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L-FUNCTIONS AS DISTRIBUTIONS

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ABSTRACT. We define an axiomatic class of L -functions extending the Selberg class. We show in particular that one can recast the traditional conditions of an Euler product, analytic continuation and functional equation in terms of distributional identities akin to Weil's explicit formula. The generality of our approach enables some new applications; for instance, we show that the L -function of any cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$ has infinitely many zeros of odd order.

1. INTRODUCTION

In [18], Selberg introduced his eponymous class of L -functions, defined as follows.

Definition 1.1. *The Selberg class \mathcal{S} is the set of functions F satisfying the following axioms:*

- (1) (*Dirichlet series*). *There are numbers $a(n) \in \mathbb{C}$ such that $F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, converging absolutely for $\Re(s) > 1$.*
- (2) (*Analytic continuation*). *There is an integer $m \geq 0$ such that $(s-1)^m F(s)$ continues to an entire function of finite order.*
- (3) (*Functional equation*). *There exist $k \in \mathbb{Z}_{\geq 0}$, $Q, \lambda_1, \dots, \lambda_k \in \mathbb{R}_{>0}$, $\mu_1, \dots, \mu_k \in \mathbb{C}$ with $\Re(\mu_j) \geq 0$ and $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$ such that the function*

$$\Phi(s) = \epsilon Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) \cdot F(s)$$

satisfies the functional equation

$$\Phi(s) = \overline{\Phi(1 - \bar{s})}.$$

- (4) (*Ramanujan hypothesis*). *For every $\varepsilon > 0$, $a(n) \ll_{\varepsilon} n^{\varepsilon}$.*
- (5) (*Euler product*). *$a(1) = 1$ and $\log F(s) = \sum_{n=2}^{\infty} b(n)n^{-s}$, where $b(n)$ is supported on prime powers, and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.*

Selberg went on to pose various conjectures about the elements of \mathcal{S} , in particular:

Conjecture 1.2 (Selberg orthogonality conjecture). *Let $F, G \in \mathcal{S}$ be primitive, in the sense that they cannot be expressed non-trivially as products of elements of \mathcal{S} , with Dirichlet coefficients $a_F(n)$ and $a_G(n)$. Then*

$$\sum_{\substack{p \text{ prime} \\ p \leq x}} \frac{a_F(p)\overline{a_G(p)} - \delta_{FG}}{p} \ll_{F,G} 1,$$

where $\delta_{FG} = 1$ if $F = G$ and 0 otherwise.

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This idea of codifying the properties of L -functions is appealing as an alternative to the Langlands program. However, one immediate problem is that it is not obvious which properties of L -functions should be taken as axioms, and which are theorems to be derived from the axioms. More to the point, Selberg’s choice of axioms does not correspond perfectly to the properties of the known L -functions, i.e. those associated to automorphic representations. For instance, the Ramanujan bound $a(n) \ll_\epsilon n^\epsilon$ remains a conjecture for most automorphic L -functions, but Conjecture 1.2 is essentially known in that context.¹ This difficulty seems inherent to the axiomatic approach and may never be resolved completely, since there are differing opinions about what properties of L -functions are essential, and it is likely impossible to avoid making at least some choices based purely on aesthetics.

Nevertheless, Selberg’s paper has been influential in shaping the way that researchers think about L -functions, and has spurred a large volume of research, both attempting to classify the elements of \mathcal{S} and studying the consequences of Conjecture 1.2. The general belief is that \mathcal{S} essentially coincides with the class of automorphic L -functions. However, Selberg’s list of axioms is in principle more flexible; for instance, the local Euler factor $\exp(\sum_{n=1}^\infty b(p^n)p^{-ns})$ can be any function of the form $e^{f(p^{-s})}$, where f is analytic on a disc of radius $p^{-\theta}$ for some $\theta < \frac{1}{2}$ and satisfies $f(0) = 0$. This is substantially more general than the factors that occur for automorphic L -functions (which are always reciprocal polynomials of p^{-s}) and permits some natural operations, such as taking square roots and quotients. On the other hand, the Γ -factors $\Gamma(\lambda s + \mu)$, while again more general than those associated to automorphic L -functions (for which we may always reduce to the case $\lambda = \frac{1}{2}$), do not seem to occur naturally when $\lambda \notin \frac{1}{2}\mathbb{Z}_{>0}$, so this offers no effective increase in generality.²

In this paper, we propose a broader set of axioms with the goal of putting the Γ -factors on the same level of generality as the other Euler factors, and as we will show, this enables some new applications. Our approach is to change language, and express everything not in terms of L -functions directly (since there is no agreement on how they should be defined anyway), but in terms of their *explicit formulae*. Following the point of view introduced by Weil [23], these are identities of distributions relating the zeros of an L -function to the coefficients of its logarithmic derivative via the Fourier transform. For instance, if $\chi \pmod{q}$ is an even primitive Dirichlet character with complete L -function $\Lambda(s, \chi) = \Gamma_{\mathbb{R}}(s)L(s, \chi)$ and $g : \mathbb{R} \rightarrow \mathbb{C}$ is a sufficiently nice test function (e.g. smooth of compact support) with Fourier

¹In full generality it is known in a slightly weaker form that includes the prime powers in the sum, and those may be removed under a mild hypothesis; see [25, 1].

