A coarse geometric approach to graph layout problems

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Abstract

We define a range of new coarse geometric invariants based on various graph-theoretic measures of complexity for finite graphs, including: treewidth, pathwidth, cutwidth, search number, topological bandwidth, bandwidth, minimal linear arrangment, sumcut, profile, vertex and edge separation. We prove that, for bounded degree graphs, these invariants can be used to define functions which satisfy a strong monotonicity property, namely they are monotonically non-decreasing with respect to regular maps, and as such have potential applications in coarse geometry and geometric group theory. On the graph-theoretic side, we prove asymptotically optimal upper bounds on the treewidth, pathwidth, cutwidth, search number, topological bandwidth, vertex separation, edge separation, minimal linear arrangement, sumcut and profile for the family of all finite subgraphs of any bounded degree graph whose separation profile is known to be of the form $r^a \log(r)^b$ for some $a > 0$. This large class includes the Diestel-Leader graph, all Cayley graphs of non-virtually cyclic polycyclic groups, uniform lattices in almost all connected unimodular Lie groups, and certain hyperbolic groups.

1 Introduction

The primary objects of study in this paper are **graph layout problems**: finding linear orderings or decompositions of graphs which minimise a specified cost function. Examples of such invariants include treewidth, cutwidth and bandwidth. There are many motivations for finding upper bounds on these invariants coming from a variety of areas including: optimization of networks for parallel computer architectures, VLSI circuit design, information retrieval, numerical analysis, computational biology, graph theory, scheduling and archaeology (cf. [DPS02], [Bod06] and references therein). Typically, realising the minimal possible value of these invariants is NP-hard (see, for example [GJ79] for cutwidth and [ACP87] for treewidth). Despite this, there are asymptotically optimal upper bounds for large classes of graphs. One noteworthy result in the context of this paper is that for every m-vertex graph H, any n-vertex graph which does not contain H as a minor has treewidth at most $m^{3/2}n^{1/2}$ [AST90, (1.8)]. The goal of this paper is to take a "large-scale" viewpoint on these invariants which will enable us to give asymptotically optimal upper and lower bounds on the growth of some of these graph layout parameters for subgraphs of particular infinite graphs of bounded degree. The motivation for working in this level of generality comes from the study of (obstructions to) regular maps.

Until recently, there were very few methods to obstruct regular maps between graphs, but in the last decade the number and variety of tools has greatly increased [BST12, HMT20, HMT22, BH21, HMT23]. This paper is another part of this proliferation of new techniques.

A map between the vertex sets of two bounded degree graphs is **regular** if it is Lipschitz (with respect to the shortest path metrics) and preimages of vertices have uniformly bounded cardinality. The inclusion of one graph as a subgraph of another is always a regular map; also, given two finitely generated groups H, G, there is a regular map from a Cayley graph of H to a Cayley graph of G whenever H admits a finite-index subgroup which is isomorphic to a subgroup of G. Thus, the statement that no regular map exists between two graphs is a strong geometric version of the statement X is not (similar to) a subgraph of Y, or H is not (similar to) a subgroup of G. For bounded degree graphs, regular maps generalise both quasi-isometric and coarse embeddings. These observations give combinatorial and algebraic motivations for determining obstructions to the existence of regular maps.

To fix notation, the vertex and edge sets of a graph Γ will be denoted by $V\Gamma$ and $E\Gamma$ respectively, and we write $\Gamma \leq \Gamma'$ to denote that Γ is a subgraph of Γ'. The maximal vertex degree of Γ (when it exists) will be denoted by $\Delta(\Gamma)$ or just Δ when there is no ambiguity.

1.1 Invariants and monotonicity under regular maps

Our main result is that many invariants related to graph layout problems can be used as obstructions to the existence of regular maps. Specifically, we will define four spectra of invariants. The first three are denoted ec^p, vc^p and ℓ^p with $p \in [1, +\infty]$: they include several well-studied invariants.

- ec^∞ is cutwidth (cw), and $\mathrm{ec}^1 = \ell^1$ is minimal linear arrangement[1] (mla).
- vc^∞ is vertex separation (vs), and vc^1 is sumcut (sc) which is also equal to profile.
- ℓ^∞ is bandwidth.

[1] also known as the optimal linear ordering, the edge sum, the minimum-1- sum, the bandwidth sum or the wirelength problem in other parts of the literature

The final class of invariants are denoted $\text{wid}^{G,p}$ and are indexed by graphs G and $p \in [1,\infty]$. These also include two well-studied invariants: when G is the disjoint union of all trees or paths, $\text{wid}^{G,\infty} -1$ equals the treewidth (tw) or pathwidth (pw) respectively. We note here that for any finite graph Γ and any G, $\text{wid}^{G,1}(\Gamma) = |V\Gamma|$, but for other finite values of p the invariants are more interesting (cf. Remark 2.5).

We now define our set of invariants

$$\mathcal{I} = \bigcup_{p\in[1,\infty]} \left(\{\text{ec}^p, \text{vc}^p, \ell^p\} \cup \bigcup_{G} \left\{ \text{wid}^{G,p} \right\} \right).$$

Theorem 1.1. *Let X and Y be bounded degree graphs and assume Y has infinitely many edges. If there is a regular map $VX \to VY$, then for every $I \in \mathcal{I}$, there is a constant C such that for all $r \in \mathbb{N}$*

$$I_X(r) \leq C I_Y(Cr + C) + C$$

where $I_X(r) = \max\{I(\Gamma) \mid \Gamma \leq X,\ |V\Gamma| \leq r\}$.

The motivation behind the definition of I_X comes from the separation profile sep_X of Benjamini-Schramm-Timár (where I is the cutsize cf. Definition 4.1) [BST12]. We will refer to I_X as the ***I*-profile of** X, which we generally consider as a $\simeq$-equivalence class of functions partially ordered by $\lesssim$, where $f \lesssim g$ if there is some constant C such that $f(r) \leq Cg(Cr + C) + C$ for all r, and $f \simeq g$ if $f \lesssim g$ and $g \lesssim f$.

1.2 Comparing Invariants

Our next goal is to compare these new invariants with each other and to other cost functions associated to graph layout problems. Firstly, we present some elementary comparisons of ℓ^p norms on functions.

