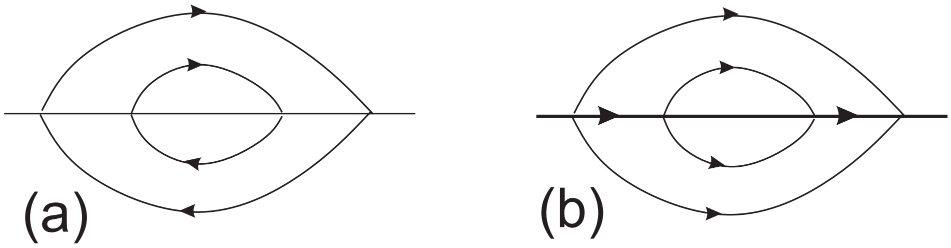

Figure 3.5: Invisible two-fold [8, Figure 68]; n, m both odd, $b_0 < 0$, $d_0/c_0 > 0$. (a) sewed centre crossing ($b_0 d_0 > 0$, $c_0 < 0$), (b) centre-like sliding in finite time ($b_0 d_0 < 0$, $c_0 > 0$).

3.4.2. sliding. If $b_0 d_0 < 0$ then $d_0 > 0$ and $c_0 > 0$ from (3.15). So by Lemma 2.2 the motion on Σ reaches the origin in finite time. Solutions leave Σ in X^- to return to it in X^+. There is a clear topological difference between the focus-like (Figure 3.4b) and centre-like (Figure 3.5b) behaviours.

Hence there are **four classes of the invisible two-fold** for analytic functions.

3.5. Invisible fold-cusp, Figure 3.6 [8, Figure 69]. We take

$$n \text{ odd}, \quad m \text{ even}, \quad b_0 > 0, \quad d_0/c_0 > 0. \tag{3.16}$$

Since $m + n$ is odd, there is always sliding: in $x \leq 0$ if $b_0 d_0 > 0$ and in $x \geq 0$ if $b_0 d_0 < 0$. The choice of $b_0 > 0$ is without loss of generality, as in subsection 3.3.

3.5.1. sliding. If $b_0 d_0 > 0$, then $d_0 > 0, c_0 > 0$. Hence Lemma 2.2 holds and the sliding region exists in $x \leq 0$ with finite time approach to the origin (Figure 3.6a).

If $b_0 d_0 < 0$, then $d_0 < 0, c_0 < 0$. There is no possibility of infinite time approach since $m \neq n$ by assumption. If $m > n$ then (3.9) implies that $P^0(x) \leq -k < 0$ and approach to the origin is in finite forwards time (Figure 3.6b). If $m < n$ then by (3.8), $P^0(x) > 0$ locally and the approach is in (backwards) finite time (Figure 3.6c).

Hence there are **three classes of the invisible fold-cusp** for analytic functions.

3.6. Two-cusp (co.), Figure 3.7 [8, Figure 70]. We take

$$n, m \text{ both even}, \quad b_0 > 0, \quad d_0/c_0 > 0. \tag{3.17}$$

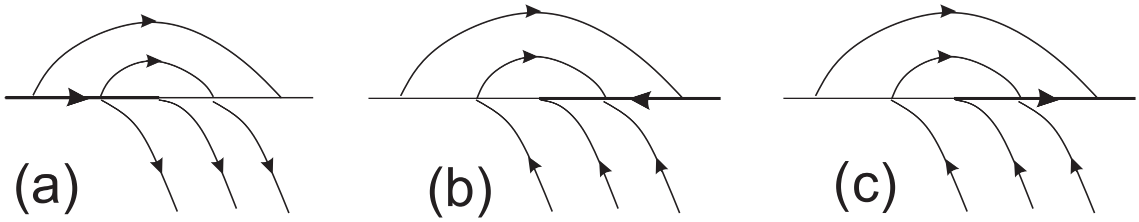

Figure 3.6: Invisible fold-cusp [8, Figure 69]; n odd, m even, $b_0 > 0$, $d_0/c_0 > 0$. (a) sliding in finite time in $x \leq 0$ ($b_0 d_0 > 0, c_0 > 0$), (b) sliding in finite time in $x \geq 0$ ($b_0 d_0 < 0, c_0 < 0, m > n$), (c) as (b) with $m < n$.

