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Balanced Walls for Random Groups

John M. Mackay & Piotr Przytycki

Abstract. We study a random group G in the Gromov density model and its Cayley complex X. For density $< \frac{5}{24}$, we define walls in X that give rise to a nontrivial action of G on a CAT(0) cube complex. This extends a result of Ollivier and Wise, whose walls could be used only for density $< \frac{1}{5}$. The strategy employed might be potentially extended in future to all densities $< \frac{1}{4}$.

1. Introduction

Following Gromov [Gro93] and Ollivier [Oll05], we study random groups in the following *Gromov density model.* Fix m letters $S = \{s_1, s_2, \dots, s_m\}$, and let S^{-1} denote the formal inverses of S. Choose a *density* $d \in (0, 1)$. A *random group* (*presentation*) *at density d and length l* is a group $G = \langle S|R \rangle$, where R is a collection of $\lfloor (2m-1)^{dl} \rfloor$ cyclically reduced words in $S \cup S^{-1}$ of length l chosen independently and uniformly at random. In our article we assume additionally that l is even. A random group (presentation) at density d has property P *with overwhelming probability* (shortly *w.o.p.*) if the probability of G having P tends to 1 as $l \to \infty$.

Gromov and Ollivier proved that for $d > \frac{1}{2}$, a random group G is w.o.p. $\mathbb{Z}/2\mathbb{Z}$ (we assumed l to be even), whereas for $d < \frac{1}{2}$, it is w.o.p. nonelementary hyperbolic with hyperbolicity constant linear in l, torsion free, and with contractible Cayley complex X [Gro93; Oll04]. For $d > \frac{1}{3}$, a random group G has w.o.p. Kazhdan's property (T), which was proved by Żuk [Żuk03] (and completed by Kotowski and Kotowski [KK13]). On the other hand, property (T) fails for $d < \frac{1}{5}$ since in that range Ollivier and Wise [OW11] proved that w.o.p. G acts nontrivially on a CAT(0) cube complex (they also proved that the action is proper for $d < \frac{1}{6}$).

Their cube complex is obtained from Sageev's construction [Sag95], using an action of G on a suitable space with walls. They use the following wall structure in the Cayley complex X of $G = \langle S|R \rangle$: Consider the graph whose vertices are edge midpoints of X and whose edges are pairs of opposite edge midpoints in the 2-cells of X. *Hypergraphs* are connected components of that graph, immersed in X in such a way that its edges are mapped to the diagonals of the 2-cells. Ollivier and Wise prove that for $d < \frac{1}{5}$, a hypergraph is w.o.p. an embedded tree, separating X essentially, with cocompact stabilizer H. Thus, possibly after replacing H with its index 2 subgroup preserving the halfspaces, the number of relative ends satisfies

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$e(G, H) > 1$, and hence the action of G on the CAT(0) cube complex given by Sageev's construction is nontrivial. However, for $d > \frac{1}{5}$, w.o.p. hypergraphs self-intersect, and thus we do not have control on H.

The aim of our paper is to introduce a new wall structure by replacing the antipodal relation inside 2-cells by a different relation, so that the resulting hypergraphs are embedded trees and we can perform Sageev's construction. Whereas our strategy is designed to work up to density $\frac{1}{4}$, the technical complications that arise force us for the moment to content ourselves with the following.

Theorem 1.1. *In the Gromov density model at density $< \frac{5}{24}$, a random group w.o.p. acts nontrivially on a* CAT(0) *cube complex and does not satisfy Kazhdan's property* (T).

The CAT(0) cube complex in Theorem 1.1 can be chosen to be finite-dimensional and cocompact; see Remark 6.3.

We assumed l to be even only to have an easy proof of Lemma 6.2, which would have been otherwise slightly more difficult and would also require $d > \frac{1}{8}$ (to which we actually could have restricted). For l odd, one subdivides the edges of X into two and replaces l with $2l$.

Strategy Outline

In the remaining part of the Introduction, we outline our strategy for the proof of Theorem 1.1. Our starting point is Figure 1, left (Figure 21 from [OW11]), explaining why at $d > \frac{1}{5}$ a hypergraph obtained from the antipodal relation in 2-cells self-intersects. The principal reason for the self-intersection to appear is the sharp turn that the hypergraph makes in the union T of the two 2-cells C, C' on the left. As d approaches $\frac{1}{4}$, the possible length $|A|$ of the common path $A = C \cap C'$ approaches $\frac{1}{2}l$. Hence, the distance in $T^{(1)}$ between the endpoints of a hypergraph segment tends to 0, see Figure 1, right. This is bad since it can easily be turned into a self-intersection by adding a third 2-cell to T as in Figure 1, left.

To remedy this, whenever $|A| > \frac{1}{4}l$, we replace the antipodal relation in one of the two 2-cells of T, say in C', by the relation $\sim$ described in Figure 2. Specifically, we consider two subpaths α_+, α_- of A of length $\lceil |A| - \frac{1}{4}l \rceil$ containing the

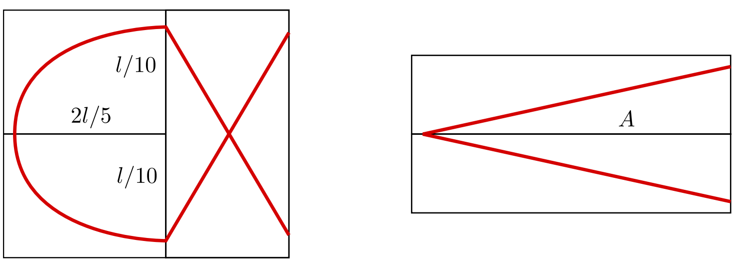

Figure 1

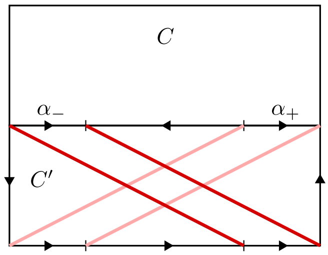

Figure 2

endpoints of A. Let $s_\pm$ be the symmetry of $\alpha_\pm$ exchanging its endpoints. If x, y are antipodal edge midpoints of C' and y lies in the interior of $\alpha_\pm$, then we put $x \sim s_\pm(y)$, otherwise let $x \sim y$.

This relation has the following advantage over the antipodal one. Let x, x' be in the same hypergraph of T. We claim that the distance between x and x' in $T^{(1)}$ is bounded below by

$$\mathrm{Bal}(T) = \frac{1}{4}(|T| + 1)l - |A|,$$

where $|T| = 2$ is the number of 2-cells in T. This value is called the *balance* of T and is bounded from above by $\frac{1}{2}l$. Notice that for $d < \frac{1}{4}$, it is also bounded below by $\frac{1}{4}l$ since $|A| < \frac{1}{2}l$.

To justify the claim, there are four cases to consider. If x, x' are antipodal in C or C', then there is nothing to prove. If x, x' are both in C' and $x' = s_\pm(y)$, where y is antipodal to x, then since the distance $|x', y|$ satisfies $|x', y| < |A| - \frac{1}{4}l$, it suffices to use the triangle inequality. Otherwise $x \in C - C'$, $x' \in C' - C$, and the hypergraph segment xx' crosses A in an edge midpoint y such that: either y is antipodal to both x and x' and lies outside of the interiors of $\alpha_\pm$; or y lies in the interior of, say, α_+ and is antipodal to x in C, whereas $s_+(y)$ is antipodal to x' in C'. In the first situation, y is at distance $\leq \frac{1}{4}l$ from the endpoints of A, so that the distance between x, x' is $\geq \frac{1}{2}l + \frac{1}{2}l - 2(\frac{1}{4}l)$, as desired. In the second situation the sum of the distances of y, $s_+(y)$ from an endpoint of A is $\leq 2|A| - (|A| - \frac{1}{4}l) = \frac{1}{4}l + |A|$, so that the distance between x, x' is $\geq \frac{1}{2}l + \frac{1}{2}l - (\frac{1}{4}l + |A|)$, which finishes the proof of the claim.

Note that this is the best estimate we can hope for: consider the edge midpoint $y \in \alpha_+$ nearly at the endpoint of A and suppose that we attempt to move $x' \sim y$ in C', which is the antipode of $s_+(y)$. Then either we decrease the distance between x' and y, or we decrease the distance between x' and the antipode x of y in C, and both these distances are nearly equal to $\mathrm{Bal}(T)$.

We call T a *tile* and the relation on the edge midpoints of T induced by the relations in C and C' a *balanced tile-wall structure.* We iterate this construction: whenever two tiles, or a tile and a 2-cell have large overlap, we change the antipodal relation into one that makes the tile-wall structure balanced. The tiles in X will not share 2-cells, except for very particular configurations, and will be used instead of 2-cells in van Kampen diagrams. One way to think about this is that since we do not see negative curvature on the original presentation level, we zoom out and look at tiles instead of 2-cells, where we are already able to define walls with negative curvature behavior.

There are two technical problems that one would need to overcome to extend the proof of Theorem 1.1 to all densities $< \frac{1}{4}$. First of all, we need to understand the combinatorial complication coming from tiles sharing 2-cells (generalization of assertions (i)–(iii) in Proposition 4.10). Secondly, even for a tile disjoint from all other ones (as in Step 1 of Construction 3.7), but glued of two tiles of size ≥ 3, we do not know in general how to define a balanced tile-wall structure, that is, how to extend Part 1 of Proposition 4.10.

