ON THE CLASSICAL LAGRANGE AND MARKOV SPECTRA: NEW RESULTS ON THE LOCAL DIMENSION AND THE GEOMETRY OF THE DIFFERENCE SET

HAROLD ERAZO, LUKE JEFFREYS, AND CARLOS GUSTAVO MOREIRA

Abstract. Let L and M denote the classical Lagrange and Markov spectra, respectively. It is known that $L \subset M$ and that $M \setminus L \neq \varnothing$. Inspired by three questions asked by the third author in previous work investigating the fractal geometric properties of the Lagrange and Markov spectra, we investigate the function $d_{loc}(t)$ that gives the local Hausdorff dimension at a point t of L'. Specifically, we construct several intervals (having non-trivial intersection with L') on which d_{loc} is non-decreasing. We also prove that the respective intersections of M' and M'' with these intervals coincide. Furthermore, we completely characterize the local dimension of both spectra when restricted to those intervals. Finally, we demonstrate the largest known elements of the difference set $M \setminus L$ and describe two new maximal gaps of M nearby.

1. Introduction

The classical Lagrange and Markov spectra are two subsets of the real line related to the study of Diophantine approximation. Given a positive real number α we define its *best constant of Diophantine approximation* to be

$$k(\alpha) := \limsup_{p,q\to\infty} \frac{1}{|q(q\alpha - p)|} = \sup\left\{ k > 0 \,:\, \left|\alpha - \frac{p}{q}\right| < \frac{1}{kq^2} \text{has infinitely many solutions } \frac{p}{q} \in \mathbb{Q} \right\}.$$

The *Lagrange spectrum* is defined to be the set

$$L := \{k(\alpha) \,:\, \alpha \in \mathbb{R} \setminus \mathbb{Q}\}.$$

The *Markov spectrum* is related to the approximation of binary quadratic forms and is defined to be

$$M := \left\{ \sup_{(p,q)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|ap^2 + bpq + cq^2|} \,:\, ax^2 + bxy + cy^2 \text{ real indefinite, } b^2 - 4ac = 1 \right\}.$$

It is known that $L \subset M \subset \mathbb{R}^+$.

In the late 1800s, Markov [14, 15] determined that

$$L \cap (0,3) = M \cap (0,3) = \left\{ \sqrt{5} < 2\sqrt{2} < \frac{\sqrt{221}}{5} < \dots \right\} = \left\{ \sqrt{9 - \frac{4}{m^2}} \,:\, m \text{ is a Markov number} \right\},$$

where a Markov number is the largest number in a triple of positive integers (x, y, z) satisfying the so-called Markov equation $x^2 + y^2 + z^2 = 3xyz$. In 1975, Freiman [3] showed that $[c_F, \infty) \subset L \subset M$ and $(\nu, c_F) \cap M = \varnothing$ where $c_F = 4.527829566\ldots$ and $\nu = 4.527829538\ldots \in M$. This ray $[c_F, \infty)$

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is known as Hall's ray after earlier work of Hall [10] (see also the intermediate results of Freiman-Judin [7], Hall [8], Freiman [5] and Schecker [24]) and c_F is now known as Freiman's constant. Hall's ray is currently the only known continuous part of the spectra. It is a long-standing and wide-open conjecture of Berstein [6] that $[4.1, 4.52] \subset L \subset M$.

Between 3 and c_F, both L and M have interesting fractal structure. Indeed, the third author proved [20] that

$$d(t) := \dim_{\mathrm{H}}(L \cap (-\infty, t)) = \dim_{\mathrm{H}}(M \cap (-\infty, t)),$$

is a continuous function with $d(3+\epsilon) > 0$ for any $\epsilon > 0$ and $d(\sqrt{12}) = 1$. In the same paper, it was also proved that

$$D(t) := \dim_{\mathrm{H}}(k^{-1}(-\infty, t]) = \dim_{\mathrm{H}}(k^{-1}(-\infty, t)),$$

is also a continuous function and that $d(t) = \min\{1, 2 \cdot D(t)\}$.

More recently, the third author in joint work with Matheus, Pollicott and Vytnova [19] determined that

$$t_1 := \min\{t \in \mathbb{R} : d(t) = 1\} = 3.334384....$$

One can also define the local dimension function $d_{loc} : L' \to [0, 1]$ by

$$d_{loc}(t) := \lim_{\epsilon \to 0} \dim_{\mathrm{H}}(L \cap (t - \epsilon, t + \epsilon)).$$

As mentioned above, it is known that $L \subset M$. Freiman [4] also showed that $M \setminus L \neq \varnothing$. Recently, in the same work of the third author, Matheus, Pollicott and Vytnova discussed above, it was shown that the Hausdorff dimension $\dim_{\mathrm{H}}(M \setminus L)$ of $M \setminus L$ satisfies

$$0.537152 < \dim_{\mathrm{H}}(M \setminus L) < 0.796445.$$

This lower bound was heuristically improved to 0.593 in recent work of the second and third authors with Matheus [9].

In [20], the third author raised the following questions for further investigation:

1) Is the function d_{loc} non-decreasing?
2) What is the geometric structure of the difference set $M \setminus L$?
3) Is $M'' = M'$?

In this work, we will investigate these questions and will answer them for certain subsets of the spectra lying within intervals of the real line that we call "good intervals" (see Definition 2.8). We will show that we have a complete understanding of the local dimension of both spectra restricted to these intervals. As a by-product we also improve the upper bound on $\dim_{\mathrm{H}}(M \setminus L)$ in these regions.

The above questions remain open in general.

1.1. **Continued fractions and shift space dynamics.** Before discussing the main results, we must introduce the modern notation used to describe the Lagrange and Markov spectra.

The values in the Lagrange and Markov spectra can be calculated using the theory of continued fractions thanks to the work of Perron [23] who proved that if we have

$$\alpha = [a_0; a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots}}},$$

then

$$k(\alpha) = \limsup_{n \to \infty} \left([a_n; a_{n-1}, \ldots, a_0] + [0; a_{n+1}, a_{n+2}, \ldots]\right).$$

This allows us to study the structure of the Lagrange spectrum using the dynamics of the bi-infinite shift space $\Sigma := \{1, 2, 3, \ldots\}^{\mathbb{Z}}$. That is, for $(a_i)_{i \in \mathbb{Z}} \in \Sigma$ we define

$$\lambda_0((a_i)_{i \in \mathbb{Z}}) := [a_0; a_1, a_2, \ldots] + [0; a_{-1}, a_{-2}, \ldots],$$

and, for $j \in \mathbb{Z}$,

$$\lambda_j((a_i)_{i\in\mathbb{Z}}) := \lambda_0(\sigma^j((a_i)_{i\in\mathbb{Z}})) = \lambda_0((a_{i+j})_{i\in\mathbb{Z}}),$$

where $\sigma : \Sigma \to \Sigma$ is the left-shift sending $(a_i)_{i\in\mathbb{Z}}$ to $(a_{i+1})_{i\in\mathbb{Z}}$. This shift map is related to the classical Gauss map

$$g([a_0; a_1, a_2, \ldots]) = [a_1; a_2, a_3, \ldots].$$

The Lagrange spectrum can then be equivalently defined as

$$L := \{\ell(\underline{a}) := \limsup_{j\to\infty} \lambda_j(\underline{a}) \mid \underline{a} \in \Sigma\}.$$

The Markov spectrum also permits a dynamical definition as

$$M := \{m(\underline{a}) := \sup_{n\in\mathbb{Z}} \lambda_n(\underline{a}) \mid \underline{a} \in \Sigma\}.$$

Given an infinite sequence $(a_i)_{i\in\mathbb{Z}} \in \Sigma$, we will often express it as $\ldots a_{-2}a_{-1}a_0^*a_1a_2\ldots$ where the asterisk denotes the 0th position. We will also use an overline to denote periodicity; e.g., $\overline{1^*23} = \ldots 1231231^*23123123\ldots$. This notation should be clear from the context as we will typically restrict to the subshift $\{1,2,3,4\}^{\mathbb{Z}}$ so, in particular, all a_i will be single digits. With this notation, Freiman's gap can be described as

$$c_F = \lambda_0(\overline{12131}322344^*3211\overline{313121}),$$

$$\nu = \lambda_0(\overline{323444}313134^*313121133\overline{313121}).$$

Furthermore, given a finite sequence $a \in \mathbb{N}^n$ for some n, inequalities of the form $\lambda_0(a) > x$ will mean that we have $\lambda_0(w) > x$ for all bi-infinite sequences w that are obtained by extending the finite sequence a on both sides.

1.2. **Local dimension.** Our first result investigates the third author's question about when the function d_{loc} is non-decreasing.

Theorem 1.1. *Consider the intervals*

- $[3.06, 3.1221)$
- $[3.1299, 3.285441)$
- $[3.28603, 3.2872)$
- $[3.29296, 3.29331)$
- $[3.333958, 3.33475)$
- $[3.359, 3.423)$
- $[3.464, 3.84)$
- $[3.87, 3.9306)$
- $[3.9362, 3.943767)$
- $[3.94405, 3.9716)$
- $[3.97995, 3.9857)$
- $[4.5207, 4.5231)$
- $[4.5251, 4.5279)$.

For each interval I above, $I \cap L' \neq \varnothing$ and for all $t \in I \cap L'$ we have

$$d_{\mathrm{loc}}(t) = \lim_{\epsilon\to 0} \dim_{\mathrm{H}}(L \cap (t-\epsilon, t+\epsilon)) = \lim_{\epsilon\to 0} \dim_{\mathrm{H}}(M \cap (t-\epsilon, t+\epsilon)) = d(t).$$

In particular $d_{loc}|_{I\cap L'}$ is non-decreasing.

To prove this, we will define the notion of a *good interval* (see Definition 2.8). Such intervals $[\nu, \mu)$ will have the property that d_{loc} is non-decreasing on $L' \cap [\nu, \mu)$. To prove that an interval is good, we will be required to understand how two shift spaces related to this interval are combinatorially related. Indeed, good intervals have strong transitivity properties for two naturally related subshifts. We analyse the properties of good intervals abstractly in Section 2. We then prove that each of the above intervals are good intervals in Section 3. These intervals are shown in blue in Figure 1.1.

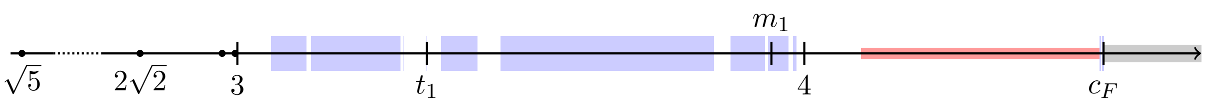

Figure 1.1. The intervals of Theorem 1.1 are depicted in blue. Hall's ray is depicted in grey. The interval $[4.1, 4.52]$ of Berstein's conjecture is depicted in red.

1.3. **Geometry of $M \setminus L$.** So far all known regions of $M \setminus L$ have the structure depicted in Figure 1.2 where the spaces represent gaps in M, $j_1 \in L'$, j_0 is isolated in M, and $(M \setminus L) \cap (j_0, j_1)$ is given as a disjoint union of the form

$$(M \setminus L) \cap (j_0, j_1) = C \sqcup D$$

where C is a Cantor set and D is an infinite set of discrete points in M. All elements of $(M \setminus L) \cap (j_0, j_1)$ connect in the past or in the future with some periodic not semi-symmetric word $\overline{w}$ of odd period. That is, the elements are given by sequences that are eventually periodic on the left or on the right with period w.

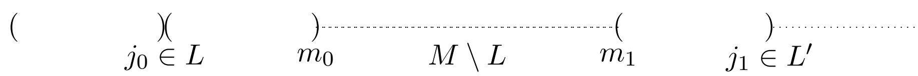

Figure 1.2. Structure of known regions of $M \setminus L$

Since (j_0, m_0) and (m_1, j_1) are gaps of M, we know that the intersection $(M \setminus L) \cap (j_0, j_1)$ is a closed set. Recently, it was shown by Lima-Matheus-Moreira-Vieira in [13] that $M \setminus L$ is not a closed set. More precisely they constructed a decreasing sequence $\widetilde{m}_k \in M \setminus L$ lying in distinct maximal gaps of L such that $\lim_{k\to\infty} \widetilde{m}_k = 1 + \frac{3}{\sqrt{2}} \in L$. However, it is not known if there is a maximal gap (ν, μ) of L such that $(M \setminus L) \cap (\nu, \mu)$ is not closed. We do not yet have examples of elements in $L \cap \overline{(M \setminus L)}$ greater than $\sqrt{12}$, in other words such that the associated sequences contain at least one 3 or 4.

It is discussed in Cusick-Flahive [2, Chapter 3], that Berstein [1] was trying to find necessary and sufficient conditions for a Markov value to not be contained in the Lagrange spectrum. Berstein gave a list of conditions and constructed some intervals within which these conditions were both necessary and sufficient for a point $m \in M$ to be outside of L.

In Section 2.4, we will demonstrate that these conditions also hold for points $m \in (M \setminus L) \cap [\nu, \mu)$ for a good interval $[\nu, \mu)$. In particular, we will establish that any such element m always connects to some non-semi-symmetric word $\overline{w}$. As a consequence, we conclude that all elements of $(M \setminus L) \cap [\nu, \mu)$ can be constructed using the method discovered by Freiman. This method and its relation with dynamical systems is very well explained in [16].

It is important to note that Berstein's intervals are constructed in an analogous way to our good intervals. That is, Berstein's proofs of the necessity and sufficiency of the conditions for $m \in M$ being in L rely on proving that one theorem [1, Theorem 2] is true for each such interval. However, and importantly so, Berstein is not considering the transitivity of two subshifts related to the interval. These subshifts and their transitivity are crucial to our proofs on local dimension and perfectness of M'.

In spite of the fact that the first digit of $\dim_H(M \setminus L)$ is still unknown, we are able to characterize completely the Hausdorff dimension of $M \setminus L$ when restricted to good intervals.

Theorem 1.2. *Let $[\nu, \mu) \subset \mathbb{R}$ be an interval. If $[\nu, \mu)$ is good, then*

(i) If (ℓ_1, ℓ_2) is a maximal gap of L contained in $[\nu, \mu)$ and such that $M' \cap (M \setminus L) \cap (\ell_1, \ell_2) \neq \varnothing$, then

$$\dim_H((M \setminus L) \cap (\ell_1, \ell_2)) = D(\ell_1) = D(\ell_2). \tag{1}$$

(ii) For all $t \in M' \cap (M \setminus L) \cap [\nu, \mu)$ we have

$$\lim_{\varepsilon \to 0} \dim_H((M \setminus L) \cap (t - \varepsilon, t + \varepsilon)) = D(t).$$

On the other hand, in recent work of the second and third authors with Matheus [9], the largest known elements of $M \setminus L$ were found near to 3.938. Using (1) we can confirm rigorously the lower bound of [9]: we have that $\dim_H((M \setminus L) \cap (3.931, 3.943)) \geq D(3.931) > 0.594179\dots$ (see Section 3.8 for details).

The values of $M \setminus L$ mentioned in that paper were discovered via a computer-assisted investigation of gaps in the Lagrange spectrum suggested by numerical approximations in the work of Delecroix, Matheus and the third author [25, Figure 5]. The algorithm used in the computer-assisted investigation is described in an appendix of the above work of Matheus and the second and third authors [9, Appendix B], but we also describe how the algorithm works in Subsection 4.1. In [9], it is also mentioned that the computer investigations suggested that there should also be a region of $M \setminus L$ near to 3.942 (see [9, Appendix A]). It was (heuristically) determined that new values of $M \setminus L$ in this range would not lead to an appreciable improvement on the lower bound of $\dim_H(M \setminus L)$ and so this region was not investigated further in that work. Our second result is a full investigation of the structure of $M \setminus L$ in this region near 3.942.

Let

$$j_0 = m(\overline{121112333^*11133232}) = 3.9420011599113414692135 48 \dots \in L.$$

Let us denote the finite word $w = 12111233311133232$ and $w^* = 121112333^*11133232$. It is better to write the period of j_0 in this way because we have the same number of digits at each side of 3^*.

We have the following result.

Theorem 1.3. *The intersection of $M \setminus L$ with $(3.942, 3.943)$ is non-empty. The largest known element of $M \setminus L$ is*

$$m_1 = \lambda_0(\overline{w}w^*w121111 1\overline{23}) = 3.9420011599 \dots \approx j_0 + 8.26 \cdot 10^{-22}.$$

The full description of $M \setminus L$ in this region is given in Theorem 4.43. The value m_1 is also indicated on Figure 1.1.

We should highlight that in the computer approximations given by Delecroix-Matheus-Moreira [25, Figure 5], the last visible gap in the approximation of L is near 3.942.

In this direction, we prove the following result that demonstrates the existence of two new maximal gaps of M near to this largest known value of $M \setminus L$. These intervals could also be contributing to the last visible gap in the computer approximations.

Theorem 1.4. *Let*

$$\nu_1 = m(\overline{23}331113^*3\overline{32}) = 3.94254\dots,$$

$$\mu_1 = m(\overline{23}3311133113212311333^*11133111333113212311331113 3\overline{32}) = 3.943304\dots,$$

$$\nu_2 = m(\overline{12}3113331113311133111 3^*33113\overline{21}) = 3.94330534\dots,$$

and

$$\mu_2 = m(\overline{21}11331113^*3311321231133111 33\overline{32}) = 3.94330716\dots.$$

The intervals (ν_i, μ_i), $i = 1, 2$, *are maximal gaps of* M*; i.e.,* $(\nu_i, \mu_i) \cap M = \varnothing$ *and* $\nu_i, \mu_i \in M$.

We prove this theorem in Section 5.

1.4. $\boldsymbol{M'}$ **and** $\boldsymbol{M''}$**.** In Section 2, we will prove the following theorem addressing the question of whether $M' = M''$ in the setting of good intervals.

Theorem 1.5. *Let* $[\nu, \mu) \subset \mathbb{R}$ *be an interval. If* $[\nu, \mu)$ *is good in the sense of Definition 2.8, then* $M' \cap [\nu, \mu) = M'' \cap [\nu, \mu)$.

This will be proved as one part of Theorem 2.11.

1.5. **Properties of** $D(t)$**.** Recall that $D(t) = \dim_{\mathrm{H}}(k^{-1}(-\infty, t])$. In [21], the third author and Villamil showed a concentration of dimension result, i.e., $D(t) = \dim_{\mathrm{H}}(k^{-1}(t))$, at points of the closure of the interior of the spectra, i.e., for $t \in \overline{\mathrm{int}(L)} = \overline{\mathrm{int}(M)}$. Moreover, they proved that $D(t)$ is strictly increasing across subintervals of the closure of the interior.

Using similar techniques and ideas, we will prove that the above results also hold on good intervals.

Proposition 1.6. *Let* $[\nu, \mu) \subset \mathbb{R}$ *be an interval. If* $[\nu, \mu)$ *is good, then for all* $t \in L' \cap [\nu, \mu)$ *we have*

$$\dim_{\mathrm{H}}(k^{-1}(t)) = D(t).$$

Moreover if $s_1 < t < s_2$ *are such that* $t \in L' \cap [\nu, \mu)$, *then*

$$D(s_1) < D(s_2).$$

Recall that it is still a wide-open conjecture whether $\mathrm{int}(L \cap (-\infty, c_F]) \neq \varnothing$.

1.6. **Open questions.** We remind the reader that the questions of the third author discussed above remain open in general; i.e., outside of good intervals.

Motivated by Figure 1.2, we also propose the following questions.

Question 1.7. *Given a maximal gap* (ν, μ) *of* L*, it is true that* $M \cap (\nu, \mu)$ *is closed in* $\mathbb{R}$*?*

Question 1.8. *Is* $M' \cap L = L'$*?*

Recalling the maximal gap (ν, c_F) before Hall's ray, we also ask:

Question 1.9. *Is* $(M \setminus L) \cap (3.945, \nu) \neq \varnothing$*? Are there points of* $M \setminus L$ *close to* ν*?*

Question 1.10. *What can be said about the structure of the sets* $\partial L \cap [4.52, \nu]$ *and* $\partial M \cap [4.52, \nu]$*? Are they uncountable?*

Acknowledgements: We would like to thank Carlos Matheus for mentioning the relation of good intervals with the previous work of Berstein.

2. Monotonicity of local dimension and good intervals

In this section, we will investigate the local dimension function $d_{loc} : L' \to [0, 1]$, define good intervals, and demonstrate that on such intervals d_{loc} is non-decreasing.

2.1. **Good intervals.** Here we introduce the notion of a "good interval." These intervals will be the settings in which we are able to address the questions of the third author discussed in the introduction. Berstein [1] produced intervals with very similar properties in his study of $M \setminus L$. We discuss how our good intervals are related to Berstein's intervals in Subsection 2.4.

The following two lemmas are from [19] and [18] respectively. The first will be used heavily in Section 3 in order to bound Markov values.

Given a subshift $\Sigma \subset (\mathbb{N}^*)^{\mathbb{Z}}$ we denote $\Sigma^+(\Sigma) = \{(a_n)_{n\geq 0} : (a_n)_{n\in\mathbb{Z}} \in \Sigma\}$ and by $K(\Sigma) = \{[0; a_1, a_2, \dots] : (a_n)_{n\in\mathbb{Z}} \in \Sigma\}$. When Σ is of finite type then $K(\Sigma)$ is called the stable Gauss-Cantor set associated to Σ.

Lemma 2.1. *Let $\Sigma(C)$ be a transitive symmetric subshift. Assume that three half-infinite sequences $v^1, v^2, v^3 \in \Sigma^+(C)$ are such that $[0; v^1] > [0; v^3] > [0; v^2]$. Then for all $\underline{a} \in \Sigma(C)$ and for all $j \leq n+1$*

$$\lambda_0\left(\sigma^j(\dots a_{-2}a_{-1}a_0\dots a_n v^3)\right) \quad \leq \quad \max\left(m(\dots a_{-2}a_{-1}a_0\dots a_n v^1), m(\dots a_{-2}a_{-1}a_0\dots a_n v^2)\right).$$

We number the sequences in the order above (i.e., $[0; v^1] > [0; v^3] > [0; v^2]$) as it resembles the proofs in Section 3 where we will bound a sequence v_B between two sequences v^1 and v^2.

The second lemma will be used in Section 3 to determine the minimum Markov values given by sequences in certain subshifts. These minima will be the right endpoints of our good intervals.

Lemma 2.2. *Let β be a finite word and θ be a symmetric finite word of even size on $\{1, 2, \dots, A\}$. Assume that, under some conditions (e.g., after forbidding a finite list of finite strings), the Markov value of an infinite word of the type $\gamma = \omega_2^T \beta^T A\theta A\beta\omega_1$ is attained at one of the A next to θ, where ω_1, ω_2 are infinite words (on $\{1, 2, \dots, A\}$). Then this Markov value is minimised when $\omega_1 = \omega_2 =: \omega$ and $[0; \beta, \omega]$ is minimal (under these conditions).*

We will need a simple criterion to determine when two subshifts $\Sigma(B)$ and $\Sigma(C)$ are transitive.

Let $\mathcal{A} = \{1, 2, \dots, A\}$ be an alphabet. Given a finite word w in $\mathcal{A}$, we write $w = w^- a$ where $a \in \mathcal{A}$. We will assume that $\mathcal{F}$ is a finite set of words in $\mathcal{A}$ that is symmetric, i.e. it contains all the transposes. Furthermore we assume that no word $w \in \mathcal{F}$ is a subword of another word $\widetilde{w} \in \mathcal{F}, w \neq \widetilde{w}$. Let

$$\Sigma = \{\underline{a} \in \mathcal{A}^{\mathbb{Z}} \mid \underline{a} \text{ has no substring from } \mathcal{F}\}$$

be a symmetric finite type subshift.

Given $\underline{a} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$ and $N \in \mathbb{Z}$ we denote $a_{-\infty,N} = \dots a_{N-2}a_{N-1}a_N$ and $a_{N,\infty} = a_N a_{N+1} a_{N+2} \dots$ the tails of $\underline{a}$.

Proposition 2.3. *Suppose there is an $a \in \mathcal{A}$ with $\overline{a} \in \Sigma$ and such that for all $w \in \mathcal{F}$ there is a finite word τ_w in $\mathcal{A}$ such that $w^- \tau_w \overline{a}$ does not contain words from $\mathcal{F}$. Then the subshift Σ is transitive.*

Proof. Let $\underline{a} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$ and $N \in \mathbb{N}^*$. We want to find a finite τ in $\mathcal{A}$ such that $a_{-\infty,N}\tau\overline{a} \in \Sigma$. Let L be the length of the longest word of $\mathcal{F}$. We start by considering $\underline{b}^{(1)} = a_{-\infty,N+L}\overline{a}$. If it does not contain words from $\mathcal{F}$, then we are done, otherwise it must contain a word $w_1 \in \mathcal{F}$. That is, there are $n_1^-, n_1^+ \in \mathbb{N}$ with $N < n_1^- \leq N + L < n_1^+$ such that $w_1 = b^{(1)}_{n_1^-} \cdots b^{(1)}_{n_1^+}$. We take w_1 with n_1^- minimal. By hypothesis there are τ_{w_1} a finite word in $\mathcal{A}$ such that $w_1^- \tau_{w_1}\overline{a}$ does not contain words from $\mathcal{F}$. Consider the bi–infinite sequence

$$\underline{b}^{(2)} = b^{(1)}_{-\infty,n_1^+-1}\tau_{w_1}\overline{a} = \dots a_{n_1^- -2}a_{n_1^- -1}w_1^-\tau_{w_1}\overline{a}.$$

If this bi–infinite sequence does not contain words from $\mathcal{F}$ we are done, otherwise it contains a word $w_2 \in \mathcal{F}$, that is there are $n_2^-, n_2^+ \in \mathbb{N}$ such that $w_2 = b^{(2)}_{n_2^-} \cdots b^{(2)}_{n_2^+}$. Take n_2^- minimal. We claim that this subword starts before w_1^- and ends after w_1^-, that is $n_2^- < n_1^- < n_1^+ \leq n_2^+$. Indeed, first note if $n_1^- \leq n_2^-$ then w_2 would be a subword of $b^{(2)}_{n_1^-,\infty} = w_1^- \tau_{w_1} \overline{a}$ which contradicts the hypothesis. If $n_2^+ < n_1^+$, then w_2 would be a subword of $b^{(2)}_{-\infty,n_1^+-1} = b^{(1)}_{-\infty,n_1^+-1}$, hence w_2 would be a forbidden subword of $\underline{b}^{(1)}$ starting at $n_2^- < n_1^-$, but this contradicts the minimality of n_1^-.

Now that the claim is proved, in particular we obtain that w_2 is longer $|w_2| = n_2^+ - n_2^- + 1 \geq n_1^+ - n_1^- + 2 = |w_1| + 1$. Now consider the bi–infinite sequence

$$\underline{b}^{(3)} = b^{(2)}_{-\infty,n_2^+-1} \tau_{w_2} \overline{a} = \dots a_{n_2^- -2} a_{n_2^- -1} w_2^- \tau_{w_2} \overline{a}.$$

Inductively we find a sequence of possible continuations $\underline{b}^{(1)}, \dots, \underline{b}^{(r)}$ each one associated with a word $|w_i| \geq |w_{i-1}| + 1$ and such that $n_i^- < n_{i-1}^- < n_{i-1}^+ \leq n_i^+$. Since $\mathcal{F}$ is finite, this algorithm must finish for some $r \leq L$ and so

$$\underline{b}^{(r+1)} = b^{(r)}_{-\infty,n_r^+-1} \tau_{w_r} \overline{a} = \dots a_{n_r^- -2} a_{n_r^- -1} w_r^- \tau_{w_r} \overline{a} \in \Sigma.$$

Finally this completes our initial aim because $n_r^- \geq n_r^+ - L \geq n_1^+ - L > N$. □

Now we introduce the notion of when an orbit connects to a transitive subshift of finite type. The first definition is from [19, Definition 3.2] and the second one is reminiscent of [21, Definition 3.1].