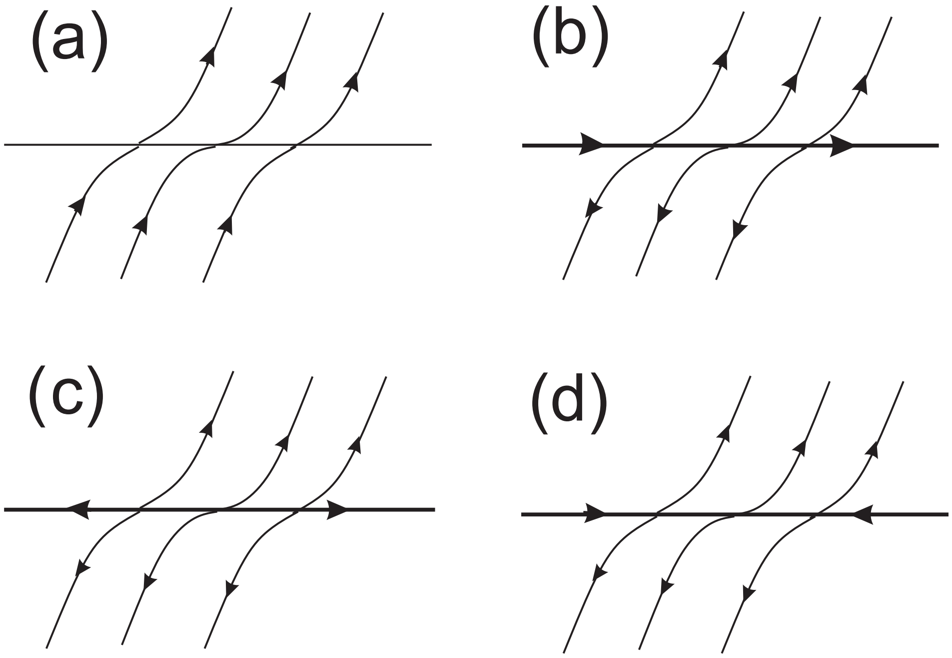

Figure 3.7: Two-cusp (co.) [8, Figure 70]; n, m both even, $b_0 > 0$, $d_0/c_0 > 0$. (a) crossing ($b_0 d_0 > 0$, $c_0 > 0$), (b) sliding ($b_0 d_0 < 0$, $c_0 < 0$) in finite time with $m \neq n$ or $m = n, d_0 - b_0 c_0 \neq 0$ *or* in infinite time with $m = n, d_0 - b_0 c_0 = 0, p$ even, (c) sliding ($b_0 d_0 < 0, c_0 < 0$) in infinite time with $m = n, d_0 - b_0 c_0 = 0, p$ odd, $A > 0$, (d) as (c) with $A < 0$.

Since $n + m$ is even, (3.6) implies that either there is crossing ($b_0 d_0 > 0$) or sliding ($b_0 d_0 < 0$) locally on Σ. The choice of $b_0 > 0$ is without loss of generality.

3.6.1. crossing. If $b_0 d_0 > 0$, then $d_0 > 0$ and $c_0 > 0$ from (3.17) and the flow is completely determined (Figure 3.7a). There is no sliding.

3.6.2. sliding. If $b_0 d_0 < 0$ then $d_0 < 0$, $c_0 < 0$. Hence Lemma 2.2 does not hold. So we have the possibility that the origin can be a pseudo-equilibrium and hence approached in infinite time, as in subsection 3.2. There are three possibilities to consider:

(a) $m < n$: $P^0(x) = 1 + O(|x|)$ from (3.8). Since $P^0(x) > 0$ locally, solutions on Σ approach the singular point in finite time (forwards in X^-, backwards in X^+). See Figure 3.7b.

(b) $m > n$: $P^0(x) = c_0 + O(|x|)$ from (3.9). In particular U may be chosen so that $P^0(x) \leq -k < 0$ and so solutions on Σ approach the singular point in finite time (backwards in X^-, forwards in X^+). Hence the case of sliding with $m > n$ is the same as the case of sliding with $m < n$, Figure 3.7b, on rotation of the phase plane by $\pm\pi$.

(c) $m = n$: $P^0(x) = (d_0 - b_0c_0)/(d_0 - b_0) + O(|x|)$ from (3.10). Since $b_0 > 0, d_0 < 0$ the denominator is negative and cannot vanish.

   (i) If $d_0 - b_0c_0 \neq 0$, then $P^0(x) \neq 0$ can take both signs and has the same sign in both $X^\pm$. The approach to the singular point is in finite time. The two signs of $P^0(x)$ give the same phase plane picture (Figure 3.7b) on rotation of the phase plane by $\pm\pi$, just as for $m < n$ and $m > n$.