Organization

In Section 2 we discuss the isoperimetric inequality for random groups. In Section 3 we define tiles, and we equip them with balanced tile-wall structures in Section 4. We then show that induced hypergraphs are embedded trees (Section 5), which are quasi-isometrically embedded (Section 6), and we conclude with the proof of Theorem 1.1.

2. Isoperimetric Inequality

In this section we recall Ollivier's isoperimetric inequality for disc diagrams in random groups, extended to uniformly bounded nonplanar complexes by Odrzygóźdź.

We always assume that 2-cells in our complexes are l-gons with l even. A *disc diagram* D is a contractible 2-complex with a fixed embedding in $\mathbb{R}^2$. Its *boundary path* ∂D is the attaching map of the cell at infinity.

Suppose that Y is a 2-complex, not necessarily a disc diagram. Let the *size* $|Y|$ denote the number of 2-cells of Y. If A is a graph and is not treated as a disc diagram of size 0, then we denote by $|A|$ the number of 1-cells in A. The *cancellation* of Y is

$$\mathrm{Cancel}(Y) = \sum_{e \text{ an edge of } Y} (\deg(e) - 1),$$

where $\deg(e)$ is the number of times that e appears as the image of an edge of the attaching map of a 2-cell of Y. Observe that if D is a disc diagram, then $\mathrm{Cancel}(D)$ counts the number of internal edges of D.

REMARK 2.1. Suppose that $Y_i \subset X$ are subcomplexes that are closures of their 2-cells, and that they do not share 2-cells. Then

$$\mathrm{Cancel}\Big(\bigcup_i Y_i\Big) \geq \sum_i \Big(\mathrm{Cancel}(Y_i) + \frac{1}{2}\Big|Y_i \cap \bigcup_{j \neq i} Y_j\Big|\Big).$$

Equality holds if and only if no triple of Y_i shares an edge.

We say that Y is *fulfilled* by a set of relators R if there is a combinatorial map from Y to the presentation complex X/G that is locally injective around edges (but not necessarily around vertices). In particular, any subcomplex of the Cayley complex X is fulfilled by R. Since X is simply connected, for any closed path α in $X^{(1)}$, there exists a disc diagram D with a map $D \to X$ such that ∂D maps to α. Moreover, by canceling some 2-cells we can assume that D is fulfilled by R. We say that $D \to X$ is a *disc diagram for* α.

THEOREM 2.2 [Oll07, Thm. 2]. *For each* $\varepsilon > 0$, *w.o.p. there is no disc diagram* D *fulfilling* R *and satisfying*

$$\mathrm{Cancel}(D) > (d+\varepsilon)|D|l.$$

(*Equivalently, every disc diagram* D *fulfilling* R *has* $|\partial D| \geq (1-2d-2\varepsilon)|D|l$.)

We deduce the following:

LEMMA 2.3. *Let* $d < \frac{1}{4}$. *Then w.o.p. there is no embedded closed path in* $X^{(1)}$ *of length* $< l$.

Proof. Otherwise, let $D \to X$ be a disc diagram for that closed path. The case $|D| = 1$ is not possible. Otherwise $|D| \geq 2$, and hence

$$\mathrm{Cancel}(D) = \frac{1}{2}(|D|l - |\partial D|) > \frac{1}{2}|D|l - \frac{1}{2}l \geq \frac{1}{4}|D|l,$$

which contradicts Theorem 2.2. □

Lemma 2.3 immediately implies the following corollaries.

COROLLARY 2.4. *Let* $d < \frac{1}{4}$. *Then w.o.p. the boundary paths of all* 2*-cells embed in* X.

COROLLARY 2.5. *Let* $d < \frac{1}{4}$. *Then w.o.p. every path* α *embedded in* $X^{(1)}$ *of length* $\leq \frac{1}{2}l$ *is geodesic in* $X^{(1)}$.

COROLLARY 2.6. *Let* $d < \frac{1}{4}$. *Then w.o.p. there is no immersed closed path* $\alpha : I \to X^{(1)}$ *with* $|\alpha(I)| < l$.

Another consequence of Theorem 2.2 is the following result of Ollivier and Wise, whose proof we include as a warm-up.

LEMMA 2.7 [OW11, Cor. 1.11]. *Let* $d < \frac{1}{4}$. *Then w.o.p. for all intersecting* 2*-cells* C, C' *of* X, *we have that* $C \cap C'$ *is connected.*

Proof. If $C \cap C'$ is not connected, then there is in $C \cup C'$ a homotopically non-trivial embedded closed path $\alpha \cup \alpha'$ of length $\leq l$ with α in C and α' in C'. This contradicts Lemma 2.3 unless $|\alpha| = |\alpha'| = \frac{1}{2}l$. By Theorem 2.2, as in the proof of Lemma 2.3, this shows that $\alpha \cup \alpha'$ bounds a disc diagram D of size $|D| = 1$, hence consisting of a single 2-cell C''. This contradicts Theorem 2.2 with $d < \frac{1}{4}$ for the diagram $C \cup C''$. □

We close with the following variant of Theorem 2.2 for uniformly bounded non-planar complexes. We say that a 2-complex Y is (K, K')-*bounded* if $|Y| \le K$ and Y is obtained from the disjoint union of its 2-cells by gluing them along $\le K'$ subpaths of their boundary paths. Note that for $d < \frac{1}{4}$, by Corollary 2.4 and Lemma 2.7, if $Y \subset X$, then $|Y| \le K$ implies that Y is $(K, \frac{1}{2}K(K-1))$-bounded.

PROPOSITION 2.8 (see [Odr14, Thm. 1.5]). *For any K, K' and $\varepsilon > 0$, w.o.p. there is no (K, K')-bounded 2-complex Y fulfilling R and satisfying*

$$\mathrm{Cancel}(Y) > (d+\varepsilon)|Y|l.$$

In fact, Odrzygóźdź proves the following stronger result. We say that Y has L *fixed paths* if we distinguish L subpaths of the boundary paths of the 2-cells in Y. We denote their union by $\mathrm{Fix}(Y)$. A *labeling* of a 2-complex Y with fixed paths is a combinatorial map from $\mathrm{Fix}(Y)$ to $X^{(1)}/G$. A *polynomial labeling scheme* is a function assigning to each 2-complex Y with fixed paths a set of labelings, where the cardinality of the set of labelings is bounded by a polynomial in l.

PROPOSITION 2.9 [Odr14, Thm. 1.5]. *Given a polynomial labeling scheme, for any K, K', L, and $\varepsilon > 0$, w.o.p. there is no (K, K')-bounded 2-complex Y with L fixed paths fulfilled by R in such a way that the combinatorial map to X/G restricts on* $\mathrm{Fix}(Y)$ *to one of the labelings assigned to Y by the scheme, and satisfying*

$$\mathrm{Cancel}(Y) + |\mathrm{Fix}(Y)| > (d+\varepsilon)|Y|l.$$

We will use only the following consequence of Proposition 2.9.

COROLLARY 2.10. *Let $d < \frac{1}{4}$. Consider 2-complexes $Y' \subset X$ with $2 \le |Y'| \le K$, fulfilled by R in such a way that exactly one 2-cell $C' \subset Y'$ is carried by the map to X/G onto the 2-cell corresponding to a specified relator r_1. With overwhelming probability, there is no such Y' satisfying*

$$\mathrm{Cancel}(Y') > \frac{1}{4}(|Y'|-1)l.$$

Proof. Let $K' = \frac{1}{2}K(K-1)$ and $L = K-1$. We apply Proposition 2.9 to the random presentation with relators $R - \{r_1\}$, which are independent from r_1. Consider the polynomial labeling scheme assigning the labelings that restrict on each of the L paths of $\mathrm{Fix}(Y)$ to subwords of the cyclic translates of r_1.

Given Y' as in the statement of Corollary 2.10, we consider $Y \subset Y'$ that is the closure of the 2-cells distinct from C'. Let $\mathrm{Fix}(Y) = C' \cap Y$. By Lemma 2.7 the intersection $C' \cap C$ is connected for any 2-cell $C \subset Y$, and thus $\mathrm{Fix}(Y)$ is a union of at most $L = K-1$ subpaths of the boundary paths of 2-cells. We have $\mathrm{Cancel}(Y') = \mathrm{Cancel}(Y) + |\mathrm{Fix}(Y)|$. Moreover, since Y' is fulfilled by R in such a way that only C' is carried by the map to X/G onto r_1, the 2-complex Y is fulfilled by $R - \{r_1\}$ in such a way that the restriction to $\mathrm{Fix}(Y)$ is one of the labelings assigned to Y by our scheme. Thus, the desired inequality follows from that in Proposition 2.9. □

3. Tiles

In this section we describe the construction of *tiles* mentioned in the Introduction. From now on we always assume that $d < \frac{1}{4}$.

DEFINITION 3.1. A *tile* T is a single 2-cell or a 2-complex T that is the closure of its 2-cells, that satisfies

$$\mathrm{Cancel}(T) > \frac{1}{4}(|T| - 1)l$$

and can be expressed as a union of two tiles that do not share a 2-cell. A tile *in* X is a tile that is a subcomplex of X.

REMARK 3.2. Let T, T' be tiles in X that do not share 2-cells. If $|T \cap T'| > \frac{1}{4}l$, then by Remark 2.1 the union $T \cup T'$ is a tile. In the case where T, T' are single 2-cells, conversely, if $T \cup T'$ is a tile, then $|T \cap T'| > \frac{1}{4}l$.