Definition 2.4. *Consider two transitive and symmetric subshifts of finite type $\Sigma(B) \subset \Sigma(C)$. Let $\underline{a} \in \Sigma(C)$ be a sequence with $m(\underline{a}) = \lambda_0(\underline{a}) = m \in M$.*

- *We say that $\underline{a}$ connects positively to B, if for every $k \in \mathbb{N}$ there exist a finite sequence τ and an infinite sequence $v \in \Sigma^+(B)$ such that for $\widetilde{\underline{a}} := \dots a_{-2} a_{-1} a_0 \dots a_k \tau v$ we have*

$$m(\widetilde{\underline{a}}) < m(\underline{a}) + 2^{-k}. \tag{2}$$

  We say that $\underline{a}$ connects negatively to B if the reversed sequence $\underline{a}^T$ connects positively to B.
- *We say that $\underline{a}$ connects positively to B before t, if for every $k \in \mathbb{N}$ there exist a finite sequence τ and an infinite sequence $v \in \Sigma^+(B)$ such that for $\widetilde{\underline{a}} := \dots a_{-2} a_{-1} a_0 \dots a_k \tau v$ we have*

$$m(\widetilde{\underline{a}}) < t + 2^{-k}.$$

  We say that $\underline{a}$ connects negatively to B before t if the reversed sequence $\underline{a}^T$ connects positively to B before t.

Remark 2.5. *Assume that $\Sigma(B) \subset \Sigma(C)$ are two transitive symmetric subshifts and let x be such that $m(\underline{b}) \leq x$ for all $\underline{b} \in \Sigma(B)$. Consider a sequence $\underline{c} \in \Sigma(C)$ with $m(\underline{c}) = \lambda_0(\underline{c}) = m > x$. Then for any finite sequence τ and half-infinite sequence $v \in \Sigma^+(B)$ directly from definition of Lagrange and Markov numbers we get*

$$\limsup_{j \to +\infty} \lambda_0(\sigma^j(\dots \gamma_{-2}\gamma_{-1}\gamma_0 \dots \gamma_k \tau v)) < m,$$

Thus, if we want to get that $m(\dots \gamma_{-2}\gamma_{-1}\gamma_0 \dots \gamma_k \tau v) < m + 2^{-k}$, then it suffices to check that

$$\lambda_0(\sigma^j(\dots \gamma_{-2}\gamma_{-1}\gamma_0 \dots \gamma_k \tau v)) < m + 2^{-k}$$

for finitely many values of j, namely, for all $0 \leq j \leq k + |\tau| + l$ where l is sufficiently large (so that $2^{1-l} \leq m - x + 2^{-k}$). This is because, for two continued fractions $\alpha = [0; a_1, a_2, \dots]$ and $\beta = [0; b_1, b_2, \dots]$ with $a_i = b_i, 1 \leq i \leq N$, and $a_{N+1} \neq b_{N+1}$, we have $|\alpha - \beta| < 2^{1-N}$.

The important consequence of connecting to both sides to a transitive subshift with smaller Markov values is given by [19, Lemma 3.6].

Lemma 2.6. *Consider two transitive and symmetric subshifts of finite type $\Sigma(B) \subset \Sigma(C)$. Let x be such that $m(\underline{b}) \leq x$ for all $\underline{b} \in \Sigma(B)$. Suppose that a sequence $\gamma \in \Sigma(C)$ satisfies $m(\gamma) = \lambda_0(\gamma) = m \geq x$ and connects positively and negatively to B. Then $m \in L$.*

Remark 2.7. *In general, if Σ is a transitive symmetric subshift with infinite cardinality then* $\sup\{m(\gamma) : \gamma \in \Sigma\} \in L'$.

In this direction, we will associate to an interval $[x, y)$ two symmetric transitive subshifts of finite type $\Sigma(B) = \Sigma(B_x) \subset \Sigma(C) = \Sigma(C_y) \subset (\mathbb{N}^*)^{\mathbb{Z}}$ such that

- $m(\underline{b}) \leq x$ for all $\underline{b} \in \Sigma(B)$; and
- for all $\underline{c} \in (\mathbb{N}^*)^{\mathbb{Z}}$ such that $m(\underline{c}) < y$ we have $\underline{c} \in \Sigma(C)$.

We will always choose $x := \sup\{m(\underline{b}) : \underline{b} \in \Sigma(B)\}$. Note that if $\Sigma(B)$ has infinite cardinality then Remark 2.7 gives us that $x \in L'$.

Definition 2.8. *Let $[\nu, \mu)$ be an interval. We say that $[\nu, \mu)$ is good if it can be covered by intervals $[x, y)$ as above such that for any $m(\underline{a}) = \lambda_0(\underline{a}) < y$ it holds: given any position $N \in \mathbb{N}$ such that there are two continuations $v^1, v^2 \in \Sigma^+(C)$ with* different *first term, then there is a continuation $v_B \in \Sigma^+(B)$ such that*

$$m(\dots a_{-1}\alpha_0 \dots a_N v_B) \leq \max\{m(\dots a_{-1}a_0 \dots a_N v^1), m(\dots a_{-1}a_0 \dots a_N v^2), x\}.$$

Note that a connected finite union of good intervals is again a good interval.

2.2. **Local dimension on good intervals.** Define $d(t) : \mathbb{R} \to [0, 1]$ by

$$d(t) = \dim_{\mathrm{H}}(L \cap (-\infty, t)) = \dim_{\mathrm{H}}(M \cap (-\infty, t)).$$

Recall that Moreira [20] proved the following formula:

$$d(t) = \min\{1, 2 \cdot D(t)\},$$

where $D(t) = \dim_{\mathrm{H}}(K_t) = \overline{\dim_{\mathrm{B}}}(K_t)$ is a continuous function and

$$K_t = \{[0; c_1, \dots, c_n, \dots] \mid \text{there exists } (c_{-n})_{n\geq 0} \in (\mathbb{N}^*)^{\mathbb{N}} \text{ such that } [c_k; c_{k+1}, \dots,] + [0; c_{k-1}, c_{k-2}, \dots] \leq t, \forall k \in \mathbb{Z}\}.$$

Note that, using the notation from the introduction, $K_t = k^{-1}((-\infty, t])$ and

$$D(t) = \dim_{\mathrm{H}}(k^{-1}(-\infty, t]) = \dim_{\mathrm{H}}(k^{-1}(-\infty, t)).$$

Define the function $d_{\mathrm{loc}} : L' \to [0, 1]$

$$d_{\mathrm{loc}}(t) = \lim_{\varepsilon \to 0} \dim_{\mathrm{H}}(L \cap (t - \varepsilon, t + \varepsilon)).$$

By using the fact that d is continuous and the monotonicity of the Hausdorff dimension, we always have $d_{\mathrm{loc}} \leq d$.

Proposition 2.9. *The function $d_{\mathrm{loc}} : L' \to [0, 1]$ is non-decreasing if and only if $d_{\mathrm{loc}} = d$ on L'.*

Proof. Suppose d_{loc} non-decreasing. If $d_{\mathrm{loc}}(t) < d(t)$ for some t, let $t_0 = \min\{s : d(s) = d(t)\}$. By definition we have that $d(t_0 - \varepsilon) < d(t_0)$ for all $\varepsilon > 0$, so we must have $d_{\mathrm{loc}}(t_0) = d(t_0)$. However since $t_0 \leq t$ by hypothesis $d(t) = d(t_0) = d_{\mathrm{loc}}(t_0) \leq d_{\mathrm{loc}}(t) < d(t)$. □

Recall that the notation Σ_t for some $t \in \mathbb{R}$ denotes the set

$$\Sigma_t := \{\underline{a} \in (\mathbb{N}^*)^{\mathbb{Z}} \,|\, m(\underline{a}) \leq t\}.$$

Theorem 2.10. *Let $[\nu, \mu) \subset \mathbb{R}$ be an interval. If $[\nu, \mu)$ is good, then $d_{\mathrm{loc}}(t) = d(t)$ for all $t \in L' \cap [\nu, \mu)$. In particular d_{loc} is non-decreasing in $[\nu, \mu)$. Moreover for all $t \in L' \cap [\nu, \mu)$ we have*

$$d_{\mathrm{loc}}(t) = \lim_{\varepsilon \to 0} \dim_{\mathrm{H}}(M \cap (t - \varepsilon, t + \varepsilon)). \tag{3}$$

Proof. Let $t \in L' \cap [x,y)$ with x,y as in Definition 2.8. We want to show $d_{\text{loc}}(t) \geq d(t)$ by connecting $\Sigma_s \to \Sigma(B) \to \Sigma_{t'} \to \Sigma(B) \to \Sigma_s$ for values of $s \in M$, $s < t$ close to t and values $t' \in L$ close to t. In other terms, given $\varepsilon > 0$ small, we would like to find a sequence $\gamma = (\gamma_i)_{i\in\mathbb{Z}}$ with $m(\gamma) = \lambda_0(\gamma) = t'$ close to t, finite subwords β_r, β_l of some sequences in $\Sigma(B)$, finite words $\underline{\tau}, \tilde{\tau}, \tau, \underline{\tilde{\tau}}$ and a complete subshift $\Sigma(A) \subset \Sigma_s$ with $\dim_{\text{H}}(K(A))$ close to $\dim_{\text{H}}(K_s)$, such that

$$|m(v_l^T \underline{\tau}\beta_l\tilde{\tau}\gamma_{-k}\dots\gamma_k\tau\beta_r\underline{\tilde{\tau}}v_r) - t| < \varepsilon, \qquad \text{for all } v_l, v_r \in \Sigma^+(A).$$

More specifically, we will prove the following: for all $\varepsilon > 0$ small enough

$$\dim_{\text{H}}((M \setminus L) \cap (t-\varepsilon, t+\varepsilon)) \leq \dim_{\text{H}}(K_{t+\varepsilon}), \tag{4}$$

while

$$\dim_H(M \cap (t-\varepsilon, t+\varepsilon)) \geq d(t) - \widetilde{\delta}, \tag{5}$$

where $\widetilde{\delta} \to 0$ as $\varepsilon \to 0$. By continuity of $t \mapsto \dim_{\text{H}}(K_t)$ we know that

$$\dim_{\text{H}}(K_{t+\varepsilon}) \leq \min\{2\dim_{\text{H}}(K_t), 1\} - c = d(t) - c$$

for some $c > 0$ and for all $\varepsilon > 0$ small. Hence (4) and (5) can only happen if $d_{\text{loc}}(t) = d(t)$. Moreover from these equations (3) also follows.

Now let $t \in (M \setminus L) \cap (x,y)$. We know that $\{\underline{a} \in (\mathbb{N}^*)^{\mathbb{Z}} : m(\underline{a}) = \lambda_0(\underline{a}) = t\}$ is finite (by Proposition 2.16), so let $\{\underline{a}^{(1)}, \dots, \underline{a}^{(r)}\}$ be the sequences such that $m(\underline{a}^{(i)}) = \lambda_0(\underline{a}^{(i)}) = t$. We claim that for any $n \in \mathbb{N}$, there is $\varepsilon > 0$ such that if $|m(\underline{a}) - t| < \varepsilon$ with $m(\underline{a}) = \lambda_0(\underline{a})$, then $a_{-n,n} = a^{(j)}_{-n,n}$ for some $1 \leq j \leq r$. Indeed, by contradiction there will be a sequence $\underline{b}^{(i)}$ with $m(\underline{b}^{(i)}) = \lambda_0(\underline{b}^{(i)})$, $|m(\underline{b}^{(i)}) - t| < 1/i$ but $b^{(i)}_{-n,n} \neq a^{(j)}_{-n,n}$ for all i,j. Therefore, passing to a subsequence of indexes, we conclude that the $\underline{b}^{(i)}$ converge to a sequence $\underline{b}$ with $m(\underline{b}) = \lambda_0(\underline{b}) = t$ but with $b_{-n,n} \neq a^{(j)}_{-n,n}$ for all $1 \leq j \leq r$, a contradiction.

As a consequence, for any $\varepsilon > 0$ small enough, it holds that given any $\underline{a}$ such that $|m(\underline{a}) - t| < \varepsilon$ with $m(\underline{a}) = \lambda_0(\underline{a})$, we will have that either $a_{0,\infty} = a^{(j)}_{0,\infty}$ or $a_{-\infty,0} = a^{(j)}_{0,\infty}$ for some $1 \leq j \leq r$. To see this, we can assume, without loss of generality, that each $\underline{a}^{(j)}$ does not connect positively to B. Thus there is a $k \in \mathbb{N}$, such that for any $N \geq k$ we will have $m(a^{(j)}_{-\infty,N}v_B) \geq t + 2^{-k}$ for any $v_B \in \Sigma^+(B)$ and all $1 \leq j \leq r$. Suppose $\varepsilon < 2^{-k}$ is small enough to guarantee that $a_{-N,N} = a^{(j)}_{-N,N}$ for some j (up to transposition of $\underline{a}$). Hence if $N_1 \geq N$ is such that $a_{N_1} \neq a^{(j)}_{N_1}$, since the interval is good there is $v_B \in \Sigma^+(B)$ such that

$$m(a^{(j)}_{-\infty,N_1-1}v_B) \leq \max\{m(\underline{a}^{(j)}), m(\underline{a})\} < t + \varepsilon < t + 2^{-k}$$

which is impossible.

Given $\varepsilon > 0$ small, by the previous argument we know that up to transposition there is a finite set X of possible continuations υ and $N \in \mathbb{N}$ such that given any $\lambda_0(\gamma) = m(\gamma) \in (M \setminus L) \cap (t-\varepsilon, t+\varepsilon)$, we have $\gamma_N\gamma_{N+1}\dots = \upsilon$ for some $\upsilon \in X$. In particular, we will have that

$$(M \setminus L) \cap (t-\varepsilon, t+\varepsilon) \subset \bigcup_{\upsilon\in X} \bigcup_{\substack{1\leq a\leq T \\ i=1,2,\dots}} (a + g^i([0;\upsilon]) + K_{t+\varepsilon}),$$

where g is the Gauss map.

Since the Hausdorff dimension of a countable union is the supremum of the Hausdorff dimensions of each set, we get $\dim_{\text{H}}((M \setminus L) \cap (t-\varepsilon, t+\varepsilon)) \leq \dim_{\text{H}}(K_{t+\varepsilon})$ for all ε small enough. This finishes the proof of (4).

Now we want to show (5). Since $t \in L'$, by the proof of [20, Theorem 3] we know that for any $\delta' > 0$ there are regular Cantor sets $K(A_1), K(A_2)$ defined by iterates of the Gauss map such that $\operatorname{diam}\big(t-\big(K(A_1)+K(A_2)\big)\big) \le \delta'$ and $K(A_1)+K(A_2) \subset L$. We claim that for all $m(\gamma)=\lambda_0(\gamma) \in K(A_1)+K(A_2)$ we have that γ connects positively and negatively with B before $\max\{m(\gamma),x\}$. Indeed, given any $k \in \mathbb{N}$, since $K(A_2)$ is a regular Cantor set, we can find some large $N \in \mathbb{N}, k \le N$ and continuations v^1, v^2 with different first term and such that $m(\dots\gamma_{-N}v^i) < m(\gamma)+2^{-N}$ for $i=1,2$. Since the interval is good we know there is $v_B \in \Sigma^+(B)$ such that $m(\dots\gamma_{-N}v_B) < \{m(\gamma),x\}+2^{-N}$. So by taking $\tau=\gamma_{k+1}\dots\gamma_N$ and $v=v_B \in \Sigma^+(B)$ we are done. The same proof applies to γ^T and $K(A_1)$. If $t>x$ we impose that $0<\delta'<t-x$ and if $t=x$ we choose $\gamma \in \Sigma(B)$ such that $\lambda_0(\gamma)=m(\gamma)=x$.

We will use the following facts that hold for any $t \in \mathbb{R}$. By [20, Lemma 2], for any $\eta \in (0,1)$ there is $\delta>0$ and a complete shift $\Sigma(A)$ such that $\Sigma(A) \subset \Sigma_{t-\delta}$ and $\dim_{\mathrm{H}}(K(A)) > (1-\eta)\dim_{\mathrm{H}}(K_t)$. Suppose that the maximum of $m(\Sigma(A))$ is attained at

$$\underline{\theta}=(\dots,\widetilde{a}_1,\widetilde{a}_0,\widetilde{a}_1,\dots), \quad \widetilde{a}_i \in A, \forall i \in \mathbb{Z}. \tag{6}$$

in a position belonging to the word $\widetilde{a}_0$. In particular $m(\underline{\theta}) \le t-\delta$. By the same proof given above we have that $\underline{\theta}$ connects positively and negatively with B before $t-\delta$.

Since $\underline{\theta}$ connects positively and negatively to B before $t-\delta$, given any $k \in \mathbb{N}^*$, there exist finite sequences $\underline{\tau},\underline{\widetilde{\tau}}$ and infinite sequences $\underline{v},\underline{\widetilde{v}} \in \Sigma^+(B)$ such that

$$m(\dots\widetilde{a}_{-2}\widetilde{a}_{-1}\widetilde{a}_0\dots\widetilde{a}_k\underline{\tau}\underline{v}) < t-\delta+2^{-k} \text{ and } m(\underline{\widetilde{v}}^T\underline{\widetilde{\tau}}\widetilde{a}_{-k}\dots\widetilde{a}_0\widetilde{a}_1\widetilde{a}_2\dots) < t-\delta+2^{-k}.$$

Similarly, since γ connects positively and negatively to B, there exist finite sequences $\tau,\widetilde{\tau}$ and infinite sequences $v,\widetilde{v} \in \Sigma^+(B)$ such that

$$m(\dots\gamma_{-2}\gamma_{-1}\gamma_0\dots\gamma_k\tau v) < m(\gamma)+2^{-k} \text{ and } m(\widetilde{v}^T\widetilde{\tau}\gamma_{-k}\dots\gamma_0\gamma_1\gamma_2\dots) < m(\gamma)+2^{-k}. \tag{7}$$

In the particular case where $m(\gamma)=\lambda_0(\gamma)=x$ we connect to B just using $\tau v=\gamma_{k+1}\gamma_{k+2}\dots$ and $\tilde{v}^T\tilde{\tau}=\dots\gamma_{-k-2}\gamma_{-k-1}$.

The Markov value of either of the continuations (7) is attained at some position $|n| \le |\tau|+|\widetilde{\tau}|+k+l$, where l is such that $2^{1-l} < m(\gamma)-x+2^{-k}$. In particular

$$|m(\widetilde{v}^T\widetilde{\tau}\gamma_{-k}\dots\gamma_{-1}\gamma_0\gamma_1\dots\gamma_k\tau v)-m(\gamma)| < 2^{1-k}, \tag{8}$$

and the Markov value is attained in a position $|n| \le |\tau|+|\widetilde{\tau}|+k+l$.

Let $v^{l+k} := v_1\dots v_{l+k}$, $\widetilde{v}^{l+k} := \widetilde{v}_{l+k}\dots\widetilde{v}_1$, $\underline{v}^{l+k} := \underline{v}_1\dots\underline{v}_{l+k}$, $\underline{\widetilde{v}}^{l+k} := \underline{\widetilde{v}}_{l+k}\dots\underline{\widetilde{v}}_1$ be the initial segments of $v,\widetilde{v}^T,\underline{v},\underline{\widetilde{v}}^T$ respectively. By transitivity of $\Sigma(B)$, there exists $\beta \in \Sigma(B)$ which contains non-overlapping occurrencies of the strings $\underline{v}^{l+k}$ and $\widetilde{v}^{l+k}$ in this order and analogously the same holds for v^{l+k} and $\underline{\widetilde{v}}^{l+k}$. Let us denote by $(\underline{v}^{l+k} * \widetilde{v}^{l+k})$ a finite substring of β which begins with $\underline{v}^{l+k}$ and terminates with $\widetilde{v}^{l+k}$ and analogously define $(v^{l+k} * \underline{\widetilde{v}}^{l+k})$.

Finally, using all the above we obtain that for $j \ge k$

$$\begin{aligned} m(\dots a_{-j-2}a_{-j-1}\widetilde{a}_{-j}\dots\widetilde{a}_0\dots\widetilde{a}_k\underline{\tau}(\underline{v}^{l+k} * \widetilde{v}^{l+k})\widetilde{\tau}\gamma_{-k}\dots\gamma_0\dots \\ \dots\gamma_k\tau(v^{l+k} * \underline{\widetilde{v}}^{l+k})\underline{\widetilde{\tau}}\widetilde{a}_k\dots\widetilde{a}_0\dots\widetilde{a}_j a_{j+1}a_{j+2}\dots) < t+2^{3-k} \end{aligned} \tag{9}$$

for any choice of $a_{|i|} \in A$ for $|i| \ge j+1$, where we used (8) and $|m(\gamma)-t|=|\lambda_0(\gamma)-t|<\delta'<\delta$. By the same reason the Markov value of the above bi-infinite sequence is attained in some position $|n| \le |\tau|+|\widetilde{\tau}|+k+l$ inside $(\underline{v}^{l+k} * \widetilde{v}^{l+k})\widetilde{\tau}\gamma_{-k}\dots\gamma_0^*\dots\gamma_k\tau(v^{l+k} * \underline{\widetilde{v}}^{l+k})$ where γ_0^* denotes the zero position. By (8), we will also have that the above Markov value is at least $t-\delta'-2^{3-k}$.

By all the above we finally obtain that $M \cap (t-\varepsilon,t+\varepsilon)$ contains a sum of the form $K+K'$ where both K and K' are Gauss-Cantor sets that are bi-Lipschitz equivalent to $K(A)$. By using

Moreira's dimension formula [22] we get

$$\dim_{\mathrm{H}}(M\cap(t-\varepsilon,t+\varepsilon))\geq\dim_{\mathrm{H}}(K+K')=\min\{2\dim_{\mathrm{H}}(K(A)),1\}$$
$$\geq\min\{2(1-\eta)\dim_{\mathrm{H}}(K_t),1\}\geq d(t)-\widetilde{\delta}.$$

□

2.3. **The topology and geometry of M' and the Hausdorff dimension of $M\setminus L$.** Here we discuss the topological and geometric properties of M' in the setting of good intervals.

As discussed in the introduction, an open question asked by the third author and concerning the topology of the spectra is whether the set M' is perfect; that is, is $M'=M''$?

We will prove in the following theorem that these two sets do coincide within good intervals.

Theorem 2.11. *Let $[\nu,\mu)\subset\mathbb{R}$ be an interval. If $[\nu,\mu)$ is good, then*

(i) If (ℓ_1,ℓ_2) is a maximal gap of L contained in $[\nu,\mu)$ and such that $M'\cap(M\setminus L)\cap(\ell_1,\ell_2)\neq\varnothing$, then

$$\dim_{\mathrm{H}}((M\setminus L)\cap(\ell_1,\ell_2))=D(\ell_1)=D(\ell_2). \tag{10}$$

(ii) For all $t\in M'\cap(M\setminus L)\cap[\nu,\mu)$ we have

$$\lim_{\varepsilon\to0}\dim_{\mathrm{H}}((M\setminus L)\cap(t-\varepsilon,t+\varepsilon))=\dim_{\mathrm{H}}(K_t)=D(t). \tag{11}$$

(iii) We have

$$M'\cap[\nu,\mu)=M''\cap[\nu,\mu). \tag{12}$$

Proof. Since $D(t)=\dim_{\mathrm{H}}(K_t)$ is constant on gaps of L (because $D(t)=\dim_{\mathrm{H}}(k^{-1}(-\infty,t))=\dim_{\mathrm{H}}(k^{-1}(-\infty,t]))$, observe that (4) already gives one side of the equalities in (10) and (11). To prove the converses inequalities, the idea is to connect $\Sigma_{t'}\to\Sigma(B)\to\Sigma_s$ where $s<t$ and t' are very close to t. More precisely, for any $\eta\in(0,1)$, $\varepsilon>0$, we will show that there is $\delta>0$ and a Gauss-Cantor set K such that $K\subset M'$, $\operatorname{diam}(t-K)<\varepsilon$ and $\dim_{\mathrm{H}}(K)>(1-\eta)\dim_{\mathrm{H}}(K_t)$. In particular if $t\in M'\cap[\nu,\mu)$ then $t\in M''$ as well and $t\in M'\cap(M\setminus L)\cap[\nu,\mu)$ then $K\subset(M\setminus L)\cap(t-\varepsilon,t+\varepsilon)$ for ε small.

Let $t\in M'\cap[x,y)$ with x,y as in Definition 2.8. Since we already know that $x\in L'=L''$, we can assume that $t\in(x,y)$. Consider a sequence t_n converging to t, $t_n\in M$, $t_n\neq t$. Choose $\underline{\theta}^{(n)}\in\Sigma$ such that $t_n=m(\underline{\theta}^{(n)})=\lambda_0(\underline{\theta}^{(n)})$. Let $\underline{\theta}^{(n)}=(b_\ell^{(n)})_{\ell\in\mathbb{Z}}$ and assume $b_\ell^{(n)}\leq4$, $\forall\,\ell$, $\forall\,n$ (which is possible since me may assume that the t_n are not in Hall's ray). Given $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ large such that $n\geq n_0\implies|m(\underline{\theta}^{(n)})-t|<\delta$. Let $N=\lceil\varepsilon^{-1}\rceil+3$. There is a sequence S such that for infinitely many values of n, S appears infinitely many times as $(b_{-N}^{(n)},b_{-N+1}^{(n)},\dots,b_N^{(n)})$, i.e. there are $n_1<n_2<\dots$ such that $S=(b_{-N}^{(n_i)},b_{-N+1}^{(n_i)},\dots,b_N^{(n_i)})$, $\forall i\geq1$. Since $t_{n_i}\neq t$ and $t_{n_i}\to t$, we have (up to transposition) that for some $N_1\geq N$ there are two continuations v^1,v^2 with different first term such that $v^1=(b_{N_1+1}^{(n_i)},b_{N_1+2}^{(n_i)},\dots)$ and $v^2=(b_{N_1+1}^{(n_j)},b_{N_1+2}^{(n_j)},\dots)$ for some $n_i\neq n_j$ and $(b_0^{(n_i)},\dots,b_{N_1}^{(n_i)})=(b_0^{(n_j)},\dots,b_{N_1}^{(n_j)})$. Define, for some $h\in\mathbb{Z}$, $\underline{\theta}_{-\infty,h}^{(n_i)}=\dots b_{h-2}^{(n_i)}b_{h-1}^{(n_i)}b_h^{(n_i)}$ the left tails of $\underline{\theta}^{(n_i)}$. Since the interval is good, there is a continuation $v_B\in\Sigma^+(B)$ such that

$$|m(\underline{\theta}_{-\infty,N_1}^{(n_i)}v_B)-t_{n_i}|=|m(\dots b_{-N_1-1}^{(n_i)}\dots b_{N_1}^{(n_i)}v_B)-t_{n_i}|<2^{-N_1}. \tag{13}$$

Now we will use the constructions of (6) and (7) with some fixed $k\geq\max\{\lceil\delta^{-1}\rceil,N_1\}$. Let $\underline{\widetilde{v}}^k:=\underline{\widetilde{v}}_k\dots\underline{\widetilde{v}}_1$ and $v_B^k=(v_B)_1\dots(v_B)_k$ be the initial segments of $\underline{\widetilde{v}}^T$ and v_B, respectively. By transitivity of $\Sigma(B)$, there exists $\beta\in\Sigma(B)$ which contains non-overlapping occurrences of the strings v_B^k and $\underset{\sim}{\widetilde{v}}^k$ in this order. Let us denote by $(v_B^k * \underset{\sim}{\widetilde{v}}^k)$ a finite substring of β which begins with v_B^k and and terminates with $\underset{\sim}{\widetilde{v}}^k$. Using all the above we obtain that

$$|m(\underline{\theta}^{(n_i)}_{-\infty,N_1}(v_B^k * \underset{\sim}{\widetilde{v}}^k)\underline{\widetilde{\tau}}\widetilde{a}_k \dots \widetilde{a}_0 \dots \widetilde{a}_m a_{m+1} a_{m+2} \dots) - t| < 2^{3-k} < \varepsilon,$$

for any choice of $a_i \in A$ for $i \geq m+1$. This finishes the proof.