²Selberg acknowledges in a footnote of his paper that we may take λ to be a half-integer in every known case. It seems likely that he did not intend for his definition to be taken as a serious attempt at generalization, but rather as a recognition of the fact that the Γ -factors are non-canonical because of the Legendre and Gauss multiplication formulas. We note that those identities have analogues at the finite places as well, e.g. the Legendre duplication formula is analogous to the “difference of squares” identity $1 - p^{-2s} = (1 - p^{-s})(1 + p^{-s})$, but this ambiguity causes no real confusion. Note also that the analogue of $\Gamma(\lambda s)$ is the generalized Dirichlet series $1/(1 - p^{-2\lambda s})$, which is not permitted under Selberg’s definition unless λ is a half-integer.

transform $h(z) = \int_{\mathbb{R}} g(x)e^{izx} dx$ satisfying $h(\mathbb{R}) \subseteq \mathbb{R}$, then the explicit formula is the identity

$$\sum_{z \in \mathbb{C}} m(z)h(z) = 2\Re \left[\int_0^{\infty} (g(0) - g(x)) \frac{e^{-x/2}}{1 - e^{-2x}} dx + \frac{1}{2} \left(\log \frac{q}{8\pi} - \gamma - \frac{\pi}{2} \right) g(0) - \sum_{n=2}^{\infty} \frac{\Lambda(n)\chi(n)}{\sqrt{n}} g(\log n) \right],$$

where $m(z) = \text{ord}_{s=\frac{1}{2}+iz} \Lambda(s, \chi)$.

Here the integral kernel $\frac{e^{-x/2}}{1-e^{-2x}}$ is related to the Γ -factor $\Gamma_{\mathbb{R}}(s)$ associated to χ (by a logarithmic derivative and Fourier transform). Since the explicit formula is additive, i.e. the formula for a product of L -functions is the sum of the individual formulas, in this language it is clear how the Γ -factors can be deformed. For instance, replacing $\Gamma_{\mathbb{R}}(s)$ by its square root amounts to dividing the kernel by 2. It is also now clear how to generalize the notion of Γ -factor—we simply consider any suitable integral kernel. Of course, which kernel functions should be considered “suitable” is again open to interpretation, but there is one essential feature of the kernels occurring in the explicit formulae of L -functions that must be present, namely a first-order singularity at 0, with residue reflecting the degree.³ All other conditions should be chosen to suit the desired applications. With that in mind, after some trial and error, we arrived at the following definition.

Definition 1.3. *An L -datum is a triple $F = (f, K, m)$, where $f : \mathbb{Z}_{>0} \rightarrow \mathbb{C}$, $K : \mathbb{R}_{>0} \rightarrow \mathbb{C}$ and $m : \mathbb{C} \rightarrow \mathbb{R}$ are functions satisfying the following axioms:*

- (A1) $f(1) \in \mathbb{R}$, $f(n) \log^k n \ll_k 1$ for all $k > 0$, and $\sum_{n \leq x} |f(n)|^2 \ll_{\varepsilon} x^{\varepsilon}$ for all $\varepsilon > 0$;
- (A2) $xK(x)$ extends to a Schwartz function on \mathbb{R} , and $\lim_{x \rightarrow 0^+} xK(x) \in \mathbb{R}$;
- (A3) $\text{supp}(m) = \{z \in \mathbb{C} : m(z) \neq 0\}$ is discrete and contained in a horizontal strip $\{z \in \mathbb{C} : |\Im(z)| \leq y\}$ for some $y \geq 0$, $\sum_{\substack{z \in \text{supp}(m) \\ |\Re(z)| \leq T}} |m(z)| \ll 1 + T^A$ for some $A \geq 0$,
and $\#\{z \in \text{supp}(m) : m(z) \notin \mathbb{Z}\} < \infty$;
- (A4) for every smooth function $g : \mathbb{R} \rightarrow \mathbb{C}$ of compact support and Fourier transform $h(z) = \int_{\mathbb{R}} g(x)e^{izx} dx$ satisfying $h(\mathbb{R}) \subseteq \mathbb{R}$, we have the equality

$$\sum_{z \in \text{supp}(m)} m(z)h(z) = 2\Re \left[\int_0^{\infty} K(x)(g(0) - g(x)) dx - \sum_{n=1}^{\infty} f(n)g(\log n) \right].$$

Given an L -datum $F = (f, K, m)$, we associate an L -function $L_F(s)$ defined by

$$L_F(s) = \sum_{n=1}^{\infty} a_F(n)n^{-s} = \exp \left(\sum_{n=2}^{\infty} \frac{f(n)}{\log n} n^{\frac{1}{2}-s} \right) \quad \text{for } \Re(s) > 1;$$

we call $d_F = 2 \lim_{x \rightarrow 0^+} xK(x)$ the degree of F and $Q_F = e^{-2f(1)}$ its analytic conductor; and we say that F is positive if there are at most finitely many $z \in \mathbb{C}$ with $m(z) < 0$.

³It is tempting to consider more general singularities as well, but we quickly find ourselves in a much larger landscape of functions that is presumably very hard to classify; for instance, the Selberg zeta-functions and their trace formulae give identities of this type with second-order singularities.

Let \mathcal{L} denote the set of all L -data and $\mathcal{L}^+ \subseteq \mathcal{L}$ the subset of positive elements. Note that \mathcal{L} is a group with respect to addition, with identity element $(0, 0, 0)$, and \mathcal{L}^+ is a monoid. For any $d \in \mathbb{R}$, let $\mathcal{L}_d = \{F \in \mathcal{L} : d_F = d\}$ and $\mathcal{L}_d^+ = \mathcal{L}_d \cap \mathcal{L}^+$.

Examples 1.4.

- (1) If $L(s) = \exp(\sum_{n=2}^{\infty} b(n)n^{-s})$ is an element of the Selberg class with complete L -function

$$\Phi(s) = \epsilon Q^s \prod_{j=1}^k \Gamma(\lambda_j s + \mu_j) \cdot L(s),$$

then there is an L -datum $F = (f, K, m) \in \mathcal{L}^+$ satisfying $d_F = 2 \sum_{j=1}^k \lambda_j$, $L_F(s) = L(s)$,

$$f(n) = \begin{cases} -\log Q - \Re \sum_{j=1}^k \lambda_j \frac{\Gamma'}{\Gamma}(\frac{\lambda_j}{2} + \mu_j) & \text{if } n = 1, \\ \frac{b(n) \log n}{\sqrt{n}} & \text{if } n > 1, \end{cases}$$

$$K(x) = \sum_{j=1}^k \frac{e^{-(\frac{1}{2} + \frac{\mu_j}{\lambda_j})x}}{1 - e^{-\frac{x}{\lambda_j}}}, \quad \text{and} \quad m(z) = \text{ord}_{s=\frac{1}{2}+iz} \Phi(s).$$

Note in particular that the estimate $\sum_{n \leq x} |f(n)|^2 \ll_{\epsilon} x^{\epsilon}$ follows from the Ramanujan hypothesis together with the bound $b(n) \ll n^{\theta}$ (see [14, Lemma in §2]).