REMARK 3.3. If T is a tile in X, by Proposition 2.8 for any K and $\varepsilon > 0$, w.o.p. if $|T| \leq K$, then we have $(d+\varepsilon)|T| > \frac{1}{4}(|T|-1)$. It follows, since $d < \frac{1}{4}$, that w.o.p. the size $|T|$ of a tile is uniformly bounded. Explicitly, if $d < \frac{1}{4}\frac{N}{N+1}$, then $|T| \leq N$ since it suffices to consider $K = 2N$ to exclude the possibility of obtaining tiles by gluing two tiles of size $\leq N$. In particular, for $d < \frac{5}{24}$, we have $|T| \leq 5$.

However, the reader will see that the tiles effectively considered in the article will have size ≤ 4.

We now generalize Lemma 2.7.

LEMMA 3.4. *Let T, T' be intersecting tiles in X that do not share 2-cells. Then $T \cap T'$ is connected.*

Before we give the proof, we deduce the following:

REMARK 3.5. By Proposition 2.8 applied to $T \cup T'$ we have

$$\begin{aligned}|T \cap T'| &= \mathrm{Cancel}(T \cup T') - \mathrm{Cancel}(T) - \mathrm{Cancel}(T') \\ &< \frac{1}{4}(|T| + |T'|)l - \frac{1}{4}(|T| - 1)l - \frac{1}{4}(|T'| - 1)l = \frac{1}{2}l.\end{aligned}$$

By Corollary 2.5, $T \cap T'$ is a forest and hence a tree by Lemma 3.4. It follows that tiles in X are contractible.

Proof of Lemma 3.4. If $T \cap T'$ is not connected, then there is in $T \cup T'$ a homotopically nontrivial embedded closed path $\alpha \cup \alpha'$ with α in T and α' in T'. Let $D \to X$ be a disc diagram for $\alpha \cup \alpha'$. By Remark 3.3 the size of $T \cup T'$ is uniformly bounded, and hence $|\alpha \cup \alpha'|$ is uniformly bounded as well. By Theorem 2.2, $|D|$ is uniformly bounded. After passing to a subdisc of D and allowing

α, α' to be immersed, we can also assume that the cells in D adjacent to α, respectively α', are not mapped to T, respectively T'.

Let Y be the union of $T \cup T'$ with the image of D in X. The size of Y is uniformly bounded, so we will be able to apply Proposition 2.8 to Y. Let $\mathcal{C}$ be the set of 2-cells of $Y - T \cup T'$. Let P be the image of ∂D in Y. We estimate $\mathrm{Cancel}(Y)$ using Remark 2.1 with $\{Y_i\} = \{T, T'\} \cup \mathcal{C}$. The edges of P contribute $\frac{1}{2}|P|$ in total to the terms with $Y_i = T, T'$. Boundary paths of the 2-cells of $\mathcal{C}$ contribute additionally $\frac{1}{2}|\mathcal{C}|l$ in total to their own terms. By Corollary 2.6 we have $|P| \geq l$. Thus,

$$\begin{aligned}\mathrm{Cancel}(Y) &\geq \mathrm{Cancel}(T) + \mathrm{Cancel}(T') + \frac{1}{2}|P| + \frac{1}{2}|\mathcal{C}|l \\ &> \frac{1}{4}(|T|-1)l + \frac{1}{4}(|T'|-1)l + \frac{1}{2}l + \frac{1}{4}|\mathcal{C}|l = \frac{1}{4}|Y|l,\end{aligned}$$

which contradicts Proposition 2.8. □

DEFINITION 3.6. A *tile assignment* $\mathcal{T}$ assigns G-equivariantly to each 2-cell C of X a tile $\mathcal{T}(C)$ in X containing C. An example of a tile assignment is $\mathcal{T}_0(C) = C$ consisting of single 2-cells. If $T = \mathcal{T}(C)$ for some C of X, then we say that T *belongs* to $\mathcal{T}$ and write $T \in \mathcal{T}$.

CONSTRUCTION 3.7. We will make use of a particular tile assignment $\mathcal{T} = \mathcal{T}_k$ obtained as a last tile assignment in a sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k$, where $\mathcal{T}_0$ is as in Definition 3.6, and $\mathcal{T}_{i+1}$ is constructed from $\mathcal{T}_i$ in the following process consisting of Step 1 and Step 2. During Step 1 of the process, every 2-cell of X will be in exactly one $T \in \mathcal{T}_{i+1}$.

STEP 1. For $i = 0, 1, \ldots$, we repeat the following construction of $\mathcal{T}_{i+1}$ while there are distinct $T, T' \in \mathcal{T}_i$ satisfying $|T| + |T'| \leq 4$ and $|T \cap T'| > \frac{1}{4}l$.

Choose T, T' so that $|T| + |T'|$ is maximal possible, which means in particular that if T is a single 2-cell, then we first consider T' consisting of two 2-cells, rather than T' that is a single cell. This will be used only later in Proposition 4.10. By Remark 3.2 the union $T \cup T'$ is a tile.

We claim that the tiles T, T' are not in the same G-orbit. Otherwise, if $T' = gT$, then let Y be the 2-complex obtained from $T \cup T'$ by identifying T with T'. In other words, Y is obtained from T by identifying for all the pairs of 2-cells C, C' of T the paths $C \cap g(C')$ and $g^{-1}(C) \cap C'$. Thus, Y is $(|T|, \frac{1}{2}(|T|(|T|-1)) + |T|^2)$-bounded. Since $\mathrm{Cancel}(Y) = \mathrm{Cancel}(T) + |T \cap T'| > \frac{1}{4}|T|l$, this contradicts Proposition 2.8, justifying the claim.

Let $\mathcal{T}_{i+1}$ be obtained from $\mathcal{T}_i$ by differing it only on gC for all $g \in G$ and $\mathcal{T}_i(C) \in \{T, T'\}$ and putting $\mathcal{T}_{i+1}(gC) = gT \cup gT'$. Loosely speaking, we glue the tiles T and T'.

The process in Step 1 terminates since the tiles have bounded size and hence there are finitely many tile orbits. Once this process terminates, we initiate the process described in Step 2:

STEP 2. Repeat the following construction of $\mathcal{T}_{i+1}$ while there are $T \in \mathcal{T}_i$ with $|T| = 2$ and a 2-cell $C = \mathcal{T}_i(C)$ such that $T' = T \cup C$ is a tile.

Note that by Step 1 we have $|C \cap T| \leq \frac{1}{4}l$. Let C be chosen so that $|C \cap T|$ is maximal possible. Consider first the case where there is a 2-cell $C' = \mathcal{T}_i(C') \neq C$ such that $|C' \cap T'| > \frac{1}{4}l$. The 2-cells C, C' cannot be in the same G-orbit; otherwise, the complex obtained from $T' \cup C'$ by identifying C with C' would violate Proposition 2.8. In that case let $\mathcal{T}_{i+1}$ be obtained from $\mathcal{T}_i$ by redefining $\mathcal{T}_{i+1}(gC) = \mathcal{T}_{i+1}(gC') = gT' \cup gC'$. In the case where there is no such C', we redefine only $\mathcal{T}_{i+1}(gC) = gT'$. Note that we keep $\mathcal{T}_{i+1}(gC'') = gT$ for a 2-cell C'' of T.

REMARK 3.8. Each tile $T \in \mathcal{T}$ obtained in Construction 3.7 contains a unique tile $T_c \in \mathcal{T}$ that also belongs to the tile assignment in which we terminate after Step 1. If $T_c \subsetneq T$, then $|T_c| = 2$. We call T_c the *core* of T. If distinct $T, T' \in \mathcal{T}$ share 2-cells, then these are the two 2-cells of $T_c = T'_c$ with $|T_c| = 2$ (because in Step 2 we worked only with $|T| = 2$).

4. Tile-Walls

In this section we will extend the hypergraph construction from the strategy outline in the Introduction to all the tiles in the tile assignment $\mathcal{T}$ from Construction 3.7. Recall our standing assumption $d < \frac{1}{4}$.

DEFINITION 4.1. Let T be a tile. A *tile-wall structure* on T is an equivalence relation $\sim_T$ on the edge midpoints of T such that:

- The relation $\sim_T$ restricts to the boundary path of each 2-cell C of T to a relation $\sim_C$ that has exactly two elements in each equivalence class.
- For each equivalence class $\mathcal{W}$ of $\sim_T$, called a *tile-wall,* consider the graph $\Gamma_{\mathcal{W}}$ in T, obtained by connecting the points of $\mathcal{W}$ in the boundary path of each 2-cell C by a diagonal in C. We call $\Gamma_{\mathcal{W}}$ the *hypergraph* of $\mathcal{W}$ and require that it is a tree.

If $x \sim_T x' \in \mathcal{W}$, then the unique path from x to x' in $\Gamma_{\mathcal{W}}$ is called the *hypergraph segment* between x and x' and is denoted by xx'.

DEFINITION 4.2. Let T be a 2-complex. The *balance* of T is the value

$$\mathrm{Bal}(T) = \frac{1}{4}(|T| + 1)l - \mathrm{Cancel}(T).$$

Note that if T is a tile, then $\mathrm{Bal}(T) \leq \frac{1}{2}l$ by Definition 3.1. Moreover, if T is a tile in X, then $\mathrm{Bal}(T) > \frac{1}{4}l$ by Proposition 2.8 since $d < \frac{1}{4}$.