Theorem 1.2. *For any bounded degree graph X and any $p \in [1,\infty]$:*

$$\text{ec}^p_X(r) \simeq \text{vc}^p_X(r).$$

Moreover, for any $p, q \in [1,\infty]$ with $p \leq q$ and any graph G:

$$\text{ec}^q_X(r) \lesssim \text{ec}^p_X(r) \lesssim r^{\frac{1}{p}-\frac{1}{q}}\, \text{ec}^q_X(r),$$
$$\ell^q_X(r) \lesssim \ell^p_X(r) \lesssim r^{\frac{1}{p}-\frac{1}{q}} \ell^q_X(r),$$
$$\text{wid}^{G,q}_X(r) \lesssim \text{wid}^{G,p}_X(r) \lesssim r^{\frac{1}{p}-\frac{1}{q}}\, \text{wid}^{G,q}_X.$$

Throughout the paper we make the convention that $\frac{1}{\infty} = 0$.

Next, using known bounds from the literature, we obtain the following comparisons. Given any finite graph Γ with maximal degree Δ:

- $\mathrm{cut}(\Gamma) \leq \mathrm{tw}(\Gamma)$, [BST12, Lemma 2.3].
- $\mathrm{tw}(\Gamma) \leq \mathrm{pw}(\Gamma)$ by definition.
- $\mathrm{pw}(\Gamma) \leq \mathrm{cw}(\Gamma)$, [Bod88, Theorem 5.4].
- $\mathrm{cw}(\Gamma) \leq \lfloor \frac{1}{2}\Delta \rfloor (\mathrm{sn}(\Gamma) - 1) + 1$, [MS89, Theorem 3.2].
- $\mathrm{sn}(\Gamma) \leq \mathrm{tbw}(\Gamma) + 1$, [MPS83, Corollary 2.2].
- $\mathrm{tbw}(\Gamma) \leq \mathrm{cw}(\Gamma)$, [MPS83, Corollary 2.1].
- $\mathrm{cw}(\Gamma) \leq \Delta\, \mathrm{pw}(\Gamma)$, [CS89].
- $\mathrm{tbw}(\Gamma) \leq \mathrm{bw}(\Gamma)$, by definition.

Here cut is the cutsize, sn is the search number and tbw is the topological bandwidth. We will not define these notions in this paper, the interested reader is recommended to check the above papers and the references therein. In order, the above inequalities prove the relations:

Theorem 1.3. *For any bounded degree graph X,*

$$\begin{aligned} \mathrm{sep}_X &\lesssim \mathrm{tw}_X \lesssim \mathrm{pw}_X \\ \mathrm{pw}_X &\lesssim \mathrm{cw}_X \lesssim \mathrm{sn}_X \lesssim \mathrm{tbw}_X \lesssim \mathrm{cw}_X \lesssim \mathrm{pw}_X \\ \mathrm{tbw}_X &\lesssim \mathrm{bw}_X \,. \end{aligned}$$

In certain cases, we know that the invariants we get definitely differ. For example, when X is the infinite 3-regular tree, $\mathrm{sep}_X(r) = \mathrm{tw}_X(r) = 1$, $\mathrm{pw}_X(r) \simeq \log(r)$ and $\mathrm{bw}_X(r) \simeq r/\log(r)$. We are not aware of an example of a bounded degree graph X where $\mathrm{sep}_X \not\simeq \mathrm{tw}_X$, this and other questions arising from this work are listed in §5.

1.3 Calculations

Our next result states that when the separation profile is a “nice” function and not too small, then all of the invariants mentioned in Theorem 1.3 (except bandwidth) are $\simeq$-equivalent.

Theorem 1.4. *Let X be a bounded degree graph such that $\mathrm{sep}_X(r) \lesssim r^a \log(r)^b$ with $a > 0$ and $b \in \mathbb{R}$. Then*

$$\mathrm{cw}_X(r) \lesssim \begin{cases} r^a \log(r)^b & \textit{if} \quad a > 0 \\ \log(r)^{b+1} & \textit{if} \quad a = 0 \end{cases}$$

In particular, if $\mathrm{sep}_X(r) \simeq r^a \log(r)^b$ with $a > 0$, then

$$\mathrm{sep}_X(r) \simeq \mathrm{tw}_X(r) \simeq \mathrm{pw}_X(r) \simeq \mathrm{cw}_X(r) \simeq \mathrm{sn}_X(r) \simeq \mathrm{tbw}_X(r).$$

Moving to lower bounds, we obtain the following result:

Theorem 1.5. *For every bounded degree graph X and every $p \in [1, \infty]$,*

$$\mathrm{vc}^p_X(r) \gtrsim r^{\frac{1}{p}} \,\mathrm{sep}_X(r).$$

Thus, if $\mathrm{sep}_X(r) \simeq r^a \log(r)^b$ with $a > 0$, then for every $p \in [1, \infty]$

$$\mathrm{vc}^p_X(r) \simeq r^{\frac{1}{p}+a} \log(r)^b.$$

The class of graphs whose separation profile is known to be of the form $r^a \log(r)^b$ with $a > 0$ is rich. As a sample, we obtain the following two results:

Corollary 1.6. *Let X be a bounded degree graph such that $\mathrm{sep}_X(r) \simeq r^{1-1/d}$ for some $d > 1$. For each $I \in \{\mathrm{tw}, \mathrm{pw}, \mathrm{cw}, \mathrm{sn}, \mathrm{tbw}\}$ there is a constant $C = C(X, I)$ such that for every $r \geq 1$*

- *each $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $I(\Gamma) \leq Cr^{1-1/d} + C$,*
- *some $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $I(\Gamma) \geq C^{-1}r^{1-1/d} - C$.*

Moreover, we obtain the following bounds for sumcut and minimal linear arrangement. For each $J \in \{\mathrm{sc}, \mathrm{mla}\}$ there is a constant $C = C(X, J)$ such that for every $r \geq 1$

- *each $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $J(\Gamma) \leq Cr^{2-1/d} + C$,*
- *some $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $J(\Gamma) \geq C^{-1}r^{2-1/d} - C$.*

Examples of such graphs include products $P \times H$ where P is a vertex transitive graph of polynomial growth of degree k, H is quasi-isometric to a rank one symmetric space $\mathbb{H}^m_{\mathbb{K}}$ and $d := k + (m + 1) \dim_{\mathbb{R}} \mathbb{K} - 2 \geq 2$ [HMT20, HMT22]. Other examples of graphs with separation profiles as above include certain fractal approximations [GS21] and more general classes of graphs with polynomial growth [GLC23].