$\square$

It is a natural question to ask whether the local dimension of M' is also monotone. More precisely, define the function $d_{\text{loc}}^M : M' \to [0,1]$ by

$$d_{\text{loc}}^M(t) := \lim_{\varepsilon \to 0} \dim_{\text{H}}(M \cap (t-\varepsilon, t+\varepsilon)).$$

By using the explicit construction of Cantor sets in $M \setminus L$ carried out in [17], one can see that the answer is that d_{loc}^M is not monotone. Recall from [17] the two numbers

$$b_\infty = [2; \overline{1,1,2,2,2,1,2}] + [0; \overline{1,2,2,2,1,1,2}] = 3.2930442439\dots$$

and

$$B_\infty = [2; 1, \overline{1,2,2,2,1,2,1,1,2,1,1,2}] + [0; 1,2,2,2,1,1,2,1,2,2,2,$$
$$1,1,2,1,2,2,\overline{1,2,2,2,1,2,1,1,2,1,1,2}] = 3.2930444814\dots.$$

We have that (b_∞, B_∞) is a maximal gap of L, hence $D(t) = \dim_{\text{H}}(k^{-1}(-\infty, t))$ satisfies $D(b_\infty) = D(B_\infty)$, however by the explicit description of the Cantor set done in [17] one has that $0 < \dim_{\text{H}}((M \setminus L) \cap (b_\infty, B_\infty)) \leq D(B_\infty) < d(B_\infty) = d(b_\infty)$ which implies that there is $s \leq b_\infty$ such that for all $t \in (M \setminus L) \cap (b_\infty, B_\infty)$ one has $d_{\text{loc}}^M(t) < d(B_\infty)$ but $d_{\text{loc}}^M(s) = d(b_\infty)$.

From the fact that the local dimension of $M' \cap (M \setminus L)$ is equal to $D(t)$ in good intervals, we can obtain an alternative proof of the fact that d_{loc}^M is not monotone and obtain a stronger conclusion:

Proposition 2.12. *Let $[\nu, \mu)$ be a good interval. For any $t \in M' \cap (M \setminus L) \cap [\nu, \mu)$, there is $s < t$ such that $d_{\text{loc}}^M(s) > d_{\text{loc}}^M(t)$.*

Proof. Let $z_1 < t < z_2$ with $z_1, z_2 \in L$ be a maximal gap containing t. Then we have that $D(z_1) = D(z_2)$ and $d_{\text{loc}}^M(t) = D(t) < d(z_2) = d(z_1)$, but on the other hand by using the same proof of Proposition 2.9 one can show that there exists $s \leq z_1$ such that $d_{\text{loc}}^M(s) = d(z_1)$. $\square$

A weaker question could be whether $d_{\text{loc}}^M : M' \to [0,1]$ is a continuous function. Using the fact that $M \setminus L$ is not closed, established in [13], we answer this question negatively. Recall from [13] that there are values $\widetilde{m}_k \in M \setminus L$ and $\widetilde{\ell}_k \in L$ such that $\widetilde{\ell}_k < \widetilde{m}_k < \widetilde{\ell}_{k-1}$ and $\lim_{k\to\infty} \widetilde{m}_k = 1 + \frac{3}{\sqrt{2}} \in L'$.

Theorem 2.13. *The function $d_{\text{loc}}^M : M' \to [0,1]$ is not continuous.*

Proof. Let $t = 1 + \frac{3}{\sqrt{2}} \approx 3.1213\dots$ and notice that it belongs to the good interval $(3.06, 3.1221)$. We claim that $\widetilde{m}_k \in M'$. This implies that $d_{\text{loc}}^M(\widetilde{m}_k) = D(\widetilde{m}_k)$ while $d_{\text{loc}}^M(t) = 2 \cdot D(t)$. Since $\widetilde{m}_k \to t$ and D is continuous, this will finish the proof.

In fact we will exhibit a small Cantor set contained in $M \setminus L$ that contains $\widetilde{m}_k$. For $k \in \mathbb{N}$ let 2_k denote a string of 2s of length k. Recall from [13] that

$$\widetilde{m}_k = m(\overline{w_k} w_k^* w_k 2_{k-2} 1 2_{2k-1} 1 1 \overline{2}) = \lambda_0(\overline{w_k} w_k^* w_k 2_{k-2} 1 2_{2k-1} 1 1 \overline{2}), \quad \widetilde{\ell}_k = m(\overline{w_{k-1}}),$$

where $w_k = 2_{k-2} 1 2_{2k+1} 1 2_{2k-1} 1 2_{k+2}$ and $w_k^* = 2_{k-2} 1 2_{2k+1} 1 2^* 2_{2k-2} 1 2_{k+2}$. Moreover by the proof of [13, Proposition 1], we know that the Markov value is only attained at 0, in fact there is $\delta_k > 0$ such that $\lambda_i(\overline{w_k} w_k^* w_k 2_{k-2} 1 2_{2k-1} 1 1 \overline{2}) \leq \widetilde{m}_k - \delta_k$ for any $i \neq 0$. Let $r_k \in \mathbb{N}^*, r_k \geq 2$ such that $2^{-r_k} < \delta$. In particular we will have that

$$\{\lambda_0(\overline{w_k} w_k^* w_k 2_{k-2} 1 2_{2k-1} 1 1 2_{r_k} \gamma_1 \gamma_2 \dots) : \gamma_i \in \{2211, 11\}, \forall i \geq 1\},$$

is a Cantor set contained in $(M \setminus L) \cap (\widetilde{\ell}_k, \widetilde{\ell}_{k-1})$ (we used that $2_{2k-2}12_{k+2}w_k2_{k-2}12_{2k-1}11$ is even and that $\lambda_0(22^*11) < 3.06$). $\square$

2.4. **Good intervals and Berstein's investigation of $M \setminus L$.**

Here we discuss how our good intervals are related to the intervals previously considered by Berstein in his work on $M \setminus L$ [1].

It is also discussed in Cusick-Flahive [2, Chapter 3], that Berstein was trying to find necessary and sufficient conditions for a Markov value to also be contained in the Lagrange spectrum. Berstein gave a list of conditions [1, Theorems 2-8] (see [2, Chapter 3, Theorem 6 and Theorem 7] for a summary) and constructed 23 intervals (listed in Appendix B) on which the conditions are both necessary and sufficient for a point $m = m(\underline{a})$ to be contained in L.

Berstein provided a list of conditions that must hold for points in $M \setminus L$. We will prove that these conditions also hold for points $m \in (M \setminus L) \cap [\nu, \mu)$ for a good interval $[\nu, \mu)$.

Definition 2.14 (ε-property). *A bi-infinite sequence $\underline{a} = (a_n)_{n\in\mathbb{Z}}$ is said to have the right (or, respectively left) ε-property if there is an integer N such that for any different $\underline{a'} = (a'_n)_{n\in\mathbb{Z}}$ which has $a_n = a'_n$ for all $n \le N$ (respectively, $n \ge N$) we have $m(\underline{a'}) > m(\underline{a}) + \varepsilon$.*

Proposition 2.15. *Let $[\nu, \mu)$ be a good interval. Then for all $t = \lambda_0(\underline{a}) = m(\underline{a}) \in (M \setminus L) \cap [\nu, \mu)$ we have that $\underline{a}$ satisfies the right or left ε-property for some $N \in \mathbb{N}^*$.*

Proof. Consider $t \in [x, y)$ as in Definition 2.8. By Remark 2.7 we have that $t \in (x, y)$. Since $m(\underline{a}) \in M \setminus L$, without loss of generality we can assume that $\underline{a}$ does not connect positively to B by Lemma 2.6. We will prove that in this case, $\underline{a}$ has the right ε-property. By the Definition 2.4, there is $k \in \mathbb{N}$ such that:

$$\forall N \ge k+2, \forall v_B \in \Sigma^+(B) \quad \text{it holds} \quad m(a_{-\infty,N}v_B) \ge t + 2^{-k}. \tag{14}$$

In particular for any continuation $v = v_{N+1}v_{N+2}\dots$ with $v_{N+1} \ne a_{N+1}$ we must have $m(a_{-\infty,N}v) \ge t + 2^{-k}$, otherwise since the interval is good there will be $v_B \in \Sigma^+(B)$ such that $m(a_{-\infty,N}v_B) \le \max\{m(\underline{a}), m(a_{-\infty,N}v)\} < t + 2^{-k}$. $\square$

Although the next proposition was already proved by Berstein ([1, Theorems 2-8]), we will give here a proof for completeness.

Proposition 2.16. *Suppose a bi-infinite sequence $\underline{a}$ satisfies the right or left ε-property for some $N \in \mathbb{N}^*$ and $m(\underline{a}) \in M \setminus L$. Then*

- *(i) $\underline{a}$ is periodic on at least one side,*
- *(ii) if $p = p_1 \dots p_l$ is the period of the previous item, then $\overline{p}$ has the ε-property to the same side as $\underline{a}$ for some $\varepsilon > m(\underline{a}) - m(\overline{p})$,*
- *(iii) the period p is not semi-symmetric,*
- *(iv) $m^{-1}(t)$ is the union of finitely many orbits and any $\underline{a} \in m^{-1}(t)$ satisfies $m(\underline{a}) = \lambda_i(\underline{a})$ for some i.*

Proof.

- (i) This is [2, Chapter 3, Lemma 3], which states that if $\underline{a}$ has the ε-property on one side, then it will be periodic on that respective side.
- (ii) It is not difficult to see that in fact this statement is equivalent to $\underline{a}$ have the ε-property.
- (iii) Suppose $\underline{a}$ has the right ε-property and write $\underline{a} = \dots a_{n-1}a_n\overline{p}$ with $a_n \ne p_l$ (since $m(\underline{a}) \notin L$). If $p = rs$ where $r^T = r$ and $s^T = s$ are palindromes, then we would have that $\dots a_{n-1}a_np^kra_na_{n-1}\dots$ has arbitrarily close Markov value with $\underline{a}$ if k is large, which contradicts the right ε-property for $\underline{a}$.

(iv) Suppose there is an infinite sequence $\underline{a}^{(i)} \in \{1, \dots, \lfloor t \rfloor\}^{\mathbb{Z}}$ with all terms distinct such that $m(\underline{a}^{(i)}) = \lambda_0(\underline{a}^{(i)}) = t$. Since the entries are bounded, we can assume that they converge to a limit, that is, there is a sequence $\underline{b}$ such that for any $n \in \mathbb{N}$ there is $i_0 = i_0(n)$ such that $a^{(i)}_{-n,n} = b_{-n,n}$ for all $i \geq i_0$. In particular we have $m(\underline{b}) = \lambda_0(\underline{b}) = t$. By the first item, we can assume that $\underline{b}$ has the right ε-property for some $N \in \mathbb{N}^*$. Fix some $j \geq i_0 = i_0(N_1)$ where $N_1 \geq N$ is such that $2^{-N_1} < \varepsilon$. Since $\underline{a}^{(i)} \neq \underline{b}$ for all i, we must have $a^{(j)}_{N_1+1,\infty} \neq b_{N_1+1,\infty}$, otherwise it will contradict the fact that $\lambda_0(\underline{a}^{(j)}) = \lambda_0(\underline{b})$ but $\underline{a}^{(j)} \neq \underline{b}$. Hence, using that $a^{(j)}_{-N_1,N_1} = b_{-N_1,N_1}$, we have for $i \geq 0$

$$\lambda_i(b_{-\infty,N_1} a^{(j)}_{N_1+1,\infty}) \leq m(\underline{a}^{(j)}) + 2^{-N_1} < t + \varepsilon,$$

while for $i < 0$

$$\lambda_i(b_{-\infty,N_1} a^{(j)}_{N_1+1,\infty}) \leq m(\underline{b}) + 2^{-N_1} < t + \varepsilon,$$

which contradicts the fact that $\underline{b}$ satisfies the right ε-property.

Hence we have finitely many bi-infinite sequences $\underline{a}^{(1)}, \dots, \underline{a}^{(r)}$ such that $m(\underline{a}^{(i)}) = \lambda_0(\underline{a}^{(i)}) = t$. Let $\underline{c} \in \{1, \dots, \lfloor t \rfloor\}^{\mathbb{Z}}$ be any other bi-infinite sequence such that $m(\underline{c}) = t$ but $\lambda_i(\underline{c}) < t$ for all $i \in \mathbb{Z}$. By definition of Markov value there is a subsequence say $n_1 < n_2 < \dots$ such that $\lambda_{n_i}(\underline{c}) \to t$. By compactness this means that, restricting to a further subsequence if necessary , that $\sigma^{n_i}(\underline{c})$ converges to a sequence which has Markov value t and that attains it precisely at the index 0. Since there are only finitely such bi-infinite sequences say it converges to to $\underline{a}^{(1)}$. However, since $\underline{a}^{(1)}$ has the ε-property to one side, the same proof above yields that $\sigma^{n_i}(\underline{c}) = \underline{a}^{(1)}$ for some i, contrary to the hypothesis.

□

Remark 2.17. *As mentioned in the introduction, it is important to note that Berstein's intervals are constructed in a similar way to our good intervals. That is, Berstein's proofs of the necessity and sufficiency of the conditions for $m(\underline{a})$ being in L depend on establishing that one theorem [1, Theorem 2] is true for each such interval. In the proof of this theorem, Berstein considers two distinct continuations of a particular sequence and constructs a third continuation giving rise to a Markov value that is no greater than the values for the original two continuations. We see that this is very reminiscent of our definition of a good interval. However, and importantly so, Berstein is not considering the transitivity of two subshifts related to the interval. These subshifts are crucial to our proofs on local dimension, etc., given above and below.*

2.5. **Properties of $D(t)$ on good intervals.** We finish this section by showing that the main results of [21] also hold on good intervals. Recall that $D(t) = \dim_{\mathrm{H}}(K_t) = \dim_{\mathrm{H}}(k^{-1}(-\infty, t])$.

Proposition 2.18. *Let $[\nu, \mu) \subset \mathbb{R}$ be an interval. If $[\nu, \mu)$ is good, then for all $t \in L' \cap [\nu, \mu)$ we have*

$$\dim_{\mathrm{H}}(k^{-1}(t)) = D(t). \tag{15}$$

Proof. Consider $t \in [x, y)$ as in Definition 2.8. Fix $\eta \in (0, 1)$ so that there is $\delta > 0$ and a complete shift $\Sigma(A)$ such that $\Sigma(A) \subset \Sigma_{t-\delta}$ and $\dim_{\mathrm{H}}(K(A)) > (1 - \eta) \dim_{\mathrm{H}}(K_t)$. As before, since $t \in L'$, for each $k \in \mathbb{N}^*$, there are regular Cantor sets $K(A_1^{(k)}), K(A_2^{(k)})$ defined by iterates of the Gauss map such that

$$\operatorname{diam}\left(t - \left(K(A_1^{(k)}) + K(A_2^{(k)})\right)\right) \leq 2^{3-k},$$

and $K(A_1^{(k)}) + K(A_2^{(k)}) \subset L$. Fix an element $m(\gamma^{(k)}) = \lambda_0(\gamma^{(k)}) \in K(A_1^{(k)}) + K(A_2^{(k)})$ or choose $\gamma^{(k)} \in \Sigma(B)$ with $\lambda_0(\gamma^{(k)}) = m(\gamma^{(k)}) = t$ if $t = x$. By repeating the same construction done in (9), there is a finite word

$$\hat{\theta}^{(k)} = \widetilde{a}_{-k}^{(k)} \dots \widetilde{a}_0^{(k)} \dots \widetilde{a}_k^{(k)} \underline{\tau}^{(n)} (\underline{v}^{l+k} * \widetilde{v}^{l+k})^{(k)} \widetilde{\tau}^{(k)} \gamma_{-k}^{(k)} \dots \gamma_0^{(k)} \dots$$
$$\dots \gamma_k^{(k)} \tau^{(k)} (v^{l+k} * \widetilde{\underline{v}}^{l+k})^{(k)} \widetilde{\underline{\tau}} \widetilde{a}_k^{(k)} \dots \widetilde{a}_0^{(k)} \dots \widetilde{a}_k^{(k)},$$

such that for any choice $a_{|i|} \in A$ for $|i| \geq 1$ one has that

$$|m(\dots a_{-2}a_{-1}\hat{\theta}^{(k)}a_1a_2\dots) - t| < 2^{4-k}. \tag{16}$$

Denote by r_k the length (in the alphabet $\{1, \dots, \lfloor t \rfloor\}$) of the finite word $\hat{\theta}^{(k)}$ and let $s_k = r_1 + \cdots + r_k$. Given $(a_1, a_2, \dots) \in \Sigma^+(A)$ where $a_i \in \{1, \dots, \lfloor t \rfloor\}$, denote $a(k) = a_{s_k!+1} \dots a_{s_{k+1}!}$. For each $z = [0; a_1, a_2, \dots] \in K(A)$, define the map $h : K(A) \to k^{-1}(t)$ by

$$h(z) = [0; a_1, \dots, a_{s_1!}, \hat{\theta}^{(1)}, a(1), \hat{\theta}^{(2)}, a(2), \hat{\theta}^{(3)}, \dots].$$

The fact that $k(h(z)) = t$ comes from (16). This map is clearly injective and continuous. Moreover, for any $\rho > 0$, one has that $|z - z'| = O(|h(z) - h(z')|^{1-\rho})$. To see this, first note that by the bounded distortion property, given any finite word $a_1 \cdots a_n$ in the alphabet $\{1, \dots, T\}$ where $T = \lfloor t \rfloor$, there are positive constants C_T and $\lambda_1(T) < \lambda_2(T) < 1$ all depending only on T such that one has

$$C_T^{-1} \lambda_1^n(T) < \mathsf{s}(a_1 \dots a_n) < C_T \lambda_2^n(T),$$

where

$$\mathsf{s}(a_1 \dots a_n) = \left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{1}{q_n(q_n + q_{n-1})},$$

and $[0; a_1, a_2, \dots, a_n] = p_n/q_n$.

Now, given $z, z \in K(A)$, let $s+1$ be the first position of the continued fractions of z and z' where z and z' differ and consider k maximal such that $s+1 \leq s_{k+1}!$. In particular, since $z, z' \in I(a_1 \dots a_s)$, we have that $|z - z'| \leq \mathsf{s}(a_1 \dots a_s)$. On the other hand

$$\begin{aligned}
|h(z) - h(z')|^{1-\rho} &\geq C_T^{1-\rho}\, \mathsf{s}(a_1 \dots a_{s_1!} \hat{\theta}^{(1)} a(1) \dots a_s)^{1-\rho} \\
&= C_T^{1-\rho}\, \mathsf{s}(a_1 \dots a_{s_1!} \hat{\theta}^{(1)} a(1) \dots a_s)\, \mathsf{s}(a_1 \dots a_{s_1!} \hat{\theta}^{(1)} a(1) \dots a_s)^{-\rho} \\
&\geq 2^{-3(k+1)} C_T^{1-\rho}\, \mathsf{s}(a_1 \dots a_s)\, \mathsf{s}(\hat{\theta}^{(1)}) \cdots \mathsf{s}(\hat{\theta}^{(k)})\, \mathsf{s}(a_1 \dots a_{s_1!} \hat{\theta}^{(1)} a(1) \dots a_s)^{-\rho} \\
&\geq 2^{-3(k+1)} C_T^{1-\rho}\, \mathsf{s}(a_1 \dots a_s) C_T^{-k} \lambda_1(T)^{s_k} C_T^{-\rho} \lambda_2(T)^{-\rho \cdot s_k!} \\
&\geq \widetilde{C}_{T,\rho}\, \mathsf{s}(a_1 \cdots a_s)
\end{aligned}$$

for some constant $\widetilde{C}_{T,\rho}$ that only depends on $T = \lfloor t \rfloor$ and ρ.

Therefore using this map h one obtains that for any $\rho > 0$

$$\dim_{\mathrm{H}}(k^{-1}(t)) \geq (1-\rho)^{-1} \dim_{\mathrm{H}}(K(A)) \geq (1-\rho)^{-1}(1-\eta) D(t).$$

Letting $\rho \to 0$ and then $\eta \to 0$ shows that $\dim_{\mathrm{H}}(k^{-1}(t)) \geq D(t)$. The other inequality follows from the fact that $D(t) = \dim_{\mathrm{H}}(k^{-1}(-\infty, t]) \geq k^{-1}(t)$. □

We will use a slight variant of [12, Lemma 2.5]. The following lemma states that if we remove one of the parts J of a Markov partition $\mathcal{P}$ of a regular Cantor set, then the maximal invariant set of $\mathcal{P} \setminus \{J\}$ has Hausdorff dimension strictly smaller than K.

Lemma 2.19. *Let $(K, \mathcal{P}, \psi)$ be a C^k-regular cantor set, $k > 1$ and $J \in \mathcal{P}$. Then the maximal invariant set*

$$C := \bigcap_{n \geq 0} \psi^{-n} \left(\bigcup_{I \in \mathcal{P}, I \neq J} I \right)$$

satisfies $\dim_{\mathrm{H}}(C) < \dim_{\mathrm{H}}(K)$.

Proposition 2.20. *Let $[\nu, \mu) \subset \mathbb{R}$ be a good interval. If $s_1 < t < s_2$ where $t \in L' \cap [\nu, \mu)$, then*

$$D(s_1) < D(s_2). \tag{17}$$

Proof. Consider $t \in [x, y)$ as in Definition 2.8. First, we will show that for any $t \in [x, y)$ and $\varepsilon > 0$, any sequence $\gamma \in \Sigma_{t-\varepsilon}$ is nonwandering in Σ_t. Assume $n \in \mathbb{N}$ is so big so that $t - \varepsilon + 2^{-n} < \min\{y, t - \varepsilon/2\}$. Consider the finite set of words

$$F(n, t-\varepsilon) = \{a_1 \dots a_{2n+1} \in \{1, \dots, \lfloor t \rfloor\}^{2n+1} : \lambda_0(a_1 \dots a_n a_{n+1}^* a_{n+2} \dots a_{2n+1}) > t - \varepsilon\}.$$

Clearly, one has that $\Sigma_{t-\varepsilon} \subset \Sigma(F) \subset \Sigma_{t-\varepsilon+2^{-n}}$ where $\Sigma(F)$ is the subshift of finite type that forbids all the words in $F(n, t-\varepsilon)$. As explained in [11, Chapters 1 and 5], to a subshift of finite type we can associate a transition matrix M that encodes the subshift. Let $\Sigma_{M_1}, \dots, \Sigma_{M_r} \subset \Sigma_M$ be the irreducible components of $\Sigma_M = \Sigma(F)$.

Since each Σ_{M_i} is a transitive subshift and the interval $[x, y)$ is good, we have that any $\gamma \in \Sigma_{M_i}$ connects positively and negatively with B before $t - \varepsilon/2$, unless Σ_{M_i} is trivial, in which case γ is a periodic orbit, which is clearly nonwandering. By [11, Observation 5.1.2], we know that for any $\gamma \in \Sigma(F)$ its positive and negative limit sets are each contained in an irreducible component of Σ_M. In particular, this implies that for any sufficiently large N, there is a finite sequence τ and a continuation $\upsilon \in \Sigma^+(B)$ such that $m(\gamma_{-\infty,N}\tau\upsilon) = m(\dots\gamma_{N-1}\gamma_N\tau\upsilon) < t - \varepsilon/4$. Similarly, there is a finite sequence $\tilde{\tau}$ and a continuation $\tilde{\upsilon} \in \Sigma^+(B)$ such that

$$m(\tilde{\upsilon}^T\tilde{\tau}^T\gamma_{-N}\dots\gamma_N\tau\upsilon) < t. \tag{18}$$

Since $\Sigma(B)$ is transitive this implies that γ is nonwandering in Σ_t.

Now we consider the subshift of finite type $\Sigma(\tilde{F})$ that forbids all words in $F(n, t)$. In particular $\Sigma_t \subset \Sigma(\tilde{F}) \subset \Sigma_{t+\varepsilon}$. By repeating the above construction, there is a transition matrix $\tilde{M}$ for this subshift $\Sigma(\tilde{F}) = \Sigma_{\tilde{M}}$ and there are irreducible components $\Sigma_{\tilde{M}_1}, \dots, \Sigma_{\tilde{M}_{\tilde{r}}} \subset \Sigma_{\tilde{M}}$. We claim that all nontrivial components Σ_{M_j} are contained in a single irreducible component of $\Sigma_{\tilde{M}}$, say $\Sigma_{\tilde{M}_i}$. First note that since $\Sigma(B)$ is transitive, it must be contained in a single irreducible component, say $\Sigma_{\tilde{M}_i}$. Now, given any $\gamma \in \Sigma_{M_j}$ with Σ_{M_j} nontrivial, by (18) we see that γ must be in the same transitive component as $\Sigma(B)$. In conclusion we have that $\bigcup_j \Sigma_{M_j} \subset \Sigma_{\tilde{M}_i} \subset \Sigma_{t+\varepsilon}$ where the union is over nontrivial components Σ_{M_j}.

Now let $\gamma \in \Sigma_t$ be such that $m(\gamma) = \lambda_0(\gamma) = t$. Since $t \in L' \cap [x, y)$, we can assume that γ connects positively and negatively to $\Sigma(B)$, so in particular belongs to the same irreducible component as $\Sigma(B)$, so it must belong to $\Sigma_{\tilde{M}_i}$. On the other hand, for some k large we will have that any sequence containing $\gamma_{-k}\dots\gamma_k$ will have Markov value at least $t - 2^{-k+1} > t - \varepsilon/2$. Hence, if we forbid this finite word from the subshift of finite type $\Sigma_{\tilde{M}_i}$, we will get a new subshift of finite type $\Sigma(\hat{F})$ that contains $\bigcup_j \Sigma_{M_j}$. By Lemma 2.19 we must have that $\dim_{\mathrm{H}}(K(\Sigma(\hat{F}))) < \dim_{\mathrm{H}}(K(\Sigma_{\tilde{M}_i}))$.

Finally,

$$\begin{aligned} D(t-\varepsilon) = \dim_{\mathrm{H}}(K_{t-\varepsilon}) &\leq \sup_j \dim_{\mathrm{H}}(K(\Sigma_{\hat{M}_j})) \\ &\leq \dim_{\mathrm{H}}(K(\Sigma(\hat{F})) < \dim_{\mathrm{H}}(K(\Sigma_{\tilde{M}_i})) \leq \dim_{\mathrm{H}}(K_{t+\varepsilon}) = D(t+\varepsilon). \end{aligned}$$

Now take $\varepsilon > 0$ small enough so that $s_1 < t - \varepsilon < t + \varepsilon < s_2$ and use the monotonicity of D. □

3. Construction of good intervals

In this section, we will prove Theorem 1.1 by demonstrating that following is a list of good intervals (in the sense of Definition 2.8). One important feature of the intervals found so far is the following: they contain almost all of the known elements of $M \setminus L$. The only exceptions are the finite set of elements $m(\gamma_k^1)$, $k \in \{1, 2, 3, 4\}$ found in [13]. The only reason that these elements do not feature, is that they lie very close to 3 which is a region of the spectra that we have not yet investigated.