DEFINITION 4.3. Let C be a 2-cell in a tile T. A tile-wall structure on T is C-*balanced* if for any tile-wall $\mathcal{W}$ and $x, x' \in \mathcal{W}$ such that the hypergraph segment xx' traverses C, the distance between x and x' in $T^{(1)}$ satisfies

$$|x, x'|_T \geq \mathrm{Bal}(T).$$

For example, a tile-wall structure on a single 2-cell C is C-balanced if and only if $\sim_C$ is the antipodal relation. We say that a tile-wall structure on T is *balanced* if it is C-balanced for every 2-cell C in T.

Before we construct balanced tile-walls in Example 4.9 and Proposition 4.10, we need a handful of lemmas.

LEMMA 4.4. *Let T, T' be tiles in X that do not share 2-cells, and suppose that T has a C-balanced tile-wall structure $\sim_T$ for some 2-cell C in T. Let $x \sim_T x'$ be such that xx' traverses C. Then at most one of x, x' lies in T'.*

In particular, if the tile-wall structure is balanced, then the conclusion holds for all distinct $x \sim_T x'$.

Proof of Lemma 4.4. If both x, x' lie in T', then by Lemma 3.4 we have $|T \cap T'| \geq |x, x'|_T \geq \mathrm{Bal}(T)$. Thus,

$$\begin{aligned}\mathrm{Cancel}(T \cup T') &= |T \cap T'| + \mathrm{Cancel}(T) + \mathrm{Cancel}(T') \\ &\geq \frac{1}{4}(|T|+1)l + \frac{1}{4}(|T'|-1)l = \frac{1}{4}(|T \cup T'|)l,\end{aligned}$$

which contradicts Proposition 2.8. □

LEMMA 4.5. *Let T, T' be tiles in X that do not share 2-cells with $|T \cap T'| \geq \frac{1}{4}l$. Suppose that T has a C-balanced tile-wall structure for some 2-cell C in T. Let $x \sim_T x'$ be such that xx' traverses C. Then*

$$|x, x'|_{T \cup T'} \geq \mathrm{Bal}(T \cup T') + |T \cap T'| - \frac{1}{4}l.$$

Proof. By Remark 3.5 we have $|T \cap T'| < \frac{1}{2}l$. Hence, by Corollary 2.5 we obtain

$$|x, x'|_{T \cup T'} = |x, x'|_T \geq \mathrm{Bal}(T).$$

On the other hand,

$$\begin{aligned}\mathrm{Bal}(T \cup T') &= \frac{1}{4}(|T \cup T'|+1)l - \mathrm{Cancel}(T \cup T') \\ &= \frac{1}{4}(|T|+1)l + \frac{1}{4}|T'|l - (\mathrm{Cancel}(T) + |T \cap T'| + \mathrm{Cancel}(T')) \\ &\leq \mathrm{Bal}(T) - |T \cap T'| + \frac{1}{4}l.\end{aligned}$$

□

LEMMA 4.6. *Let T, T' be tiles in X that do not share 2-cells with tile-wall structures that are C-(respectively C'-)balanced. Let α be an embedded path in $T \cap T'$ of length $\leq \frac{1}{4}l$ such that $T \cap T'$ is contained in the $\frac{1}{4}l$-neighborhood of α. Let $s : \alpha \to \alpha$ be the symmetry exchanging the endpoints of α. Suppose that we have edge midpoints $x \in T$, $x' \in T'$, $y \in \alpha$ such that $x \sim_T y$, $x' \sim_{T'} s(y)$, where xy, $x's(y)$ traverse C, C', respectively. Then*

$$|x, x'|_{T \cup T'} \geq \mathrm{Bal}(T \cup T').$$

In the proof we need the following:

SUBLEMMA 4.7. *Let A be a tree, and $\alpha \subset A$ a path such that A is contained in the q-neighborhood of α. Let s be the symmetry of α exchanging its endpoints. Then for any points $z, z' \in A$ and $y \in \alpha$, we have*

$$|y, z|_A + |s(y), z'|_A \leq |A| + \max\{|\alpha|, q\}.$$

Proof. First, consider the case where the paths yz, $s(y)z'$ in A intersect outside α. Then they leave α in the same point, and hence $|yz \cap \alpha| + |s(y)z' \cap \alpha| \leq |\alpha|$. Their length outside α is bounded by both q and $|A| - |\alpha|$. Thus, $|y, z|_A + |s(y), z'|_A \leq |\alpha| + (q + |A| - |\alpha|)$, as desired. In the second case, where yz, $s(y)z'$ are allowed to intersect only in α, we have $|y, z|_A + |s(y), z'|_A \leq 2|\alpha| + (|A| - |\alpha|)$. □

Proof of Lemma 4.6. We apply Sublemma 4.7 with $A = T \cap T'$. The upper bound from Sublemma 4.7 is $\leq |A| + \frac{1}{4}l$. Let z, z' be the closest point projections to A of x, x' in the 1-skeleton of $T \cup T'$. By Sublemma 4.7 we have $|y, z| + |s(y), z'| \leq |A| + \frac{1}{4}l$. Then

$$\begin{aligned}
&|x, x'|_{T\cup T'} \\
&\quad \geq |x, z|_T + |x', z'|_{T'} \geq |x, y|_T - |y, z|_T + |x', s(y)|_{T'} - |s(y), z'|_{T'} \\
&\quad \geq \frac{1}{4}(|T| + 1)l - \mathrm{Cancel}(T) + \frac{1}{4}(|T'| + 1)l - \mathrm{Cancel}(T') - \left(|A| + \frac{1}{4}l\right) \\
&\quad = \frac{1}{4}(|T \cup T'| + 1)l - \mathrm{Cancel}(T \cup T'),
\end{aligned}$$

as desired (the last equality comes from Remark 2.1). □

Applying Lemma 4.6 with α equal to a point y, we obtain the following. Note that the distance condition on y is satisfied automatically if $|T \cap T'| \leq \frac{1}{4}l$.

COROLLARY 4.8. *Let T, T' be tiles in X that do not share 2-cells with tile-wall structures that are C-(respectively C'-)balanced. Let $y \in T \cap T'$ be an edge midpoint such that $T \cap T'$ is contained in the $\frac{1}{4}l$-neighborhood of y. Suppose that we have edge midpoints $x \in T$, $x' \in T'$ satisfying $x \sim_T y$, $x' \sim_{T'} y$, where xy, $x'y$ traverse C, C', respectively. Then*

$$|x, x'|_{T\cup T'} \geq \mathrm{Bal}(T \cup T').$$

The following warm-up example generalizes the balanced tile-wall construction from the Introduction.

EXAMPLE 4.9. Let T be a tile, and let T' be a complex obtained by gluing to T a 2-cell C along a path A of length $\frac{1}{4}l < |A| < \frac{1}{2}l$. Suppose that T has a balanced tile-wall structure $\sim_T$. We can then extend $\sim_T$ to the following balanced tile-wall structure $\sim_{T'}$.

Let $\alpha_+, \alpha_- \subset A$ be subpaths of length $\lceil |A| - \frac{1}{4}l \rceil$ starting at the endpoints of A. The paths $\alpha_\pm$ are disjoint since $|A| < \frac{1}{2}l$. Let β_+, β_- be the images in ∂C of α_+,

α_- under the antipodal map. Note that $\beta_\pm$ are outside T since $|A| < \frac{1}{2}l$. Let s_+ be the symmetry of α_+ exchanging its endpoints, and let s_- be the symmetry of α_- exchanging its endpoints.

We define $\sim_C$ to be the antipodal relation outside the union of the interiors of $\alpha_+, \alpha_-, \beta_+$, and β_-. For an edge midpoint x in the interior of $\beta_\pm$ and its antipode $y \in \alpha_\pm$, we define $x \sim_C s_\pm(y)$. By Lemma 4.4 for each pair of edge midpoints related by $\sim_T$, at most one of them lies in A, and by construction the same holds for $\sim_C$. Thus, the relation $\sim_{T'}$ generated by $\sim_T$ and $\sim_C$ is a tile-wall structure.

Now we show that the relation $\sim_{T'}$ is balanced. Consider distinct $x \sim_{T'} x'$. If $x, x' \in T$, then by Lemma 4.5 we have $|x, x'|_{T'} \geq \mathrm{Bal}(T')$, as desired. Secondly, consider the case where $x, x' \in C$. If x, x' are not antipodal, then one of them, say x, lies in $\alpha_\pm$, so the antipode of x' is $s_\pm(x)$. By Lemma 4.5 we thus have $|s_\pm(x), x'|_{T'} \geq \mathrm{Bal}(T') + |A| - \frac{1}{4}l > \mathrm{Bal}(T') + |x, s_\pm(x)|_{T'}$, and by the triangle equality we obtain the desired bound on $|x, x'|_{T'}$.

Finally, consider the case where $x \in T - C$, $x' \in C - T$. Thus, there is $y \in A$ with $x \sim_T y$ and $y \sim_C x'$. If $y \in \alpha_\pm$, then the required estimate follows from Lemma 4.6. Otherwise, y and x' are antipodal, and we use Corollary 4.8.

Now there follows the key result of the article, where we construct C-balanced tile-wall structures on all the tiles from the tile assignment in Construction 3.7. In Part 1, we consider the tiles obtained in Step 1 of that construction, extending Example 4.9. In Part 2, we need to deal with tiles obtained in Step 2, which might share 2-cells according to Remark 3.8. To deal with this complication in later sections, we need to record additional ad hoc properties (ii)–(iii) in Proposition 4.10, which we recommend to ignore at a first reading.