   (ii) If[5] $d_0 - b_0c_0 = 0$ then (3.11) holds and $P^0(x) = Ax^p + O(|x|^{p+1})$ for some $p \geq 1$. The origin is now a pseudequilibrium and the approach to the singular point is in infinite time (either forwards or backwards).
   If p is even, the infinite time approach is in the same direction in $X^\pm$, depending on the sign of A. The two classes are identical, up to a rotation of the phase plane by $\pm\pi$ (Figure 3.7b).
   If p is odd, the infinite time approach is in opposite directions in $X^\pm$ and depends on the sign of A. The two classes are shown in Figure 3.7c and Figure 3.7d.

We see that, for analytic functions, Figure 3.7b can be split into two different classes of finite and infinite approach, whereas Figure 3.7c and Figure 3.7d are only possible with infinite time approach to the origin.

Hence there are **five classes of the two-cusp (co.)** for analytic functions.

3.7. Two-cusp (anti.), Figure 3.8 [8, Figure 71]. We take

$$n, m \text{ both even}, \quad b_0 > 0, \quad d_0/c_0 < 0. \tag{3.18}$$

The analysis is identical to that of the visible two-fold in subsection 3.1, since $n + m$ is even in this case also. The phase portraits are shown in Figure 3.8.

Hence there are **two classes of the two-cusp (anti.)** for analytic functions.

3.8. Analytic $P^\pm$, $Q^\pm$: summary. From Figures 3.1 to 3.8, we have 22 different classes of type 3 singular points corresponding to the 22 panels in the figures. This number is in agreement with [8, p.222]. We have also seen how three of these figures (Figure 3.2b, Figure 3.2c and Figure 3.7b) can have both finite and infinite sliding arrival times at the singular point, depending on parameter values. Hence when $P^\pm, Q^\pm$ are analytic, there are a total of 25 classes.

Even though our approach can systematically enumerate all possible classes, it does not directly prove that all possible topologies of the vector fields have been considered. This is

[5] Since $d_0 < 0, b_0c_0 < 0$, the numerator in $P^0(x)$ can vanish.

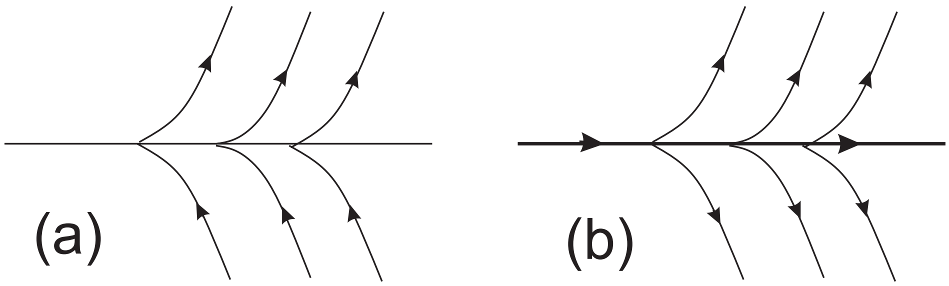

Figure 3.8: Two-cusp (anti.) [8, Figure 71]; n, m both even, $b_0 > 0$, $d_0/c_0 < 0$. (a) crossing ($b_0 d_0 > 0$, $c_0 < 0$), (b) sliding in finite time ($b_0 d_0 < 0$, $c_0 > 0$).

the exercise that Filippov carried out. So it might be useful to briefly describe his approach[6].

To begin with, 8 figures are given [8, Figures 64-71, p.221], in the absence of any direction of time in both flows, and with no indication of sliding. Filippov [8, §17] had already established a topological classification of singular points, that divides the neighbourhood of a singular point into a finite number of sectors. The sector boundaries are trajectories (separatrices) that either pass through, or tend to, the singular point.

Considering all possible combinations of trajectories through the singular point in $U^{\pm}$, together with the occurrence (or otherwise) of sliding in $X^{\pm}$, allowed Filippov to claim that there are 22 topological classes. He was aware that there were some classes where the singular point could be reached in both finite and infinite time. But he gives no details of which of the 22 cases were involved. Nevertheless he [8, p.222] claims a total of 24 classes, but giving no details of the extra two cases.

Our method finds the same number (22) of topological classes as Filippov, whilst at the same time explicitly obtaining those parameter values that change the sliding arrival times in 3 of these cases. We are unable to say which class was missed, but it must be one of Figure 3.2b, Figure 3.2c and Figure 3.7b with both finite and infinite sliding arrival times at the singular point.

4. $P^{\pm}, Q^{\pm}$ non-analytic.

The 25 classes for $P^{\pm}, Q^{\pm}$ analytic in section 3 are all realizable for C^1_* systems. Some of them cannot be altered by considering systems that are C^1_* but not C^{ω}_*: those which involve crossing ($Q^+Q^-(x) > 0$ if $x \neq 0$) and those whose class is determined by invoking Lemma 2.2, which applies generally.