Theorem 3.1. *The following are good intervals (in the sense of Definition 2.8).*

- $[3.0508, 3.1221)$
- $[3.1299, 3.285441)$
- $[3.28603, 3.2872)$
- $[3.29296, 3.29331)$
- $[3.333958, 3.33475)$
- $[3.359, 3.423)$
- $[3.464, 3.84)$
- $[3.87, 3.9306)$
- $[3.9362, 3.943767)$
- $[3.94405, 3.9716)$
- $[3.97995, 3.9857)$
- $[4.5207, 4.5231)$
- $[4.5251, 4.5279)$.

Our method of proof will be to construct two subshifts $\Sigma(B)$ and $\Sigma(C)$ and calculate (or bound) the maximum and minimum Markov values for sequences in $\Sigma(B)$ and $\Sigma(C)$, respectively. Then, given two continuations $\dots a_{-1}a_0a_1\dots a_Nv^i$ of a sequence $\underline{a} = (a_i)_{i\in\mathbb{Z}}$ satisfying the hypotheses of the definition of a good interval, we will find a continuation $\dots a_{-1}a_0a_1\dots a_Nv_B$ with $v_B \in \Sigma^+(B)$ and

$$[0; v^1] > [0; v_B] > [0; v^2].$$

We will then make use of Lemma 2.1 and the argument discussed in Remark 2.5 to bound the Markov value of the continuation $\dots a_{-1}a_0a_1\dots a_Nv_B$ and, in doing so, prove that the continuation v_B works as intended.

We will argue the first few intervals in full detail before dropping some details once the method has become clear to the reader.

Intervals before $t_1 = 3.334384\dots$

Note here that, for $t \in L' \cap [x, y)$ for a good interval $[x, y)$, we will have $d_{\text{loc}}(t) = d(t) < 1$. Moreover, all sequences here will be from $\{1, 2\}^{\mathbb{Z}}$.

3.1. **Interval [3.0508, 3.122183).**

Let $\Sigma(C) = \{\underline{a} \in \{1, 2\}^{\mathbb{Z}}$: 121, 122212 and 212221 are not substrings of $\underline{a}\}$. The minimum word containing 121 is $m(\overline{12^*1}) = 3.162\dots$ (indeed, if $\underline{1212}$ is forbidden then a continued fraction in $\{1, 2\}$ that begins with $[2; 1]$ is minimized when $[2; \overline{1, 1, 2}]$). Note that

$$\lambda_0(1222^*122) \geq 3.1216$$

which is currently below our claimed right endpoint of the interval. We claim that the right endpoint of the interval can be pushed to

$$m(\overline{1222^*12212221}) = 3.122183\dots$$

which corresponds to the minimum Markov value of a word containing 1222122.

Remark 3.2. *The minimum is a palindrome.*

Indeed, since $\lambda_0(1222^*1222) > 3.1257$, $\lambda_0(1222^*12211) > 3.1228$, $\lambda_0(212^*2212) > 3.1248$, $\lambda_0(111222^*12) > 3.1225$ are greater than the above candidate and since 121 is forbidden, the subword 1222122 must extend to 2211222122122. Since $\lambda_0(2211222^*1221221) > 3.1222$ and $\lambda_0(2211222^*12212222) > 3.122187$, this subword must extend to 221122212212221 and using the previous forbidden words it extends to 221122212212221122. Hence it has the form $w_1^t\beta^t 2^*\theta 2\beta w_2$ where $\theta = 1221$ is an even palindrome, $\beta = 221122$ and $w_1, w_2 \in \{1,2\}^{\mathbb{N}}$. Hence, by Lemma 2.2, this is minimized when $w_1 = w_2$ and $[0;\beta, w_1]$ is minimal. Using the fact that 121, 12221222, 122212211, 22112221221221, 221122212212222 are forbidden, a continued fraction with coefficients in $\{1,2\}$ that begins with $[0;2,2,1,1,2,2]$ is minimized when $[0;\overline{2,2,1,1,2,2,2,1,2,2,1,2}]$.

Remark 3.3. *In order to explain the choice of 122212 as a forbidden word, observe that the word* $\overline{1222}$ *is the word with the maximum Markov value over words without 121* ($m(\overline{12^*22}) > 3.1298$)*. So, when building* $\Sigma(C)$*, we need to avoid this word somehow. We do this by forbidding arbitrarily large subsequences of* $\overline{1222}$*. We begin with the word 122212 because it is the first one with dangerous positions. Since* 121 *is forbidden, it extends to 1222122.*

Let $\Sigma(B) = \{\underline{a} \in \{1,2\}^{\mathbb{Z}} \colon 121, 212$ are no substrings of $\underline{a}\}$. We have that if $\underline{b} \in \Sigma(B)$ then $m(\underline{b}) \le m(\overline{1112^*22}) = 3.050816\ldots$ (in fact according to Cusick-Flahive [2, Table 1, Chapter 5], we have a gap $(m(\overline{1112^*22}), m(\overline{12^*2})) = (3.050816\ldots, 3.073\ldots))$.

So we set $[x,y) = [m(\overline{1112^*22}), m(\overline{1222^*12212221})) = [3.050816\ldots, 3.122183\ldots)$.

We will prove that the interval $[x,y)$ is good by using $\Sigma(B)$ and $\Sigma(C)$ for every $m := m(\underline{a}) < y$. That is, we will not require local intervals of transitivity.

Let $m := m(\underline{a}) = \lambda_0(\underline{a}) < y$, then $\underline{a} \in \Sigma(C)$. Suppose that, as in the definition of a good interval, we have two distinct continuations $\ldots a_{-1}a_0a_1\ldots a_N v^i$ of $\underline{a}$ with $m(\ldots a_{-1}a_0a_1\ldots a_N v^i) < y$. Let $v^i = v_1^i v_2^i \ldots$. We may assume that $v_1^1 \ne v_1^2$. Since $\Sigma(C) \subset \{1,2\}^{\mathbb{Z}}$, we may further assume that $v_1^1 = 1$ and $v_1^2 = 2$. We observe that, since 121 and 122212 are both forbidden,

$$[0;v^1] = [0;1,v_2^1,v_3^1,\ldots] \ge [0;1,1,\overline{2,2,2,1,1,1}] =: [0;v_B] \in K(B).$$

So either $v^1 = v_B \in \Sigma^+(B)$ and we had nothing to prove, or since $v_1^2 = 2$ we have

$$[0;v^1] > [0;v_B] > [0;v^2].$$

Hence, by Lemma 2.1, we have

$$\lambda_j(\ldots a_{-1}a_0a_1\ldots a_N v_B) \le \max_{i=1,2} m(\ldots a_{-1}a_0a_1\ldots a_N v^i), \quad j \le N+1,$$

and since for positions $j > N+1$ the value is small because $\lambda_0(112^*2) \le x$, it follows that

$$m(\ldots a_{-1}\alpha_0\ldots a_N v_B) \le \max\{m(\ldots a_{-1}a_0\ldots a_N v^1), m(\ldots a_{-1}a_0\ldots a_N v^2), x\},$$

as required.

Hence the interval $[x,y) = [m(\overline{1112^*22}), m(\overline{1222^*12212221})) = [3.050816\ldots, 3.122183\ldots)$ is good.

3.2. **Interval [3.1299, 3.2854441).**

We will prove that this interval is good by showing that the intervals [3.1299, 3.2811) and [3.2659, 3.2854441) are both good. We will again give full details.

3.2.1. *Interval* $[3.12984\ldots, 3.2811\ldots)$.

Let $\Sigma(C) = \{\underline{a} \in \{1,2\}^{\mathbb{Z}} \colon 1212$ and 2121 are not substrings of $\underline{a}\}$. The minimum Markov value of a bi-infinite sequence containing 2121 (or its transpose) corresponds to $m(\overline{1212212^*1}) = 3.2811\ldots$. In fact, this point is an isolated point between $m(\overline{2^*111}) = 3.2659\ldots$ and $m(\overline{212}212112^*1\overline{2212}) = 3.2812\ldots$. This was proved by Matheus-Moreira-Vytnova [18, Subsection 3.1.5]. Hence we take $y = m(\overline{1212212^*1}) = 3.2811\ldots$.

Let $\Sigma(B) = \{\underline{a} \in \{1,2\}^{\mathbb{Z}} \colon 121$ is not a substring of $\underline{a}\}$. If $\underline{b} \in \Sigma(B)$ then, as discussed in Remark 3.3, $m(\underline{b}) \le m(\overline{12^*22}) = 3.12984\ldots$.

So we set $[x,y) = [m(\overline{12^*22}), m(\overline{1212212^*1})) = [3.12984\ldots, 3.2811\ldots)$.

Let $m := m(\underline{a}) = \lambda_0(\underline{a}) < y$, then $\underline{a} \in \Sigma(C)$. Suppose that, as in the definition of a good interval, we have two distinct continuations $\ldots a_{-1}a_0a_1\ldots a_N v^i$ of $\underline{a}$ with $m(\ldots a_{-1}a_0a_1\ldots a_N v^i) < y$. Let

$v^i = v_1^i v_2^i \dots$. We may assume that $v_1^1 \neq v_1^2$. Since $\Sigma(C) \subset \{1,2\}^{\mathbb{Z}}$, we may further assume that $v_1^1 = 1$ and $v_1^2 = 2$. We observe that, since 1212 and 2121 are both forbidden,

$$[0; v^2] = [0; 2, v_2^2, v_3^2, \dots] \leq [0; 2, 2, \overline{1, 2, 2, 2}] =: [0; v_B] \in K(B).$$

So either $v^2 = v_B \in \Sigma^+(B)$ and we had nothing to prove, or since $v_1^1 = 1$ we have

$$[0; v^1] > [0; v_B] > [0; v^2].$$

Hence, by Lemma 2.1, we have

$$\lambda_j(\dots a_{-1} a_0 a_1 \dots a_N v_B) < \max_{i=1,2} m(\dots a_{-1} a_0 a_1 \dots a_N v^i), \quad j \leq N+1,$$

so, by the argument in Remark 2.5, it follows from that fact that $\lambda_0(22^*\overline{1222}), \lambda_0(2212^*22\overline{1222}) \leq m(\overline{12^*22}) = x$ that

$$m(\dots a_{-1} a_0 a_1 \dots a_N v_B) < \max\{m(\dots a_{-1} a_0 \dots a_N v^1), m(\dots a_{-1} a_0 \dots a_N v^2), x\},$$

as required.

Hence the interval $[x, y) = [m(\overline{12^*22}), m(\overline{1212212^*1})) = [3.12984\dots, 3.2811\dots)$ is good.

3.2.2. *Interval* $[3.2811\dots, 3.28544419\dots)$.

Let $F_{C_0} = \{21212, 212111, 12121, 2121122, 12221211211\}$.

Now consider

$$\Sigma(C_0) = \{\underline{a} \in \{1,2\}^{\mathbb{Z}} : \forall w \in F_{C_0},\ w \text{ and its transpose are not substrings of } \underline{a}\}.$$

Observe that

$$\begin{aligned} \lambda_0(12^*121) &> 3.297 \\ \lambda_0(212^*12) &> 3.4 \\ \lambda_0(212^*111) &> 3.314 \\ \lambda_0(212^*1122) &> 3.2884 \\ \lambda_0(122212^*11211) &> 3.2872 \end{aligned}$$

The right extreme of this region is determined by $m(\overline{12112}^*1222\overline{1121}) = 3.2872978\dots$, which corresponds to the minimum Markov value of a string containing 12221211211. Indeed, one only has to use that 212111 and 2121122 are forbidden.

Let

$$\Sigma(B_0) = \{\underline{a} \in \{1,2\}^{\mathbb{Z}} : 1212, 2121 \text{ are not substrings of } \underline{a}\}$$

Thus we have that $m(\underline{b}) \leq m(\overline{2^*111}) = 3.2659\dots$ for all $\underline{b} \in \Sigma(B_0)$. Indeed, one just need to use that if 2121, 1212 are forbidden then a continued fraction in $\{1,2\}$ that begins with $[2; 1, 1]$ is maximized with $[2; \overline{1, 1, 1, 2}]$ and the fact that $\lambda_0(12^*2) < 3.16$ and $\lambda_0(22^*2) < 3$.

So $\Sigma(B_0)$ and $\Sigma(C_0)$ can be used to analyse the interval $[3.2659\dots, 3.28729\dots)$. We will start our analysis here before specialising since we will also use this preliminary analysis for the interval in the next subsection.

So let $m = m(\underline{a}) = \lambda_0(\underline{a}) < 3.28729\dots$ and suppose that we have two continuations $v^1 = 1v_2^1\dots$ and $v^2 = 2v_2^2\dots$ satisfying the hypothesis of the definition of a good interval. If $a_N = 1$, then since 111212 is forbidden

$$[0; v^1] = [0; 1, v_2^1] \geq [0; 1, \overline{1, 2, 1, 1}] \in K(B_0).$$

So either $v^1 \in \Sigma^+(B_0)$ or, since $\lambda_0(112^*11) < 3.268$, we can use the continuation $1\overline{1211} \in \Sigma^+(B_0)$. So from here on we only need to consider the situation with $a_N = 2$.

If $a_N = 2$ and $a_{N-1} = 2$, then since 2211212 is forbidden we can again show that

$$[0; v^1] \geq [0; 1, \overline{1, 2, 1, 1}] \in K(B_0),$$

so that the continuation $1\overline{1211}$ can be used.

So, going forward, we need only consider the case $a_{N-1}a_N = 12$. At this point, however, if one tries to lower bound $[0; v^1]$ or upper bound $[0; v^2]$ then one is led to continuations that are not contained in $\Sigma^+(B_0)$, and so similar arguments to the above do not follow immediately. Instead, we must further restrict the Cantor sets.

Now consider the $\Sigma(C) = \{\underline{a} \in \Sigma(C_0) : 1222121 \text{ and } 1212221 \text{ is not a substring of } \underline{a}\}$. Note that if a sequence in $\{1, 2\}^{\mathbb{Z}}$ contains the subword 1222121, then this subword must extend to 1222121121 since 21212, 212111 and 2121122 are forbidden. Now observe that

$$\lambda_0(122212^*112122) > 3.2854.$$

In fact, the minimum Markov value of a sequence containing 122212112122 is

$$m(\overline{121}1222212112^*1222\overline{1121}) = 3.28544419\ldots$$

Indeed, since $\lambda_0(122212^*1121221) > 3.2855$ and $\lambda_0(122212^*11212222) > 3.285447$ are greater than the above candidate, if 122212112122 is a subword then it must extend as 12221211212221. Using Lemma 2.2 and the fact that 2121122 and 212111 are forbidden, one confirms that the above is the minimum.

We set $\Sigma(B) = \Sigma(B_0)$.

So, we set $[x, y) = [m(\overline{2^*111}), m(\overline{121}1222212112^*1222\overline{1121})) = [3.2659\ldots, 3.28544419\ldots)$.

Let $m = m(\underline{a}) = \lambda_0(\underline{a}) < y$. By the above discussion, we need only consider the continuations $v^1 = 1\ldots$ and $v^2 = 2\ldots$ with $a_{N-1}a_N = 12$. In such a case, since 1222121 is forbidden we have

$$[0; v^2] = [0; 2, v^2_2, \ldots] \le [0; \overline{2, 2, 1, 2}] \in K(B).$$

So, since $\lambda_0(222^*122) < 3.14$, we can use the continuation $\overline{2212} \in \Sigma^+(B)$.

Hence, the interval $[3.2659\ldots, 3.28544419\ldots)$ is good. Therefore, the larger interval

$$[m(\overline{12^*22}), m(\overline{121}1222212112^*1222\overline{1121})) = [3.12984\ldots, 3.28544419\ldots)$$

is good.

3.3. **Interval [3.28603, 3.28729).**

Consider again the Cantor set $\Sigma(C_0)$ from the previous subsection.

We will set

$$\Sigma(C) = \{\underline{a} \in \Sigma(C_0) : 22212112112 \text{ and } 21121121222 \text{ are not substrings of } \underline{a}\}$$

Observe that

$$\lambda_0(2112112^*1222) > 3.2879$$

The right extreme of this interval is determined by $y = m(\overline{12112}^*1222\overline{1121}) = 3.2872978\ldots$, which corresponds to the minimum Markov value of a word containing 11211212221. Indeed, one only has to use that 212111 and 2121122 are forbidden.

We will set

$$\Sigma(B) = \{\underline{a} \in \Sigma(C_0) : 1121121222, 221112112212112111212,$$
$$2111211221211211212211, 111211121122121121112122122 \text{ and their transposes}$$
$$\text{are not substrings of } \underline{a}\}.$$

Suppose that $\underline{b} \in \Sigma(B)$ satisfies $m(\underline{b}) = \lambda_0(\underline{b})$. Since $\lambda_0(12^*2) < 3.16$, we must have $b_{-1}b_0^*b_1 = 12^*1$. Since $\lambda_0(112^*11) < 3.268$ we can assume that $b_{-1}b_0^*b_1b_2 = 12^*12$. Since 21212, 212111, 2121122, 12121 are forbidden it is forced to 12112*122. Since $\lambda_0(212112^*122) < 3.28588$ we must continue as 112112*122. Since 1121121222 is forbidden we are forced to 112112*1221. Since $\lambda_0(112112^*12212) < 3.2846$, $\lambda_0(1112112^*1221) < 3.28518$, $\lambda_0(112112^*122111) < 3.2856298$,

$\lambda_0(2112112^*1221122) < 3.28598$, we should continue as $2112112^*1221121$. Since 2121122 is forbidden we are forced to $2112112^*12211211$. Since $\lambda_0(22112112^*1221) < 3.28587$ and $\lambda_0(112112112^*12211211) < 3.28599$, we should continue as $212112112^*12211211$. Since 12121 and 1121121222 are forbidden we are forced to $12212112112^*12211211$. Since we have $\lambda_0(12212112112^*122112112) < 3.286015$, $\lambda_0(12212112112^*1221121111) < 3.286026$, we should continue as $12212112112^*1221121112$. Since 22111211221211211212 and 212111 are forbidden in $\Sigma(B)$ we are forced to $12212112112^*122112111211$. Since 2111211221211211212211 is forbidden and $\lambda_0(212212112112^*1221121112112) < 3.28602838$, we should continue as $212212112112^*1221121112111$. Since 11121112112212112112122122 is forbidden we must continue as $1212212112112^*1221121112111$.

Since 12121, 21212, 212111, 2121122, 2221211211 are forbidden, a continued fraction in $\{1, 2\}$ is maximized when

$$[0; 1, 2, 1, 2, 2, 2, 1, 2, 1, 1, 2, 1, 2, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2, 2, 1, 1, \overline{2, 1, 1, 1}]$$

On the other hand, since 21211 is forbidden, a continued fraction in $\{1, 2\}$ that begins with $[0; 1, 1, 1]$ is maximized when $[0; \overline{1, 1, 1, 2}]$.

Therefore if $\underline{b} = \ldots 1212212112112^*1221121112111\ldots$, then $\lambda_0(\underline{b})$ is maximized when

$$m(\overline{1112}1122121121121221211212221211212212112112^*12211\overline{2111}) = 3.2860284\ldots$$

Remark 3.4. *Note that the middle word is a palindrome and the period 2111 is semi-symmetric.*

Hence we set

$$x = m(\overline{1112}1122121121121221211212221211212212112112^*12211\overline{2111}) = 3.2860284\ldots.$$

Let $m = m(\underline{a}) = \lambda_0(\underline{a}) \in< y$. From the arguments of the previous subsection, we need only consider the situation where we have two continuations $v^1 = 1\ldots$ and $v^2 = 2\ldots$ and $a_{N-1}a_N = 12$. In such a case, by considering all of the forbidden words giving $\Sigma(C)$, we get

$$[0; v^2] = [0; 2, v^2_2, \ldots] \leq [0; 2, 2, 1, 2, 1, 1, 2, 1, 2, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2, 2, 1, 1, 2, \overline{1, 1, 1, 2}] \in K(B).$$

So, since $\lambda_0(122212^*112122) < 3.28589$ and $\lambda_0(122212112^*122) < 3.2858$, by arguments similar to the previous subsections, we can use the continuation $221211212212112112122112\overline{1112} \in \Sigma^+(B)$.

An interval containing Freiman's second example

3.4. **Interval [3.29296, 3.29331).**

In [2, Page 44] it is mentioned that the list of Berstein intervals (see Appendix B) does not cover Freiman's second example $\alpha_\infty = m(\overline{2122211}2^*12221121\overline{2}) = 3.293044265\cdots \in M \setminus L$. Here we construct an interval that does.

We set

$$\Sigma(C) = \{\underline{a} \in \{1, 2\}^{\mathbb{Z}} : 21212,\ 212111,\ 12121,\ 2221211221,\ 21222121122 \text{ and their transposes are not substrings of } \underline{a}\}$$

Observe that

$$\lambda_0(22212^*11221) > 3.2935$$
$$\lambda_0(2122212^*1122) > 3.2933$$

The right extreme of the interval can be pushed to 3.29331. Indeed, we claim that the minimum value of a word containing 21222121122 is

$$m(\overline{121221}22212^*1122212112222\overline{2111}) = 3.293315\ldots$$

where we used the forbidden words 111212, 1221121222, 12221211222212, 21212.

Now let

$$\Sigma(B) = \{\underline{a} \in \Sigma(C) : 2211212221 \text{ and } 1222121122 \text{ are not substrings of } \underline{a}\}$$

Suppose that $\underline{b} \in \Sigma(B_1)$ satisfies $m(\underline{b}) = \lambda_0(\underline{b})$. Since $\lambda_0(12^*2) < 3.16$, we must have $b_{-1}b_0^*b_1 = 12^*1$. Since $\lambda_0(112^*11) < 3.268$ we can assume that $b_{-1}b_0^*b_1b_2 = 12^*12$. Since 21212, 212111, 12121 are forbidden it is forced to 2112^*122. Since $\lambda_0(2112^*1221) < 3.2921$, $\lambda_0(12112^*122) < 3.2891$ and 2211212221, 1221121222 are forbidden in $\Sigma(B)$ we can assume that its equal to 222112^*12222. Now we use the fact if 12121, 212111, 2221211221 are forbidden a continued fraction that begins with $[0; 1, 2, 2, 2, 2]$ is maximized with $[0; 1, 2, \overline{2, 2, 2, 2, 1, 2, 1, 1}]$ and similarly a continued fraction that begins with $[0; 1, 1, 2, 2, 2]$ is maximized with $[0; \overline{1, 1, 2, 2, 2, 2, 1, 2}]$. This shows that

$$m(\underline{b}) \leq m(\overline{21222211}2^*12222212\overline{11222212}) = 3.292954\ldots$$

In fact this Markov value corresponds to the left endpoint of a gap. Recall the numbers

$$b_\infty = [2; \overline{1, 1, 2, 2, 2, 1, 2}] + [0; \overline{1, 2, 2, 2, 1, 1, 2}] = 3.2930442439\ldots$$

and

$$\begin{aligned} B_\infty = {}&[2; 1, \overline{1, 2, 2, 2, 1, 2, 1, 1, 2, 1, 1, 2}] + [0; 1, 2, 2, 2, 1, 1, 2, 1, 2, 2, 2, \\ &1, 1, 2, 1, 2, 2, \overline{1, 2, 2, 2, 1, 2, 1, 1, 2, 1, 1, 2}] = 3.2930444814\ldots. \end{aligned}$$

discussed in Section 2. It can be demonstrated that the minimum Markov value of a word containing 2211212221 is

$$m(\overline{2^*112221}) = 3.2930442439\cdots = b_\infty.$$

Indeed, it follows from [17, Lemma 3.7] and [17, Lemma 3.8] that any bi-infinite sequence $\underline{c} \in \{1, 2\}^{\mathbb{Z}}$ containing 2211212221 with $m(\underline{c}) < B_\infty = 3.2930444814\ldots$ must contain $221\overline{2112221}$. Now since

$$\lambda_0(122121122212^*1122212112) > b_\infty$$
$$\lambda_0(2222121122212^*112221211222) > b_\infty$$

this subsequence must extend to $12221\overline{2112221}$. Applying [17, Lemma 3.7] inductively yields that $\underline{c} = \overline{2112221} = b_\infty$ (in particular b_∞ is isolated). So there is a gap between $3.292954\ldots$ and b_∞.

So we set

$$\begin{aligned} [x, y) &= [m(\overline{21222211}2^*12222212\overline{11222212}), m(\overline{12122}122212^*1122212112222\overline{2111})) \\ &= [3.292954\ldots, 3.293315\ldots). \end{aligned}$$

Let $m = m(\underline{a}) = \lambda_0(\underline{a}) < y$, and suppose that we have two continuations $v^1 = 1\ldots$ and $v^2 = 2\ldots$. If $a_N = 1$, since 111212 is forbidden, we have

$$[0; v^1] \geq [0; \overline{1, 1, 2, 1}] \in K(B)$$

so since $\lambda_0(112^*11) < 3.269$ we have the continuation $\overline{1121} \in \Sigma^+(B)$. So we need only consider the case $a_N = 2$.

If $a_{N-1} = 1$, then since 12121, 21212, 212111 and 21222121122 are forbidden, we can choose the continuation $\overline{112122212112}$ which is allowed because $\lambda_0(12112^*122) < 3.28907$.

So we need only consider the case where $a_{N-1}a_N = 22$. In such a case, since 21212, 212111, 2221211221 are forbidden, we have

$$[0; v^2] = [0; 2, v_2^2, \ldots] \leq [0; \overline{2, 2, 1, 2, 1, 1, 2, 2}] \in K(B),$$

and since

$$\begin{aligned}\lambda_0(\dots a_{N-1}a_N\overline{2212^*1122}) &= [2;\overline{1,1,2,2,2,2,1,2}]+[0;1,2,2,2,2,\dots]\\ &\le [2;\overline{1,1,2,2,2,2,1,2}]+[0;1,2,\overline{2,2,2,2,1,2,1,1}]\\ &= m(\overline{21222211}2^*12222212\overline{11222212}) = x < m\end{aligned}$$

we can use the continuation $\overline{22121122}\in\Sigma^+(B)$.

An interval containing t_1

Since $t_1 = \min\{t\in\mathbb{R} : d(t)=1\}$, we see that d_{loc} starts below 1 in the interval below, but reaches 1 by the end of the interval.

Remark 3.5. *We want to highlight that we found this good interval by constructing two transitive subshifts that work for one of Berstein's intervals.*

3.5. **Interval [3.333958, 3.33475).**

We set

$$\Sigma(C) = \{\underline{a}\in\{1,2\}^{\mathbb{Z}} : 21212, 21112121, 111112121, 211121222, 12111212,\\ 111212111, 21122121112 \text{ and their transposes are not substrings of } \underline{a}\}.$$

Observe that

$$\begin{aligned}\lambda_0(212^*12) &> 3.4\\ \lambda_0(21112^*121) &> 3.35\\ \lambda_0(111112^*121) &> 3.337\\ \lambda_0(21112^*1222) &> 3.337\\ \lambda_0(121112^*12) &> 3.335\\ \lambda_0(1112^*12111) &> 3.335\\ \lambda_0(2112212^*1112) &> 3.3347\end{aligned}$$

The minimum Markov value of a bi-infinite sequence containing 21122121112 is

$$m(\overline{121111121222}2211121221112122112212^*111221211122\overline{222121111121}) = 3.3347525256\dots$$

where we used that 21212, 111112121, 21211121, 222121112, 1122121112221 are forbidden, as well as Lemma 2.2. So we set $y = m(\overline{121111121222}2211121221112122112212^*111221211122\overline{222121111121}) = 3.3347525256\dots$.