PROPOSITION 4.10. *For the collection of tiles T belonging to the tile assignment $\mathcal{T}$ from Construction 3.7, there are tile-wall structures $\sim_T$ that are C-balanced for each C with $T = \mathcal{T}(C)$. Moreover*:

(i) *The relation $\sim_T$ on $T \in \mathcal{T}$ restricts to $\sim_{T_c}$ on the core $T_c \in \mathcal{T}$ from Remark 3.8.*

(ii) *If C' is a 2-cell of $T - T_c$ with $x \sim_{C'} y$ distinct and not antipodal in C', then one of x, y, say y, lies in T_c, and the edge midpoint y' antipodal to x also lies in T_c.*

(iii) *If C' is a 2-cell of $T - T_c$ with $x \sim_T y \sim_T w$, where $x \neq y \in C'$ and $w \in T_c$, then one of x, y lies in T_c, and the other lies in no other 2-cells of T except for C'.*

Proof. Recall that in Construction 3.7 we obtain $\mathcal{T} = \mathcal{T}_k$ as the last of a sequence of tile assignments $(\mathcal{T}_i)$. We will construct inductively relations $\sim^i_T$ on the tiles $T \in \mathcal{T}_i$ satisfying required conditions for $\mathcal{T} = \mathcal{T}_i$. More precisely, for all 2-cells of X, we will construct $\sim^i_C$ generating $\sim^i_T$; in particular, assertion (i) will be automatic. Note that for $\mathcal{T} = \mathcal{T}_0$, where $\mathcal{T}_0(C) = C$, it suffices to consider the antipodal relation.

Part 1. During Step 1 of Construction 3.7, distinct tiles in $\mathcal{T}_{i+1}$ do not share 2-cells, and for each 2-cell C of $T \in \mathcal{T}_{i+1}$, we have $\mathcal{T}_{i+1}(C) = T$. Thus, if $T, T' \in \mathcal{T}_i$ are as in Step 1 of Construction 3.7, we only need to construct a tile-wall structure on $T \cup T'$ that is balanced (assertions (ii)–(iii) are void). If $|T| + |T'| \leq 3$, then at least one of T, T' is a single cell, and such a tile-wall structure is given in Example 4.9.

Now assume that in Step 1 we have $|T| = |T'| = 2$. Without loss of generality assume that T appeared for smaller i in $\mathcal{T}_i$ than T'. Denote the 2-cells of T' by C_1, C_2. Note that the intersection path $\alpha_j = C_j \cap T$ cannot have length $> \frac{1}{4}l$: otherwise, by the maximality condition in Step 1, instead of gluing C_1 to C_2 to obtain T', we would have had to glue C_j to T. In particular, the intersection $T \cap T'$ has the form of a (possibly degenerate) tripod $\alpha_1 \cup \alpha_2$, where an endpoint of α_1 coincides with an endpoint of α_2, and the other endpoint u_1 of α_1 (respectively u_2 of α_2) is outside α_2 (respectively α_1). Moreover, the complement in $\alpha_1 \cup \alpha_2$ of the $\frac{1}{4}l$-neighborhood of u_2 (respectively u_1) is either empty or is a path containing u_1 (u_2) disjoint from α_2 (α_1). This path is an edge-path if l is divisible by 4; otherwise, it ends with a half-edge. Its span, which is an edge-path, will be called α_+ (α_-).

We change the relation $\sim^i_{C_1}$ (which does not have to be antipodal at this stage) to $\sim^{i+1}_{C_1}$ in the following way. Let s_+ be the symmetry of α_+ exchanging its endpoints. If we have distinct $x \sim^i_{C_1} y$ with y in the interior of α_+, then we replace it with $x \sim^{i+1}_{C_1} s_+(y)$. Analogically, let s_- be the symmetry of α_- exchanging its endpoints. If we have distinct $x \sim^i_{C_2} y$ with y in the interior of α_-, then we replace it with $x \sim^{i+1}_{C_2} s_-(y)$. All other relations remain unchanged.

By Lemma 4.4 the relation $\sim^{i+1}_{T\cup T'}$ generated by $\sim^i_T$, $\sim^{i+1}_{C_1}$, and $\sim^{i+1}_{C_2}$ is a tile-wall structure. It is balanced by Lemmas 4.5, 4.6, and Corollary 4.8 by considering the same four cases as in Example 4.9. This closes the construction of tile-walls for the tiles in $\mathcal{T}_{i+1}$ from Step 1 of Construction 3.7.

Part 2. Now consider $T, C, C' \in \mathcal{T}_i$ as in Step 2 of Construction 3.7 (C' might not be defined). Note that by the process in Step 1 we have, when defined, all $|T \cap C|, |T \cap C'|, |C \cap C'| \leq \frac{1}{4}l$. Consequently, $|T \cap (C \cup C')|, |C \cap (C' \cup T)| \leq \frac{1}{2}l$.

We first claim that the tile-wall structure $\sim^{i+1}_{T'}$ on $T' = T \cup C$ generated by $\sim^i_T$ and the antipodal relation $\sim^i_C$ is C-balanced. Indeed, suppose that $x \sim^{i+1}_{T'} x'$ and that xx' traverses C. Then without loss of generality we have $x' \in C - T$. If $x \in T - C$, then $|x, x'|_{T'} \geq \mathrm{Bal}(T')$ by Corollary 4.8. Otherwise, x is the antipode of x' in C, so we have trivially $|x, x'|_{T'} = \frac{1}{2}l$, which is $\geq \mathrm{Bal}(T')$ by Definition 4.2. This justifies the claim.

If we continue to glue a 2-cell C' to T', and let A be the path $C' \cap T'$ of length $|A| > \frac{1}{4}l$. Note that by Lemma 3.4 the path A consists of three segments, the first one in $T - C$, the second one (possibly degenerate) in $T \cap C$, and the third one in $C - T$. Let $\alpha \subset A$ be the subpath of length $\lceil |A| - \frac{1}{4}l \rceil$ containing the endpoint of A that lies in T. Since $A \cap C = C' \cap C$ has length $\leq \frac{1}{4}l$, the interior of the path α

is disjoint from C. We set $\sim_{C'}^{i+1}$ to be antipodal except in the interior of α and its antipodal image β, where for antipodal $x \in \beta$, $y \in \alpha$, we put $x \sim_{C'}^{i+1} s(y)$, where s is the symmetry of α exchanging its endpoints.

Let $T'' = T' \cup C'$. By Lemma 4.4 the relation $\sim_{T''}^{i+1}$ generated by $\sim_{T'}^{i+1}$ and $\sim_{C'}^{i+1}$ is a tile-wall structure. We now prove that $\sim_{T''}^{i+1}$ is C-balanced and C'-balanced. Let $x \sim_{T''}^{i+1} x'$ with the hypergraph segment xx' traversing C or C'. If xx' is contained in T' or C', then the required estimate follows from the previous claim and from Lemma 4.5, as in the first two cases of Example 4.9.

Otherwise, we can assume that $x \in T' - C'$, $x' \in C' - T'$, and there is $y \in A \cap xx'$. Note that if the neighborhood of y in yx lies in C, then we have $yx \subset C$ since the length of $|C \cap (T \cup C')|$ is $\leq \frac{1}{2}l$ and $\sim_C^{i+1} = \sim_C^i$ was antipodal. In this case, $|x,x'|_{T''} = |x,x'|_{C \cup C'}$ by Corollary 2.5, and the latter is $\geq \frac{1}{2}l$ by Corollary 4.8 applied with C and C' playing the roles of T, T'.

Otherwise, the neighborhood of y in yx lies in T. If, nevertheless, yx traverses C, then since $\sim_{T'}^{i+1}$ is C-balanced, we can apply Lemma 4.6 and Corollary 4.8 as in the last two cases of Example 4.9 to obtain $|x,x'|_{T''} \geq \mathrm{Bal}(T'')$.

It remains to consider the situation where $yx \subset T$. By Lemma 4.6 and Corollary 4.8 we obtain $|x,x'|_{T \cup C'} \geq \mathrm{Bal}(T \cup C')$. By Corollary 2.5 we have $|x,x'|_{T''} = |x,x'|_{T \cup C'}$. By the process in Step 2 we have $|C \cap T| \geq |C' \cap T|$, so that $|C \cap (C' \cup T)| \geq |C' \cap T'| > \frac{1}{4}l$. Thus,

$$\mathrm{Bal}(T'') = \mathrm{Bal}(T \cup C') + \frac{1}{4}l - |C \cap (C' \cup T)| < \mathrm{Bal}(T \cup C') \leq |x,x'|_{T''},$$

as desired.

Assertions (ii) and (iii) follow immediately from the construction. $\square$

In the next section we will operate on tiles that we will need to make disjoint. To do this, we will sometimes replace them by single 2-cells, according to the behavior of the wall in which we will be interested:

DEFINITION 4.11. Let $\mathcal{T}$ be the tile assignment from Construction 3.7. Let C be a 2-cell of X and $\mathcal{W}$ a wall of $T = \mathcal{T}(C)$ intersecting C. We assign to each such pair $(C, \mathcal{W})$, the *augmented* tile denoted by $\mathcal{T}(C, \mathcal{W})$ that equals

- T if $\mathcal{W}$ intersects the core T_c of T and
- C otherwise.