- (2) If π is a unitary cuspidal automorphic representation of $\text{GL}_d(\mathbb{A}_{\mathbb{Q}})$ with conductor q ,

$$L(s, \pi_{\infty}) = \prod_{j=1}^d \Gamma_{\mathbb{R}}(s + \mu_j), \quad -\frac{L'}{L}(s, \pi) = \sum_{n=2}^{\infty} c_n n^{-s} \quad \text{and} \quad \Lambda(s, \pi) = L(s, \pi_{\infty})L(s, \pi),$$

then there is an L -datum $F = (f, K, m) \in \mathcal{L}_d^+$ satisfying $L_F(s) = L(s, \pi)$,

$$f(n) = \begin{cases} -\frac{1}{2} \log q - \Re \sum_{j=1}^d \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\frac{1}{2} + \mu_j) & \text{if } n = 1, \\ \frac{c_n}{\sqrt{n}} & \text{if } n > 1, \end{cases}$$

$$K(x) = \sum_{j=1}^d \frac{e^{-(\frac{1}{2} + \mu_j)x}}{1 - e^{-2x}}, \quad \text{and} \quad m(z) = \text{ord}_{s=\frac{1}{2}+iz} \Lambda(s, \pi).$$

In this case, the estimate $\sum_{n \leq x} |f(n)|^2 \ll \log^2 x$ for $x \geq 2$ follows from the Rankin-Selberg method (see [16, (2.24)]), and the other conditions on f and K follow from partial results toward the Ramanujan conjecture [13].

- (3) If $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_d(\mathbb{C})$ is an Artin representation then there is an L -datum $F = (f, K, m) \in \mathcal{L}_d$ with $L_F(s) = L(s, \rho)$, and f, K, m defined similarly to the case of automorphic L -functions above. The Artin conjecture asserts that F is positive.

Remarks 1.5.

- (1) Note that we do not require an Euler product, and in fact the primes make no appearance in Definition 1.3. What effectively replaces this is the assumption of non-vanishing outside the critical strip, which is implied by the absolute convergence of $\log L_F(s)$ for $\Re(s) > 1$. In [4] it was shown in wide generality that this condition essentially characterizes the Euler products among all Dirichlet series associated to automorphic forms. For instance, it follows from [4, Theorem 1.1] and Theorem 1.6

below that if $f \in S_k(\Gamma_1(N))$ is a classical holomorphic modular form then there is an L -datum $F \in \mathcal{L}_2^+$ with $L_F(s) = L(s + \frac{k-1}{2}, f)$ if and only if f is a normalized newform and Hecke eigenform.

- (2) We have not imposed the Ramanujan bound $a_F(n) \ll_\varepsilon n^\varepsilon$, largely to avoid excluding most of the automorphic L -functions. However, this has the side effect of including some examples which might be deemed undesirable, e.g. $\zeta(2s - \frac{1}{2})$ is the L -function of some element of \mathcal{L}_2^+ . We take the view that it is better to include a few misfits in our definition than to throw out the baby with the bath water, and one can always pass to a restricted subclass if this becomes problematic.⁴
- (3) Definition 1.3 is arguably simpler than Definition 1.1, since it makes no mention of Euler products, the Γ -function or analytic continuation. It is also more concrete, compared to the rather intangible notion of analytic continuation, since one can study axiom (A4) as an identity of unknowns to be solved for. This was the essential point of [2, Proposition 4.2], which may be viewed as a prototype for our Theorem 1.7 below.
- (4) The notion of analytic conductor as a measure of complexity of an L -function was introduced in [8]. Our formulation is similar (but not identical) to that of *log conductor* in [6]. We make no claims that this formulation is the most suitable in all contexts, but it at least has the feature of being canonically defined, as shown by Theorem 1.6 below.

1.1. Main results. The map $F \mapsto L_F$ defines a homomorphism from \mathcal{L} to the multiplicative group of non-vanishing holomorphic functions on $\{s \in \mathbb{C} : \Re(s) > 1\}$. Our first result shows that this map is injective, i.e. each L -datum is determined by its L -function, in the following strong sense.

Theorem 1.6 (Multiplicity one). *For $F = (f, K, m) \in \mathcal{L}$, the following are equivalent:*

- (i) $F = (0, 0, 0)$;
- (ii) $\sum_{n=2}^{\infty} \frac{|f(n)|}{\log n} < \infty$;
- (iii) $\sum_{n=1}^{\infty} \frac{|a_F(n)|}{\sqrt{n}} < \infty$;
- (iv) $L_F(s)$ is a ratio of Dirichlet polynomials;
- (v) $\sum_{\substack{z \in \text{supp}(m) \\ |\Re(z)| \leq T}} |m(z)| = o(T)$.

Thus, although we have chosen to promote the three components f , K and m of our definition equally, without loss of generality one can focus only on the L -functions, as in the Selberg class.

Next, we show that the classification of the degree $d < \frac{5}{3}$ elements of the Selberg class, begun by Conrey–Ghosh [7] and continued and refined by Kaczorowski–Perelli [9, 10] and Soundararajan [19], can be adapted to our setting. (We speculate that Kaczorowski and Perelli’s very intricate extension [11] to degree < 2 could be adapted as well, but have not attempted to do so.)

⁴There is also an argument in favor of keeping examples like $\zeta(2s - \frac{1}{2})$ in the definition: Shimura’s integral representation for the symmetric square L -function could be viewed as an extension of the Rankin–Selberg method to this example, and that in turn was a key ingredient in the proof of the Gelbart–Jacquet lift.