Remark 3.6. *The minimum is a palindrome and the period is semi-symmetric.*

Set

$$\Sigma(B) = \{\underline{a}\in\Sigma(C) : 121111212112 \text{ and its transpose is no substrings of } \underline{a}\}.$$

Suppose that $\underline{b}\in\Sigma(B)$ satisfies $m(\underline{b}) = \lambda_0(\underline{b})$. Since $\lambda_0(12^*2) < 3.16$, we must have $b_{-1}b_0^*b_1 = 12^*1$. Since $\lambda_0(112^*11) < 3.268$ and 21212 is forbidden we can assume that $b_{-2}b_{-1}b_0^*b_1b_2 = 112^*12$. Since $\lambda_0(2112^*12) < 3.32$ we must continue as 1112^*12. If it continues as 1112^*121, then since 21112121, 111112121, 21212, 111212111 are forbidden and $\lambda_0(1211112^*12112) < 3.33383$ we must continue as 1211112^*12112, which is forbidden in $\Sigma(B)$. Hence we must continue as 1112^*122. Since $\lambda_0(11112^*122) < 3.329$ and 211121222 is forbidden we must continue as 21112^*1221. Since 12111212, 2111212211 are forbidden we must continue as 221112^*12212. Hence we have that

$$\lambda_0(221112^*12212) \le m(\overline{211111212221}112^*1221221211\overline{122212111112}) = 3.333958\dots$$

We set x to be this value.

Let $m = m(\underline{a}) = \lambda_0(\underline{a}) < y$ and suppose that we have two continuations $v^1 = 1\ldots$ and $v^2 = 2\ldots$. If $a_N = 2$, then since 21212, 222121112 and 111112121 are forbidden, we have

$$[0; v^2] = [0; 2, v^2_2, \ldots] \leq [0; \overline{2, 2, 1, 2, 1, 1, 1, 1, 1, 2, 1, 2}] \in K(B),$$

so that, since $\lambda_0(2212^*1111) < 3.3282$, we can choose the continuation $\overline{221211111212} \in \Sigma^+(B)$.

So we need only consider the case where $a_N = 1$. If $a_{N-1} = 2$, then, since 21212, 21211121, 22212112 and 111112121 are forbidden, we have

$$[0; v^2] = [0; 2, v^2_2, \ldots] \leq [0; 2, 2, 1, 2, 1, 1, \overline{1, 2, 2, 2, 1, 2, 1, 1, 1, 1, 1, 2}] \in K(B)$$

so that we can take the continuation $221211\overline{122212111112} \in \Sigma^+(B)$, which is allowed because $\lambda_0(2212^*1111) < 3.3282$ and

$$\lambda_0(a_{N-1}a_N 2212^*11\overline{122212111112}) \leq m(\overline{211111212221}11212212212^*11\overline{122212111112}) = 3.333958\ldots.$$

So, we are left considering the case of $a_{N-1}a_N = 11$. If $a_{N-2} = 2$, then since 21212, 21122121112 and 111112121 are forbidden, we have

$$[0; v^2] = [0; 2, v^2_2, \ldots] \leq [0; \overline{2, 2, 1, 2, 1, 1, 1, 1, 1, 2, 1, 2}] \in K(B)$$

and so, since $\lambda_0(2212^*1111) < 3.3282$, we have the continuation $\overline{221211111212} \in \Sigma^+(B)$. If $a_{N-2} = 1$, then since 111112121, 21212 and 222121112 are forbidden, we have

$$[0; v^1] = [0; 1, v^1_2] \geq [0; \overline{1, 1, 2, 1, 2, 2, 2, 1, 2, 1, 1, 1}] \in K(B)$$

and, since $\lambda_0(2212^*1111) < 3.3282$, we can use the continuation $\overline{112122212111} \in \Sigma^+(B)$.

An interval between t_1 and $\sqrt{12} = 3.4641\ldots$

Recall that $t_1 = \min\{t \in \mathbb{R} : d(t) = 1\}$, so now that we are above t_1 we have that $d_{loc}(t) = 1$ for all $t \in L' \cap [\nu, \mu)$ where $[\nu, \mu)$ is a good interval. We are still in the situation where all of our Cantor sets will correspond to subshifts of $\{1, 2\}^{\mathbb{Z}}$ since we remain below $\sqrt{12}$.

3.6. **Interval [3.359, 3.423).**

Consider $\Sigma(C) = \{\underline{a} \in \{1, 2\}^{\mathbb{Z}} : 121212, 212121 \text{ are not substrings of } \underline{a}\}$.

The minimum Markov value of a bi-infinite sequence containing 121212 is

$$m(\overline{121212212^*121}) = 3.42339101\ldots$$

This Markov value was calculated using the algorithm discussed in Appendix A.

Remark 3.7. *The period is a palindrome.*

Indeed, since $\lambda_0(1212^*121) > 3.44$, $\lambda_0(1212^*1222) > 3.427$, $\lambda_0(1212^*12211) > 3.424$, $\lambda_0(1212^*122122) > 3.4236$, $\lambda_0(1212^*1221211) > 3.4234$, $\lambda_0(1212^*1221212) > 3.423397$ are all greater than the above candidate the subword must extend to the word 121212212121 which has the form $\beta^T 2\theta 2\beta$ with $\beta = 121$ and the even palindrome $\theta = 1221$. Hence by Lemma 2.2 and the forbidden words 1212121, 12121222, 121212211, 1212122122, 12121221211, 121212212122, the minimum is attained at the above candidate. So we can set $y = m(\overline{121212212^*121}) = 3.42339101\ldots$.

Let $\Sigma(B) = \{\underline{a} \in \{1, 2\}^{\mathbb{Z}} : 21212 \text{ is not a substring of } \underline{a}\}$. From Cusick-Flahive [2, Chapter 5, Table 1], we know that $m(\underline{b}) \leq m(\overline{1112^*12}) = 3.35871\ldots$ for all $\underline{b} \in \Sigma(B)$. So we set $x = m(\overline{1112^*12}) = 3.35871\ldots$.

Suppose there are two continuations $v^1 = 1\ldots$ and $v^2 = 2\ldots$. Since, 121212 is forbidden, we have

$$[0; v^1] = [0; 1, v^1_2, \ldots] \geq [0; \overline{1, 1, 2, 1, 2, 1}] \in K(B),$$

so that, since $\lambda_0(112^*121) \leq x$, we can use the continuation $\overline{112121} \in \Sigma^+(B)$.

Intervals after $\sqrt{12} = 3.464\ldots$

Since we are now above $\sqrt{12}$, our shifts will be subshifts of $\{1,2,3\}^{\mathbb{Z}}$ not contained in $\{1,2\}^{\mathbb{Z}}$; that is, our sequences must now contain 3s.

3.7. **Interval $[\sqrt{12}, 3.8568)$.**

This interval was done in the paper [19] but we will give the details here for completeness. The subshifts used are $\Sigma(C) = \{\underline{a} \in \{1,2,3\}^{\mathbb{Z}} : 13, 31$ are no substrings of $\underline{a}\}$ and $\Sigma(B) = \{1,2\}^{\mathbb{Z}}$. The right extreme of the interval corresponds to the minimum Markov value of a bi-infinite sequence containing 13, namely $m(\overline{3^*113}) = 3.8465462\ldots$. In fact this Markov value is isolated: note that the words 131, 132, 312, 313, 1332, 1333, 3111, 1133112 all produce Markov values greater than 3.8488 so if a bi-infinite sequence contains 13 then it must be equal to $\overline{1331}$ and on the other hand the maximal Markov value of a bi-infinite sequence that does not contain 13 is $m(\overline{2}11223223^*232232\overline{332322}) = 3.84654305\ldots$.

The minimum Markov value of a bi-infinite sequence containing 1133112 is

$$m(\overline{2}113^*31\overline{12}) = 3.856886\ldots$$

where we used that 1113 and 312 are forbidden, as well as Lemma 2.2.

The minimum Markov value of a bi-infinite sequence containing 1332 is

$$m(\overline{12}23233113^*3232\overline{21}) = 3.856958\ldots$$

Indeed, since the words 312, 313, 1113, 21132, 131, 231, 33311, 13311332 all produce larger Markov values than the above candidate, the subword 1332 must extend to 23311332. Now use Lemma 2.2 and the forbidden words 132, 312, 32323.

Remark 3.8. *In both cases the minimum has middle word palindrome and period semi-symmetric.*

If we have two continuations $v^1 = 1\ldots$ and $v^2 = 3\ldots$, then we can choose $v_B = \overline{21}$ since we will have

$$[0; v^1] > [0; v_B] > [0; v^2],$$

so that, since the Markov value must occur at a position of the sequence with a 3 and $v_B \in \{1,2\}^{\mathbb{N}}$, v_B is allowed.

For this reason, for the remainder of the section, we need only consider continuations of the form $v^1 = 1\ldots$ and $v^2 = 2\ldots$, or continuations $v^1 = 2\ldots$ and $v^2 = 3\ldots$.

In the former case, since 13 and 31 are forbidden, we will have

$$[0; v^1] \geq [0; 1\overline{1,2}] \in K(B)$$

so the continuation $v_B = 1, \overline{12} \in \Sigma^+(B)$ can be used. In the latter case, we will have

$$[0; v^1] \geq [0; \overline{2,1}] \in K(B)$$

so the continuation $v_B = \overline{21} \in \Sigma^+(B)$ can be used.

3.8. **Interval $[3.873, 3.930691)$.**

Consider $\Sigma(C) = \{\underline{a} \in \{1,2,3\}^{\mathbb{Z}} :$ 132, 231, 312, 213, 313, 131 are no substrings of $\underline{a}\}$. Note that

$$\lambda_0(3^*12) \geq [3; 1, 2, \overline{3,1}] + [0; \overline{3,1}] > 3.95$$
$$\lambda_0(3^*13) \geq [3; 1, 3, \overline{3,1}] + [0; \overline{3,1}] > 4.02$$
$$\lambda_0(13^*1) \geq [3; 1, \overline{1,3}] + [0; 1, \overline{1,3}] > 4.11$$

Assuming that 312, 313 and 131 are forbidden, we have that the least Markov value of a word containing 231 is $m(\overline{12}3^*113\overline{21}) = 3.930691\ldots$.

Remark 3.9. *Note that the middle word 3113 is palindromic.*

Indeed, the Markov value of a word $m(\gamma) = \lambda_0(\gamma) \in (3.9, 4)$ must have the form 23^*1 (since 131 is forbidden). Note that

$$\lambda_0(13^*23) \geq [3; 2, 3, \overline{3, 1}] + [0; 1, \overline{1, 3}] > 3.99$$
$$\lambda_0(13^*22) \geq [3; 2, 2, \overline{3, 1}] + [0; 1, \overline{1, 3}] > 3.967$$

So 23^*1 must continue as 123^*11. Since 312 is forbidden, the existence of the above word implies that it must continue as 123^*11321 so it is $x^t123^*11321y$ for some $x, y \in \{1, 2, 3\}^{\mathbb{N}}$. By the Lemma 2.2 we have that this minimum value is attained when $[0; 2, 1, x] = [0; 2, 1, y]$ is minimum, which implies $x = y = \overline{21}$ since 312 is forbidden.

We can set $y = m(\overline{123^*11321}) = 3.930691\ldots$. In particular we will have that $\Sigma(C) \subset \Sigma_y$. In [19, Table 6], it was rigorously proved that $\dim_{\mathrm{H}}(K(C)) = 0.594179\ldots$, so this shows that $D(3.931) \geq D(y) \geq 0.594179\ldots$.

On the other hand we can use

$$\Sigma(B) = \{\underline{a} \in \{1, 2, 3\}^{\mathbb{Z}} : 13,\ 31 \text{ are no substrings of } \underline{a}\}.$$

We have that $m(\underline{b}) \leq x = m(\overline{32}) = 3.8729\ldots$.

If we have two continuations $v^1 = 1\ldots$ and $v^2 = 2\ldots$, then, since 231 is forbidden, we have

$$[0; v^2] \leq [0; \overline{2, 3}] \in K(B)$$

so the continuation $v_B = \overline{23} \in \Sigma^+(B)$ can be used since $\lambda_0(23^*2) \leq x$. In the case $v^1 = 2\ldots$ and $v^2 = 3\ldots$, since 213 is forbidden, we have

$$[0; v^1] \geq [0; \overline{2, 1}] \in K(B)$$

so we can use the continuation $v_B = \overline{21} \in \Sigma^+(B)$.

3.9. **Interval [3.93616, 3.943767).**

We will build this interval as the union of the intervals [3.9362, 3.9373), (3.93726, 3.94) and (3.93931, 3.943767).

First, consider the subshift

$$\Sigma(C_0) = \{\underline{a} \in \{1, 2, 3\}^{\mathbb{Z}} : 313,\ 131,\ 1322,\ 1323,\ 312,\ 11132 \text{ and their transposes are not substrings of } \underline{a}\}$$

As above, if $v^1 = 2\ldots$ and $v^2 = 3\ldots$, we can use the continuation $v^B = \overline{21}$. So we need only consider the case $v^1 = 1\ldots$ and $v^2 = 2\ldots$. If $a_N \in \{2, 3\}$ then, since 3231 and 2231 are forbidden, we can show that

$$[0; v^2] \leq [0; \overline{2, 3}]$$

so the continuation $v_B = \overline{23}$ can again be used.

So we are left to consider the case $a_N = 1$. Since 312 is forbidden and v^2 begins with 2, we must have $a_{N-1}a_N \in \{21, 11\}$.

At this point, we have

$$[0; v^1] \geq [0; 1, 1, \overline{3, 3, 3, 1, 1, 1}]$$

and

$$[0; v^2] \leq [0; \overline{2, 3, 1, 1}]$$

so we will construct several $\Sigma(C)$ and $\Sigma(B)$ by forbidding longer subwords of these periods.

3.9.1. *Interval [3.93616, 3.9373).*

Consider the subshift

$$\Sigma(C)=\{\underline{a}\in\Sigma(C_0): 23112, 11231, 3111333, 3211133311121 \text{ and their transposes are not substrings of } \underline{a}\}.$$

Observe that

$$\begin{aligned}
\lambda_0(1123^*112)&>3.954\\
\lambda_0(1123^*1133)&>3.944\\
\lambda_0(1123^*1132)&>3.941\\
\lambda_0(2123^*112)&>3.941\\
\lambda_0(31113^*33)&>3.94\\
\lambda_0(32111333^*11121)&>3.9375
\end{aligned}$$

Note that if 23112 is a subword, then it must extend to either 2123112 or 1123112 because 1323, 1322 and 312 are forbidden in $\Sigma(C_0)$. Similarly 11231 must extend to 11231132, 11231133 or 1123112 because 313, 312 and 131 are forbidden in $\Sigma(C_0)$.

We set

$$\Sigma(B)=\{\alpha\in\Sigma(C): 33311121, 321231133, 2212311331, 1211133311122, 2211133311122 \text{ and their transposes are no substrings of } \alpha\}$$

We claim that for all $\underline{b}\in\Sigma(B)$

$$m(\underline{b})\le m(\overline{12}33111332111333^*1112\overline{23})=3.936154\ldots$$

Indeed, let $\underline{b}\in\Sigma(B)$ such that $m(\underline{b})=\lambda_0(\underline{b})$ assumes the maximum of $\Sigma(B)$, so we have either $b_{-1}b_0^*b_1$ is equal to 33^*1 or 23^*1. If it is equal to 33^*1, then since 313 and 312 are forbidden it is forced to 33^*11. Since $\lambda_0(33^*11b_3)<3.9$ for $b_2\in\{2,3\}$ we can assume that it continues as 33^*111. Since $\lambda_0(33^*1111)<3.928$, $\lambda_0(233^*1112)<3.933$, $\lambda_0(233^*11133)<3.9365$ and $\lambda_0(133^*11)<3.923$ we see that we can assume that it continues as 333^*111. Since $\lambda_0(333^*1111)<3.928$, $\lambda_0(333^*11123)<3.9357$ and 3331113, 33311121 are forbidden, we can assume it continues as 333^*11122. Since $\lambda_0(3333^*11122)<3.934$, $\lambda_0(2333^*11122)<3.9349$, $\lambda_0(311333^*11122)<3.9358$, $\lambda_0(211333^*11122)<3.936$ and 213, 313, 3111333 are forbidden, we must continue with either 2111333^*11122 or $\lambda_0(1111333^*11122)\le\lambda_0(\overline{31}111333^*1112\overline{23})<3.9361$ (where we used that 2231 is forbidden). Since 1211133311122 and 2211133311122 are forbidden in $\Sigma(B)$ we can assume the maximum has the form 32111333^*11122. Finally, we use that if 11132, 131, 213, 313, 3111333 are forbidden then a continued fraction in $\{1,2,3\}$ that begins with $[3;3,3,1,1,1,2,3]$ is maximized with $[3;3,3,1,1,1,2,3,3,1,1,1,3,3,\overline{2,1}]$ and similarly if 2231 is forbidden a continued fraction in $\{1,2,3\}$ that begins with $[0;1,1,1,2,2]$ is maximized with $[0;1,1,1,2,\overline{2,3}]$.

Now assume that $b_{-1}b_0^*b_1=23^*1$. Since 2231, 3231, 312, 313, 23112, 23111, 11231 are forbidden, we have that 23^*1 is forced to 2123^*113. Since $\lambda_0(2123^*1132)<3.93605$ and 131 is forbidden we must continue as 2123^*1133. Since $\lambda_0(12123^*1133)<3.9343$ and 321231133, 2212311331 are forbidden, and moreover $\lambda_0(22123^*11333)<3.9359$, $\lambda_0(22123^*11332)<3.9361$, we see that this continuations leads to smaller Markov values.

Recall that we have $v^1=1\ldots$, $v^2=2\ldots$, and $a_{N-1}a_N\in\{21,11\}$. If $a_{N-1}=1$, since 11231 and 3231 are forbidden, we have

$$[0;v^2]\le[0;\overline{2,3}]$$

and the continuation $v_B=\overline{23}$ can again be used.

So we have $a_{N-1}a_N = 21$. Since 131, 313, 312, 23111, 23112 and 3111333 are forbidden, we have

$$[0; v^2] \leq [0; 2, 3, 1, 1, 3, 3, 1, 1, 1, 3, 3, \overline{2, 1}]$$

and, since $\lambda_0(a_{N-2}2123^*1133111332\overline{1}) < 3.3936$ when $a_{N-2} \in \{1, 2\}$, we can use the continuation $v_B = 231133111332\overline{1}$ if $a_{N-2} \neq 3$. Otherwise, since 11132, 313, 312, 3331113, 3211133311121 and 2231 are forbidden, we have

$$[0; v^1] \geq [0; 1, 1, 3, 3, 3, 1, 1, 1, 2, \overline{2, 3}]$$

so that, since $\lambda_0(2111333^*11122) < 3.9362$, the continuation $v_B = 11333111\overline{223}$ can be used.

3.9.2. *Interval (3.93726, 3.94).*

Now, consider the subshift

$$\Sigma(C) = \{\underline{a} \in \Sigma(C_0) : 23112, 11231, 3111333 \text{ and their transposes are not substrings of } \underline{a}\}.$$

Observe that

$$\lambda_0(31113^*33) > 3.94$$

Now we set $\Sigma(B)$ so that the above argument works. We choose

$$\Sigma(B) = \{\underline{a} \in \Sigma(C_1) : 33311121, 13331112 \text{ are no substrings of } \underline{a}\}$$

We claim that for all $\underline{b} \in \Sigma(B)$

$$m(\underline{b}) \leq m(\overline{1}2331113311321231133111332\overline{1}) =: x = 3.93726\ldots \tag{19}$$

Indeed, let $\underline{b} \in \Sigma(B)$ such that $m(\underline{b}) = \lambda_0(\underline{b})$ assumes the maximum of $\Sigma(B)$, so we have either $b_{-1}b_0^*b_1$ is equal to 33^*1 or 23^*1. If it is equal to 33^*1, then since 313 and 312 are forbidden it is forced to 33^*11. Since $\lambda_0(33^*11b_3) < 3.9$ for $b_2 \in \{2, 3\}$ we can assume that it continues as 33^*111. Since $\lambda_0(33^*1111) < 3.928$, $\lambda_0(233^*1112) < 3.933$, $\lambda_0(233^*11133) < 3.9365$ and $\lambda_0(133^*11) < 3.923$ we see that we can assume that it continues as 333^*111. Since 3331113 is forbidden, we can assume it continues as 333^*1112. Since $\lambda_0(b_{-3}333^*1112b_5) < 3.935$ for $b_{-3}, b_5 \in \{2, 3\}$ and 13331112 and 33311121 are forbidden, we can see that this continuations are smaller than the above candidate.

Now assume that $b_{-1}b_0^*b_1 = 23^*1$. Since 2231, 3231, 312, 313, 23112, 23111, 11231 are forbidden, we have that 23^*1 is forced to 2123^*113. If 313, 312, 11132, 3111333 are forbidden a continued fraction in $\{1, 2, 3\}$ that begins in $[0; 1, 1, 3]$ is maximized with $[0, 1, 1, 3, 3, 1, 1, 1, 3, 3, \overline{2, 1}]$ and similarly since 313, 312, 23111, 23112, 3111333 are forbidden, a continued fraction in $\{1, 2, 3\}$ that begins with $[0; 2, 1, 2]$ is maximized with $[0; 2, 1, 2, 3, 1, 1, 3, 3, 1, 1, 1, 3, 3, \overline{2, 1}]$.

The continuations can be argued as in the case of the previous interval. However, in the case of $a_{N-1}a_N = 21$ we can now use the continuation $v_B = 231133111332\overline{1}$ for any a_{N-2} as it can be checked that

$$\lambda_0(a_{N-2}2123^*1133111332\overline{1}) \leq x$$

for any $a_{N-2} \in \{1, 2, 3\}$.

3.9.3. *Interval (3.93931, 3.943767).*

Now let

$$\Sigma(C) = \{\underline{a} \in \Sigma(C_0) : 1113331113, 3111333111 \text{ are not substrings of } \underline{a}\}.$$

Observe that

$$\lambda_0(111333^*1113) > 3.9435$$

The minimum Markov value of a word containing 21113331113 is the same as that of a word containing the palindrome 21113331113311133311112 which is

$$m(\overline{1}231132111333^*111331113331112311332\overline{1}) = 3.943767\ldots$$

where we used the forbidden words 11132, 1123112, 11231133, 213, 313. So we can choose $y = 3.943767\ldots$.

Remark 3.10. *The middle word is palindrome an the period is semi-symmetric.*

Now we define $\Sigma(B)$ by

$$\Sigma(B) = \{\underline{a} \in \Sigma(C_1) : 23112, 11231, 3331113 \text{ is no substring of } \underline{a}\}.$$

We claim that $m(\underline{b}) \leq m(\overline{21}11333^*11\overline{12}) = 3.939301\ldots$ for all $\underline{b} \in \Sigma(B)$. Indeed, let $\underline{b} \in \Sigma(B)$ such that $m(\underline{b}) = \lambda_0(\underline{b})$ assumes the maximum of $\Sigma(B)$, so we have either $b_{-1}b_0^*b_1$ is equal to 33^*1 or 23^*1. Since 2231, 3231, 312, 313, 23112, 23111, 11231 are forbidden in $\Sigma(B)$, we have that 23^*1 is forced to $\lambda_0(2123^*113) < 3.937672$. Hence we can assume that $b_{-1}b_0^*b_1 = 33^*1$. Since 312 and 313 are forbidden we are forced to 33^*11. Since $\lambda_0(b_{-2}33^*11) < 3.9396$ for $b_{-2} \in \{1, 2\}$ and $\lambda_0(33^*11b_3) < 3.9$ for $b_3 \in \{2, 3\}$, we can assume that it continues as 333^*111. Since $\lambda_0(333^*1111) < 3.928$ and 3331113 is forbidden, we can assume that it continues as 333^*1112. Since $\lambda_0(b_{-2}333^*1112) < 3.9385$ for $b_{-2} \in \{2, 3\}$ and since 312 and 313 are forbidden, we can assume that it continues as 11333^*1112. Since 312 is forbidden, a continued fraction in $\{1, 2, 3\}$ that begins with $[0; 1, 1, 1, 2]$ is maximized with $[0; 1, 1, \overline{1, 2}]$. Since $\lambda_0(b_{-5}11333^*11\overline{12}) < 3.939$ for $b_{-5} \in \{2, 3\}$, we can assume that it continues as 111333^*1112. Finally since 3331113 and 312 are forbidden, a continued fraction in $\{1, 2, 3\}$ that begins with $[3; 3, 3, 1, 1, 1]$ is maximized with $[3; 3, 3, 1, 1, \overline{1, 2}]$.

Recall that we need only consider continuations of the form $v^1 = 1\ldots, v^2 = 2\ldots$, with $a_{N-1}a_N \in \{21, 11\}$. Therefore, since 131, 11132, 313, 312 and 1113331113 are forbidden, we have

$$[0; v^1] \geq [0; 1, 1, 3, 3, 3, 1, 1, \overline{1, 2}]$$

and, since $\lambda_0(11333^*111212) < 3.9394$ and $\lambda_0(21113^*3311\overline{12}) < 3.93987$, we can use the continuation $v_B = 11333111\overline{2}$.

3.10. **Interval [3.94405, 3.971606).**

Before specialising to this interval, we consider subshifts for the region [3.94405, 3.9857).

Consider the subshift $\Sigma(C_0)$ defined by the forbidden subwords

$$131, 313, 2132, 111322, 1323, 1213, 33312, 23312, 211132, 311132, 1123111, 223112.$$

Observe that

$$\begin{aligned} \lambda_0(213^*2) &> 4.05 & \lambda_0(1123^*111) &> 3.987 \\ \lambda_0(1113^*22) &> 4.01 & \lambda_0(21113^*21) &> 3.987 \\ \lambda_0(223^*1123) &> 3.998 & \lambda_0(223^*112) &> 3.9855 \\ \lambda_0(333^*12) &> 3.996 & \lambda_0(1213^*3) &> 3.982 \\ \lambda_0(31113^*2) &> 3.996 & \lambda_0(233^*12) &> 3.984 \end{aligned}$$

The word that (allegedly) is determining the right extreme of the region is 12133, but we will see that in fact it is 23312.

We choose

$$\Sigma(B_0) = \{\alpha \in \Sigma(C_0) : 312, 23112, 11231 \text{ and their transposes are no substrings of } \alpha\} \tag{20}$$

We claim that $m(\underline{b}) \leq m(\overline{111333^*}) = 3.94405\ldots$ for all $\underline{b} \in \Sigma(B_0)$. Indeed, let $\underline{b} \in \Sigma(B_0)$ such that $m(\underline{b}) = \lambda_0(\underline{b})$ assumes the maximum of $\Sigma(B_0)$, so we have either $b_{-1}b_0^*b_1$ is equal to 33^*1 or 23^*1. If $b_{-1}b_0^*b_1 = 23^*1$, then since 312, 313, 2231, 3231 are forbidden it is forced to 123^*11. Since 23111, 23112, 11231, 131, 312 are forbidden in $\Sigma(B_0)$, it is forced to $\lambda_0(2123^*113) < 3.9377$.

If $b_{-1}b_0^*b_1 = 33^*1$ then since 312 and 313 are forbidden in $\Sigma(B_0)$ it must extend to 33^*11. If 131, 311132, 312 are forbidden, a continued fraction in $\{1, 2, 3\}$ that begins with $[3; 1, 1]$ is maximized with $[3;\overline{1,1,1,3,3,3}]$ and similarly a continued fraction in $\{1,2,3\}$ that begins with $[3;3]$ is maximized with $[3;\overline{3,3,1,1,1,3}]$. This proves the claim.