If $\gamma \subset C$ is a hypergraph segment of a wall $\mathcal{W}$ of T, then we denote the augmented tile $\mathcal{T}(C, \mathcal{W})$ also by $\mathcal{T}(C, \gamma)$.

REMARK 4.12. Suppose that we have a C-balanced tile-wall structure on T satisfying Proposition 4.10(ii). If $\mathcal{T}(C, \mathcal{W}) = C$, then the two points of $\mathcal{W}$ in C are antipodal. Thus, in general, any $x, y \in \mathcal{W}$ in $T' = \mathcal{T}(C, \mathcal{W})$ such that xy traverses C satisfy $|x, y|_{T'} \geq \mathrm{Bal}(T')$.

5. Walls

DEFINITION 5.1. Suppose that on each 2-cell C of the Cayley complex X we have a relation $\sim_C$ on edge midpoints that has exactly two elements in each equivalence class. A *wall structure* on X is the equivalence relation $\sim$ on edge midpoints of X generated by such $\sim_C$.

For an equivalence class $\mathfrak{W}$ of $\sim$, called a *wall,* consider the *hypergraph* $\Gamma_{\mathfrak{W}}$, immersed in X, obtained by connecting the points $x \sim_C x' \in \mathfrak{W}$ in each 2-cell C by a diagonal in C. A *hypergraph segment* is an edge-path in $\Gamma_{\mathfrak{W}}$.

We consider tile-wall structures $\sim_T$ on the tiles in the tile assignment $\mathcal{T}$ from Construction 3.7 satisfying Proposition 4.10. By Proposition 4.10(i) they restrict to consistent $\sim_C$ on 2-cells and thus give rise to a wall structure $\sim$ on X, which we fix from now on.

THEOREM 5.2. *If $d < \frac{5}{24}$, then w.o.p. all hypergraphs are embedded trees.*

The proof of Theorem 5.2 is divided into two parts. In the current section we prove its weaker version: hypergraphs of a priori bounded length are embedded trees. We will complete the proof in Section 6.

There is a technical preliminary step to perform. To have control over a (self-intersecting) hypergraph segment and the tiles it traverses, we will decompose it into particular subsegments in Definition 5.3. Next, we will improve the decomposition so that it becomes tight; see Definition 5.8 and Proposition 5.9. We recommend the reader to skip the proof of the latter at a first reading.

DEFINITION 5.3. A concatenation $\gamma_1 \cdots \gamma_n$ forming a hypergraph segment is its *decomposition of length n* if for each $i = 1, \ldots, n$, for one of the 2-cells C_i traversed by γ_i, γ_i is contained in $T_i = \mathcal{T}(C_i, \gamma_i \cap C_i)$, which is the augmented tile for C_i.

REMARK 5.4. Given a decomposition $\gamma_1 \cdots \gamma_n$, the tile T_i (but not C_i) is uniquely determined by γ_i. Otherwise, if we had $\gamma_i \subset T_i' \neq T_i$ with $T_i' = \mathcal{T}(C_i', \gamma_i \cap C_i')$, by Definition 4.11 and Remark 3.8 we would have $T_i, T_i' \in \mathcal{T}$ and $T_i \cap T_i' = T_c$, which is their common core. Consequently, we would have $\gamma_i \subset T_c = C_i \cup C_i'$, which would yield $T_i = \mathcal{T}(C_i, \gamma_i \cap C_i) = T_c = \mathcal{T}(C_i', \gamma_i \cap C_i') = T_i'$, a contradiction.

DEFINITION 5.5. Consider a decomposition $\gamma = \gamma_1 \cdots \gamma_n$. Denote the endpoints of γ by x_0 and x_n. We say that $\gamma_1 \cdots \gamma_n$ is *returning at T_0* for a tile $T_0 \in \mathcal{T}$ if $x_0, x_n \in T_0$ and there is no T_i that contains $\bigcup_j T_j$.

The main goal of this section is to prove the following:

PROPOSITION 5.6. *Let $d < \frac{5}{24}$. For each N, w.o.p. there is no decomposition $\gamma_1 \cdots \gamma_n$ returning at tile in $\mathcal{T}$ with $n \leq N$.*

Proposition 5.6 implies the aforementioned weak version of Theorem 5.2 in view of the following observation.

Remark 5.7. If a hypergraph segment γ self-intersects in a 2-cell C_0 of X, then let $C_0, C_1, \ldots, C_n, C_{n+1} = C_0$ be consecutive 2-cells traversed by the diagonals $\gamma_i \subset \gamma$. Let $T_0 = \mathcal{T}(C_0)$, and for $1 \leq i \leq n$, let $T_i = \mathcal{T}(C_i, \gamma_i)$. No T_i contains all the others since hypergraphs in tiles are embedded trees. Thus, $\gamma_1 \cdots \gamma_n$ is returning at T_0.

The tiles $T_1, \ldots, T_n$ of a decomposition $\gamma_1 \cdots \gamma_n$ can share 2-cells. Before we begin the proof of Proposition 5.6, we will modify the decomposition so that T_i overlap in the following controlled way.

Definition 5.8. A decomposition $\gamma_1 \cdots \gamma_n$ is *tight* if for $1 \leq i < j \leq n$, the tiles T_i, T_j share no 2-cells, except for possibly some pairs $T_i, T_{i+1} \in \mathcal{T}$ with common core T_c, in which case γ_{i+1} is exactly a diagonal of C_{i+1} and intersects T_c only at its starting point.

If the decomposition is returning at T_0, then it is *tight* if the same holds for $0 \leq i < j \leq n$.

Note that in a tight (returning) decomposition, if T_i, T_{i+1} share 2-cells, then T_{i+1}, T_{i+2} do not share 2-cells. Indeed, otherwise the core T_c of T_{i+1} would also be the core of T_i and T_{i+2}; hence, T_i and T_{i+2} would also share 2-cells.

Proposition 5.9.

(i) *For any hypergraph segment with a returning decomposition of length $\leq N$, there is a tight returning decomposition of length $\leq N$ of a subsegment of it* (*possibly with reversed orientation*).
(ii) *Any hypergraph segment with decomposition of length $\leq N$ has a tight decomposition of length $\leq N$, or there is a returning decomposition of length $\leq N$ of a subsegment of it.*

Proof. We prove (i) by contradiction. Let $\gamma_1 \cdots \gamma_n$ be a returning decomposition of a subsegment of the given segment with minimal length $n \leq N$. Let T_0 be the tile at which it is returning. Denote the endpoints of γ_i by x_{i-1} and x_i. After shortening γ_1 or γ_n, we can assume that the neighborhood of x_0 in γ_1 intersects T_0 only at x_0 and that the neighborhood of x_n in γ_n intersects T_0 only at x_n.

We analyze in what situation T_i and T_j might share 2-cells, where $i < j$. Suppose first that $T_j \subset T_i$. If $j = i + 1 > 1$, then we could have merged $\gamma_i \cup \gamma_{i+1}$ into one segment in T_i to decrease n. If $j = 1$ and $i = 0$, then this would contradict the assumption on x_0. If $j > i + 1$, then we could have replaced $\gamma_1 \cdots \gamma_n$ with $\gamma_{i+1} \cdots \gamma_{j-1}$ and T_0 with T_i to decrease n as well. If $T_i \subset T_j$, then the argument is the same, except when $i = 0$. If $j = 1$, then we could have passed to $\gamma_2 \cdots \gamma_n$ replacing T_0 with T_1. If $j > 1$, then we could restrict to $\gamma_1 \cdots \gamma_{j-1}$ replacing T_0 with T_j.

This shows that if T_i, T_j share 2-cells, then neither is contained in the other. Hence, they cannot be single 2-cells and thus are tiles of $\mathcal{T}$ by Definition 4.11. By Remark 3.8, T_i, T_j share their core T_c. Note that C_j is outside T_c since otherwise we would have $T_j = T_c \subset T_i$. By Proposition 4.10(iii) there is a unique edge midpoint $x \in \gamma_j$ in $C_j \cap T_c$.

Consider first $j = i + 1$. Without loss of generality we can assume that C_{i+1} is the first 2-cell in $T_{i+1} - T_c$ traversed by γ_{i+1}. Moreover, we can move to γ_i the part of γ_{i+1} preceding $\gamma_{i+1} \cap C_{i+1}$, which lies in T_c. Then $x = x_i$; otherwise, we could replace T_0 with T_i and pass to $x_i \cdots x \subset \gamma_{i+1}$, which decreases n unless $i = 0$, $n = 1$, and $x = x_1$, in which case we just interchange x_0 with x_1. By Proposition 4.10(iii) the edge midpoint $y \in \gamma_{i+1} \cap C_{i+1}$ distinct from x equals x_{i+1}. Hence, γ_{i+1} is a diagonal of C_{i+1} as in Definition 5.8.

If $j > i + 1$, then passing to the hypergraph segment $\gamma_{i+1} \cdots x$, we obtain a contradiction with minimality of n, unless $i = 0$, $j = n$, and $x = x_n$. Then again by Proposition 4.10(iii), γ_n is a diagonal of C_n. Note that, unless $n = 1$, it cannot simultaneously happen that T_0, T_1 share 2-cells and T_0, T_n share 2-cells. Otherwise, T_1 and T_n would also share the 2-cells of the common core T_c, which would yield $n = 2$ and $x = C_2 \cap T_c$ being simultaneously equal to x_1 and x_2, a contradiction. In particular, by possibly reversing at the beginning of the procedure the order of γ_i, we can assume that T_0 and T_n do not share 2-cells, unless $n = 1$, which case was discussed before. Thus, the returning decomposition $\gamma_1 \cdots \gamma_n$ we obtained is tight, as desired.