Theorem 1.7 (Converse theorem). *Let $F \in \mathcal{L}_d^+$ for some $d < \frac{5}{3}$. Then either $d = 0$ and $L_F(s) = 1$, or $d = 1$ and there is a primitive Dirichlet character χ and $t \in \mathbb{R}$ such that $L_F(s) = L(s + it, \chi)$.*

1.2. Applications. We describe three applications of Theorem 1.7. The first two concern the zeros of automorphic L -functions.

Corollary 1.8. *Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$. Then its complete L -function $\Lambda(s, \pi)$ has infinitely many zeros of odd order.*

Proof. Let $F = (f, K, m) \in \mathcal{L}_3^+$ be the L -datum associated to π . If $\Lambda(s, \pi)$ has at most finitely many zeros of odd order then $m(z)$ is an even integer for all but at most finitely many z , and thus $\frac{1}{2}F \in \mathcal{L}_{3/2}^+$, in contradiction to Theorem 1.7. \square

Corollary 1.9. *For $j = 1, 2$, let π_j be a unitary cuspidal automorphic representation of $\mathrm{GL}_{d_j}(\mathbb{A}_{\mathbb{Q}})$ with complete L -function $\Lambda(s, \pi_j)$. If $d_2 - d_1 \leq 1$ and $\pi_1 \not\cong \pi_2$ then $\Lambda(s, \pi_2)/\Lambda(s, \pi_1)$ has infinitely many poles.*

Proof. If $d_2 < d_1$ then the conclusion follows by counting zeros, so we may assume that $d_2 \in \{d_1, d_1 + 1\}$. Let $F \in \mathcal{L}$ be the L -datum with L -function $L_F(s) = L(s, \pi_2)/L(s, \pi_1)$, so that $d_F \in \{0, 1\}$. If $\Lambda(s, \pi_2)/\Lambda(s, \pi_1)$ has at most finitely many poles then F is positive, so by Theorem 1.7, either $L_F(s) = 1$ or $L_F(s) = L(s + it, \chi)$ for some primitive Dirichlet character χ and $t \in \mathbb{R}$. However, neither of these is possible since $\pi_1 \not\cong \pi_2$ and π_2 is cuspidal. \square

Remarks 1.10.

- (1) The assumption of cuspidality is only for ease of presentation, and one could formulate versions of both of the above results for products of cuspidal L -functions.
- (2) If π_1 and π_2 are unitary cuspidal automorphic representations of $\mathrm{GL}_{d_1}(\mathbb{A}_{\mathbb{Q}})$ and $\mathrm{GL}_{d_2}(\mathbb{A}_{\mathbb{Q}})$ with $\pi_1 \not\cong \pi_2$, the Grand Simplicity Hypothesis predicts that $\Lambda(s, \pi_1)\Lambda(s, \pi_2)$ has at most finitely many non-simple zeros. Corollaries 1.8 and 1.9 give some modest evidence in that direction. The fact that these results are new is testimony of the difficulty of proving anything about the zeros of high degree L -functions!
- (3) Corollary 1.9 could likely be strengthened to $d_2 - d_1 \leq 2$ by combining the methods of this paper with those of [3]. Some special cases along these lines were demonstrated by Raghunathan [15].

Our third application generalizes a result of Lemke-Oliver [12], who considered totally multiplicative functions $f : \mathbb{Z}_{>0} \rightarrow D = \{z \in \mathbb{C} : |z| \leq 1\}$ whose summatory functions exhibit better than square-root cancellation relative to their mean-square size, i.e.

$$(1.1) \quad \sum_{n \leq x} |f(n)|^2 \gg x \quad \text{and} \quad \sum_{n \leq x} f(n) \ll x^{\frac{1}{2} - \delta} \quad \text{for some } \delta > 0.$$

Lemke-Oliver noted that this holds if f is a non-trivial Dirichlet character, and asked if that is essentially the only example. Although the problem appears to be intractable in full generality, he was able to make progress for the subclass of f that are *dictated by Artin symbols*, in the sense that there is a Galois extension K/\mathbb{Q} such that for every prime p that does not ramify in K , $f(p)$ depends only on the Frobenius conjugacy class Frob_p at p . His proof shows that for such f there is a decomposition

$$f(p) = \sum_x a_x \chi(\mathrm{Frob}_p)$$

for all unramified primes p , where χ ranges over the characters of the irreducible representations of $\text{Gal}(K/\mathbb{Q})$, and $a_\chi \in \mathbb{Q}$. Thus, the Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ behaves like an Artin L -function of degree $d = \sum_{\chi} a_\chi$, which is the value of f at split primes.

Lemke-Oliver concluded that f must agree with a Dirichlet character for almost all p under the assumption that $d = 1$, by adapting Soundararajan's proof [19] of the classification of degree 1 elements of the Selberg class. Note that $f(p)$ assumes only finitely many values for unramified p , each occurring with positive density, so by the Selberg–Delange method, we expect that the lower bound in (1.1) is only possible if $f(p) \in \partial D = \{z \in \mathbb{C} : |z| = 1\}$. If that is indeed the case then we must have $d = 1$, but rather than attempting to justify that heuristic, it suffices to appeal to Theorem 1.7; in fact, we obtain following stronger result.

Corollary 1.11. *Let $f : \mathbb{Z}_{>0} \rightarrow \mathbb{C}$ be a totally multiplicative function dictated by Artin symbols, with $|f(p)| < \frac{5}{3}$ for all primes p . If f satisfies (1.1) then there is a primitive Dirichlet character χ such that $f(p) = \chi(p)$ for all but at most finitely many primes p .*

1.3. Concluding remarks. Above we described three applications of our expanded notion of L -functions. However, our results so far have relied on essentially the same arguments as those already applied to the Selberg class, and it is natural to wonder whether this train of thought might also lead to insights that go beyond those arguments, perhaps as far as a classification of \mathcal{L}_d^+ for some $d \geq 2$. In particular, can we improve on the converse theorem for classical modular forms?