The only classes worth considering for possible extra topological classes are those with

$$\text{both} \quad P^+P^- < 0 \quad \text{and a sliding segment.} \tag{4.1}$$

[6] The original version of Filippov's book was published in Russian in 1985, with the English translation following in 1988 [8]. It is a very difficult and demanding read. The writing style varies dramatically between sections. Definitions are often missing or duplicated, details of proofs are sometimes sketchy or omitted, dense terse text is poorly signposted, both equation and subsection numbering begins afresh in each new section and the index is minimal. It is perhaps for these reasons that many of Filippov's results are still not widely appreciated. Yet it remains the most complete source of fundamental results in the field of piecewise smooth systems.

These are easily identifiable from section 3, since neither condition depends on analytic properties of $P^\pm, Q^\pm$. We shall address the question of whether non-analytic functions can change the sliding arrival times from those found for analytic functions.

In *all* the classes where a new topological class is possible, it is straightforward to construct a C^1_* example. Thus we do not have to be drawn into complicated theorems about the non-existence of certain flows. Instead we simply confirm that a C^1_* example can be found where such a possibility exists. Let $H(x)$ denote the Heaviside[7] step function

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

The observation that makes construction simple is that if $n_i, m_i \geq 2$, $i = 1, 2$ then the functions

$$\begin{aligned} P^+(x,y) &= 1 + a_1 x \\ Q^+(x,y) &= b_1 x^{n_1} H(x) + b_2 x^{n_2} H(-x) \\ P^-(x,y) &= c_0 \\ Q^-(x,y) &= d_1 x^{m_1} H(x) + d_2 x^{m_2} H(-x) \end{aligned} \tag{4.2}$$

define a C^1_* piecewise smooth system, and if c_0, b_i and d_i are all non-zero[8] then the origin is an isolated type 3 singular point[9]. If $M = \min\{n_i, m_i\} \geq 2$ then (4.2) is C^{M-1}_*.

4.1. Non-analytic visible two-fold, Figure 3.1 [8, Figure 64]. The analytic classes are shown in Figure 3.1. Here if $P^+P^- = c_0 < 0$, we have crossing. Similarly, when we have sliding, we have $P^+P^- = c_0 > 0$. Hence condition (4.1) is not satisfied.

Hence there are **no extra classes of the visible two-fold** for non-analytic functions.

4.2. Non-analytic visible-invisible two-fold, Figure 3.2 [8, Figure 65]. The analytic classes are shown in Figure 3.2. We can get no new non-analytic classes from Figure 3.2a. Both Figure 3.2b and Figure 3.2c have either finite or infinite time approach to the origin in both $X^\pm$. Figure 3.2d only has infinite time approach to the origin in both $X^\pm$. Five new non-analytic classes come from:

- in Figure 3.2b and Figure 3.2c, allowing finite time approach in $X^\pm$ and infinite time approach in $X^\mp$ (two new classes, up to suitable symmetries).
- in Figure 3.2d, allowing finite time approach in both $X^\pm$, finite time approach in X^+ and infinite time approach in X^-, infinite time approach in X^+ and finite time approach in X^- (three new classes, up to suitable symmetries).

To show that all five new topological classes exist in a non-analytic system, take $n_i = m_i = 3$ in (4.2) and define the vector fields

$$\begin{aligned} P^+(x,y) &= 1 + a_1 x, & Q^+(x,y) &= b_1 x^3 H(x) + b_2 x^3 H(-x) \\ P^-(x,y) &= c_0, & Q^-(x,y) &= d_0 x^3 \end{aligned} \tag{4.3}$$

with $a_1 \neq 0$ and

$$b_1, b_2 > 0, \quad c_0 < 0 \quad \text{and} \quad d_0 < 0. \tag{4.4}$$

[7] In the sequel, $H(x)$ is always multiplied by positive power of x, so its definition at $x = 0$ is not relevant.
[8] Subject to suitable non-degeneracy conditions, similar to (3.2).
[9] In fact, $n_i, m_i > 1$ but we have no need of non-integer powers.