The proof of (ii) is similar. Firstly, by merging some of the γ_i as before we can assume that T_i does not contain T_j for $i \neq j$. Indeed, if $j = i + 1$, then we can merge γ_i with γ_{i+1}. If $j > i + 1$, then we obtain a decomposition $\gamma_{i+1} \cdots \gamma_{j-1}$ returning at T_i, and we are done. Secondly, we can also assume that T_i and T_j share 2-cells only if $j = i + 1$ since otherwise we also obtain a returning decomposition of a subsegment. Finally, for T_i, T_{i+1} sharing 2-cells, similarly as in the proof of (i), after moving part of γ_{i+1} to γ_i, γ_{i+1} is a diagonal of C_{i+1} satisfying the condition in Definition 5.8. □

Here is the final piece of terminology used in the proof of Proposition 5.6.

Definition 5.10. Let $\gamma_1 \cdots \gamma_n$ be a decomposition returning at $T_0 \in \mathcal{T}$. Denote the endpoints of γ_i by x_{i-1} and x_i. A disc diagram $D \to X$ *bounded* by $\gamma_1 \cdots \gamma_n$ returning at T_0 is a disc diagram for $\alpha_0\alpha_1 \cdots \alpha_n$, where α_i is mapped to T_i, and for $i \neq 0$, its endpoints are mapped to x_{i-1}, x_i. Thus, we allow half-edge spurs at ∂D.

Likewise, a disc diagram $D \to X$ *bounded* by a decomposition $\gamma_1 \cdots \gamma_n$ and a path α in $X^{(1)}$ is a disc diagram for $\alpha\alpha_1 \cdots \alpha_n$, where α_i is mapped to T_i, and its endpoints are mapped to x_{i-1}, x_i.

Remark 5.11. Any decomposition $\gamma_1 \cdots \gamma_n$ returning at T_0 bounds a disc diagram: It suffices to consider arbitrary paths α_i embedded in T_i joining x_{i-1}, x_i (modulo $n + 1$) and a disc diagram for $\alpha_0\alpha_1 \cdots \alpha_n$.

Similarly, for any path α embedded in $X^{(1)}$ joining the endpoints of a decomposition $\gamma_1 \cdots \gamma_n$, there is a disc diagram bounded by $\gamma_1 \cdots \gamma_n$ and α.

Proof of Proposition 5.6. By Proposition 5.9(i) it suffices to show that for any $n \leq N$, there is no tight decomposition $\gamma_1 \cdots \gamma_n$ returning at a tile $T_0 \in \mathcal{T}$. Suppose that there is such a decomposition. By Remark 5.11 it is bounded by a diagram $D \to X$. After passing to a subdiagram, we can assume that there is no 2-cell in D mapped to T_i adjacent to α_i.

For every T_{i+1} sharing a 2-cell with T_i, replace the tile T_{i+1} with $T'_{i+1} = C_{i+1}$ and call it *shrunk*. Otherwise, if T_{i+1} shares no 2-cells with T_i, then we define $T'_{i+1} = T_{i+1}$.

Let $Y \subset X$ be the subcomplex that is the union of $T_0, T'_1, \ldots, T'_n$ and the image of D. Let $\mathcal{C}$ be the 2-cells of Y outside $T_0, T'_1, \ldots, T'_n$. For $i = 1, \ldots, n$, let $P_i \subset T'_i$ be the span of the image of α_i.

For nonshrunk $T'_i = T_i$, by Remark 4.12 we have $|P_i| \geq \mathrm{Bal}(T'_i)$. If $T'_i = C_i$ is shrunk, then the same is true except for the case where the edge midpoints $x_{i-1}, x_i \in C_i$ are at distance $< \frac{1}{2}l$. In that case, however, by Proposition 4.10(ii) the edge midpoint $x'_{i-1} \in C_i$ antipodal to x_i lies in T_{i-1}, coinciding with T'_{i-1} if $i > 1$. We then append P_i by an edge-path joining x'_{i-1} to x_{i-1} in $\partial C_i \cap T_{i-1}$, for which we keep the notation P_i and which has now at least $\frac{1}{2}l$ edges, so that we trivially have $|P_i| \geq \mathrm{Bal}(T'_i)$.

CLAIM 1. *We have* $|\mathcal{C}| = 0$ *and* $|Y| \leq 5$.

Proof. We bound the cancellation in Y from below using Remark 2.1 with $\{Y_i\} = \{T_0\} \cup \{T'_i\} \cup \mathcal{C}$:

$$\begin{aligned}
\mathrm{Cancel}(Y) &\geq \mathrm{Cancel}(T_0) + \sum_{i=1}^{n} \mathrm{Cancel}(T'_i) + \frac{1}{2}\left(\sum_{i=1}^{n} |P_i| + |\mathcal{C}|l\right) \\
&\geq \mathrm{Cancel}(T_0) + \sum_{i=1}^{n}\left(\frac{1}{2}\mathrm{Bal}(T'_i) + \mathrm{Cancel}(T'_i)\right) + \frac{1}{2}|\mathcal{C}|l \\
&= \mathrm{Cancel}(T_0) + \sum_{i=1}^{n}\left(\frac{1}{8}(|T'_i| + 1)l + \frac{1}{2}\mathrm{Cancel}(T'_i)\right) + \frac{1}{2}|\mathcal{C}|l \\
&\geq \frac{1}{4}(|T_0| - 1)l + \frac{1}{8}\sum_{i=1}^{n}((|T'_i| + 1)l + (|T'_i| - 1)l) + \frac{1}{2}|\mathcal{C}|l \\
&= \frac{1}{4}(|Y| - 1 + |\mathcal{C}|)l.
\end{aligned}$$

Since $n \leq N$ and tiles of $\mathcal{T}$ have size ≤ 4, the quantity $|\partial D|/l$ is uniformly bounded. By Theorem 2.2 the size $|D|$ is uniformly bounded, and so is $|Y|$. By Proposition 2.8 we have $|\mathcal{C}| = 0$, and by the calculation in Remark 3.3 we have $|Y| \leq 5$, as desired. □

CLAIM 2. *We have* $|D| = 0$, *that is*, D *is a tree.*

Proof. Otherwise, consider a component T in D of the 2-cells in the preimage of some T_i'. In the calculation in the proof of Claim 1 we can thus replace P_i with the image of ∂T in Y. By Corollary 2.6 we have now $|P_i| \geq l$. Thus, $\mathrm{Cancel}(T_i') + \frac{1}{2}|P_i| \geq \frac{1}{4}(|T_i'| + 1)l$. On the other hand, the term $\mathrm{Cancel}(T_i') + \frac{1}{2}|P_i|$ was estimated in the proof of Claim 1 only by $\frac{1}{4}|T_i'|l$, which gives an extra $\frac{1}{4}l$, which violates Proposition 2.8. $\square$

Note that $n > 1$ by Lemma 4.4 (and Proposition 4.10(ii) if T_1' is shrunk).

CLAIM 3. *We have* $n > 2$.

Proof. Otherwise, by Claim 2 the disc diagram D is a tripod. If $|T_1' \cap T_2'| \geq \frac{1}{4}l$, then since $|T_1'| + |T_2'| \leq 4$, the tiles T_1' and T_2' (one of which could be shrunk) would have been glued into one tile in Construction 3.7, which is a contradiction. Otherwise,

$$\begin{aligned}\mathrm{Cancel}(Y) &\geq \mathrm{Cancel}(T_0) + \sum_{i=1}^{2} \mathrm{Cancel}(T_i') + \sum_{i=1}^{2} |P_i| - \frac{1}{4}l \\ &\geq \frac{1}{4}(|T_0| - 1)l + \sum_{i=1}^{2} \frac{1}{4}(|T_i'| + 1)l - \frac{1}{4}l = \frac{1}{4}|Y|l,\end{aligned}$$

which contradicts Proposition 2.8. $\square$

CLAIM 4. *There are* $1 \leq i \leq n$ *and* $j = i \pm 1$ (*modulo* $n + 1$) *such that* $T_i' \cup T_j'$ *is a tile.*

Proof. By Claim 1 we have $|Y| \leq 5$, and by Claim 3 we have $n \geq 3$, so that the values of $|T_0|$, $|T_i'|$ are 1 or 2. In particular, none of T_i' is shrunk, and we do not append P_i. By Claim 2 the disc diagram D is a tree. Thus, there is $1 \leq i \leq n$ such that α_i is contained in $\alpha_{i-1} \cup \alpha_{i+1}$. We choose $j \in \{i-1, i+1\}$ with larger $|P_i \cap P_j|$. By Remark 4.12 we have $|P_i \cap P_j| \geq \frac{1}{2}\,\mathrm{Bal}(T_i')$.