While we are unable to give any definitive answers to this question, we offer a few philosophical remarks and suggestions for future work.

Remarks 1.12. Let \mathcal{L}^{aut} be the subgroup of \mathcal{L} generated by the L -data associated to unitary cuspidal automorphic representations of $\text{GL}_d(\mathbb{A}_{\mathbb{Q}})$ for all d . Presumably \mathcal{L}^{aut} coincides with the subset of $F \in \mathcal{L}$ satisfying the Ramanujan bound $a_F(n) \ll_{\varepsilon} n^{\varepsilon}$, but at present we cannot prove an inclusion in either direction. The elephant in the room is that \mathcal{L}^{aut} is not only a group, but has the additional structure of a commutative ring, at least conjecturally. Precisely, if $F_1, F_2 \in \mathcal{L}^{\text{aut}}$ are generators corresponding to cuspidal representations π_1, π_2 , then the Langlands functoriality conjecture predicts that there is an automorphic representation with L -function equal to the Rankin–Selberg product $L(s, \pi_1 \times \pi_2)$, and we define the product $F_1 F_2 \in \mathcal{L}^{\text{aut}}$ to be the L -datum with that L -function.

The approach to classifying the elements of the Selberg class taken so far purposefully ignores most of this structure and relies essentially on Fourier analysis, which amounts to considering twists by n^{-it} , i.e. multiplication (in the above sense) by $F \in \mathcal{L}$ with $L_F(s) = \zeta(s + it)$. Note that such F are units in \mathcal{L}^{aut} , as are the L -data corresponding to $L(s + it, \chi)$ for primitive Dirichlet characters χ .

Put in these terms, one cannot help but wonder whether it would be more natural to build stability under twist into the definition, at least by all of the units, i.e. to consider the subclass of $F \in \mathcal{L}$ which have a twist $F_\chi \in \mathcal{L}$ for every primitive character χ . For this subclass, it seems likely that one could adapt the existing converse theorems for classical holomorphic and Maass modular forms to classify the positive elements of degree 2. (In fact, it might only be necessary to assume that F is positive, and not all of the twists F_χ , by following the method of [3].) Moreover, Cogdell and Piatetski-Shapiro conjectured [5, p. 166] that the analytic properties of twists by characters should in general suffice to characterize

the automorphic representations among all irreducible admissible representations, so there is at least some hope of eventually classifying everything of integral degree this way.

However, there are a few subtleties that need to be considered before this can be carried out. First, in all known versions of the converse theorem for degree at least 2 (beginning with Weil [24]), knowledge of the relationship between the root numbers and conductors of a given L -series and its twists is essential in the proof. On the other hand, Definition 1.3 does not even mention the root number (it makes only a brief appearance in the proof of Proposition 2.1, as a constant of integration), and as our results demonstrate, it plays no role in the classification of low-degree elements of \mathcal{L}^+ .

Second, the role of the Euler product in the converse theorem is similarly hazy. It has been conjectured that the degree 2 L -functions with Euler products can be characterized by a converse theorem without any twists, but this is known to be false if one drops the Euler product assumption. (In the other direction, with the added information from twists, Weil's converse theorem does not require an Euler product.) More generally, there are examples of Dirichlet series (e.g. certain Shintani zeta-functions [22]) which, together with all of their character twists, have meromorphic continuation and satisfy a functional equation, but are not associated to automorphic representations. These examples do not contradict Cogdell and Piatetski-Shapiro's conjecture since they lack Euler products. Thus, the Euler product seems to be an important hypothesis for characterizing automorphic representations with minimal analytic data, but it is far from clear why this is so.

In our definition we offered a weaker alternative (non-vanishing outside the critical strip, implied by axiom (A1)) as a possible substitute, and we speculate that it may help to shed light on the matter. In any case, we find it likely that in order to make progress on the classification for degree 2 and beyond, one must first clarify the roles that the Euler product and root number play in the converse theorem. As tentative steps in this direction, we issue the following challenges:

- (1) Prove a converse theorem for classical holomorphic modular forms, assuming that all character twists satisfy the expected analytic properties, but *without knowledge of the root number*.
- (2) Prove a converse theorem for automorphic representations of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$, assuming axiom (A1) and that all character twists have the expected analytic properties, but *without requiring an Euler product*.

Of course it might be that one or both of these is impossible, in which case a proof that there is no such result would be even more interesting!

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2. BASIC PROPERTIES

In this section we establish the basic properties of the L -functions associated to elements of \mathcal{L} , culminating in the proof of Theorem 1.6. First, we show that the derivation of the

explicit formula for L -functions can be inverted to prove that for any $F \in \mathcal{L}$, a suitably “completed” form of $L_F(s)$ has meromorphic continuation and satisfies a functional equation. In particular, we construct a canonical notion of “ Γ -factor” associated to F , as follows.

Proposition 2.1 (Meromorphic continuation and functional equation). *For every $F = (f, K, m) \in \mathcal{L}$, there is a function $\gamma_F(s)$, defined uniquely up to scaling by elements of \mathbb{R}^\times , with the following properties:*

- (i) $\log \gamma_F(s)$ is holomorphic for $\Re(s) > \frac{1}{2}$, and $\frac{d^n}{ds^n} \log \gamma_F(s)$ extends continuously to $\Re(s) \geq \frac{1}{2}$ for each $n \geq 0$;
- (ii) there are constants $d, c_{-1} \in \mathbb{R}$ and $\mu, c_0, c_1, \dots \in \mathbb{C}$ such that

$$\log \gamma_F(s) = \left(s - \frac{1}{2}\right) \left(\frac{d}{2} \log \frac{s}{e} + c_{-1}\right) + \frac{\mu}{2} \log \frac{s}{e} + \sum_{j=0}^{n-1} \frac{c_j}{s^j} + O_n(|s|^{-n}),$$

uniformly for $\Re(s) \geq \frac{1}{2}$ and any fixed $n \geq 0$;