Now that we have chosen $\Sigma(B_0)$ and $\Sigma(C_0)$, we shall consider possible continuations v^1 and v^2 for strings in $\Sigma(C_0)$.

Firstly, suppose that $v^1 = 2\dots$ and $v^2 = 3\dots$. If $a_N = 1$, then since 1213 is forbidden, we have

$$[0;v^1] = [0;2,\dots] \geq [0;\overline{2,1}] \in K(B_0).$$

If $a_N \neq 1$, then since 313, 23312, 33312, 131 and 311132 are forbidden, we have

$$[0;v^2] = [0;3,\dots] \leq [0;3,3,\overline{1,1,1,3,3,3}] \in K(B_0),$$

which is allowed since $\lambda_0(33^*11) \leq m(\overline{333^*111}) = 3.94405\dots$.

So we now consider $v^1 = 1\dots$ and $v^2 = 2\dots$. If $v^2 = 22\dots$ or $v^2 = 21\dots$, then

$$[0;v^2] < [0;\overline{2,3}] \in K(B_0),$$

and so we can assume that $v^2 = 23\dots$. If $v^1 = 13\dots$, $v^1 = 12\dots$, $v^1 = 111\dots$ or $v^1 = 112\dots$, then

$$[0;v^1] \geq [0;1,1,\overline{2,1}] \in K(B_0),$$

and so we can assume that $v^1 = 113\dots$. If v^1 continues as $v^1 = 1133\dots$ then, since 313, 33312, 131 and 311132 are forbidden, we have that

$$[0;v^1] = [0;1,1,3,3,\dots] \geq [0;1,1,\overline{3,3,3,1,1,1}] \in K(B_0)$$

which is allowed by arguments from above. Moreover, if $v^2 = 232\dots$ or $v^2 = 233\dots$, then we have

$$[0;v^2] \leq [0;2,\overline{3,2}] \in K(B_0),$$

which is again allowed.

So in summary, and since 131, 313 and 2312 are forbidden, we must have

$$v^1 = 1132\dots \quad \text{and} \quad v^2 = 2311\dots. \tag{21}$$

Now specialising to the interval (3.94405, 3.971606). Consider the subshift $\Sigma(C)$ defined by the forbidden subwords

$$131, 313, 2132, 1323, 1213, 33312, 23312, 23111, 2231.$$

Observe that

$$\lambda_0(213^*2) > 4.05 \qquad \lambda_0(1213^*3) > 3.982$$
$$\lambda_0(333^*12) > 3.996 \qquad \lambda_0(223^*1) > 3.967$$
$$\lambda_0(233^*12) > 3.984 \qquad \lambda_0(23^*111) > 3.967$$

For this interval we keep $\Sigma(B) = \Sigma(B_0)$ defined in (20).

The minimum Markov value of a word containing 23111 is

$$m(\overline{12}2221111331 23^*111132133111122\overline{21}) = 3.9716067\dots$$

Remark 3.11. *The middle word is palindrome and the period is semi-symmetric.*

Indeed, since 1323, 231112, 231113 are forbidden and $\lambda_0(223^*111) > 4$, we must extend the subword to 1231111. Since $\lambda_0(1123^*111) > 3.987$, $\lambda_0(2123^*111) > 3.974$ are greater than the above candidate and 131, 2312, 33312, 23312 are forbidden, we must extend as 1331231111. Since $\lambda_0(23^*11111) > 3.974$, $\lambda_0(33123^*11112) > 3.973$, $\lambda_0(133123^*111133) > 3.9717$ and 131 is forbidden, we must extend as 133123111132. By the previous arguments we further extend to 1331231111321331. Since 313 is forbidden and $\lambda_0(13312311113^*213312) > 3.9717$ we must extend as 113312311113213311. Since $\lambda_0(2133^*113) > 3.973$, $\lambda_0(2133^*112) > 3.973$ we must extend as 11133123111132133111. Thus the word has the form $x^t\beta^t 3\theta 3\beta y$ where $\theta = 1111$ is an even palindrome, $\beta = 2133111$ and $x, y \in \{1,2\}^{\mathbb{N}}$. Hence by Lemma 2.2 this is minimized when $x = y$ and

$[0;\beta,y]$ is minimal. Using the fact that $\lambda_0(11113213^*311113) > 3.9717$, $\lambda_0(311113213^*3111121) > 3.97164$, $\lambda_0(12311113213^*31111223) > 3.9716069$ and the fact that 22213 is forbidden (use 131, 2132, 21333, 21332, 313 and $\lambda_0(2213^*311) > 3.976$, $\lambda_0(22213312) > 3.974$), we see that $x = y = 122\overline{21}$, which confirms the candidate above.

The minimum value of a word containing 2231 is

$$m(\overline{23^*1132}) = 3.97402\ldots$$

Remark 3.12. *The minimum is a palindrome.*

Indeed, since 313, 2312 are forbidden, $\lambda_0(223^*111) > 4$ and $\lambda_0(223^*112) > 3.985$, we must extend the subword to 223113. Since $\lambda_0(223^*1133) > 3.975$ and since 131, 1323 is forbidden, we must extend the subword to 2231132. Note that if 313, 2312, 23111, 223112 are forbidden, a continued fraction that begins with $[0;2,2]$ is minimized with $[0;\overline{2,2,3,1,1,3}]$. In particular

$$\lambda_0(223^*11321) \geq [3;\overline{2,2,3,1,1,3}] + [0;1,1,3,2,1,\ldots] > [3;\overline{2,2,3,1,1,3}] + [0;1,1,3,2,2,\ldots]$$

so we must extend the subword to 22311322 since 1323 is forbidden. Thus the word has the form $x^t\beta^t 3\theta 3\beta y$ where $\theta = 11$ is an even palindrome, $\beta = 22$ and $x, y \in \{1,2\}^{\mathbb{N}}$. Hence by Lemma 2.2 this is minimized when $x = y$ and $[0;\beta,y]$ is minimal, which proves that indeed the above candidate is the minimum.

Recall that the only continuations that need to be considered at this stage are $v^1 = 1132\ldots$ and $v^2 = 2311\ldots$.

Since 3231 and 22311 are forbidden and $a_N v^2 = a_N 2311\ldots$ we must have $a_N = 1$. However, by looking at $a_{N-1}a_N v^1 = a_{N-1}11132\ldots$ we see that we cannot have these continuations at all since 11132 is forbidden.

3.11. Interval [3.97995, 3.9857).

For this interval we keep $\Sigma(C) = \Sigma(C_0)$ defined in (3.10).

We choose

$$\Sigma(B) = \{\underline{a} \in \Sigma(C) : 23312, 33121, 2231, 23111, 213312 \text{ are no substrings of } \underline{a}\}$$

We claim that $m(\underline{b}) \leq m(\overline{113^*2}) = 3.9799\ldots$ for all $\underline{b} \in \Sigma(B)$. Indeed, let $\underline{b} \in \Sigma(B)$ such that $m(\underline{b}) = \lambda_0(\underline{b})$ assumes the maximum of $\Sigma(B)$, so we have either $b_{-1}b_0^*b_1$ is equal to 33^*1 or 23^*1. As before if $b_{-1}b_0^*b_1 = 33^*1$ extends to 33^*11, then since 131, 311132, 33312 are forbidden we will have that $m(\underline{b}) \leq m(\overline{111333^*}) = 3.944\ldots$. If 33^*1 extends to 33^*12, then since 33312, 23312, 213312, 313 are forbidden it must extend to 1133^*12. Then since 33121 and 1133122 are forbidden in $\Sigma(B)$ it is forced to $\lambda_0(1133^*123) < 3.9786$.

Now assume that $b_{-1}b_0^*b_1 = 23^*1$. Since 313, 2312, 3231, 22311 are forbidden in $\Sigma(B)$, it must continue as 123^*11. Since $\lambda_0(123^*113) < 3.958$ and since 123111 is forbidden, we can assume it continues as 123^*112. If 313, 2312, 31123111 are forbidden, a continued fraction that begins with $[3;1,1,2]$ is maximized with $[3;\overline{1,1,2,3}]$ and similarly if 131, 1323, 3211322 are forbidden, if it begins with $[3;2,1]$ then it is maximized with $[3;\overline{2,1,1,3}]$. This finishes the claim.

Now that we have chosen $\Sigma(B)$ and $\Sigma(C)$, by (21) we have $v^1 = 1132\ldots$ and $v^2 = 2311\ldots$. Since 3231 is forbidden we have $a_N \neq 3$. If $a_N = 1$, then since 311132, 211132, we have $a_{N-1} = 1$. Since 1123111, 313, 2312 are forbidden, we see that v^2 is connecting to $\overline{2311}$ which is allowed because if 131, 1323, 211322 are forbidden a continued fraction that begins with $[3;2,1,1]$ is maximized with $[3;\overline{2,1,1,3}]$, so $\lambda_0(1123^*11\overline{2311}) \leq m(\overline{1123^*}) = 3.97994\ldots$

Now assume $a_N = 2$. Since 1323 and 211322 are forbidden we must continue as $v^1 = 11321\ldots$. Since 131, 1323, 211322 are forbidden we see that v^1 is connecting to $\overline{1132}$ which is allowed because if 313, 2312, 1123111 are forbidden, a continued fraction that begins with $[3;1,1,2]$ is maximized with $[3;\overline{1,1,2,3}]$, so $\lambda_0(2\overline{113^*2}) \leq m(\overline{113^*2}) = 3.97994\ldots$.

The right extreme of the interval can be pushed to 3.9857. We claim that the minimum value of a word containing 23312 is

$$m(\overline{23}3^*12311111321331\overline{223}) = 3.98575\ldots$$

Remark 3.13. *This minimum is very asymmetric in the sense that the middle is not palindrome.*

Note that all the words defining $\Sigma(C)$ but 12133 and 223112 automatically produce larger values than the above candidate. The minimum of 223112 is easy to compute because 1213, 2132, 111322 are forbidden and corresponds to $m(\overline{21}13^*2231\overline{12}) = 3.9874\ldots$. The minimum of the word 12133 necessarily has to extend to 112331211 (otherwise λ_0 is bigger than some candidate containing 12133). Note that $\lambda_0(11213^*31211) > 3.987$ is bigger than the minimum of 23312. Similarly 223112 has to extend to 22311213, which extends to 223112133 by the forbidden words 131 and 2132, however $\lambda_0(223^*112133) > 3.986$ is bigger than the minimum of 23312, so the right endpoint of the interval is determined by 23312.

First note that $\lambda_0(23^*312) < 3.72$. Clearly to minimize 233^*12 to the left, the smallest possible continuation is $\overline{23}3^*12\ldots$ which is contained in the above candidate, so we only have to minimize to the right. Since 233121 and 233122 are forbidden it must extend as 233^*123. Because of the candidate word given above we must continue as 233^*1231. Since 2312 and 131 are forbidden it must continue as $233^*123111$. Since 1231113 and 231112 are forbidden it must continue as $233^*12311111$. Since 131 is forbidden it should continue as $233^*1231111132$. Since 111322, 1323, 1123111, 3111113212 are forbidden this is forced to $233^*123111113213$. Hence it should continue as $233^*12311111321331$ because of the candidate above. Since 313 and 2133121 are forbidden it should continue as $233^*12311111321331223$. If there is 1 to the right, since 313, 2132, 1223112, 223111, 131, 312231132, 312231133 are forbidden then the word would have no extensions (in other words 312231 is forbidden). Thus it must continue as $233^*123111113213312232$, so it is minimized with $\overline{32}$ because 1323 is forbidden.

Remark 3.14. *The minimum of 12133 is the same as the palindrome 12133121 which is*

$$m(\overline{333111}\beta^T3^*3\beta\overline{111333}) = 3.98827021\ldots$$

where

$$\beta = 121132233123112221331132212212113223113221331122311331222.$$

The middle of this word is palindromic and the period is semi–symmetric.

Intervals near to and containing Freiman's constant c_F

In this setting, our subshifts will be subshifts of $\{1,2,3,4\}^{\mathbb{Z}}$. Recall that Freiman's constant $c_F = \lambda_0(\overline{121313}22344^*3211\overline{313121}) = 4.52782956616\ldots$ is the beginning of Hall's ray $[c_F,\infty) \subset L \subset M$.

3.12. **Interval [4.520781, 4.523103).**

Consider the subshift $\Sigma(C)$ defined by the forbidden subwords

$$41, 42, 334, 343, 434, 234, 31313, 131312131312, 231312131312, 331312131312$$

Observe that

$\lambda_0(4^*1) > 4.75$	$\lambda_0(3234^*4323) > 4.5235$
$\lambda_0(4^*2) > 4.56$	$\lambda_0(33131213^*1312) > 4.5228$
$\lambda_0(344^*34) > 4.54$	
$\lambda_0(334^*43) > 4.533$	$\lambda_0(34^*3) > 4.5224$
$\lambda_0(313^*13) > 4.524$	

Observe that if 334 is a subword, since $\lambda_0(334^*3) > 4.56$ and 41, 42 are forbidden, then it must extend to 33443 or 33444 which are both forbidden. Similarly, if 434 is a subword then it must extend to 4434, so $\lambda_0(4^*34)$ is at least $\lambda_0(344^*34) > 4.54$. Is easy to see that if 234 is a subword, then it must extend to 3234^*4323 (otherwise the value is very large) and we can bound by below the minimum of this subword by $\lambda_0(3234^*4323) > 4.5235$.

We choose

$$\Sigma(B) = \{\alpha \in \Sigma(C) : 1312131312 \text{ is no substring of } \alpha\}$$

We claim that $m(\underline{b}) \leq m(\overline{313111313^*121}) = 4.520780004\ldots$ for all $\underline{b} \in \Sigma(B)$. Indeed, let $\underline{b} \in \Sigma(B)$ such that $m(\underline{b}) = \lambda_0(\underline{b})$ assumes the maximum of $\Sigma(B)$, so we have either b_0^* is equal to 3^* or 4^*. First suppose $b_0^* = 4^*$. Since 41, 42, 343 are forbidden, we have to extend to 34^*4. Since 434, 334, 234 are forbidden we must extend to $\lambda_0(134^*4) < 4.52$. Now suppose $b_0^* = 3^*$. Since $\lambda_0(3^*4) < 4.07$, $\lambda_0(3^*3) < 4.15$, $\lambda_0(3^*2) < 4.29$, we must continue as 13^*1. Since $\lambda_0(13^*11) < 4.48$, $\lambda_0(213^*12) < 4.48$ and 31313, 14 are forbidden we must continue as 313^*12. Finally since 41, 31313, 1312131312 are forbidden we have

$$\lambda_0(313^*12) \leq m(\overline{313111313^*121}) = 4.520780004\ldots$$

As in the cases considered above, if we have two continuations $v^1 = v_1^1 v_2^1$ and $v^2 = v_1^2 v_2^2$ with $|v_1^1 - v_1^2| > 1$ (e.g., $v^1 = 1\ldots$ and $v^2 = 3\ldots$) then we can always find a continuation using a $v_B = v_1 v_2 \ldots$ with, say, $v_1^1 > v_1 > v_1^2$.

Hence, we need only consider the cases $v^1 = 1\ldots, v^2 = 2\ldots$, or $v^1 = 2\ldots, v^2 = 3\ldots$, or $v^1 = 3\ldots, v^2 = 4\ldots$.

In the first case, since 24, 14, 31313 and 231312131312 are forbidden, we have

$$[0; v^2] = [0; 2, \ldots] \leq [0; 2, 3, \overline{1, 3, 1, 2, 1, 3, 1, 3, 1, 1, 1, 3}]$$

which is allowed since $\lambda_0(113^*1) < 4.48$ and

$$\lambda_0(11313^*1213) \leq m(\overline{313111313^*121}) = 4.520780004\ldots.$$

In the second case, 42 being forbidden forces $a_N \neq 4$. If $a_N \in \{2, 3\}$, then since 234, 334, 41, 31313 and 331312131312 are forbidden, we have

$$[0; v^2] = [0; 3, \ldots] \leq [0; 3, 3, \overline{1, 3, 1, 2, 1, 3, 1, 3, 1, 1, 1, 3}],$$

which is again allowed. Whereas if $a_N = 1$, since 41, 42, 343, 434, 433, 432, 31313, and 131312131312 are forbidden, we have

$$[0; v^2] = [0; 3, \ldots] \leq [0; 3, 4, 4, 4, 3, 1, \overline{1, 3, 1, 3, 1, 2, 1, 3, 1, 3, 1, 1}$$

which is also allowed. In the final case, since 14 and 24 are forbidden, we have $a_N \in \{3, 4\}$. Hence, since 41, 42, 434, 433, 432, 31313 and 131312131312 are forbidden, we have

$$[0; v^2] = [0; 4, \ldots] \leq [0; 4, 4, 3, 1, \overline{1, 3, 1, 3, 1, 2, 1, 3, 1, 3, 1, 1}]$$

which is again allowed.

The minimum Markov value of a bi-infinite sequence containing 331312131312 is

$$m(\overline{111313121313}213131213133131213^*1312\overline{313121313111}) = 4.5231035\ldots$$

Remark 3.15. *The middle word is palindrome and the period is semi-symmetric.*

The minimum Markov value of a bi-infinite sequence containing 231312131312 is

$$m(\overline{3131213^*1312}) = 4.523119130\ldots$$

Remark 3.16. *This period is **not** semi-symmetric.*

The minimum Markov value of a bi-infinite sequence containing 131312131312 is

$$m(\overline{21313121313213}13121313113131213^*1\overline{312313121313 12}) = 4.52314985\ldots$$

Remark 3.17. *The middle word is palindrome but the period is **not** semi-symmetric.*

The minimum Markov value of a bi-infinite sequence containing 343 is

$$m(\overline{3134^*31}) = \sqrt{82}/2 = 4.52769\dots$$

Remark 3.18. *The period is semi-symmetric.*

Indeed, if 41, 313131, 313132, 4313133, 43131344 are forbidden, then a continued fraction in $\{1,2,3,4\}$ that begins with $[4;3]$ is minimized with $[4;\overline{3,1,3,1,3,4}]$.

The minimum Markov value of a bi-infinite sequence containing 31313 is the same as the minimum of 343 which is

$$m(\overline{4313^*13}) = \sqrt{82}/2 = 4.52769\dots$$

Indeed, since 313131, 313132, 3313133, 3313134 are forbidden the subword 31313 must extend to 4313134. Now since 41, 42 and 43131344 are forbidden it must extend to 343131343. Moreover, since 3434, 3433, 3432, 34311, 34312, 14, 343134, 343133, 343132, 3431311 431313431312 are forbidden this subword extends to 31313431313431313. In other words, we have self-replication so in fact 31313 extends to $\overline{431313}$.

Similarly, if 343 is a subword, then it must extend to 343131343131343, where we used the forbidden words of the previous paragraph plus 21313431312, 31313431312.

3.13. **Interval [4.5251, 4.5279).**

We realise the interval [4.5251, 4.5279) as a union of two good intervals; namely, [4.5251, 4.52769) and (4.52753, 4.5279).

3.13.1. *Interval [4.5251, 4.52769).*

Consider the subshift

$$\Sigma(C) = \{\underline{a} \in \{1,2,3,4\}^{\mathbb{Z}} : 41, 42, 334, 31313 \text{ and their transposes are not subtrings of } \underline{a}\}.$$

We choose

$$\Sigma(B) = \{\underline{a} \in \Sigma(C) : 343, 234, 434 \text{ are no substrings of } \underline{a}\}$$

We claim that $m(\underline{b}) \le m(\overline{13^*1312}) = 4.52509\dots$ for all $\underline{b} \in \Sigma(B)$. Indeed, let $\underline{b} \in \Sigma(B)$ such that $m(\underline{b}) = \lambda_0(\underline{b})$ assumes the maximum of $\Sigma(B)$, so we have either b_0^* is equal to 3^* or 4^*. First suppose $b_0^* = 4^*$. Since 41, 42, 343 are forbidden, we have to extend to 34^*4. Since 434, 334, 234 are forbidden we must extend to $\lambda_0(134^*4) < 4.52$. Now suppose $b_0^* = 3^*$. Since $\lambda_0(3^*4) < 4.07$, $\lambda_0(3^*3) < 4.15$, $\lambda_0(3^*2) < 4.29$, we must continue as 13^*1. Now since 131313 and 14 are forbidden, we see that $\lambda_0(13^*1) \le m(\overline{13^*1312}) = 4.52509\dots$.

As above, we need only consider the cases $v^1 = 1\dots, v^2 = 2\dots$, or $v^1 = 2\dots, v^2 = 3\dots$, or $v^1 = 3\dots, v^2 = 4\dots$.

In the first case, since 41, 14 and 31313 are forbidden, we have

$$[0; v^1] \ge [0; 1, 1, \overline{3, 1, 3, 1, 2, 1}]$$

so, since $\lambda_0(13^*1) \le m(\overline{13^*1312}) = 4.52509\dots$, we can use the continuation $v_B = 11\overline{313121}$.

In the second case, as similar argument shows that

$$[0; v^1] \ge [0; \overline{2, 1, 3, 1, 3, 1}]$$

so that $v_B = \overline{213131}$ can be used.

Finally, when $v^1 = 3\dots$ and $v^2 = 4\dots$, since 14 and 24 are forbidden, we must have $a_N \in \{3,4\}$ and so we can show that

$$[0; v^1] \ge [0; \overline{3, 1, 3, 1, 2, 1}]$$

so that the continuation $v_B = \overline{313121}$ can be used.

3.13.2. *Interval (4.52753, 4.5279).*

Now consider the subshift $\Sigma(C)$ defined by the forbidden subwords

$$41, 42, 334, 313131, 313132, 313133, 4434, 2343, 444321, 444322$$

and their transposes.

Observe that

$$\lambda_0(234^*3) > 4.55$$
$$\lambda_0(313^*131) > 4.542$$
$$\lambda_0(344^*34) > 4.54$$
$$\lambda_0(313^*132) > 4.532$$
$$\lambda_0(444^*321) > 4.53$$
$$\lambda_0(34313^*133) > 4.529$$
$$\lambda_0(444^*322) > 4.5279$$

Note that if 313133 is a subword, then since 41 and 42 and forbidden we have that $\lambda_0(313^*133)$ is at least $\lambda_0(34313^*133) > 4.529$. Similarly if 4434 is a subword, then $\lambda_0(44^*34)$ is at least $\lambda_0(344^*34) > 4.54$.

Now set

$$\Sigma(B) = \{\underline{a} \in \Sigma(C) : 343, 23443 \text{ are no substrings of } \underline{a}\}$$

We claim that $m(\underline{b}) \leq m(\overline{23444^*3}) = 4.52752\ldots$ for all $\underline{b} \in \Sigma(B)$. Indeed, let $\underline{b} \in \Sigma(B)$ such that $m(\underline{b}) = \lambda_0(\underline{b})$ assumes the maximum of $\Sigma(B)$, so we have either b_0^* is equal to 3^* or 4^*. First suppose $b_0^* = 3^*$. Since $\lambda_0(3^*4) < 4.07$, $\lambda_0(3^*3) < 4.15$, $\lambda_0(3^*2) < 4.29$, we must continue as 13^*1. Now since 131313 and 14 are forbidden, we see that $\lambda_0(13^*1) \leq m(\overline{13^*1312}) = 4.52509\ldots$. Now suppose $b_0^* = 4^*$. Since $\lambda_0(44^*4) < 4.48$ and 41, 42, 343 are forbidden, we have to extend to 34^*4. Since 4434, 334 are forbidden and $\lambda_0(134^*4) < 4.52$, we must extend to 234^*4. Since 23443 is forbidden it extends to 234^*44. Finally using that 444321, 444322, 24, 41, 42, 2343, 4434, 433 are forbidden this is maximized with $\lambda_0(234^*44) \leq m(\overline{234^*443}) = 4.5275206\ldots$

The arguments of the previous interval will allow us to only have to consider the continuations $v^1 = 3\ldots$, $v^2 = 4\ldots$, with $a_N \in \{3, 4\}$. If $a_N = 3$, then, since 331313 is now forbidden, we will have

$$[0; v^1] \geq [0; \overline{3, 1, 3, 1, 2, 1}]$$

so the continuation $v_B = \overline{313121}$ can be used. Otherwise, since 41, 42, 4434, 433, 444321, 444322 and 2343 are forbidden, we have

$$[0; v^2] \leq [0; \overline{4, 4, 3, 2, 3, 4}]$$

so that, since

$$\lambda_0(444^*\overline{323444}) \leq m(\overline{444^*323}) = 4.5275206\ldots$$

we can use the continuation $v_B = \overline{443234}$.

Remark 3.19. *Even though this fact is not used here, we note that Freiman's gap (ν, c_F), satisfies*

$$\nu = \lambda_0(\overline{323444}313134^*313121133\overline{313121}) = \lim_{n\to\infty} \lambda_0((\overline{323444})^n 313134^*313121133(\overline{313121})^n 1) \in L'.$$

Indeed, one only needs to use that $m(\overline{3131211}\overline{323444}) = m(\overline{323444^*}) = 4.5275206\cdots < \nu$.

Remark 3.20. *As mentioned at the beginning of this section, we wish to point out again that all but finitely many of the currently known regions of $M \setminus L$ occur inside good intervals. It is an open question as to whether or not there exist elements of $M \setminus L$ close to ν. The fact that one can place ν within a good interval is an interesting observation.*

4.1. **Brief description of the algorithm.**

Here we give a description of the algorithm used in the computer investigations used to find new regions of $M \setminus L$. See also [9].

Let w be a (finite) odd not semi–symmetric word and $j_0 := m(\overline{w})$. By rewriting the period of this Markov value we can write $w^* = \eta_1 3^* \eta_2$ where both η_1, η_2 have the same length and where $m(\overline{w}) = \lambda_0(\overline{w^*})$. The first step is to establish local uniqueness: we want to find a $\varepsilon > 0$ such that if $\lambda_0(b) \in (j_0 - \varepsilon, j_0 + \varepsilon)$ for some $b \in \{1,2,3\}^{\mathbb{Z}}$, then we must have $b = \eta_1 w^* \eta_2$. The next step is to prove self-replication, that is: there is $\varepsilon > \delta > 0$ such that if $\lambda_0(b) \in (j_0 - \delta, j_0 + \delta)$ for some $b \in \{1,2,3\}^{\mathbb{Z}}$, then $b = \eta_1' w w^* w \eta_2'$ where either $\eta_1' = \eta_1$ and η_2' is a prefix of η_2 or $\eta_2' = \eta_2$ and η_1' is a suffix of η_1. In particular we must have that either $b = \overline{w} w^* w \eta_2'$ or $b = \eta_1' w w^* w \overline{w}$. To prove this it is **essential** to just use forbidden words: that is any words of the form $\tau_1 \eta_1 w \eta_2 \tau_2$, where either $\tau_1 \eta_1$ is a suffix of ww and $\eta_2 \tau_2$ minus the last digit is prefix of ww but $\eta_2 \tau_2$ is not, or $\eta_2 \tau_2$ is a prefix of ww and $\tau_1 \eta_1$ minus the first digit is a suffix of ww but $\eta_1 \tau_1$ is not. In particular $\tau_1 \eta_1 w \eta_2 \tau_2$ is not a subword of $wwwww$. This will give us a finite sequence of forbidden words $f_1, f_2, \dots$ such that $\lambda_0(f_i) > j_0 + \varepsilon_i$ with $\varepsilon_i < \varepsilon$. In turns out that there should exists a minimal forbidden word $f_{\min}$ that has the lowest lower bound $\varepsilon_{\min} = \delta$. Next we consider $t_{\min}$, which is the minimum Markov value of a bi-infinite sequence that contains $f_{\min}$. If we have luck, we will have that $t_{\min} \in L$. Otherwise we minimize the next minimal forbidden word until this minimum belong to L. We call $j_1 = j_1(w) \in L$ to be the right border of the maximal gap $(j_0, j_1) \cap L = \varnothing$.