To prove that $T_i' \cup T_j'$ is a tile, we compute its cancellation:

$$\begin{aligned}\mathrm{Cancel}(T_i' \cup T_j') &\geq \mathrm{Cancel}(T_j') + \frac{1}{2}\,\mathrm{Bal}(T_i') + \mathrm{Cancel}(T_i') \\ &\geq \frac{1}{4}(|T_j'| - 1)l + \frac{1}{8}(|T_i'| + 1)l + \frac{1}{2}\,\mathrm{Cancel}(T_i') \\ &\geq \frac{1}{4}(|T_j'| - 1)l + \frac{1}{4}|T_i'|l.\end{aligned}$$

$\square$

To obtain the final contradiction, it suffices to observe that Claim 4 saying that $T_i' \cup T_j'$ is a tile, whereas $|T_i'| + |T_j'| \leq 3$, contradicts Construction 3.7. $\square$

6. Quasi-Convexity

In this section we complete the proof of Theorem 5.2 using Theorem 6.1 saying that hypergraphs are quasi-isometrically embedded in X. It is intriguing that we will not use Theorem 6.1 directly in the proof of Theorem 1.1 but it is difficult to circumvent to obtain Theorem 5.2.

Let γ be a hypergraph segment. Consider the path metric on γ where all diagonals have length $\frac{1}{2}l$. Let $\mathfrak{G}$ be the vertex set of γ with the restricted metric.

THEOREM 6.1. *Let $d < \frac{5}{24}$. There are constants Λ, c such that w.o.p. the map from the vertex set $\mathfrak{G}$ of any hypergraph segment to $X^{(1)}$ is a (Λ, cl)-quasi-isometric embedding.*

Before we prove Theorem 6.1, we give the following.

Proof of Theorem 5.2. Suppose that we have a hypergraph segment xx' that self-intersects, that is, starts and ends at the same 2-cell, so that $|x, x'|_{X^{(1)}} \leq \frac{1}{2}l$. Denote by n the number of 2-cells traversed by xx'. By Theorem 6.1 we have

$$|x, x'|_{X^{(1)}} \geq \frac{1}{\Lambda}\left(n\frac{1}{2}l\right) - cl.$$

Hence, we have an a priori bound $n \leq (2c+1)\Lambda$. Thus, by Remark 5.7 it suffices to apply Proposition 5.6 with $N = (2c+1)\Lambda$. □

Proof of Theorem 6.1. The Cayley graph $X^{(1)}$ of a random group at a fixed density $d < \frac{1}{2}$ is w.o.p. hyperbolic with the hyperbolicity constant a linear function of l. We thus rescale the metric on $X^{(1)}$ by $\frac{1}{l}$, so that the constant is uniform. We also rescale by $\frac{1}{l}$ the metric on $\gamma \supset \mathfrak{G}$. We appeal to [GdlH90, Thm. 5.21] implying that local quasigeodesics are quasigeodesics. More precisely, to prove Theorem 6.1, it suffices to find λ such that for some sufficiently large $N = N(\lambda)$, the map to $X^{(1)}$ from any $\mathfrak{G}$ of cardinality $\leq N$ is λ-bi-Lipschitz. We will do that for $\lambda = \frac{1}{1-4d}$.

Let γ be a hypergraph segment with vertex set $\mathfrak{G}$ of cardinality $\leq N$, and let α be a geodesic in $X^{(1)}$ joining the endpoints of γ.

By taking for C_i consecutive 2-cells traversed by γ we see that γ has a decomposition $\gamma_1 \cdots \gamma_n$, and by Propositions 5.9(ii) and 5.6 we can assume that it is tight. We put $T_1' = T_1$, and as in the proof of Proposition 5.6, for every T_{i+1} sharing a 2-cell with T_i, we define $T_{i+1}' = C_{i+1}$ and call it *shrunk,* and otherwise we take $T_{i+1}' = T_{i+1}$. By Remark 5.11 there is a disc diagram $D \to X$ bounded by $\gamma_1 \cdots \gamma_n$ and α. After passing to a subdiagram, we can assume that there is no 2-cell in D mapped to T_i adjacent to α_i.

Let $Y \subset X$ be the subcomplex that is the union of T_i' and the image of D. Let $\mathcal{C}$ be the 2-cells of Y outside $\bigcup_i T_i'$. Let $P_i \subset T_i'$ be the span of the image of α_i, which we append as in the proof of Proposition 5.6 for T_i' shrunk. Thus, $|P_i| \geq \mathrm{Bal}(T_i')$. We estimate the cancellation in Y using Remark 2.1 (recall that we rescaled the length of α by factor l):

$$\begin{aligned}\mathrm{Cancel}(Y) &\geq \sum_{i=1}^{n} \mathrm{Cancel}(T_i') + \frac{1}{2}\left(\sum_{i=1}^{n} |P_i| + |\mathcal{C}|l - |\alpha|l\right)\\ &\geq \sum_{i=1}^{n}\left(\frac{1}{2}\,\mathrm{Bal}(T_i') + \mathrm{Cancel}(T_i')\right) + \frac{1}{2}|\mathcal{C}|l - \frac{1}{2}|\alpha|l\\ &\geq \frac{1}{4}|Y|l - \frac{1}{2}|\alpha|l.\end{aligned}$$

Note that the value $N = N(\lambda)$ gives a uniform bound on the length of α and consequently by Theorem 2.2 a uniform bound on the size of D and Y. We can thus apply Proposition 2.8 to Y, which yields

$$2\left(\frac{1}{4} - d\right)|Y| \leq |\alpha|.$$

The distance $|\gamma|$ between the endpoints of γ in $\mathfrak{G}$ is $\leq \frac{1}{2}|Y|$; thus, we obtain $\frac{1}{\lambda}|\gamma| \leq |\alpha|$. Since consecutive points of $\mathfrak{G}$ are at distance $\leq \frac{1}{2}$ in $X^{(1)}$, we also have $|\alpha| \leq |\gamma|$. Thus, $\mathfrak{G} \to X^{(1)}$ is λ-bi-Lipschitz, as desired. □

Before we prove Theorem 1.1, we need the following lemma.

LEMMA 6.2. *Let $d < \frac{5}{24}$. With overwhelming probability, there are a hypergraph in X and an element of its stabilizer in G exchanging its two complementary components.*

Actually, G acts transitively on the set of complementary components of all hypergraphs, but we do not need this.

Proof of Lemma 6.2. We will prove that there is a relator $r \in R$ such that

(1) there are two antipodal occurrences of a letter s in r and
(2) the relation $\sim_C$ on some (hence any) 2-cell C corresponding to r is antipodal.

This suffices to prove the lemma since if a hypergraph contains a diagonal γ connecting the midpoints of the directed edges e, e' of C labeled by same letter, then there exists $g \in G$ with $ge = e'$, and so g stabilizes that hypergraph. Since inside C, the edges e, e' cross γ in opposite directions, we see that g exchanges the complementary components of this hypergraph.

We claim that w.o.p. the first relator $r_1 \in R$ satisfies condition (1). Since there are $2m$ letters in $S \cup S^{-1}$, the probability that a fixed antipodal edge pair is labeled by the same letter is nearly $\frac{1}{2m}$ as $l \to \infty$ and, conditioned on the event that a preceding antipodal edge pair is not labeled by the same letter, is nearly $\frac{2m-2}{(2m-1)^2}$. However, there are $\frac{1}{2}l$ antipodal edge pairs in r_1, so the probability that none of them is labeled by the same letter tends to 0 as $l \to \infty$, justifying the claim.

By Construction 3.7 the map from a tile in $\mathcal{T}$ to X/G maps distinct 2-cells to distinct 2-cells. In particular, if a 2-cell of X corresponding to r_1 lay in a tile of size > 1, then this would contradict Corollary 2.10. Thus, condition (2) is satisfied as well. □

Proof of Theorem 1.1. Let $F \subset G$ be the stabilizer of a hypergraph Γ in X satisfying Lemma 6.2. Note that F acts on Γ cocompactly since $\sim_C$ are G-invariant.

We claim that the components of $X - \Gamma$ are *essential,* that is, they are not at finite distance from Γ. Otherwise, since they are exchanged by F, both components of $X - \Gamma$ are at finite distance from Γ. Thus, F acts cocompactly on X, and consequently G is quasi-isometric to F and hence to Γ, which is a tree by Theorem 5.2. Recall that G is w.o.p. torsion free since X is contractible [Gro93; Oll04]. Thus, by Stallings' theorem [Sta68], the group G is free, and hence $\chi(G) \leq 0$. But, on the other hand, we have $\chi(G) = 1 - m + \lfloor (2m-1)^{dl} \rfloor > 0$ for l large enough, which is a contradiction.

This justifies the claim that the components of $X - \Gamma$ are essential. Let $F' \subset F$ be the index 2 subgroup preserving the components of $X - \Gamma$. Then the number of relative ends $e(G, F')$ is greater than 1. Thus, Sageev's construction ([Sag95, Thm. 3.1], see also [Ger97] and [NR98]) gives rise to a nontrivial action of G on a CAT(0) cube complex. By [NR98] the group G does not satisfy Kazhdan's property (T). □

REMARK 6.3. By Theorem 6.1 the subgroup F' in the previous proof is quasi-isometrically embedded in the hyperbolic group G. Thus, by [Sag97, Thm. 3.1] or [GMRS98] there is a CAT(0) cube complex satisfying Theorem 1.1 for which the action of G is cocompact and the complex is finite dimensional.

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School of Mathematics
University of Bristol
University Walk
Bristol, BS8 1TW
UK

john.mackay@bristol.ac.uk

Dep. of Math. & Stat.
McGill University
Burnside Hall, Room 1005
805 Sherbrooke St. W
Montreal, Quebec H3A 2K6
Canada
and
Inst. of Math.
Polish Academy of Sciences
Śniadeckich 8, 00-656 Warsaw
Poland

piotr.przytycki@mcgill.ca