- (iii) the product $\Lambda_F(s) = \gamma_F(s)L_F(s)$ continues meromorphically to

$$\Omega = \mathbb{C} \setminus \bigcup_{\{z \in \mathbb{C}: m(z) \notin \mathbb{Z}\}} \left[\left(\frac{1}{2} + i(-\infty, \Re(z))\right) \cup \left(\left[\frac{1}{2} - |\Im(z)|, \frac{1}{2} + |\Im(z)|\right] + i\Re(z)\right) \right]$$

and has meromorphic finite order, i.e. $\Lambda_F(s) = h_1(s)/h_2(s)$, where h_1 and h_2 are holomorphic on Ω , and there is a number $A \geq 0$ such that for any closed subset $E \subseteq \Omega$ we have $h_1(s), h_2(s) \ll_E \exp(|s|^A)$ for all $s \in E$;

- (iv) the functional equation $\Lambda_F(s) = \overline{\Lambda_F(1 - \bar{s})}$ holds as an identity of meromorphic functions on Ω ;
- (v) $\frac{\Lambda'_F}{\Lambda_F}(s)$ continues meromorphically to \mathbb{C} , with at most simple poles, and satisfies

$$\operatorname{Res}_{s=\frac{1}{2}+iz} \frac{\Lambda'_F}{\Lambda_F}(s) = m(z) \quad \text{for all } z \in \mathbb{C}.$$

In particular, $\operatorname{supp}(m) \subseteq \{z \in \mathbb{C} : |\Im(z)| \leq \frac{1}{2}\}$.

Remark 2.2. The proof of Proposition 2.1 shows that the number d appearing in (ii) is the degree $d_F = 2 \lim_{x \rightarrow 0^+} xK(x)$, and $\mu = -2 \lim_{x \rightarrow 0^+} \frac{d}{dx}(xK(x))$.

2.1. Lemmas. We begin with a few lemmas, the first of which establishes the basic equivalence between distributional identities and functions possessing analytic continuation and a functional equation. In what follows we denote by H the set of entire functions h such that $h(\mathbb{R}) \subseteq \mathbb{R}$ and the Fourier transform $g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(t)e^{-ixt} dt$ is smooth of compact support.

Lemma 2.3. *Let φ be a holomorphic function on $\{z \in \mathbb{C} : \Im(z) < y\}$ for some $y \in \mathbb{R}$, and suppose that $\varphi(z)$ has at most polynomial growth as $|z| \rightarrow \infty$ in any fixed horizontal strip $\{z \in \mathbb{C} : \Im(z) \in [a, b]\}$ with $a < b < y$. Fix $c < y$, and suppose that $\int_{\Im(z)=c} \varphi(z)h(z) dz \in \mathbb{R}$ for every $h \in H$. Then φ continues to an entire function, with at most polynomial growth in horizontal strips, and satisfies the functional equation $\varphi(\bar{z}) = \overline{\varphi(z)}$.*

Proof. Consider the integral

$$u(x) = \frac{1}{2\pi} \int_{\Im(z)=c} \varphi(z)e^{-z^2-ixz} dz.$$

Since $\varphi(z)e^{-z^2}$ is holomorphic and of rapid decay in horizontal strips for $\Im(z) < y$, $u(x)$ is independent of the value of c . For any fixed $x \in \mathbb{R}$, $e^{-z^2} \cos(xz)$ and $e^{-z^2} \sin(xz)$ are real valued for $z \in \mathbb{R}$, so it follows by a standard approximation argument that

$$u(x) + u(-x) = \frac{1}{\pi} \int_{\Im(z)=c} \varphi(z) e^{-z^2} \cos(xz) dz$$

and

$$i(u(x) - u(-x)) = \frac{1}{\pi} \int_{\Im(z)=c} \varphi(z) e^{-z^2} \sin(xz) dz$$

are real valued, so that $u(-x) = \overline{u(x)}$. Combining this with the trivial estimate $u(x) \ll_c e^{cx}$, we get $u(x) \ll_c e^{c|x|}$ for all $c < y$.

Together with the Fourier inversion formula

$$\varphi(z) e^{-z^2} = \int_{-\infty}^{\infty} u(x) e^{ixz} dx,$$

this shows that $\varphi(z)$ continues to an entire function, has finite order in any fixed horizontal strip, and satisfies $\varphi(\bar{z}) = \overline{\varphi(z)}$. Finally, the Phragmén–Lindelöf convexity principle applied to φ on the strip $\{z \in \mathbb{C} : |\Im(z)| \leq 1 + |y|\}$ shows that φ has at most polynomial growth in horizontal strips. \square

Lemma 2.4. *Let $(f, K, m) \in \mathcal{L}$. Then $m(\bar{z}) = m(z)$ for all $z \in \mathbb{C}$.*

Proof. Put $m'(z) = m(z) - m(\bar{z})$. Clearly $m'(z) = 0$ for all $z \in \mathbb{R}$, and we aim to show that this holds for all $z \in \mathbb{C}$. By axiom (A3), there is a positive integer M such that $\sum_{z \in \text{supp}(m')} \left| \frac{m'(z)}{z^M} \right| < \infty$. With this choice of M , let

$$q(z) = \sum_{z_0 \in \text{supp}(m')} \frac{m'(z_0)}{z - z_0} \left(\frac{z}{z_0} \right)^{M-1}.$$

Then q is meromorphic on \mathbb{C} with at most simple poles, satisfies $\text{Res}_{z=z_0} q(z) = m'(z_0)$ for all $z_0 \in \mathbb{C}$, and $q(\bar{z}) = -\overline{q(z)}$. Further, setting $y = \sup\{|\Im(z)| : z \in \text{supp}(m')\}$, q is holomorphic for $\Im(z) < -y$ and has at most polynomial growth in any strip $\{z \in \mathbb{C} : \Im(z) \in [a, b]\}$ with $a < b < -y$.