The sliding vector field $P^0(x)$ is given by

$$P^0(x) = \begin{cases} \frac{d_0-b_1c_0}{d_0-b_1} + \frac{a_1d_0}{d_0-b_1}x & \text{if } x > 0 \\ \frac{d_0-b_2c_0}{d_0-b_2} + \frac{a_1d_0}{d_0-b_1}x & \text{if } x < 0 \end{cases} \quad (4.5)$$

from (2.2). We have three classes to consider:

(a) Assume that both $d_0 - b_1c_0 \neq 0$ and $d_0 - b_2c_0 \neq 0$. Since $d_0 - b_i < 0$ and both d_0 and b_ic_0 have the same sign by (4.4), the (leading order) constant terms of $P^0(x)$ may take any pair of signs independently. Thus we may obtain finite time approach in any combination of directions. Two of the four possibilities already occur in the analytic case (where the constant terms have the same sign, Figure 3.2b and Figure 3.2c). This leaves two classes where the constant terms have opposite signs, corresponding to finite time approach to the origin in forwards (or backwards) time in both $X^\pm$. These two classes are related by symmetry. So we obtain one new class, within Figure 3.2d, for non-analytic functions, under the assumption that both $d_0-b_1c_0 \neq 0$ and $d_0-b_2c_0 \neq 0$.

(b) Now assume that $d_0 - b_1c_0 = 0$ and $d_0 - b_2c_0 \neq 0$. We have

$$P^0(x) = \begin{cases} \frac{a_1c_0}{c_0-1}x & \text{if } x > 0 \\ \frac{d_0-b_2c_0}{d_0-b_2} + O(|x|) & \text{if } x < 0 \end{cases} \quad (4.6)$$

corresponding to infinite time approach in X^+, governed only by the sign of the term a_1 (since $c_0 < 0$) and finite time approach in X^-, governed by the sign of the constant term $(d_0-b_2c_0)/(d_0-b_2)$. Again there are four possible sign combinations. When the signs are both positive, this corresponds to a new class within Figure 3.2b. When the signs are both negative, this corresponds to a new class within Figure 3.2c. When the signs are opposite, this corresponds to two new topological classes within Figure 3.2d, since these classes are not related by symmetry. So we obtain four additional new topological classes within Figure 3.2.

(c) If $d_0 - b_1c_0 \neq 0$ and $d_0 - b_2c_0 = 0$, we obtain the same four classes as in case (b).

(d) When both $a_0d_0 - b_1c_0 = 0$ and $a_0d_0 - b_2c_0 = 0$, we recover Figure 3.2d.

Hence there are **five extra classes of the visible-invisible two-fold** for non-analytic functions. The finite time approach to the origin is due to the fact that $P^0(x)$ cannot be extended to a continuous function at 0. The new topological classes are a consequence of the fact that $P^0(x)$ is in C^1_*.

4.3. Non-analytic visible fold-cusp, Figure 3.3 [8, Figure 66]. There are three classes for analytic functions, shown in Figure 3.3. In the case shown in Figure 3.3c, $P^+P^- > 0$. So Lemma 2.2 holds and there are no new classes when the functions are non-analytic. It suffices, then, to consider the cases shown in Figure 3.3a and Figure 3.3b; in the analytic case the sliding surface is a half-line with finite time approach to the origin, so the question is whether non-analytic classes can be found with infinite time approach. Let

$$\begin{aligned} P^+(x,y) &= 1 + a_1x, & Q^+(x,y) &= b_0x^3 \\ P^-(x,y) &= c_0, & Q^-(x,y) &= d_0x^3H(x) - d_0x^3H(-x) \end{aligned} \quad (4.7)$$

with

$$b_0 > 0, \quad c_0 < 0, \quad d_0 > 0, \tag{4.8}$$

to ensure a visible fold-cusp. Note that the solutions in $y < 0$ lie, topologically, on generalised cubics notwithstanding the odd power of x in the expression for Q^-, because of the discontinuity at $x = 0$.

Then $Q^+Q^-(x) \leq 0$ on Σ if $x \leq 0$. The sliding flow there is

$$P^0(x) = \frac{d_0 + b_0c_0}{d_0 + b_0} + \frac{a_1d_0}{d_0 + b_0}x, \quad x \leq 0. \tag{4.9}$$

Our construction (4.7) and (4.8) means that $d_0 + b_0 > 0$ and we can choose $d_0 = -b_0c_0 > 0$ so that the first term in (4.9) vanishes. Then the second term depends only on the sign of a_1 and represents infinite time approach to the origin as $t \to \infty$ if $a_1 < 0$ and as $t \to -\infty$ if $a_1 > 0$.

Hence there are **two extra classes of the visible fold-cusp** for non-analytic functions.

4.4. Non-analytic invisible two-fold, Figures 3.4 and 3.5 [8, Figures 67-68]. The analytic classes are shown in Figure 3.4 and Figure 3.5. Neither of the classes in Figure 3.5, nor the focus-like sliding in Figure 3.4b, need further consideration. The only question is whether, in Figure 3.4a, it is possible to construct a finite time approach to the singular point for non-analytic functions. We have shown previously that this is possible [10, Theorem 3]. Details are reproduced in Appendix B, for ease of reference.