Corollary 1.7. *Let Y be a bounded degree graph such that $\mathrm{sep}_Y(r) \simeq r/\log(1+r)$. For each $I \in \{\mathrm{tw}, \mathrm{pw}, \mathrm{cw}, \mathrm{sn}, \mathrm{tbw}\}$ there is a constant $C = C(Y, I)$ such that for every $r \geq 1$*

- *each $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $I(\Gamma) \leq Cr/\log(1 + r) + C$,*
- *some $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $I(\Gamma) \geq C^{-1}r/\log(1 + r) - C$.*

Moreover, for each $J \in \{\mathrm{sc}, \mathrm{mla}\}$ there is a constant $C = C(Y, J)$ such that for every $r \geq 1$

- *each $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $J(\Gamma) \leq Cr^2/\log(1 + r) + C$,*
- *some $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $J(\Gamma) \geq C^{-1}r^2/\log(1 + r) - C$.*

Examples of such graphs include all Cayley graphs of polycyclic groups with exponential growth and all uniform lattices in semisimple Lie groups whose noncompact factor has rank at least 2, these and further examples can be found in [HMT22]. More generally, any graph Y which contains a Diestel-Leader graph and has finite Assouad-Nagata dimension will also satisfy $\text{sep}_Y(r) \simeq r/\log(1+r)$ [Hum17].

When $a = 0$ we still find upper bounds and they are optimal in the case of the 3-regular tree. However, we suspect that they are not optimal in the following case.

Corollary 1.8. *Let Z be the graph of the tessellation of the hyperbolic plane by right-angled octagons. For each $I \in \{\text{tw}, \text{pw}, \text{cw}, \text{sn}, \text{tbw}\}$ there is a constant $C = C(Z, I)$ such that for every $r \geq 1$, each $\Gamma \leq X$ with $|\Gamma| \leq r$ satisfies $I(\Gamma) \leq C \log(1 + r)^2 + C$.*

Note that when Γ is a metric ball in Z, then a stronger upper bound $I(\Gamma) \leq C \log(1 + r) + C$ is proved in [KMP01, Proposition 2.3].

More generally, upper and lower bounds can be found in all cases where such bounds exist for the separation profile, see for instance [HM20, Coz20] for cases where the separation profile is very small, and very large respectively.

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2 Defining and comparing invariants

2.1 Graph decompositions

Definition 2.1. Let G and Γ be graphs. A **G-decomposition of Γ** is a collection $\{X_g \mid g \in VG\}$ of subsets of $V\Gamma$ satisfying the following three conditions:

1. $\bigcup_{g \in VG} X_g = V\Gamma$,
2. for every edge $vw \in E\Gamma$ there is some $g \in VG$ such that $\{v, w\} \subseteq X_g$,

3. for each $v \in V\Gamma$, the full subgraph of $\{g \in VG \mid v \in X_g\}$ is connected.

We now introduce a cost associated to G-decompositions.

Definition 2.2. For $p \in [1,\infty]$, the ***p*-width** of a G-decomposition of Γ is the ℓ^p norm of the function $VG \to \mathbb{R}$ given by $g \mapsto |X_g|$, i.e. for $p \in [1,\infty)$, the p-width is

$$\left(\sum_{g \in VG} |X_g|^p\right)^{\frac{1}{p}},$$

and the ∞-width is $\max_{g \in VG} |X_g|$.

Remark 2.3. It would be more consistent with the literature to consider the cost function $g \mapsto |X_g|-1$. Our reasons for not doing this are that in the ∞-case the difference of 1 does not change the cost up to $\simeq$, and in the case $p \in [1,\infty)$ we do not need to exclude "small" decompositions (where $|X_g| = 1$ occurs for "most" $g \in VG$) in the monotonicity result.

Now we present the first collection of invariants in $\mathcal{I}$.

Definition 2.4. Let G be any graph. The (G,p)-width of a graph Γ is the minimal p-width over the set of all G-decompositions of Γ. We denote this by $\mathrm{wid}^{G,p}(\Gamma)$.

Commonly studied examples include treewidth $\mathrm{tw}(\Gamma) = \mathrm{wid}^{T,\infty}(\Gamma) - 1$ and pathwidth $\mathrm{pw}(\Gamma) = \mathrm{wid}^{P,\infty}(\Gamma) - 1$, where T is the disjoint union of all finite trees, and P is the disjoint union of all finite paths.

Remark 2.5. We note that for any G and any Γ, $\mathrm{wid}^{G,1}(\Gamma) = |V\Gamma|$. The lower bound is a consequence of the requirement that $\bigcup_{g \in VG} X_g = V\Gamma$, and the upper bound is obtained from any decomposition where $X_g = V\Gamma$ for some $g \in VG$ and $X_{g'} = \emptyset$ for all other $g' \in VG$. For other finite values of p the invariants are more interesting. For instance, if Γ is the subgraph of $\mathbb{Z}^3$ with vertex set $\{1,\dots,k\}^3$ and edges $(a,b,c)(a',b',c')$ whenever $|a-a'|+|b-b'|+|c-c'| = 1$, then

$$X_{(x,y)} = \begin{cases} \{(a,b,c) \mid |a-x|+|b-y| \leq 1\} & \text{if } 1 \leq x,y \leq k, \\ \emptyset & \text{otherwise} \end{cases}$$

defines a $\mathbb{Z}^2$-decomposition of Γ such that

$$\left(\sum_{(x,y) \in \mathbb{Z}^2} |X_{(x,y)}|^p\right)^{1/p} \leq 5k^{1+2/p} = 5|V\Gamma|^{\frac{1}{3}+\frac{2}{3p}}.$$

2.2 Linear arrangements

Definition 2.6. A linear arrangement of a finite graph Γ is a bijective function $f : V\Gamma \to \{1, \dots, |V\Gamma|\}$.

We will associate three cost functions to a linear arrangement.

Definition 2.7. We define the following functions $\{1, \dots, |V\Gamma|\} \to \mathbb{R}$:

- **edge cut**: $i \mapsto \mathrm{ec}_f(i) = |\{vw \in E\Gamma \mid f(v) \le i < f(w)\}|$,
- **vertex cut**: $i \mapsto \mathrm{vc}_f(i) = |\{v \in V\Gamma \mid \exists vw \in E\Gamma,\ f(v) \le i < f(w)\}|$.

We also define a function $E\Gamma \to \mathbb{R}$ called **length**: $vw \mapsto |f(v) - f(w)|$.

Note that edge cut and vertex cut are related in the following way.