4.2. **Local uniqueness.**

We denote throughout this section $w = 12111233311133232$ and $w^* = 121112333^*11133232$.

Lemma 4.1 (Forbidden words I)**.** *Let $b \in \{1,2,3\}^{\mathbb{Z}}$.*

(i) If $b = 13^*1$, 3^*13, 3^*12, 323^*1, 223^*1, 23^*111, 1123^*112, 2123^*1123, 2123^*1122, 2123^*11211, 2123^*11212, 31123^*1, 21123^*1, then $\lambda_0(b) > j_0 + 10^{-4}$.

(ii) $b = 1333^*1113$, 32333^*11133, 22333^*11133, 3331113^*33211 then $\lambda_0(b) > j_0 + 10^{-5}$.

(iii) If $b = 1123^*1133$, 1123^*11321 then $\lambda_0(b) > j_0 + 10^{-4}$.

Corollary 4.2. *If $m(b) < j_0 + 10^{-4}$, then b does not contain 23112, 11231.*

Lemma 4.3. *Let $b \in \{1,2,3\}^{\mathbb{Z}}$ be such that $\lambda_0(b) < j_0 + 10^{-6}$. Then b or b^T must be of the form:*

(i) $b = 1^*, 2^*$ and $\lambda_0(b) < j_0 - 10^{-1}$.

(ii) $b = 33^*3, 33^*2, 23^*2$ and $\lambda_0(b) < j_0 - 10^{-2}$.

(iii) $b = 2123^*113$ and $\lambda_0(b) < j_0 - 10^{-3}$.

(iv) $b = 133^*11, 233^*11, 333^*112, 333^*113, 333^*1112, 333^*1111$ and $\lambda_0(b) < j_0 - 10^{-3}$.

(v) $b = 3333^*11133$ and $\lambda_0(b) < j_0 - 10^{-4}$.

(vi) $b = 12333^*11133$.

Proof. Lets assume that $b = 3^*$. If b is not of the forms in item (ii), without loss of generality we can assume that $b = 3^*1$. Since 313 and 312 are forbidden we must extend this word to $b = 3^*11$. Since 131 is forbidden we have that b is either $b = 23^*11$ or $b = 33^*11$.

If $b = 23^*11$, then since 23111, 23112, 3231, 2231, 312 and 11231 are forbidden by Corollary 4.2 and Lemma 4.1, it is forced to $b = 2123^*113$ and $\lambda_0(b) < j_0 - 10^{-3}$.

Assume $b = 33^*11$. If b is not of the forms in item (iv), then $b = 333^*1113$. Since 131 and 11132 are forbidden we have $b = 333^*11133$. Since 13331113 is forbidden we can assume that $b = 2333^*11133$. Since 323331113 and 223331113 are forbidden, we must extend to $b = 12333^*11133$. □

By the previous lemma and since 312 is forbidden, it suffices to analyse the extensions to the left of 12333^*11133, i.e. 212333^*11133, 112333^*11133.

4.2.1. *Extensions of the word* 212333*11133.

Lemma 4.4.

(i) *If* $b = 212333^*111331, 212333^*111332$ *then* $\lambda_0(b) < j_0 - 10^{-5}$.
(ii) *If* $b = 3212333^*111333$ *then* $\lambda_0(b) > j_0 + 10^{-6}$.
(iii) *If* $b = 1212333^*1113332, 1212333^*1113333$ *then* $\lambda_0(b) < j_0 - 10^{-6}$.
(iv) *If* $b = 2212333^*111333212$ *then* $\lambda_0(b) > j_0 + 10^{-7}$.

Corollary 4.5. *If* $m(b) < j_0 + 10^{-7}$*, then* b *does not contain 22123331113332.*

Proof. Use the forbidden words 311133322, 311133323, 333111333211, 312 and the previous lemma. □

Since 31113331 is forbidden, by the previous lemma it suffices to analyse the extensions of and 2212333*1113333.

Lemma 4.6. *If* $b = 2212333^*1113333$ *then* $\lambda_0(b) < j_0 - 10^{-8}$.

Proof.
$$\lambda_0(2212333^*1113333) \leq \lambda_0(\overline{13}2212333^*111333333111\overline{13}) < j_0 - 10^{-8}.$$
□

Lemma 4.7.

(i) *If* $b = 32212333^*1113333$ *then* $\lambda_0(b) < j_0 - 10^{-6}$.
(ii) *If* $b = 12212333^*11133333, 12212333^*11133333$ *then* $\lambda_0(b) > j_0 + 10^{-7}$.
(iii) *If* $b = 212212333^*111333$ *then* $\lambda_0(b) > j_0 + 10^{-6}$.
(iv) *If* $b = 1112212333^*111333311, 2112212333^*111333311$ *then* $\lambda_0(b) > j_0 + 10^{-7}$.

Corollary 4.8. *If* $m(b) < j_0 + 10^{-8}$*, then* b *does not contain 12212333111333.*

Proof. If b contains the subword 12212333111333, then using the previous lemma, Corollary 4.5 and the forbidden words 31113331, 312, 313, 23112, 131 this subword must extend to the word 33112212333111333311. Now using that 131 and 11132 are forbidden, we have the inequality
$$\lambda_0(33112212333^*111333311) \geq \lambda_0(\overline{1}333112212333^*11133331113\overline{331}) > j_0 + 10^{-8}.$$
□

Corollary 4.9. *If* $b \in \{1,2,3\}^{\mathbb{Z}}$ *is such that* $b = 212333^*11133$*, then* $|m(b) - j_0| > 10^{-8}$.

4.2.2. *Local uniqueness up to* w^*. Now we continue to analyzing the extensions of the candidate 112333*11133, which is the one that will converge to the non semi-symmetric word we are looking for (the other branches are already discarded).

Lemma 4.10.

(i) *If* $b = 112333^*111333$ *then* $\lambda_0(b) > j_0 + 10^{-5}$.
(ii) *If* $b = 2112333^*1113311, 1112333^*1113311$ *then* $\lambda_0(b) < j_0 - 10^{-6}$.
(iii) *If* $b = 3112333^*11133113, 3112333^*111332, 2112333^*111332$ *then* $\lambda_0(b) > j_0 + 10^{-6}$.
(iv) *If* $b = 33112333^*11133112, 33112333^*11133111$ *then* $\lambda_0(b) < j_0 - 10^{-7}$.

By the previous lemma and since 312, 313, 23112 are forbidden, it suffices to analyse the extensions of 112333*111332.

Lemma 4.11.

(i) *If* $b = 3112333^*111332, 2112333^*111332$ *then* $\lambda_0(b) > j_0 + 10^{-5}$.
(ii) *If* $b = 112333^*1113321$ *then* $\lambda_0(b) > j_0 + 10^{-5}$.
(iii) *If* $b = 1112333^*1113322$ *then* $m(b) > j_0 + 10^{-7}$.

Proof. If $b = 1112333^*1113322$ then $\lambda_0(b) \geq \lambda_0(\overline{13}331112333^*11133221\overline{3}) > j_0 + 10^{-7}$ where we used that 131, 11132 are forbidden. □

By the previous lemma, it suffices to analyse the extensions of $1112333^*1113323$.

Lemma 4.12.

(i) If $b = 11112333^*111332$ then $\lambda_0(b) > j_0 + 10^{-5}$.
(ii) If $b = 31112333^*1113323$ then $\lambda_0(b) < j_0 - 10^{-6}$.
(iii) If $b = 321112333^*111332, 221112333^*111332$ then $m(b) > j_0 + 10^{-6}$.
(iv) If $b = 1121112333^*11133233, 2121112333^*11133233$ then $\lambda_0(b) > j_0 + 10^{-7}$.

Proof. If $b = 221112333^*111332$ then $\lambda(b) \geq \lambda_0(\overline{13}221112333^*11133232\overline{31}) > j_0 + 10^{-6}$ where we used that 3231 is forbidden. □

Corollary 4.13. *If $m(b) < j_0 + 10^{-7}$ then b does not contain 2111233311133233.*

Since 3231 is forbidden we obtain

Corollary 4.14. *Let $b \in \{1,2,3\}^{\mathbb{Z}}$. If $|\lambda_0(b) - j_0| < 10^{-8}$ then $b = w^*$.*

Since 312 is forbidden, it suffices to analyze the extensions of $1w^*$ and $2w^*$.

Lemma 4.15.

(i) If $b = 1w^*1, 1w^*2, 31w^*3, 21w^*3$ then $\lambda_0(b) > j_0 + 10^{-8}$.
(ii) If $b = 111w^*3, 211w^*3$ then $m(b) > j_0 + 10^{-8}$.
(iii) If $b = 311w^*32$ then $\lambda_0(b) < j_0 - 10^{-8}$.
(iv) If $b = 311w^*33$ then $m(b) > j_0 + 10^{-9}$.

Proof. If $b = 211w^*3$, then since 1323 is forbidden $\lambda_0(b) \geq \lambda_0(211w^*32) > j_0 + 10^{-8}$.

If $b = 3311w^*33$, then since 23111 and 131 are forbidden, we will have $\lambda_0(b) \geq \lambda_0(3311w^*33) > j_0 + 10^{-9}$. □

Corollary 4.16. *If $m(b) < j_0 + 10^{-9}$ then b does not contain $1w33$.*

Corollary 4.17. *Let $b \in \{1,2,3\}^{\mathbb{Z}}$. If $b = 1w^*$ then $|\lambda_0(b) - j_0| > 10^{-9}$.*

Lemma 4.18.

(i) If $b = 2w^*2, 2w^*3, 2w^*11, 12w^*1, 22w^*12$ then $\lambda_0(b) < j_0 - 10^{-8}$.
(ii) If $b = 32w^*123, 32w^*122$ then $\lambda_0(b) < j_0 - 10^{-8}$.
(iii) If $b = 132w^*121$ then $\lambda_0(b) > j_0 + 10^{-9}$.
(iv) If $b = 332w^*1211, 332w^*1212$ then $\lambda_0(b) < j_0 - 10^{-9}$.
(v) If $b = 232w^*1212$ then $\lambda_0(b) > j_0 + 10^{-10}$.
(vi) If $b = 232w^*12112, 232w^*12113, 1232w^*12111$ then $\lambda_0(b) < j_0 - 10^{-9}$.
(vii) If $b = 2232w^*12111$ then $\lambda_0(b) < j_0 - 10^{-11}$.
(viii) If $b = 3232w^*121111$ then $\lambda_0(b) < j_0 - 10^{-11}$.
(ix) If $b = 3232w^*121113, 3232w^*1211121, 3232w^*1211122, 23232w^*121112$ then $\lambda_0(b) > j_0 + 10^{-11}$.

Proof. If $b = 32w^*122$ then $\lambda_0(b) \leq \lambda_0(1132w^*122) < j_0 + 10^{-8}$ where we used that 312 and 313 are forbidden.

If $b = 232w^*1212$ then $\lambda_0(b) \geq \lambda_0(21232w^*1212) > j_0 + 10^{-10}$ where we used that 312 is forbidden.

If $b = 2232w^*12111$ then $\lambda_0(b) \leq \lambda_0(\overline{12}232w^*1211133\overline{31}) < j_0 - 10^{-11}$ where we used that 312, 131, 11132 are forbidden. □

By the previous lemma, it suffices to analyse the extensions of $33232w1211123$.

Lemma 4.19.

(i) *If* $b = 333232w^*1211123$ *then* $m(b) > j_0 + 10^{-12}$.

(ii) *If* $b = 233232w^*12111232, 233232w^*12111233$ *then* $\lambda_0(b) > j_0 + 10^{-12}$.

Proof. If $b = 333232w^*1211123$ then $\lambda_0(b) \geq \lambda_0(333232w^*121112311) > j_0 + 10^{-12}$ where we used that 313, 312 are forbidden. □

Corollary 4.20. *If* $m(b) < j_0 + 10^{-12}$*, then* b *does not contain 233232w1211123.*

Proof. Use the forbidden word 11231 and the previous lemma. □

Corollary 4.21. *Let* $b \in \{1,2,3\}^{\mathbb{Z}}$ *such that* $m(b) = \lambda_0(b)$*. If* $|m(b) - j_0| < 10^{-12}$ *then either* $b = 1133232w^*12111232$ *or* $b = 1133232w^*12111233$

Proof. Use the previous lemma and that 313, 312, 11231 are forbidden. □

4.2.3. *Extensions of the word* $\tau = 1133232w12111232$. Denote $\tau^* = 1133232w^*12111232$.

Lemma 4.22.

(i) *If* $b = 1\tau^*, \tau^*2, \tau^*3$ *then* $\lambda_0(b) < j_0 - 10^{-12}$.

(ii) *If* $b = 3\tau^*12$ *then* $\lambda_0(b) > j_0 + 10^{-12}$.

(iii) *If* $b = 23\tau^*112, 23\tau^*113, 33\tau^*112, 33\tau^*113, 33\tau^*1111$ *then* $\lambda_0(b) < j_0 - 10^{-14}$.

(iv) *If* $b = 23\tau^*1112, 23\tau^*1113, 33\tau^*1113, 33\tau^*1112$ *then* $\lambda_0(b) > j_0 + 10^{-15}$.

(v) *If* $b = 2123\tau^*11112, 23\tau^*11113,$ *then* $\lambda_0(b) < j_0 - 10^{-14}$.

(vi) *If* $b = 2123\tau^*11111$ *then* $\lambda_0(b) > j_0 + 10^{-14}$.

Proof. If $b = 33\tau^*1112$ then $\lambda(b) \geq \lambda_0(\overline{13}3\tau^*111\overline{23}) > j_0 + 10^{-15}$ where we used that 11231 and 3231 are forbidden. □

Using that 131 is forbidden we get

Corollary 4.23. *If* $m(b) < j_0 + 10^{-15}$ *then* b *does not contain* $3\tau1112$ *neither* $2\tau1113$.

Corollary 4.24. *If* $b = 3\tau^*$ *then* $|m(b) - j_0| > 10^{-15}$.

Proof. Use the forbidden words 3231, 2231, 11231, 312 we see that $b = 23\tau^*$ must extend to $2123\tau^*$ and then we use the previous lemma. The case $b = 33\tau^*$ follows from items 1 to 4 of the previous lemma. □

By the previous lemma, it suffices to analyse the extensions of $2\tau1$.

Lemma 4.25.

(i) *If* $b = 2\tau^*11, 22\tau^*12, 32\tau^*12, 12\tau^*123, 212\tau^*122, 112\tau^*122$ *then* $\lambda_0(b) < j_0 - 10^{-14}$.

(ii) *If* $b = 112\tau^*1211, 1212\tau^*1211, 2212\tau^*1211, 3212\tau^*12112, 3212\tau^*12111, 23212\tau^*121133, 33212\tau^*121133, 2113212\tau^*121133, 3113212\tau^*121133$ *then* $\lambda_0(b) > j_0 + 10^{-15}$.

Corollary 4.26. *If* $m(b) < j_0 + 10^{-15}$ *then* b *does not contain* $12\tau^*1211$.

Corollary 4.27. *If* $b = 2\tau^*1211$ *then* $|m(b) - j_0| > j_0 + 10^{-15}$.

By the previous lemma and since 312 is forbidden, it suffices to analyse the extensions of $12\tau1212$.

Lemma 4.28.

(i) *If* $b = 212\tau^*1212$ *then* $\lambda_0(b) > j_0 + 10^{-13}$.

(ii) *If* $b = 2112\tau^*1212, 3112\tau^*1212, 1112\tau^*12123, 1112\tau^*12122, 11112\tau^*12121$ *then* $\lambda_0(b) < j_0 - 10^{-15}$.

(iii) *If* $b = 31112\tau^*12121$ *then* $\lambda_0(b) > j_0 + 10^{-15}$.

(iv) *If* $b = 221112\tau^*121211, 221112\tau^*121212$ *then* $\lambda_0(b) < j_0 - 10^{-15}$.

Proof. If $b = 1112\tau^*12122$ then $\lambda_0(b) \le \lambda_0(\overline{1}3331112\tau^*121223\overline{1}) < j_0 - 10^{-15}$ where we used the forbidden words 131, 23111. □

By the previous lemma, it suffices to analyse the extensions of $121112\tau12121$.

Lemma 4.29.

(i) If $b = 1121112\tau^*121211$ then $\lambda_0(b) < j_0 - 10^{-15}$.

(ii) If $b = 2121112\tau^*121212, 1121112\tau^*1212121, 1121112\tau^*1212122$ then $m(b) > j_0 + 10^{-17}$.

(iii) If $b = 31121112\tau^*1212123$ then $\lambda_0(b) < j_0 - 10^{-16}$.

(iv) If $b = 11121112\tau^*121212, 321121112\tau^*121212, 221121112\tau^*121212$ then $\lambda_0(b) > j_0 + 10^{-16}$.

(v) If $b = 1121121112\tau^*121212, 22121121112\tau^*121212, 32121121112\tau^*121212, 112121121112\tau^*121212$ then $m(b) > j_0 + 10^{-19}$.

(vi) If $b = 21121112\tau^*12121232, 21121112\tau^*12121233$ then $\lambda_0(b) > j_0 + 10^{-17}$.

(vii) If $b = 212121121112\tau^*12121231133, 212121121112\tau^*12121231132$ then $\lambda_0(b) < j_0 - 10^{-19}$.

(viii) If $b = 32121112\tau^*121211, 22121112\tau^*121211, 12121112\tau^*1212112, 12121112\tau^*1212113, 12121112\tau^*12121111, 12121112\tau^*12121112$ then $\lambda_0(b) < j_0 - 10^{-16}$.

(ix) If $b = 212121112\tau^*12121113$ then $\lambda_0(b) > j_0 + 10^{-17}$.

(x) If $b = 112121112\tau^*121211133$ then $\lambda_0(b) < j_0 - 10^{-18}$.

Proof. If $b = 1121112\tau^*1212122$ then $\lambda(b) \ge \lambda_0(\overline{1}3331121112\tau^*1212122\overline{13}) > j_0 + 10^{-16}$ where we used that 131, 23112 are forbidden. If $b = 1121121112\tau^*121212$ then

$$\lambda(b) \ge \lambda_0(\overline{31}1121121112\tau^*121212311\overline{13}) > j_0 + 10^{-17}$$

where we used that 312 and 313 are forbidden.

If $b = 22121121112\tau^*121212, 32121121112\tau^*121212, 112121121112\tau^*121212$ then

$$\lambda_0(b) \ge \lambda_0(\overline{31}112121121112\tau^*1212123113\overline{31}) > j_0 + 10^{-17}$$

where we used that 313, 312, 23111, 23112 are forbidden. □

Corollary 4.30. *If $b = \tau^*$ then $|m(b) - j_0| > 10^{-19}$.*

Proof. This is consequence of Corollary 4.14, Corollary 4.21 and Corollary 4.30. □

Now we collect all of the forbidden words and the self-replicating word $w_r := 1133232w12111233$ from Corollary 4.21. Define F as the following set words and their transposes:

- 131, 313, 312, 3231, 2231, 23111, 23112, 11231.
- 13331113, 323331113, 223331113, 112333111333.
- 3212333111333, 22123331113332, 12212333111333, 112333111333, 311233311133113, 3112333111332,2112333111332, 3112333111332, 2112333111332, 1123331113321, 1123331113322, 11112333111332, 321112333111332, 221112333111332, 2111233311133233.
- $1w1$, $1w2$, $31w$, $21w$, $111w$, $211w$, $1w33$, $132w121$, $232w1212$, $3232w121113$, $3232w1211121$, $3232w1211122$, $23232w121112$, $333232w1211123$, $233232w1211123$.
- $3\tau12, 3\tau1112, 2\tau1113, 2123\tau11111, 12\tau1211, 212\tau1212, 31112\tau12121, 2121112\tau121212, 1121112\tau1212121, 1121112\tau1212122, 11121112\tau121212, 321121112\tau121212, 221121112\tau121212, 1121121112\tau121212, 22121121112\tau121212, 32121121112\tau121212, 112121121112\tau121212, 21121112\tau12121232, 21121112\tau12121233, 212121112\tau12121113$.
- $w_r = 1133232w12111233$.

All the results of this section imply the following:

Corollary 4.31. *If $m(b) < j_0 + 10^{-19}$ then b does not contain words from $F \setminus \{w_r, w_r^T\}$.*

Corollary 4.32 (Local uniqueness)**.** *Let $b \in \{1,2,3\}^{\mathbb{Z}}$ be such that $j_0 - 10^{-19} < \lambda_0(b)$. If b does not contain words from $F \setminus \{w_r, w_r^T\}$ then up to transposition we have*

$$b_{-15} \dots b_0^* \dots b_{16} = 1133232w^*12111233.$$

4.3. **Self-replication.** Denote $w_r = 1133232w12111233$ and $w_r^* = 1133232w^*12111233$. By the previous section we know that if $m(b) < j_0 + 10^{-19}$ and $j_0 \le \lambda_0(b)$ then $b = w_r^*$.

Lemma 4.33 (Forbidden words III)**.**

(i) *If* $b = 2w_r^*, 3w_r^*, 1w_r^*1, 1w_r^*2, 11w_r^*3, 21w_r^*3, 331w_r^*33, 331w_r^*32$ *then* $\lambda_0(b) > j_0 + 10^{-13}$.

(ii) *If* $b = 331w_r^*3113, 331w_r^*3112, 1331w_r^*3111, 2331w_r^*3111, 3331w_r^*31111, 3331w_r^*31112$ *then* $\lambda_0(b) > j_0 + 10^{-16}$.

(iii) *If* $b = 33331w_r^*311133$ *then* $\lambda_0(b) > j_0 + 10^{-16}$.

(iv) *If* $b = 2123331w_r^*3111331, 11123331w_r^*3111331, 2123331w_r^*3111332, 311123331w_r^*311133233$ *then* $\lambda_0(b) > j_0 + 10^{-17}$.

(v) *If* $b = 2ww^*w3, 2ww^*w2, 12ww^*w1, 22ww^*w1, 32ww^*w11$ *then* $m(b) > j_0 + 5 \cdot 10^{-20}$.

(vi) *If* $b = 32ww^*w123, 32ww^*w122, 332ww^*w1211, 232ww^*w12113, 232ww^*w12112$ *then* $\lambda_0(b) > j_0 + 2 \cdot 10^{-21}$.

(vii) *If* $b = 1232ww^*w12111$ *then* $\lambda_0(b) > j_0 + 2 \cdot 10^{-21}$.

(viii) *If* $b = 2232ww^*w121111$ *then* $\lambda_0(b) > j_0 + 10^{-21}$.

(ix) *If* $b = 2232ww^*w121112$ *then* $\lambda_0(b) > j_0 + 2 \cdot 10^{-22}$.

Proof. If $b = 22ww^*w1$ then $\lambda(b) \ge \lambda_0(\overline{21}22ww^*w\overline{12}) > j_0 + 5 \cdot 10^{-20}$ where we used that 213 is forbidden.

If $b = 32ww^*w122$ then $\lambda_0(b) \ge \lambda_0(\overline{31}331132ww^*w122\overline{32}) > j_0 + 7 \cdot 10^{-21}$ where we used that 213, 313, 11132, 21132 are forbidden. □

Remark 4.34. *The lower bound is decreasing on each item. It is essential for self-replication that there exists a minimal forbidden word. In this case is given by the word at item (ix).*

We need some auxiliary forbidden words that will help us in the characterisation of $M \setminus L$ but are not needed in the description of the Cantor set.

Lemma 4.35.

(i) *If* $b = 232ww^*w1211113, 33232ww^*w1211112, 33232ww^*w12111111$ *then* $\lambda_0(b) > j_0 + 10^{-21}$.

(ii) *If* $b = 133232ww^*w121111$ *then* $\lambda_0(b) > j_0 + 10^{-22}$.

(iii) *If* $b = 133232ww^*w12111231$ *then* $\lambda_0(b) > j_0 + 10^{-23}$.

(iv) *If* $b = 11133232ww^*w12111232$ *then* $\lambda_0(b) > j_0 + 10^{-24}$.

Lemma 4.36 (Self-replication)**.** *Let* $b \in \{1,2,3\}^{\mathbb{Z}}$ *be such that* $m(b) < j_0 + 10^{-19}$ *and suppose that for some index* $i \in \mathbb{Z}$ *we have* $b_{i-15} \dots b_i^* \dots b_{i+16} = 1133232w^*12111233 = w_r^*$. *Then one has*

(i) *If* $\lambda_0(b) < j_0 + 2 \cdot 10^{-21}$ *then*

$$b_{i-32} \dots b_i^* \dots b_{i+30} = 232ww^*w12111.$$

(ii) *If* $\lambda_0(b) < j_0 + 10^{-21}$ *and does not contain* $2232www121112$ *then*

$$b_{i-32} \dots b_i^* \dots b_{i+30} = 1133232ww^*w12111.$$

In particular

$$\dots b_i^* \dots b_{i+30} = \overline{w}w^*w12111.$$

(iii) *If* $\lambda_0(b) < j_0 + 10^{-23}$ *and does not contain* $11133232www12111232$ *then*

$$b_{i-32} \dots b_i^* \dots b_{i+33} = 1133232ww^*w12111233.$$

In this particular case $b = \overline{w}$.

Proof. By the Lemma 4.33, this subword must extend to $b = 31w_r^*3$. Since 23111 and 131 are forbidden, we are forced to $b = 331w_r^*3$. By Lemma 4.33 and since 313, 312 are forbidden, it must extend to $b = 3331w_r^*31113$. Using that 11132 and 131 are forbidden it is forced to $b = 3331w_r^*311133$. By Lemma 4.33 and since 13331113, 323331113, 223331113 are forbidden, it must extend to $b = 123331w_r^*311133$. By Lemma 4.33 and since 112333111333, 3112333111332, 2112333111332 are forbidden, it must extend to $b = 11123331w_r^*3111332$. Using the forbidden words 1123331113321, 11123331113322, 12111233311133233, 3231, 11112333111332, 321112333111332, 221112333111332 and Lemma 4.33 it must extend to $b = ww^*w$. Now using that $1w1$, 312 are forbidden and Lemma 4.33 it must extend to $b = 32ww^*w121$. Now using that $232w1212$, 213, $132w121$, are forbidden and Lemma 4.33 it must extend to $b = 232ww^*w12111$.

By using the last three forbidden words of Lemma 4.33 and since $3232w121113$ is forbidden, we must continued as $b = 3232ww^*w12111$. Finally using the forbidden words $23232w121112$, 1323, $333232w1211123$, $233232w1211123$, 313, 213 it is forced to $b = 1133232ww^*w12111$.

Now to prove the last claim, just use the inequalities of Lemma 4.35 and the forbidden words $32323w121113$, $2w_r$ and $3w_r$. □

Corollary 4.37. *The point $j_0 = m(\overline{w})$ is an isolated point in M.*

The reason we stop extending the word in item 2 of Lemma 4.36 is because we want to apply an inductive argument, namely we know that we have local uniqueness as $\eta_1 w\eta_2$, so to self replicate we need to extend this subword until we reach a word of the form $\eta_1 www\eta_2'$ or $\eta_1' www\eta_2$ where η_i' is a **strict** suffix or prefix of η_i.

4.4. **Description of $M\setminus L$.** Define the constant j_1 to be the minimum Markov value of a bi-infinite sequence $b \in \{1,2,3\}^{\mathbb{Z}}$ containing the word $2232www121112$ and m_0 to be the minimum Markov value of a bi-infinite sequence $b \in \{1,2,3\}^{\mathbb{Z}}$ containing the word $11133232www12111232$.