Hence there is **one extra case of the invisible two-fold** for non-analytic functions.

4.5. Non-analytic invisible fold-cusp, Figure 3.6 [8, Figure 69]. This case is almost identical to the non-analytic visible fold-cusp in subsection 4.3 except that the invisible fold now requires $d_0/c_0 > 0$. We will not present the details here.

Hence there are **two extra classes of the invisible fold-cusp** for non-analytic functions.

4.6. Non-analytic two-cusp (co.), Figure 3.7 [8, Figure 70]. This case is almost identical to the visible-invisible two-fold in subsection 4.2 except that we take

$$\begin{aligned} P^+(x,y) &= 1 + a_1x, & Q^+(x,y) &= b_1x^4H(x) + b_2x^4H(-x) \\ P^-(x,y) &= c_0, & Q^-(x,y) &= d_0x^4 \end{aligned} \tag{4.10}$$

with $a_1 \neq 0$ and

$$b_1, b_2 > 0, \quad c_0 < 0 \quad \text{and} \quad d_0 < 0 \tag{4.11}$$

for compatability with (3.17). We can arrange to have two new finite time sliding classes and three new finite/infinite time classes (up to suitable symmetries), as follows. Specifically these correspond to

- Figure 3.7b: finite time in $X^\pm$ with infinite time in $X^\mp$,
- Figure 3.7c: finite time in both $X^\pm$ and finite time in $X^\pm$ with infinite time in $X^\mp$,
- Figure 3.7d: finite time in both $X^\pm$ and finite time in $X^\pm$ with infinite time in $X^\mp$.

Since the sliding flow is precisely (4.5), we shall not repeat the details.

Hence there are **five extra classes of the two-cusp (co.)** for non-analytic functions.

4.7. Non-analytic two-cusp (anti.), Figure 3.8 [8, Figure 71]. The analytic classes are shown in Figure 3.8. Here if $P^+P^- = c_0 < 0$, we have crossing. Similarly, when we have sliding, we have $P^+P^- = c_0 > 0$. Hence condition (4.1) is not satisfied.

Hence there are **no extra classes of the two-cusp (anti.)** for non-analytic functions.

4.8. Non-analytic $P^\pm$, $Q^\pm$: summary. In subsection 4.1-subsection 4.7, we have shown that there are 15 extra classes when $P^\pm, Q^\pm$ are non-analytic, all with changes to the arrival times of the sliding flow at the singular point. This number agrees with that stated by Filippov [8]. He gave neither proof of his claim nor details of the new classes.

5. Conclusion. Piecewise smooth systems are of great importance both physically and mathematically. They occur in a wide class of practical problems and present significant theoretical challenges. In particular, the solution of a piecewise smooth system may not be unique. This non-uniqueness property allows singular points of piecewise smooth systems to be reached in finite time. In this paper, we have presented the classification of type 3 singular points, in bimodal planar piecewise smooth systems. For isolated analytic type 3 singular points, we have shown that there exist 25 topological classes. 12 of these classes involve finite time sliding to the singular point and a further six involve infinite time sliding, via the creation of a pseudo-equilibrium. For isolated non-analytic type 3 singular points, there are 40 topological classes, with the extra classes occurring when finite time sliding becomes infinite time sliding (or vice versa) or when mixed finite/infinite time sliding occurs. Whilst we based our work on that of Filippov [8], the details of the classification process are shown here for the first time. In addition, our new results include precise details of each class, for analytic functions in section 3 and for non-analytic functions in section 4. Note that all isolated type 3 singular points are structurally unstable, a result proved in Filippov [8, Lemma 1, p.222].

Appendix A. Nature of tangencies. Working in U^+ we have

$$y^{(1)}(x,y) = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{Q^+(x,y)}{P^+(x,y)}.$$

Thus $\frac{dy}{dx}(0,0) = y^{(1)}(0,0) = 0$ by (2.3). Now

$$\text{(A.1)} \qquad y^{(2)}(x,y) = \frac{d^2y}{dx^2} = \frac{Q_x^+(x,y) + Q_y^+(x,y)y^{(1)}}{P^+(x,y)} - \frac{(P_x^+(x,y) + P_y^+(x,y)y^{(1)})Q^+(x,y)}{[P^+(x,y)]^2}$$

and so

$$\text{(A.2)} \qquad y^{(2)}(0,0) = \frac{Q_x^+(0,0)}{P^+(0,0)}.$$

If $n = 1$, then $Q_x^+(0,0) \neq 0$, so $y^{(2)}(0,0) = b_0$. Hence for $n = 1$, the integral curve is the parabola