Lemma 2.8. *Let Γ be a finite graph with maximal degree Δ. For every linear arrangement f and every i,*

$$\frac{1}{\Delta}\mathrm{vc}_f(i) \le \mathrm{ec}_f(i) \le \Delta\,\mathrm{vc}_f(i)$$

As with graph decompositions, for each finite graph Γ and each $p \in [1, \infty]$ we define the p-edge cut $\mathrm{ec}^p(\Gamma)$ and p-length $\ell^p(\Gamma)$ to be the minimal ℓ^p norm of the respective function over all linear arrangements of Γ.

Definition 2.9. Let X be a graph and let $p \in [1, \infty]$. We define

$$\begin{aligned}
\mathrm{ec}^p_X(r) &= \max\{\mathrm{ec}^p(\Gamma) \mid \Gamma \le X,\ |V\Gamma| \le r\} \\
\mathrm{vc}^p_X(r) &= \max\{\mathrm{vc}^p(\Gamma) \mid \Gamma \le X,\ |V\Gamma| \le r\} \\
\ell^p_X(r) &= \max\{\ell^p(\Gamma) \mid \Gamma \le X,\ |V\Gamma| \le r\}.
\end{aligned}$$

and call these the p-edge cut, p-vertex cut and p-length profiles of X respectively.

Remark 2.10. As noted in the introduction, the ℓ^1 and ℓ^∞ versions of the above invariants are typically known by other names. ∞-length is known as bandwidth bw, 1-length is minimal linear arrangement, ∞-edge cut is cutwidth cw, ∞-vertex cut is vertex separation, 1-vertex cut is sumcut (which is itself equal to profile) and 1-edge cut is equal to 1-bandwidth, cf. [DPS02, §2].

2.3 Modified cutwidth

Another cost associated to a linear arrangement is modified cutwidth (mcw). This invariant is not part of our collection $\mathcal{I}$, but is naturally comparable to ec^1 in the following way.

Lemma 2.11. *For any finite graph Γ,*

$$\mathrm{ec}^1(\Gamma) = \mathrm{mcw}(\Gamma) + |E\Gamma|$$

where $\mathrm{mcw}(\Gamma)$ is the modified cutwidth - the minimal 1-norm of the function $i \mapsto |\{vw \in E\Gamma \mid f(v) < i < f(w)\}|$ over all linear arrangements f.

Proof. For any linear arrangement f:

$$\sum_i |\{vw \in E\Gamma \mid f(v) \leq i < f(w)\} \setminus \{vw \in E\Gamma \mid f(v) < i < f(w)\}|$$
$$= \sum_i |\{vw \in E\Gamma \mid f(v) = i < f(w)\}| = |E\Gamma|.$$ □

3 Monotonicity theorems

The goal of this section is to prove Theorem 1.1.

3.1 Width of graph decompositions

We begin with a useful lemma.

Lemma 3.1. *Let $\{X_g \mid g \in VG\}$ be a G-decomposition of Γ. For every subset $A \subseteq V\Gamma$ which induces a connected subgraph of Γ,*

$$\bigcup_{a \in A} \{g \in VG \mid a \in X_g\}$$

induces a connected subgraph of G.

Proof. Set $G_a = \{g \in VG \mid a \in X_g\}$. As $\{X_g \mid g \in VG\}$ be a G-decomposition of Γ, each G_a induces a connected subgraph of G. Now for each distinct pair $b, b' \in \bigcup_{a \in A} \{g \in VG \mid a \in X_g\}$ there exist $a, a' \in A$ such that $b \in G_a$ and $b' \in G_{a'}$. Let $a = a_0, a_1, \ldots, a_k = a'$ be the vertex set of a path in A. Since $a_{i-1}a_i \in E\Gamma$ for all i, there is some $g \in VG$ such that $\{a_{i-1}, a_i\} \subseteq X_g$. Thus, $g \in G_{a_{i-1}} \cap G_{a_i}$. Therefore,

$$\bigcup_{i=0}^{k} G_{a_i} \subseteq \bigcup_{a \in A} \{g \in VG \mid a \in X_g\}$$

induces a connected subgraph of G which contains b, b'. As this holds for each pair b, b', the lemma follows. □

Now we prove that, for any graph G and any $p \in [1, \infty]$, the (G, p)-width profile behaves monotonically under regular maps.

Proposition 3.2. *Let X and Y be bounded degree graphs. If there is a κ-regular map $\phi : VX \to VY$ then for every graph G and every $p \in [1, \infty]$,*

$$\mathrm{wid}_X^{G,p}(r) \lesssim \mathrm{wid}_Y^{G,p}(r).$$

Proof. Let $\Gamma \leq X$ with $|V\Gamma| \leq r$. For each $xy \in E\Gamma$, let P_{xy} be a path of length at most 2κ in Y connecting $\phi(x)$ to $\phi(y)$. Define

$$\Gamma' = \phi(V\Gamma) \cup \bigcup_{xy \in E\Gamma} P_{xy} \leq Y$$

Given $x \in V\Gamma$, we denote by $\overline{\phi}(x)$ the set of all vertices in

$$\{\phi(x)\} \cup \bigcup_{\{vw \in E\Gamma \ \mid \ x \in \{v,w\}\}} P_{xy}.$$

We have $|V\Gamma'| \leq |V\Gamma| + 2\kappa|E(\Gamma)| \leq |V\Gamma|(1+\kappa\Delta(X))$

Fix a G-decomposition $\left\{X'_g \ \mid g \in G\right\}$ of Γ' with p-width at most $\mathrm{wid}_Y^{G,p}((1+\kappa\Delta(X))r)$.

Now define $X_g = \left\{x \in V\Gamma \ \middle|\ \overline{\phi}(x) \cap X'_g \neq \emptyset\right\}$. We now prove that $\{X_g\}$ is a G-decomposition of Γ. For every $x \in V\Gamma$ $\phi(x) \in V\Gamma'$, so there is some $g \in G$ such that $\phi(x) \in X'_g$. Therefore, $x \in X'_g$, so condition (i) holds. Now let $xy \in V\Gamma$ and let w be any vertex on the path P_{xy}. There is some $g \in G$ such that $w \in X'_g$ and $w \in \overline{\phi}(x) \cap \overline{\phi}(y)$, so $x, y \in X_g$ for the same g. Thus (ii) holds. By construction, for each $x \in V\Gamma$,

$$\{g \in VG \ \mid x \in X_g\} = \bigcup_{y \in \overline{\phi}(x)} \left\{g \in VG \ \middle|\ y \in X'_g\right\}. \tag{3.3}$$

As $\overline{\phi}(v)$ is connected, by Lemma 3.1, the full subgraph of G with vertex set (3.3) is connected, so $\{X_g\}$ is a G-decomposition of Γ.