Next, let $h \in H$. By axiom (A4), we have $\sum_{z \in \mathbb{C}} m(z)h(z) \in \mathbb{R}$, and hence

$$\sum_{z \in \mathbb{C}} m(z)h(z) = \overline{\sum_{z \in \mathbb{C}} m(z)h(z)} = \sum_{z \in \mathbb{C}} m(z)h(\bar{z}) = \sum_{z \in \mathbb{C}} m(\bar{z})h(z).$$

Thus, for any $c > y$,

$$\begin{aligned} 0 &= \sum_{z \in \mathbb{C}} m'(z)h(z) = \frac{1}{2\pi i} \int_{\Im(z)=-c} q(z)h(z) dz - \frac{1}{2\pi i} \int_{\Im(z)=c} q(z)h(z) dz \\ &= \frac{1}{\pi i} \Re \int_{\Im(z)=-c} q(z)h(z) dz, \end{aligned}$$

i.e. $\int_{\Im(z)=-c} q(z)h(z) dz \in i\mathbb{R}$. Hence, by Lemma 2.3 with $\varphi(z) = iq(z)$, q is entire, and $m'(z) = 0$ identically. \square

2.2. Proof of Proposition 2.1. Set $\mu = -2 \lim_{x \rightarrow 0^+} \frac{d}{dx}(xK(x))$ and $K_1(x) = K(x) - \frac{d_F}{4 \sinh(x/2)} + \frac{\mu}{2 \cosh(x/2)}$. Then $K_1(x)/x$ extends to a Schwartz function on \mathbb{R} , so

$$(2.1) \quad k(z) = i \int_0^\infty \frac{K_1(x)}{x} e^{-ixz} dx$$

is well defined for $\Im(z) \leq 0$ and holomorphic for $\Im(z) < 0$. For any $n \geq 0$ we have

$$k^{(n)}(z) = \int_0^\infty K_1(x) (-ix)^{n-1} e^{-ixz} dx$$

for $\Im(z) < 0$, and by a standard argument based on Lebesgue's dominated convergence theorem, this extends continuously to $\Im(z) \leq 0$.

Next let $h \in H$ with Fourier transform g . By Plancherel's theorem, we have

$$\int_0^\infty K_1(x) g(x) dx = \frac{1}{2\pi} \int_{\mathbb{R}} k'(t) h(t) dt = \frac{1}{2\pi} \int_{\Im(z)=-c} k'(z) h(z) dz$$

for any $c \geq 0$. Together with the identities

$$\int_0^\infty \frac{g(0) - g(x)}{2 \sinh(x/2)} dx = g(0) \log(4e^\gamma) + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + it \right) h(t) dt$$

and

$$\int_0^\infty \frac{g(0) - g(x)}{2 \cosh(x/2)} dx = g(0) \frac{\pi}{2} + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{2} \left[\frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{it}{2} \right) - \frac{\Gamma'}{\Gamma} \left(\frac{3}{4} + \frac{it}{2} \right) \right] h(t) dt,$$

this yields

$$\Re \int_0^\infty K(x) (g(0) - g(x)) dx = \frac{1}{2\pi} \Re \int_{\Im(z)=-c} \varphi(z) h(z) dz,$$

where

$$\varphi(z) = \frac{d_F}{2} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + iz \right) + \frac{\mu}{2} \left[\frac{\Gamma'}{\Gamma} \left(\frac{3}{4} + \frac{iz}{2} \right) - \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{iz}{2} \right) \right] - k'(z) + C$$

and $C = \frac{d_F}{2} \log(4e^\gamma) + \Re(k'(0) - \frac{\pi\mu}{2})$.

Let $\omega \in \mathbb{C}^\times$ be a constant of modulus 1, to be determined below, and set

$$(2.2) \quad \gamma_F(s) = \omega \Gamma(s)^{\frac{d_F}{2}} \left(\frac{\Gamma((s+1)/2)}{\Gamma(s/2)} \right)^\mu \exp[(C - f(1))(s - \frac{1}{2}) - ik(-i(s - \frac{1}{2}))],$$

where for any $z \in \mathbb{C}$ we define $\Gamma(s)^z = \exp(z \log \Gamma(s))$ for $\Re(s) > 0$ using the principal branch of $\log \Gamma$. Note that $\frac{\gamma'_F}{\gamma_F}(\frac{1}{2} + iz) = \varphi(z) - f(1)$, and by the above we see that $\log \gamma_F(s)$ and all of its derivatives are holomorphic for $\Re(s) > \frac{1}{2}$ and extend continuously to $\Re(s) \geq \frac{1}{2}$, which establishes (i).

To see (ii), we rewrite (2.1) in the form

$$k(z) = i \int_0^\infty \frac{K_1(x)}{x} e^{x/2} e^{-(\frac{1}{2} + iz)x} dx$$

and apply integration by parts repeatedly to see that

$$k(z) = \sum_{j=1}^{n-1} \frac{c_j}{(\frac{1}{2} + iz)^j} + O_n(|\frac{1}{2} + iz|^{-n})$$

for some constants c_j . Using this together with Stirling's formula in (2.2) yields (ii).