$$y = y_0 + \frac{1}{2}b_0x^2 + O(|x|^3, |y|)$$

for $|x|$ and $|y|$ small. If $n > 1$ then $y^{(2)}(0,0) = 0$ and higher order terms are needed. We work inductively. Suppose

$$y^{(r)}(0,0) = 0, \quad r = 1, \dots, k, \quad (k \geq 2).$$

A straightforward induction argument based on (A.1) implies that

$$\begin{aligned} y^{(k+1)}(x,y) = \;& \frac{\frac{\partial^k Q^+}{\partial x^k}(x,y)}{P^+(x,y)} + \left(\sum_{r=1}^{k} f_r(x,y,y^{(1)},\dots y^{(r)})y^{(r)}(x,y)\right) \\ & + G(x,y,y^{(1)},\dots y^{(k)})Q^+(x,y). \end{aligned}$$

If $n > k$, then $\frac{\partial^k Q^+}{\partial x^k}(0,0) = 0$ and so $y^{(k+1)}(0,0) = 0$. If $n = k$ then $\frac{\partial^n Q^+}{\partial x^n}(0,0) = b_0 n!$. Hence

$$y^{(n+1)}(0,0) = \frac{\frac{\partial^n Q^+}{\partial x^n}(0,0)}{P^+(0,0)} = b_0 n!,$$

and so

$$y = y_0 + \frac{1}{(n+1)!} y^{(n+1)}(0,0) x^{n+1} + O(|x|^{n+2}) = y_0 + \frac{b_0}{(n+1)} x^{n+1} + O(|x|^{n+2})$$

as expected.

Appendix B. Non-analytic sewed focus.

In this section, we construct a finite time approach to the singular point for the sewed focus case of the non-analytic invisible two-fold, a result given in [10, Theorem 3]. The details are repeated here for ease of reference. Consider the vector fields

$$\begin{aligned} P^+(x,y) &= -P^-(-x,-y) = 1, \\ Q^+(x,y) &= -Q^-(-x,-y) = -4x^3 H(x) - 8x^7 H(-x), \quad y > 0. \end{aligned} \tag{B.1}$$

This example is C^2_* and has a (stable) sewed focus at the origin[10]. There is no sliding. Given an initial point $(-x_0, 0)$ with $x_0 > 0$ sufficiently small, the solution will intersect Σ at points $(x_1, 0), (-x_2, 0), \dots$ with $x_i > 0$. The total time taken is given by

$$T = x_0 + 2\sum_{r=1}^{\infty} x_r. \tag{B.2}$$

If $T < \infty$ we have the existence of a new topological class that is not possible in the analytic case.

If $y > 0$ then integral curves are given by

$$y = c^- - x^8, \quad x < 0, \quad y = c^+ - x^4, \quad x > 0 \tag{B.3}$$

where the constants $c^\pm$ are determined by initial conditions and integral curves in $y < 0$ are determined by symmetry.

Thus a solution starting at $(-x_0, 0)$, $x_0 > 0$ lies on the integral curve

$$y = x_0^8 - x^8$$

[10]Note that by replacing $4x^3$ and $8x^7$ in (B.1) by $2kx^{2k-1}$ and $4kx^{4k-1}$, $k \geq 3$, respectively the relation between subsequent intersections is unchanged and so this example can be made C^r_* for any finite r.

and intersects the y-axis after time x_0 at $y = x_0^8$. It now continues on an integral curve of the form $y = c^+ - x^4$ and since $y = x_0^8$ when $x = 0$, $c^+ = x_0^8$ and the integral curve strikes the x-axis at $(x_1, 0)$ where $x_0^8 - x_1^4 = 0$, i.e. $x_1 = x_0^2$ after a further x_1 units of time.

Now the process starts again in $y < 0$ which by symmetry is effectively equivalent, so the solution through $(x_1, 0)$ intersects the x-axis at $(-x_2, 0)$ where $x_2 = x_1^2 = x_0^4$ after time $x_1 + x_2$. Induction now establishes that the infinite sequence of intersections are at points $((-1)^k x_k, 0)$ with

$$x_k = x_0^{2^k}$$

and

$$T = x_0 + 2\sum_{k=1}^{\infty} x_0^{2^k}$$

The sum is certainly less than twice the sum of all powers, which converges if $x_0 < 1$ as it is a geometric progression, so this sum also converges to a finite value.

Acknowledgment. The authors are grateful to the anonymous reviewer who helped us clarify our original subsection 4.3.