Finally, for a fixed $y \in V\Gamma'$, if $y \in \overline{\phi}(x)$ then $\phi(x)$ is contained in the ball of radius 2κ centred at y. As $|\phi^{-1}(z)| \leq \kappa$ for every $z \in VY$, we deduce that there are at most $\kappa(1+\Delta(X))^{2\kappa}$ possibilities for $x \in V\Gamma$. Thus, for each $g \in G$,

$$|X_g| \leq \kappa(1+\Delta(X))^{2\kappa}|X'_g|$$

so, for all $p \in [1, \infty]$,

$$\mathrm{wid}^{G,p}(\Gamma) \leq \kappa(1+\Delta(X))^{2\kappa}\,\mathrm{wid}^{G,p}(\Gamma').$$

As this holds for all finite $\Gamma \leq X$, we have

$$\mathrm{wid}_X^{G,p}(r) \leq K\,\mathrm{wid}_Y^{G,p}(Kr)$$

where $K = \kappa(1+\Delta(X))^{2\kappa}$. □

3.2 Linear arrangement costs

Proposition 3.4. *Let X, Y be bounded degree graphs where Y has infinitely many edges. If there is a regular map $VX \to VY$ then for every $p \in [1, \infty]$,*

$$\ell_X^p(r) \lesssim \ell_Y^p(r) \tag{3.5}$$

$$\mathrm{ec}_X^p(r) \lesssim \mathrm{ec}_Y^p(r). \tag{3.6}$$

Proof. Note that as Y has infinitely many edges, then for every r there is a subgraph $\Gamma_r \leq Y$ with at most r vertices and at least $\lfloor \frac{r}{2} \rfloor$ edges. Thus,

$$\min\{\ell_Y^p(r), \mathrm{ec}_Y^p(r)\} \geq \left\lfloor \frac{r}{2} \right\rfloor^{1/p}. \tag{3.7}$$

Therefore, we may assume that $\ell_Y^p(r) \gtrsim r^{1/p}$ and $\mathrm{ec}_Y^p(r) \gtrsim r^{1/p}$.

Since there is a regular map $\phi : VX \to VY$, there is some $\kappa \geq 1$ such that

- $d_Y(\phi(x), \phi(x')) \leq \kappa(1 + d_X(x, x'))$ for all $x, x' \in VX$, and
- $|\phi^{-1}(y)| \leq \kappa$ for all $y \in VY$.

Let $\Gamma \leq X$ with $|V\Gamma| \leq r$. Define a subgraph $\Gamma' \leq Y$ which contains $\phi(V\Gamma)$ and such that whenever $xy \in E\Gamma$, there is a path of length at most 2κ connecting $\phi(x)$ to $\phi(y)$ in Γ'. We may ensure that $\frac{1}{\kappa}|V\Gamma| \leq |V\Gamma'| \leq 2\kappa|E\Gamma| \leq \kappa\Delta(X)|V\Gamma|$.

Given a proper numbering $g : V\Gamma' \to \{1, \ldots, |V\Gamma'|\}$, choose a proper numbering $f : V\Gamma \to \{1, \ldots, |V\Gamma|\}$ with the property that $f(v) \leq f(v')$ implies $g(\phi(v)) \leq g(\phi(v'))$. In particular, this means that the sequence $a_i = g(\phi(f^{-1}(i)))$ is monotonically non-decreasing.

We claim that for every edge $vv' \in E\Gamma$ such that $|f(v) - f(v')| > 2\kappa^2$, there is some edge $ww' \in E\Gamma'$ satisfying

- $d_Y(\phi(v), w)) \leq 2\kappa$, and
- $|f(v) - f(v')| \leq 2\kappa^2|g(w) - g(w')|$.

Given the claim we complete the proof as follows. By the first condition above there is a uniform bound (call it C) on the number of edges vv' which yield the same edge ww'. Thus for $p \in [1, \infty)$,

$$\begin{aligned}
\ell^p(\Gamma) &= \min_f \left(\sum_{vv' \in E\Gamma} |f(v) - f(v')|^p \right)^{1/p} \\
&\leq \min_g \left(C \sum_{ww' \in E\Gamma'} (2\kappa^2|g(w) - g(w')|)^p \right)^{1/p} + \left(\sum_{vv' \in E\Gamma} (2\kappa^2)^p \right)^{1/p} \\
&\leq 2\kappa^2 \left(C^{1/p}\ell^p(\Gamma') + |E\Gamma|^{1/p} \right) \\
&\lesssim \ell^p_Y(\max\{|V\Gamma'|, |E\Gamma|\}).
\end{aligned}$$

Note that the last step uses the assumption that $\ell^p_Y(r) \gtrsim r^{1/p}$. For $p = \infty$,

$$\begin{aligned}
\ell^\infty(\Gamma) &= \min_f \max_{vv' \in E\Gamma} |f(v) - f(v')| \\
&\leq \min_g \max_{ww' \in E\Gamma'} 2\kappa^2|g(w) - g(w')| + 2\kappa^2 \\
&\leq 2\kappa^2 \left(\ell^\infty(\Gamma') + 1 \right).
\end{aligned}$$

It remains to prove the claim. Suppose $|f(v) - f(v')| > 2\kappa^2$. Without loss of generality, assume $k = f(v) < f(v') = k + l$. Since f and g are injective, we have

$$\left| g(\phi(\{f^{-1}(k), \ldots, f^{-1}(k + \ell)\})) \right| \geq \ell/\kappa$$

and $g(\phi(f^{-1}(k))) \leq \ldots \leq g(\phi(f^{-1}(k+\ell)))$, so $g(\phi(f^{-1}(k+\ell))) - g(\phi(f^{-1}(k))) \geq \ell/\kappa$. In particular, $\phi(v) \neq \phi(v')$.

Since $vv' \in E\Gamma$, $d_Y(\phi(v), \phi(v')) \leq 2\kappa$. Let $\phi(v) = w_0, w_1, \ldots, w_m = \phi(v')$ be the vertices of a path in Γ' with $m \leq 2\kappa$. Since $g(w_m) - g(w_0) \geq \ell/\kappa$, there is some i such that $|g(w_{i+1}) - g(w_i)| \geq \ell/(2\kappa^2)$. We choose $w = w_i$ and $w' = w_{i+1}$. By construction $ww' \in E\Gamma'$, $d_Y(\phi(v), w) \leq 2\kappa$ and $|f(v) - f(v')| \leq 2\kappa^2|g(w) - g(w')|$ completing the claim.