The reason to define j_1 is the following. First note that this word has the smallest lower bound in Lemma 4.33. In fact since we have that $\lambda_0(2232ww^*w121112) < j_0 + 1.8\cdot 10^{-21}$, there should exists a Markov value smaller than all other forbidden forbidden words. This minimal Markov value should be the one that is determining the right extreme of the gap of L, provided it belong to L.

Lemma 4.38.

$$j_1 = m(\overline{12}2232wwww1211111\overline{23}) = \lambda_0(\overline{12}2232ww^*ww1211111\overline{23}) \in M.$$

$$m_0 = m(\overline{w}121112\overline{3}\overline{21}) = \lambda_0(\overline{w}w^*w121112\overline{3}\overline{21}) \in M.$$

Numerically we have $j_1 \approx j_0 + 8.326\cdot 10^{-22}$ and $m_0 \approx j_0 + 5.9\cdot 10^{-24}$.

Proof. Let $b \in \{1,2,3\}^{\mathbb{Z}}$ be such that $b = 2232ww^*w121112$. Since $3232w1211121, 3232w1211122$ are forbidden we must extend to $b = 2232ww^*w1211123$. Since 312 is forbidden, we have $\lambda_0(b) \geq \lambda_0(\overline{1}2232ww^*w1211123)$. Hence at each step we have to minimize the right with respect to this position avoiding forbidden words. So we have to continue as $b = 2232ww^*w12111233$. In particular, because of Lemma 4.36 item 1 (note we can assume $m(b) < j_0 + 10^{-22}$ because of the above candidate) we have self-replication so the word extends to $b = 2232ww^*ww12111$. Now we have to be careful because of the dangerous positions inside of each copy of w. Since we are minimizing we should continue as $b = 2232ww^*ww121111$. Note that $\lambda_0(3232w^*121111) < j_0 - 10^{-11}$ and $\lambda_0(2232w^*12111) < j_0 - 10^{-11}$ so the only dangerous position remaining is inside the third copy of w. Now since $33232www1211112, 232www1211113$ are forbidden by Lemma 4.33, we must extend to $b = 2232ww^*ww1211111$. Since $33232www12111111$ is forbidden by Lemma 4.33, we should extend as $b = 2232ww^*ww12111112$, which is good because now

$$\lambda_0(www^*w12111112) < j_0 + 8.31\cdot 10^{-22} < \lambda_0(21212232ww^*ww12111112),$$

so we have no longer dangerous positions. Now using that 11231 and 3231 are forbidden, we see that $\lambda_0(2232ww^*ww12111112)$ is minimized with the above candidate.

Now let $b \in \{1,2,3\}^{\mathbb{Z}}$ be such that $b = 11133232ww^*w12111232$. By Lemma 4.36 we must have that $b = \overline{w}w^*w12111232$. Now we only have to minimize $\lambda_0(b)$, which is achieved with the above candidate because 213 is forbidden. $\square$

In particular for all $m(b) \in (j_0, j_1)$ we have that b is connecting in the past with $\overline{w}$.

Corollary 4.39. *Let $b \in \{1,2,3\}^{\mathbb{Z}}$ be a sequence such that $m(b) < j_0 + 10^{-19}$ and $\lambda_0(b) \in (j_0, j_1)$. Then one has*

$$\dots b_0^* \dots b_{30} = \overline{w}w^*w12111.$$

Proof. Just use Corollary 4.32 and Lemma 4.36. $\square$

Lemma 4.40.

$m_1 := \max(M \setminus L) \cap (j_0, j_1) = \lambda_0(\overline{w}w^*w121111\overline{23}) = 3.9420011599\dots \approx j_0 + 8.26 \cdot 10^{-22}$.

Proof. By the previous corollary we have $b = \overline{w}w^*w12111$. Since we want to maximize $\lambda_0(b)$, we should continue as $b = \overline{w}w^*w121111$. Now using the inequalities of Lemma 4.35 this word must be extended to $b = \overline{w}w^*w1211111$. Since we want to maximize and since $33232www12111111$ is forbidden by that same lemma we should continue with $b = \overline{w}w^*w12111112$. Now using that 11231 and 3231 are forbidden, we have that $\lambda_0(b)$ is maximized with the given candidate. $\square$

Corollary 4.41. *The intervals (j_0, m_0) and (m_1, j_1) are maximal gaps of M.*

Corollary 4.42. *The interval (j_0, j_1) is a maximal gap of L and $j_1 \in L'$.*

Proof. We will use the fact that Lagrange values of periodic words are dense in L. Suppose that $\ell(\overline{a}) \in (j_0, j_1)$ where $a \in \{1,2,3\}^N$ is a finite word in $\{1,2,3\}$ that attains its Markov value at $\ell(\overline{a}) = m(\overline{a}) = \lambda_0(\overline{a_1^* \dots a_N})$. By Corollary 4.32 and Corollary 4.39 we must have that

$$\overline{a}a_1^* \dots = \overline{w}w^*w12111\dots$$

Hence we must have $\overline{a} = \overline{w}$ and so $\ell(\overline{a}) = \ell(\overline{w}) = j_0$, a contradiction.

Now we must prove that $j_1 \in L'$. For this we use the same characterisation of L, by proving that it is the limit of a strictly decreasing sequence of Markov values of periodic words:

$$j_1 = \lim_{n\to\infty} \lambda_0\left(\overline{2(12)^n232ww^*ww1211111(23)^n}\right) = \lim_{n\to\infty} m\left(\overline{2(12)^n232ww^*ww1211111(23)^n}\right).$$

We only need to check that the Markov value of these periodic words is really attained at the position $*$. We only have three potential dangerous positions (each one inside each copy of w)

- $\lambda_0(1232w^*1211) < j_0 + 10^{-9}$
- $\lambda_0(232w^*121111) < j_0 + 10^{-11}$
- $\lambda_0(12232www^*w121111123) < 8.31 \cdot 10^{-22} < j_1$.

$\square$

Finally we can prove the complete characterization of $M \setminus L$ in this region.

Theorem 4.43. *We have that*

$$(M \setminus L) \cap (j_0, j_1) = C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2 \cup D_3 \cup X,$$

where

$$C_1 = \{\lambda_0(\overline{w}w^*w12111112\gamma) : 111112\gamma \in \{1,2,3\}^{\mathbb{N}} \textit{ not contain any word from } F\},$$

$$C_2 = \{\lambda_0(\overline{w}w^*w12111113\gamma) : 111113\gamma \in \{1,2,3\}^{\mathbb{N}} \textit{ not contain any word from } F\},$$

$$C_3 = \{\lambda_0(\overline{w}w^*w12111232\gamma) : 2111232\gamma \in \{1,2,3\}^{\mathbb{N}} \textit{ not contain any word from } F\},$$

are Cantor sets and

$$D_1 = \{\lambda_0(\overline{w}w^*w1211111\theta1111121\overline{w^T}) : \theta \textit{ finite word in 1,2,3}, \theta_1 \neq 1, (\theta^T)_1 \neq 1, [0;\theta] \geq [0;\theta^T], \textit{ and } 11111\theta11111 \textit{ not contain any word from } F\},$$

$$D_2 = \{\lambda_0(\overline{w}w^*w1211111\theta23211121\overline{w^T}) : \theta \textit{ finite word in 1,2,3}, \theta_1 \neq 1 \textit{ and } 11111\theta232 \textit{ not contain any word from } F\},$$

$$D_3 = \{\lambda_0(\overline{w}w^*w12111232\theta23211121\overline{w^T}) : \theta \textit{ finite word in 1,2,3}, [0;\theta] \leq [0;\theta^T], \textit{ and } 2111232\theta2321112 \textit{ not contain any word from } F\},$$

$$X = \Big\{ m\left(\overline{w}121111121111121\overline{w^T}\right), m\left(\overline{w}121111121\overline{w^T}\right), m\left(\overline{w}121111123211121\overline{w^T}\right), m\left(\overline{w}121112321111121\overline{w^T}\right), m\left(\overline{w}121112323211121\overline{w^T}\right), m\left(\overline{w}1211123211121\overline{w^T}\right) \Big\},$$

are sets of isolated points in M.

Proof. Let $b \in \{1,2,3\}^{\mathbb{Z}}$ be such that $m(b) = \lambda_0(b) \in (j_0, j_1)$. By Corollary 4.32 and Corollary 4.39 we must have that $b = \overline{w}w^*w12111\widetilde{\gamma}$ with $\widetilde{\gamma} \in \{1,2,3\}^{\mathbb{N}^*}$. Since $3232w121113$ is forbidden we have $\widetilde{\gamma}_1 \neq 3$. In case that $\widetilde{\gamma}_1 = 2$, then since $3232w1211121, 3232w1211122$ are forbidden we must continue as $b = \overline{w}w^*w1211123$. Since 11231 is forbidden we have that either $b = \overline{w}w^*w12111232$ or $b = \overline{w}w^*w1211123$. In the latter case using Lemma 4.36 this implies $b = \overline{w}w^*ww12111$, but since $j_0 = \lambda_0(\overline{w}w^*ww12111) < j_0 + 10^{-35} < m_0$ we must have $\lambda_0(b) = j_0$. In case that $\widetilde{\gamma}_1 = 1$, since $33232www1211112$, $33232www1211113$, $33232www12111111$ are forbidden we are forced to $b = \overline{w}w^*w1211111\widetilde{\gamma}_3 \ldots$ with $\widetilde{\gamma}_3 \neq 1$.

Note that in any case $\widetilde{\gamma}$ can not contain w_r, because by self-replication (Corollary 4.39) we would get that $\overline{w}$ contains 11111, a contradiction. Thus $\gamma := \widetilde{\gamma}_3\widetilde{\gamma}_4 \ldots$ does not contain w_r. Observe that all the words in F do not contain 211111, 111112 nor 1τ so if there are forbidden words they can only appear in 11111γ or in 2111232γ. In particular if γ does not contain w_r^T then $\lambda_0(b) \in C$.

If γ contains w_r^T, pick N minimal such that $\gamma_N\gamma_{N+1}\ldots\gamma_{N+31} = w_r^T$, then since $m(b) = m(b^T) < j_1$, by applying Lemma 4.36 to b^T we get that either $\gamma_{N-14}\gamma_{N-13}\cdots = 11121\overline{w^T}$ and so $b = \overline{w}w^*w12111\tau_L\widetilde{\theta}11121\overline{w^T}$ where $\tau_L \in \{112, 113, 232\}$ with $\widetilde{\theta} = \gamma_1 \ldots \gamma_{N-15}$ or that $\lambda_0(b) \in X$.

In the former case, using the transpose of the forbidden words above we know that $\widetilde{\theta}$ must end with 1 or 2. If it ends with 2 then it must end with 32, and since 13211 is forbidden and by the minimality of N we must have that $\widetilde{\theta} = \theta232$. In case that $\widetilde{\theta}$ ends with 1, then by using the transpose of the above forbidden words we must have $\widetilde{\theta} = \theta11$ with the last digit of θ different from 1. Since 211111, 111112 and $23211121w^T23233111 = \tau^T1$ are no subwords of any word in F, we know that forbidden words can only appear in $11111\theta11111$, $11111\theta232$ or $2111232\theta2321112$ respectively. So $\lambda_0(b) \in D_1 \cup D_2 \cup D_3$.

The above argument shows that $(M \setminus L) \cap (j_0, j_1) \subset C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2 \cup D_3 \cup X$. Conversely given $\lambda_0(b) \in C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2 \cup D_3$ (by direct computation we can check that $X \subset (j_0, j_1)$), first note that we have $j_0 = \lambda_0(\overline{w}w^*\overline{w}) < \lambda_0(\overline{w}w^*w1211111\ldots) = \lambda_0(b) \leq m_1 < j_1$ (because of the forbidden words 11231 and 3231). Now we need to show that $\lambda_0(b) = m(b) \in M$. It suffices to guarantee that the supremum is attained at the given position. Suppose that for some $N \in \mathbb{N}^*$ we have that $\lambda_0(b) < \lambda_N(b)$. Since $j_0 < \lambda_0(b)$ and because of Corollary 4.32, we will have that $b_{N-15}\ldots b_N \ldots b_{N+16}$ is equal to w_r or w_r^T. But by the definition of the sets C_i, D_i, X and since

$w_r \in F$, we will have that is equal to w_r^T, that $\lambda_0(b) \notin C_i$ and moreover the position N must occur after θ, more specifically inside the period $\overline{w^T}$. Note that we have the inequalities

- $\lambda_0(www^*ww) < j_0 + 10^{-32} < m_0$,
- $\lambda_0(ww^*121111) < j_0$,
- $\lambda_0(ww^*12111232) < j_0$.

Hence we must have $b_{N-30}\dots b_N^* b_{N+1}\dots = 11121w^T(w^*)^T\overline{w^T}$ and either $b_{N-33}b_{N-32}b_{N-31} \in \{232, 211, 311\}$ (since we assume $\lambda_0(b) \notin X$). In the case $\lambda_0(b) \in D_2$ we use the inequality $\lambda_0(\overline{w}w^*w121112) < \lambda_0(\overline{w}w^*w121111)$ and in the case that $\lambda_0(b) \in D_1$ the condition $[0;\theta] \geq [0;\theta^T]$ guarantees that $\lambda_N(b) \leq \lambda_0(b)$.

Finally we prove that $D_1 \cup D_2 \cup D_3$ is an isolated set in M. Given $\lambda_0(b) \in D_1 \cup D_2 \cup D_3$, suppose we have a sequence $\lambda_0(a^{(n)}) = m(a^{(n)})$ that converges to $\lambda_0(b)$. By Corollary 4.39 we have $a^{(n)} = \overline{w}w^*w12111\dots$. Hence the non-positive part of those sequences all coincide with the non-positive part of b, i.e. $\dots a_{-1}^{(n)}a_0^{(n)} = \dots b_{-1}b_0$. In particular using a basic inequality in continued fractions (see [20, Lemma A.1.]) we must have for n sufficiently large

- $a^{(n)} = \overline{w}w^*w1211111\theta1111121w^Tw^Tw^T\dots$
- $a^{(n)} = \overline{w}w^*w1211111\theta23211121w^Tw^Tw^T\dots$

In all cases the word w_r^T is appearing, so by using Lemma 4.36 we will obtain that $a^{(n)}$ is connecting to $\overline{w^T}$, that is $a^{(n)} = b$ for all large n. The same proof applies to all elements of X. □

Remark 4.44. *By using Proposition 2.3, one can show that the set*

$$K = \{[0; 1, 1, 1, 3, 3, 2, 3, 2, w, 1, 2, 1, 1, 1, 1, 1, \gamma] : 11111\gamma \in \{1,2,3\}^{\mathbb{N}} \text{ not contain any word from } F \text{ and } \gamma_1 \neq 1\}$$

is a topologically mixing dynamically defined Cantor set.

4.5. **Characterisation of the region associated with $w = 21133311121$.** As discussed in the introduction, the second and third authors in joint work with Matheus [9] recently studied a region of $M \setminus L$ associated with the odd non-semi-symmetric word $w = 21133311121$. We take this opportunity to state the full description of this region.

Let $j_0 = m(\overline{w}) = 3.938776241981\dots \in L$ and $j_1 \in L'$ given by

$$j_1 = m(\overline{21}23311133113212311133311121ww^*w22\overline{32}) \approx 9.52145 \cdot 10^{-12}.$$

Let F be the following set of words and their transposes:

- 131, 312, 313, 2231, 3231, 23111, 23112, 11231, 3331113, 2111333111212,
- 1113331112121, 1113331112122, 11111333111212, 21111333111212, 11133311121232,
- 11133311121233, 133311121212, 21231133311121212,
- $11w21$, $2w21$, $3w21$, $21w212$, $221w21$, $321w21$, $w2111$, $w2112$, $1121w21$.

Theorem 4.45. *We have that*

$$(M \setminus L) \cap (j_0, j_1) = C \cup \bigcup_{i,j} D_{i,j} \cup X$$

where

$$C_1 = \{\lambda_0(\gamma2121ww^*\overline{w}) : \gamma21212 \in \{1,2,3\}^{\mathbb{N}} \text{ not contain any word from } F\},$$

$$C_2 = \{\lambda_0(\gamma21231133311121w^*\overline{w}) : \gamma2123 \in \{1,2,3\}^{\mathbb{N}} \text{ not contain any word from } F\},$$

$$C_3 = \{\lambda_0(\gamma331133311121w^*\overline{w}) : \gamma331133 \in \{1,2,3\}^{\mathbb{N}} \text{ not contain any word from } F\},$$

are Cantor sets, where

$$D_{i,j} = \{\lambda_0(\overline{w^T}w^T\tau_L\theta\tau_R w^*\overline{w}) : \theta \text{ finite word in 1,2,3, } [0;\tau_R^T,\theta^T] \geq [0;\tau_L,\theta], \text{ and } \tau_L\theta\tau_R \text{ not contain any word from } F\},$$

are discrete sets of isolated points with $(\tau_1,\tau_2,\tau_3) = (2121w, 21231133311121, 331133311121)$ *and* $(\tau_R,\tau_L^T) = (\tau_i,\tau_j)$ *and finally where*

$$X = \Big\{ m\left(\overline{w^T}1211133311333113331121\overline{w}\right), m\left(\overline{w^T}12111333113331133311121\overline{w}\right),$$
$$m\left(\overline{w^T}121113331133311121\overline{w}\right), m\left(\overline{w^T}1211133311321231133311121\overline{w}\right), m\left(\overline{w^T}121113331132121\overline{w}\right),$$
$$m\left(\overline{w^T}1212311333111121\overline{w}\right), m\left(\overline{w^T}1212121\overline{w}\right), m\left(\overline{w^T}12121\overline{w}\right), m\left(\overline{w^T}121\overline{w}\right)\Big\}$$

is a finite set of isolated points.

Remark 4.46. *By using Proposition 2.3, one can show that each of the sets*

$$K_i = \{[0;3,3,1,1,2,\tau_i^T,\gamma] : \gamma^T\tau_i \in \{1,2,3\}^{\mathbb{N}} \text{ does not contain any word from } F\},$$

is a topologically mixing dynamically defined Cantor set.

5. New maximal gaps in M near 3.943

We now demonstrate the existence of two new maximal gaps in the Markov spectrum M. These are in the vicinity of the last visible gaps in the computer approximations produced by Delecroix, Matheus and the third author [25, Figure 5].

5.1. First gap.

There is gap in the Markov spectrum with left endpoint given by

$$m(\overline{2}3331113^*3\overline{32}) = 3.94254\ldots$$

and right endpoint given by

$$m(\overline{2}3331113311321231133 3^*1113311133311321231133111333\overline{2}) = 3.943304\ldots$$

Lemma 5.1.

(i) If $b = 13^*1$, 3^*13, 3^*12, 323^*1, 223^*1, 23^*111, 13^*2112, 13^*2113, 33113^*2111, 23113^*21111, 123113^*21112 *then* $\lambda_0(b) > 3.9438$

(ii) If $b = 331113^*331$ *then* $\lambda_0(b) > 3.94317$.

Suppose $m(\underline{b}) \leq 3.94317$. Since 131 is forbidden, we have that either $b_{-1}b_0^*b_1$ is equal to 13^*2 or 13^*3. Suppose first it is equal to 13^*3. Since 313, 213 are forbidden we are forced to 113^*3. Since $\lambda_0(3113^*3) < 3.9$, $\lambda_0(2113^*3) < 3.9$, $\lambda_0(113^*31) < 3.93$, $\lambda_0(113^*32) < 3.94$, $\lambda_0(11113^*33) < 3.93$, $\lambda_0(21113^*33) < 3.941$ we should continue as 31113^*33. Since 131, 23111, 331113331 are forbidden we just have the options $\lambda_0(331113^*333) < 3.942$ and $\lambda_0(331113^*332) < 3.9427$. Actually one has

$$\lambda_0(331113^*332) \leq m(\overline{2}3331113^*3\overline{32}) = 3.94254\ldots$$

where we used that 3231 and 133311133 are forbidden.

Now suppose $b_{-1}b_0^*b_1 = 13^*2$. Since 313, 213, 1322, 1323 are forbidden we are forced to 113^*21. Since 11132 is forbidden, we have two cases:

- If it continues as 2113^*21, then since 312 is forbidden we have

$$\lambda_0(2113^*21) \geq m(\overline{2}113^*\overline{21}) = 3.94337\ldots$$

- If it continues as 3113^*21 then since $\lambda_0(3113^*212) < 3.938$ and 213 is forbidden we have to continue as 3113^*211. Now using that 132113, 132112, 331132111, 131, 2231, 3231, 2311321111, 2311321112, 131, 11132 are forbidden, we are forced to

$$\lambda_0(123113^*211133) > 3.943227.$$

The minimum Markov value of a bi-infinite sequence that contains 123113211133 is the same as the palindrome 3311123113211133 which is

$$m(\overline{21}1133311123^*1132111333111\overline{12}) = 3.94342\ldots$$

where we used that if the words 313, 312, 1113331113 are forbidden, then a continued fraction in $\{1,2,3\}$ that begins with $[0;2,1,1,1,3,3]$ is minimized when $[0;2,1,1,1,3,3,3,1,1,\overline{1,2}]$.

Remark 5.2. *The middle word is a palindrome and the period is semi-symmetric.*

The minimum Markov value of a bi-infinite sequence that contains 331113331 is the same as the palindrome 113331113311133311 which is

$$m(\overline{23}33111331132123113331113311113^*3311321231133111333\overline{32}) = 3.943304\ldots$$

where we used that if the words 131, 1322, 1323, 33113211, 23111, 32123112, 311331113331 are forbidden, then a continued fraction in $\{1,2,3\}$ that begins with $[0;3,3,1,1]$ is minimized when $[0;3,3,1,1,3,2,1,2,3,1,1,3,3,1,1,1,3,3,\overline{3,2}]$.

Remark 5.3. *The middle word is a palindrome and the period is semi-symmetric.*

5.2. **Second gap.** The above argument shows that we have a local uniqueness on $(3.9433, 3.94337)$. Indeed, we have that $b_{-5}\ldots b_0^*\ldots b_4 = 331113^*3311$.

Lemma 5.4. *If* $b = 2331113^*3311$, 3331113^*3311, 331113^*33111, 331113^*33112, $11331113^*331133$, $311331113^*331132$, $211331113^*331132$, *or* $1111331113^*3311321$ *then* $\lambda_0(b) > 3.94331$.

Recall that we have that $b_{-5}\ldots b_0^*\ldots b_4 = 331113^*3311$. Using the above forbidden words and 1323, 1322, 131, 313, 312, 33113211 we get that it must extend to $111331113^*33113212$. Since $\lambda_0(3111331113^*33113212) < 3.9433055$ and 11113311133311321 is forbidden we should continue as $2111331113^*33113212$ which satisfies $\lambda_0(2111331113^*33113212) > 3.9433068$. In fact we have

$$\lambda_0(3111331113^*33113212) \le m(\overline{12}3113331113311133111113^*33113\overline{21}) = 3.94330534\ldots$$

This is the left endpoint of the gap.

The right endpoint of the gap is given by the minimum Markov value of a bi-infinite sequence that contains 2111331113331132 12, which is

$$m(\overline{21}11331113^*331132123113311133\overline{32}) = 3.94330716\ldots$$

where we used that if 313, 312, 23111, 32123112, 311331113331 are forbidden then a continued fraction in $\{1,2,3\}$ that begins with $[0;3,2,1,2]$ is minimized with $[0;3,2,1,2,3,1,1,3,3,1,1,1,3,3,\overline{3,2}]$.

Appendix A. Markov value algorithm

We want to compute the Markov value of a doubly periodic sequence of the form $\omega = \overline{p_1}\tau\overline{p_2}$.

(i) The first step is to write ω in the simplest terms.
   (a) First, one writes each p_i with its minimal period.
   (b) If τ and p_1 start with the same digit, then we apply a cyclic shift to p_1 and erase the first digit of τ. Do this until it is no longer possible.
   (c) If τ is non-empty, repeat the same process with p_2.

(d) If τ is non-empty, then we are done. Otherwise, we check if $p_1 = p_2$. If $p_1 \neq p_2$ and they begin with the same first digit, then we do a cyclic shift in both until we arrive at p_1, p_2 with different first term.

(ii) We claim that the Markov value of $\omega = \overline{p_1}\tau\overline{p_2}$ can be found on the maximum between the positions $p_1p_1\tau p_2p_2$ and the periodic Markov values $\overline{p_1}, \overline{p_2}$. Indeed, observe that for any position inside a p_1, say p_1^* the left will always be $\overline{p_1}$, while for the right we have two choices: p_1 or $\tau\overline{p_2}$. By the simplification above at least one of them **must win**. If p_1 is even, then p_1 always wins and the value at this position is maximized with $\overline{p_1}$ to the right, that is

$$\lambda_i(\overline{p_1}p_1^* \cdots) \leq \lambda_i(\overline{p_1}p_1^*\overline{p_1}) \leq m(\overline{p_1}).$$

In case p_1 is odd then we maximize with p_1, but in the next question p_1 **must lose**, so the position is maximized with

$$\lambda_i(\overline{p_1}p_1^* \cdots) \leq \lambda_i(\overline{p_1}p_1^*p_1\tau\overline{p_2}).$$

The other side is analogous.

Remark A.1. *The step (i).(a) is not always necessary, unless one is in the special situation described in step (i).(d), namely, when the word has the form* $\omega = \overline{p_1}\ \overline{p_2}$.

Appendix B. Berstein's intervals

Here, for completeness, we list the intervals that Berstein was investigating in [1]. They can be found in Table 3, pp. 117-120, of the book containing that article.

They are

- (3.001494, 3.001889)
- (3.011905, 3.012844)
- (3.026437, 3.032218)
- (3.043396, 3.049177)
- (3.049177, 3.049296)
- (3.072944, 3.091490)
- (3.116089, 3.121575)
- (3.122174, 3.122890)
- (3.123669, 3.123758)
- (3.128940, 3.129192)
- (3.166577, 3.265590)
- (3.280963, 3.284892)
- (3.284892, 3.285169)
- (3.289043, 3.292022)
- (3.325167, 3.326240)
- (3.332493, 3.333233)
- (3.333971, 3.334734)
- (3.341564, 3.351944)
- (3.405792, 3.415638)
- (3.432835, 3.437413)
- (3.440922, 3.449666)
- (3.459882, 3.460606)
- (3.462426, 3.463064).

We will not check whether Berstein's intervals are indeed good intervals. If we plot these intervals in green on our Figure 1.1 (we restrict now to the region between 3 and 4), we obtain the following:

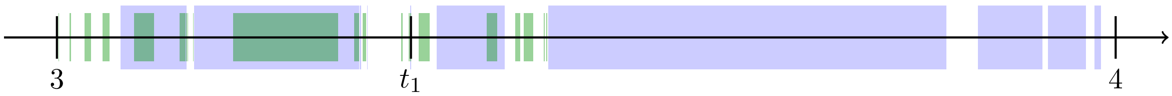

Figure B.1. The good intervals produced in this paper before 4 are depicted in blue. Berstein's intervals are depicted in green.

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(Harold Erazo) IMPA, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro, Brazil
Email address: harolderaz@gmail.com

(Luke Jeffreys) School of Mathematics , University of Bristol, Fry Building, Woodland Road, Bristol BS8 1UG, UK
Email address: luke.jeffreys@bristol.ac.uk

(Carlos Gustavo Moreira) SUSTech International Center for Mathematics, Shenzhen, Guangdong, People's Republic of China;
IMPA, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro, Brazil
Email address: gugu@impa.br