REFERENCES

[1] A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of Oscillators*, Dover Publications Inc, New York, 1966, https://doi.org/10.1016/C2013-0-06631-5.

[2] M. Antali and G. Stépán, *Nonsmooth analysis of three-dimensional slipping and rolling in the presence of dry friction*, Nonlinear Dynamics, 97 (2019), pp. 1799–1817, https://doi.org/10.1007/s11071-019-04913-x.

[3] I. Belykh, R. Kuske, M. Porfiri, and D. J. W. Simpson, *Beyond the Bristol book: Advances and perspectives in non-smooth dynamics and applications*, Chaos, 33 (2023), p. 010402, https://doi.org/10.1063/5.0138169.

[4] M. Bernardo, C. Budd, A. R. Champneys, and P. Kowalczyk, *Piecewise-smooth dynamical systems: theory and applications*, vol. 163, Springer, 2008, https://doi.org/10.1007/978-1-84628-708-4.

[5] B. Brogliato, *Nonsmooth Mechanics: Models, Dynamics and Control*, Springer, 3rd ed., 2016, https://doi.org/10.1007/978-3-319-28664-8.

[6] S. Coombes, R. Thul, and K. C. A. Wedgwood, *Nonsmooth dynamics in spiking neuron models*, Physica D: Nonlinear Phenomena, 241 (2012), pp. 2042–2057, https://doi.org/https://doi.org/10.1016/j.physd.2011.05.012.

[7] M. Şayli, A. C. Skeldon, R. Thul, R. Nicks, and S. Coombes, *The two-process model for sleep-wake regulation: A nonsmooth dynamics perspective*, Physica D: Nonlinear Phenomena, 444 (2023), p. 133595, https://doi.org/https://doi.org/10.1016/j.physd.2022.133595.

[8] A. F. Filippov, *Differential equations with discontinuous righthand sides*, vol. 18, Springer, 1988, https://doi.org/10.1007/978-94-015-7793-9.

[9] I. Flügge-Lotz, *Discontinuous Automatic Control*, Princeton University Press, Princeton, New Jersey, 1953.

[10] P. Glendinning, S. J. Hogan, M. E. Homer, M. R. Jeffrey, and R. Szalai, *Uncountably many cases of filippov's sewed focus*, J. Nonlinear Sci., 33 (2023), p. 52, https://doi.org/10.1007/s00332-023-09910-4.

[11] M. R. Jeffrey, *Hidden dynamics: the mathematics of switches, decisions and other discontinuous behaviour*, Springer, 2018, https://doi.org/10.1007/978-3-030-02107-8.

[12] K. U. Kristiansen and S. J. Hogan, *On the use of blowup to study regularizations of singularities of piecewise smooth dynamical systems in* $\mathbb{R}^3$, SIAM Journal on Applied Dynamical Systems, 14 (2015), pp. 382–422, https://doi.org/10.1137/140980995.

[13] K. U. Kristiansen and S. J. Hogan, *Regularizations of two-fold bifurcations in planar piecewise smooth systems using blowup*, SIAM J. Applied Dynamical Systems, 14 (2015), pp. 1731–1786, https://doi.org/10.1137/15M1009731.

[14] K. U. Kristiansen and S. J. Hogan, *On the interpretation of the piecewise smooth visible–invisible two-fold singularity in $\mathbb{R}^3$ using regularization and blowup*, J. Nonlinear Sci., 29 (2019), pp. 723–787, https://doi.org/10.1007/s00332-018-9502-x.

[15] Y. A. Kuznetsov, S. Rinaldi, and A. Gragnani, *One-parameter bifurcations in planar Filippov systems*, Int. J. Bifur. Chaos, 13 (2003), pp. 2157–2188, https://doi.org/10.1142/S0218127403007874.

[16] S. Li, W. Ma, W. Zhang, and Y. Hao, *Melnikov method for a class of planar hybrid piecewise-smooth systems*, Int. J. Bifur. Chaos, 26 (2016), p. 1650030, https://doi.org/10.1142/S0218127416500309.

[17] V. Noel, S. Vakulenko, and O. Radulescu, *Algorithm for identification of piecewise smooth hybrid systems: Application to eukaryotic cell cycle regulation*, in Algorithms in Bioinformatics, T. M. Przytycka and M.-F. Sagot, eds., Berlin, Heidelberg, 2011, Springer Berlin Heidelberg, pp. 225–236, https://doi.org/https://doi.org/10.1007/978-3-642-23038-7_20.

[18] V. Utkin, *Sliding Modes in Control Optimization*, Springer-Verlag, New York, 1992, https://doi.org/10.1007/978-3-642-84379-2.