For edge cut we use the same proper numbering f. Fix $1 < k \leq |V\Gamma|$. For convenience define

$$F_k = \{vv' \in E\Gamma \mid f(v) \leq k < f(v')\}$$
$$G_l = \{ww' \in E\Gamma' \mid g(w) \leq l < g(w')\}$$

and write $f_k = |F_k|$, $g_l = |G_l|$. We claim that $f_k \leq Cg_{a_k} + \kappa\Delta(X)$ for some constant C which depends only on κ and $\Delta(Y)$. We recall that $a_k = g(\phi(f^{-1}(k)))$.

Given the claim, we see that for $p = \infty$,

$$\begin{aligned}
\mathrm{ec}^\infty(\Gamma) &= \min_f \max_k f_k \\
&\leq \min_g \max_l Cg_l + \kappa\Delta(X) \\
&\leq C\,\mathrm{ec}^\infty(\Gamma') + \kappa\Delta(X).
\end{aligned}$$

When $p \in [1, \infty)$, using the fact that for any l there are at most κ values of k such that $a_k = l$, we have

$$\begin{aligned}
\mathrm{ec}^p(\Gamma) &= \min_f \left(\sum_k f_k^p\right)^{1/p} \\
&\leq \min_g \left(\sum_l \kappa\left(Cg_l + \kappa\Delta(X)\right)^p\right)^{1/p} \\
&\leq C\kappa^{1/p} \min_g \left(\sum_l (g_l)^p\right)^{1/p} + C\kappa^{1+1/p}\Delta(X)|V\Gamma'|^{1/p} \\
&\leq 2C\kappa^{1/p} \max\left\{\min_g \left(\sum_l (g_l)^p\right)^{1/p}, \kappa\Delta(X)|V\Gamma'|^{1/p}\right\} \\
&\lesssim \mathrm{ec}^p_Y(|V\Gamma'|).
\end{aligned}$$

Note that the final step uses the assumption made at the start of the proof that $\mathrm{ec}^p_Y(r) \gtrsim r^{1/p}$.

Now we prove the claim. For each edge in the set F_k, we have $g(\phi(v)) \leq a_k \leq g(\phi(v'))$. Since ϕ has preimages of cardinality at most κ there are at most $\kappa\Delta(X)$ edges in $E\Gamma$ with an end vertex w such that $g(\phi(w)) = a_k$.

If $vv' \in F_k$ is not one of these edges, then $g(\phi(v)) \leq a_k < g(\phi(v'))$. There is a path in Γ' connecting $\phi(v)$ to $\phi(v')$ with length at most 2κ, and there is some edge ww' on this path such that $g(w) \leq a_k < g(w')$.

As in the previous argument, since $d_Y(\phi(v), w)) \leq 2\kappa$ there is a uniform bound C on the number of edges vv' which yield the same edge vv' via this process. Thus,

$$g_{a_k} \geq \frac{f_k - \kappa\Delta(X)}{C},$$

as required. $\square$

4 Calculations

4.1 Proof of Theorems 1.4 and 1.5

We briefly recall the definition of the separation profile.

Definition 4.1. Let Γ be a finite graph and let $\varepsilon \in (0,1)$. A subset $C \subseteq V\Gamma$ is a ε**-cutset** if every connected component of $\Gamma \setminus C$ contains at most $\varepsilon|V\Gamma|$ vertices. The ε**-cutsize** of a finite graph Γ, $(\mathrm{cut}^\varepsilon(\Gamma))$ is the minimal cardinality of a ε-cutset.

The ε**-separation profile** of a graph X is the function

$$\mathrm{sep}^\varepsilon_X(r) = \max\left\{\mathrm{cut}^\varepsilon(\Gamma) \ \mid \Gamma \leq X, \ |V\Gamma| \leq r\right\}$$

One important result from [BST12] is that for all $\varepsilon, \varepsilon' \in (0,1)$,

$$\mathrm{sep}^\varepsilon_X(r) \simeq \mathrm{sep}^{\varepsilon'}_X(r).$$

The key observation here is the following. A similar argument appears in [DV03].

Lemma 4.2. *Let X be a bounded degree graph and let $\varepsilon \in (0,1)$. Then*

$$\mathrm{ec}^\infty_X(r) \leq \mathrm{ec}^\infty_X(\varepsilon r) + \Delta(X)\,\mathrm{sep}^\varepsilon_X(r).$$

Proof. Let $\Gamma \leq X$ with $|\Gamma| \leq r$. Let C be a ε-cutset of Γ satisfying $|\Gamma| \leq \mathrm{sep}^\varepsilon_X(r)$, and let $A_1, \ldots, A_k$ be the connected components of $\Gamma - C$. Choose bijections $\sigma_i : A_i^0 \to \{1, \ldots, |A_i|\}$ such that $\max_i |\{vw \in EA_i : \ \sigma_i(v) \leq i < \sigma_i(w)\}| = \mathrm{ec}^\infty(A_i)$. Now we define a linear arrangement σ of $V\Gamma'$. For each $v \in A_i$, define

$$\sigma(v) = |A_1| + \ldots + |A_{i-1}| + \sigma_i(v)$$

and map the vertices in C to $\{\sum_{i=0}^k |A_i| + 1, \ldots, |V\Gamma|\}$ in any bijective way.

For all $|A_1| + \ldots + |A_{i-1}| \leq j < |A_1| + \ldots + |A_i|$, we have

$$\begin{aligned} |\{vw \in E\Gamma : \ \sigma(v) \leq j < \sigma(w)\}| &\leq \mathrm{ec}^\infty(A_i) + |\{vw \in E\Gamma : \{v,w\} \cap C \neq \emptyset\}| \\ &\leq \mathrm{ec}^\infty(A_i) + \Delta(X)\,\mathrm{sep}^\varepsilon_X(r). \end{aligned}$$

Moreover, for $j > |A_1| + \ldots + |A_k|$,

$$|\{vw \in E\Gamma : \ \sigma(v) \leq j < \sigma(w)\}| \leq \Delta(X)\,\mathrm{sep}^\varepsilon_X(r).$$

As this holds for all Γ, and $|A_i| \leq \varepsilon r$ for all i, we see that

$$\mathrm{cw}_X(r) \leq \mathrm{ec}^\infty_X(\varepsilon r) + \Delta(X)\,\mathrm{sep}^\varepsilon_X(r). \qquad \square$$

Repeatedly applying the above lemma (with $\varepsilon = \frac{1}{2}$) we get a strong control on ec^∞ in terms of separation.

Corollary 4.3. *Let X be a bounded degree graph. Then*

$$\mathrm{ec}_X^\infty(r) \leq \Delta(X) \sum_{i=0}^{\lceil \log_2(r) \rceil} \mathrm{sep}(2^{-i}r) \leq \Delta(X)(1 + \lceil \log_2(r) \rceil)\, \mathrm{sep}_X(r).$$

Proof. Repeatedly applying Lemma 4.2, we see that

$$\mathrm{ec}_X^\infty(r) \leq \mathrm{ec}_X^\infty\left(\frac{r}{2^{\lceil \log_2(r) \rceil}}\right) + \Delta(X) \sum_{i=0}^{\lceil \log_2(r) \rceil} \mathrm{sep}(2^{-i}r).$$

As $2^{\lceil \log_2(r) \rceil} \geq r$ and the cutwidth of a graph with at most 1 vertex is 0, the result follows. □

When the separation profile is known to be sufficiently nice, the above bound can be improved.

Corollary 4.4. *Let X be a bounded degree graph such that $\mathrm{sep}_X(r) \lesssim r^a \log(r)^b$ where $a \in [0,1]$ and $b \in \mathbb{R}$. Then*

$$\mathrm{ec}_X^\infty(r) \lesssim \begin{cases} r^a \log(r)^b & \text{if} \quad a > 0, \\ \log(r)^{b+1} & \text{if} \quad a = 0. \end{cases}$$

Proof. By Corollary 4.3

$$\begin{aligned} \mathrm{ec}_X^\infty(r) &\leq \Delta(X) \sum_{i=0}^{\lceil \log_2(r) \rceil} \mathrm{sep}(2^{-i}r) \\ &\leq \Delta(X) \sum_{i=0}^{\lceil \log_2(r) \rceil} C(C2^{-i}r)^a \log(C2^{-i}r)^b \\ &\leq \Delta(X) C^{1+a} \log(Cr)^b \sum_{i=0}^{\lceil \log_2(r) \rceil} (2^{-i}r)^a. \end{aligned}$$

If $a > 0$, then $\sum_{i=0}^{\lceil \log_2(r) \rceil} (2^{-i}r)^a \leq \frac{r^a}{1-2^{-a}}$, so

$$\mathrm{ec}_X^\infty(r) \leq C' r^a \log(Cr)^b.$$

If $a = 0$, then $\sum_{i=0}^{\lceil \log_2(r) \rceil} (2^{-i}r)^a = 1 + \lceil \log_2(r) \rceil$, so

$$\mathrm{ec}_X^\infty(r) \leq \Delta(X) C \log(Cr)^b (1 + \lceil \log_2(r) \rceil) \lesssim \log(r)^{b+1}.$$ □

Proof of Theorem 1.4. By Corollary 4.4, if $\mathrm{sep}_X(r) \lesssim r^a \log(r)^b$, then

$$\mathrm{cw}_X(r) \simeq \mathrm{ec}_X^\infty(r) \lesssim r^a \log(r)^b.$$

If, in addition $\mathrm{sep}_X(r) \gtrsim r^a \log(r)^b$, then by Theorem 1.3

$$r^a \log(r)^b \lesssim \mathrm{sep}_X(r) \lesssim \mathrm{cw}_X(r) \lesssim r^a \log(r)^b.$$ □

Finally, we prove Theorem 1.5, which immediately follows from the following:

Proposition 4.5. *Let Γ be a finite graph and let $p \in [1, \infty)$. We have*

$$\mathrm{vc}^p(\Gamma) \geq \left\lfloor \frac{|V\Gamma|}{3} \right\rfloor^{\frac{1}{p}} \mathrm{cut}^{\frac{2}{3}}(\Gamma).$$

Proof. Choose a bijection $f : V\Gamma \to \{1, \ldots, |V\Gamma|\}$, and note that for each $\frac{|V\Gamma|}{3} \leq i \leq \frac{2|V\Gamma|}{3}$,

$$C_i := \{v \in V\Gamma \mid \exists vw \in E\Gamma,\ f(v) \leq i < f(w)\}$$

is a $\frac{2}{3}$ cutset of Γ, since no connected component of $\Gamma \setminus C_i$ can contain a vertex in both $f^{-1}(\{1, \ldots, i\})$ and $f^{-1}(\{i+1, \ldots, |V\Gamma|\})$. As there are at least $\left\lfloor \frac{|V\Gamma|}{3} \right\rfloor$ such values of i, we have

$$\mathrm{vc}^p(\Gamma) \geq \left\lfloor \frac{|V\Gamma|}{3} \right\rfloor^{\frac{1}{p}} \mathrm{cut}^{\frac{2}{3}}(\Gamma). \qquad \square$$

4.2 Coarse wirings and graph width

We recall the definition of coarse wirings and the coarse wiring profile from [BH21]. The definition given here is slightly different to allow situations where the image graph does not have bounded degree. In the bounded degree case, the two definitions are equivalent up to a multiplicative error depending only on maximal degree.

Definition 4.6. Let Γ, Γ' be graphs. A **wiring** of Γ into Γ' is a continuous map $f : \Gamma \to \Gamma'$ which maps vertices to vertices and each edge xy to a walk W_{xy} which starts at $f(x)$ and ends at $f(y)$.

A wiring f is a **coarse k-wiring** if the preimage of each vertex of $V\Gamma'$ contains at most k vertices in $V\Gamma$, and each vertex $v \in V\Gamma'$ is contained in at most k of the walks in $\mathcal{W} = \{W_{xy} \mid xy \in E\Gamma\}$.

We consider the **image** of a wiring $\mathrm{im}(f)$ to be the graph

$$\phi(V\Gamma) \cup \bigcup_{xy \in E\Gamma} W_{xy} \leq \Gamma'.$$

The **volume** of a wiring $\mathrm{vol}(f)$ is the number of vertices in its image.

Let Γ be a finite graph and let Y be a graph. We denote by $\mathrm{wir}^k(\Gamma \to Y)$ the minimal volume of a coarse k-wiring of Γ into Y. If no such coarse k-wiring exists, we say $\mathrm{wir}^k(\Gamma \to Y) = +\infty$.

Remark 4.7. The definition of a coarse k-wiring in [BH21] requires that each **edge** $e \in E\Gamma'$ is contained in at most k of the walks in $\mathcal{W} = \{W_{xy} \mid xy \in E\Gamma\}$. Any k-coarse wiring in the sense of [BH21] is a $2k$-coarse wiring as defined here. When Γ' has bounded degree, any k-coarse wiring as defined here is a $k\Delta(\Gamma')$-coarse wiring in the sense of [BH21]. We write the definition in this way here as we will want to allow Γ' to have unbounded degree.

We associate two cost functions to the difficulty of coarse wiring one graph into another.

Definition 4.8. Let X and Y be graphs. We define $\mathrm{par}_{X\to Y}(r)$ to be the minimal k such that every $\Gamma \leq X$ with $|V\Gamma| \leq r$ admits a k-coarse wiring into Y.

For each k, we define $\mathrm{wir}^k_{X\to Y}(r) = \max\left\{\mathrm{wir}^k(\Gamma \to Y) \ \mid \Gamma \leq X,\ |V\Gamma| \leq r\right\}$.

Note that when X has bounded degree, the upper bound $\mathrm{par}_{X\to Y}(r) \leq \Delta(X)r$ is obtained by mapping all of Γ to a single vertex.

Given a graph G, coarse wiring costs can be compared with $\mathrm{wid}^{G,\infty}$ in the following way:

Proposition 4.9. *For any bounded degree graph X and any graph G,*

$$\mathrm{wid}^{G,\infty}_X(r) \leq 2\,\mathrm{par}_{X\to G}(r).$$

Proof. Let $\Gamma \leq X$ with $|V\Gamma| \leq r$ and let $f : \Gamma \to G$ be a k-coarse wiring with $k \leq \mathrm{par}_{X\to G}(r)$.

For each $x \in V\Gamma$, set $\overline{f}(x)$ to be the set of all vertices in $\bigcup_{xy\in E\Gamma} W_{x,y}$. We claim that $X_g = \left\{x \in V\Gamma \ \middle|\ g \in \overline{f}(x)\right\}$ defines a G-decomposition of Γ.

For each edge $xy \in E\Gamma$, $\{x, y\} \subset X_g$ for any $g \in VW_{xy} \neq \emptyset$. Next, by construction, for each $x \in V\Gamma$,

$$\{g \in G \ \mid x \in X_g\} = \overline{f}(x)$$

which induces a connected subgraph $(\bigcup_{xy\in E\Gamma} W_{x,y})$ of G.

Finally, for each g, there are at most k edges $xy \in E\Gamma$ such that $g \in W_{x,y}$ and therefore at most $2k$ vertices $x \in E\Gamma$, such that $g \in \overline{f}(x)$. □

We define one more informative variation of $\mathrm{wid}^{G,p}$ which is more closely related to the volume of coarse wirings.

Definition 4.10. Fix ℓ. Given a graph X and a graph G, we define $\mathrm{wid}^{G,p,\ell}(\Gamma)$ to be the minimal p-norm of a G-decomposition (X_g) of Γ with $\max_g |X_g| \leq \ell$.

Now, given a graph X and a function $f : \mathbb{N} \to \mathbb{N}$, we define

$$\mathrm{wid}^{G,p,f}_X(r) = \max\left\{\mathrm{wid}^{G,p,f(r)}(\Gamma) \ \mid \Gamma \leq X,\ |V\Gamma| \leq r\right\}.$$

Repeating the proof of Proposition 3.2 we get the following:

Proposition 4.11. *If there is a κ-regular map $VX \to VY$ then there is a constant $C = C(\kappa, \Delta(X))$ such that for every graph G and function f,*

$$\mathrm{wid}^{G,p,f}_X(r) \lesssim \mathrm{wid}^{G,p,Cf}_Y(r).$$

We can also compare this variant of 1-width with the coarse wiring profile by repeating the argument of Proposition 4.9.

Proposition 4.12. *Let X be a bounded degree graph and let G be a graph. For every $k \in \mathbb{N}$ there is some $\ell \in \mathbb{N}$ such that*

$$\mathrm{wid}_X^{G,1,\ell}(r) \lesssim \mathrm{wir}_{X\to G}^k(r).$$

We note that the $\simeq$-class of $\mathrm{wir}_{X\to G}^k(r)$ can vary depending on the parameter k, as demonstrated in [Rai23].

5 Questions

Here we list a few questions which naturally arise from this work.

Question 5.1. Is there a bounded degree graph X such that $\mathrm{sep}_X(r) \not\simeq \mathrm{tw}_X(r)$?

In addition to those situations where $\mathrm{sep}_X(r) \simeq \mathrm{tw}_X(r)$ given by this paper, Benjamini-Schramm-Timár prove that $\mathrm{sep}_X(r) \simeq 1$ if and only if $\mathrm{tw}_X(r) \simeq 1$ [BST12]. We are also unaware of any bounded degree graphs with any of the following properties:

Question 5.2. For $I \in \{\mathrm{ec}, \ell\}$, is there a bounded degree graph X and $p \in [1, +\infty)$ such that $I_X^p(r) \not\simeq r^{1/p} I_X^\infty(r)$?

More generally, the only examples we know of the following phenomenon are for $I = \mathrm{wid}^G$, $p = 1$ and $q = \infty$.

Question 5.3. For $I \in \{\mathrm{ec}, \ell, \mathrm{wid}^G\}$, are there bounded degree graphs X and Y, and $p, q \in [1, +\infty]$ such that $I_X^p(r) \simeq I_Y^p(r)$ but $I_X^q(r) \not\simeq I_Y^q(r)$?

We give one natural example of a graph where the cutwidth profile is not known.

Question 5.4. Let X be the graph of a tesselation of the hyperbolic plane (say by regular right-angled octagons). Is $\mathrm{cw}_X(n) \simeq \log(n)$?

From the results of this paper we know that $\log(n) \lesssim \mathrm{cw}_X(n) \lesssim \log(n)^2$ and by [KMP01, Proposition 2.3] all balls in planar hyperbolic graphs have cutwidth which grows at most logarithmically in the number of vertices contained in the ball.

It is natural to ask under which circumstances a converse to Proposition 4.12 holds.

Question 5.5. For which graphs X and G is it true that for every $\ell \in \mathbb{N}$ there is some $k \in \mathbb{N}$ such that

$$\mathrm{wir}_{X\to G}^k(r) \lesssim \mathrm{wid}_X^{G,1,f}(r)?$$

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