Next let $\Lambda_F(s) = \gamma_F(s)L_F(s)$ and $\Phi(z) = \Lambda_F(\frac{1}{2}+iz)$. Then $\Phi(z)$ is analytic for $\Im(z) < -\frac{1}{2}$, where it satisfies

$$-i \frac{\Phi'}{\Phi}(z) = \varphi(z) - \sum_{n=1}^{\infty} \frac{f(n)}{n^{iz}}.$$

Thus, for any $c > \frac{1}{2}$ we have

$$(2.3) \quad \begin{aligned} \frac{1}{\pi} \Im \int_{\Im(z)=-c} \frac{\Phi'}{\Phi}(z) h(z) dz &= 2\Re \left[\frac{1}{2\pi} \int_{\Im(z)=-c} \varphi(z) h(z) dz - \sum_{n=1}^{\infty} \frac{f(n)}{2\pi} \int_{\Im(z)=-c} h(z) n^{-iz} dz \right] \\ &= 2\Re \left[\int_0^{\infty} K(x)(g(0) - g(x)) dx - \sum_{n=1}^{\infty} f(n)g(\log n) \right]. \end{aligned}$$

As in the proof of the Lemma 2.4, we define

$$q(z) = \frac{m(0)}{z} + \sum_{z_0 \in \text{supp}(m) \setminus \{0\}} \frac{m(z_0)}{z - z_0} \left(\frac{z}{z_0} \right)^{M-1}$$

for a suitable positive integer M . Lemma 2.4 shows that $m(\bar{z}) = m(z)$, so that $q(\bar{z}) = \overline{q(z)}$. Let $y = \sup\{|\Im(z)| : z \in \text{supp}(m)\}$ and $h \in H$ with Fourier transform g . Then for any $c > \max(y, \frac{1}{2})$, we have

$$\begin{aligned} \sum_{z \in \mathbb{C}} m(z) h(z) &= \frac{1}{2\pi i} \int_{\Im(z)=-c} q(z) h(z) dz - \frac{1}{2\pi i} \int_{\Im(z)=c} q(z) h(z) dz \\ &= \frac{1}{\pi} \Im \int_{\Im(z)=-c} q(z) h(z) dz. \end{aligned}$$

On the other hand, by axiom (A4) and (2.3), this equals

$$2\Re \left[\int_0^{\infty} K(x)(g(0) - g(x)) dx - \sum_{n=1}^{\infty} f(n)g(\log n) \right] = \frac{1}{\pi} \Im \int_{\Im(z)=-c} \frac{\Phi'}{\Phi}(z) h(z) dz.$$

Thus, $\int_{\Im(z)=-c} (\frac{\Phi'}{\Phi}(z) - q(z)) h(z) dz$ is real valued, so by Lemma 2.3 it follows that $\frac{\Phi'}{\Phi}(z) - q(z)$ continues to an entire function with at most polynomial growth in horizontal strips, and we have the functional equation $\frac{\Phi'}{\Phi}(\bar{z}) = \overline{\frac{\Phi'}{\Phi}(z)}$. This yields the residue formula

$$m(z_0) = \text{Res}_{z=z_0} q(z) = \text{Res}_{z=z_0} \frac{\Phi'}{\Phi}(z) = \text{Res}_{s=\frac{1}{2}+iz_0} \frac{\Lambda'_F(s)}{\Lambda_F(s)}$$

and establishes (v).

Define

$$l(z) = \begin{cases} \log(z-i) + \log(z+i) & \text{if } \Re(z) > 0 \text{ or } |\Im(z)| > 1, \\ \log(z^2+1) + 2\pi i & \text{if } \Re(z) < 0 \text{ and } \Im(z) > 0, \\ \log(z^2+1) - 2\pi i & \text{if } \Re(z) < 0 \text{ and } \Im(z) < 0, \end{cases}$$

where each log refers to the principal branch. One can check that the definitions agree where they overlap, so l is analytic on $\mathbb{C} \setminus ((-\infty, 0] \cup i[-1, 1])$ and satisfies $\exp(l(z)) = z^2 + 1$. For

any $z_0 \in \mathbb{C}$ we set

$$l_{z_0}(z) = \begin{cases} 2 \log(z - z_0) & \text{if } z_0 \in \mathbb{R}, \\ 2 \log |\Im(z_0)| + l\left(\frac{z - \Re(z_0)}{|\Im(z_0)|}\right) & \text{if } z_0 \notin \mathbb{R}, \end{cases}$$

so that l_{z_0} is analytic for $z - \Re(z_0) \in \mathbb{C} \setminus ((-\infty, 0] \cup i[-|\Im(z_0)|, |\Im(z_0)|])$ and satisfies $\exp(l_{z_0}(z)) = (z - z_0)(z - \bar{z}_0)$.

Next let Ω be as in the statement of the proposition and define

$$Q(z) = z^{m(0)} \prod_{z_0 \in \text{supp}(m) \setminus \{0\}} \exp\left(m(z_0) \sum_{n=1}^{M-1} \frac{(z/z_0)^n}{n}\right) \begin{cases} \exp(\frac{1}{2}m(z_0)l_{z_0}(z)) & \text{if } m(z_0) \notin \mathbb{Z}, \\ (z - z_0)^{m(z_0)} & \text{if } m(z_0) \in \mathbb{Z}, \end{cases}$$

where $z^{m(0)}$ means $\exp(m(0) \log z)$ if $m(0) \notin \mathbb{Z}$. Since $m(\bar{z}) = m(z)$ for all z , we see that Q is meromorphic on $\{z \in \mathbb{C} : \frac{1}{2} + iz \in \Omega\}$ and satisfies $\frac{Q'}{Q}(z) = q(z)$ in that region. By the above we conclude that $\Phi(z)/Q(z)$ continues to an entire, non-vanishing function of finite order. It follows that $\Lambda_F(s)$ continues meromorphically to Ω and has meromorphic finite order, which establishes (iii).

Since Ω is simply connected, by integrating the functional equation for $\frac{\Lambda'_F}{\Lambda_F}(s)$ we get $\Lambda_F(s) = c\omega^2 \overline{\Lambda_F(1 - \bar{s})}$ for some constant $c \in \mathbb{C}^\times$. Consideration of this equation for $\Re(s) = \frac{1}{2}$ shows that $|c| = 1$, and we choose ω to satisfy $c\omega^2 = 1$. This establishes (iv).

It remains only to see that γ_F is unique up to multiplication by a non-zero real scalar. To that end, suppose that $\tilde{\gamma}_F(s)$ is another function with the same properties, and consider the ratio $r(s) = \frac{\tilde{\gamma}_F(s)}{\gamma_F(s)}$. Since we may also write $r(s)$ in the form $\frac{\tilde{\gamma}_F(s)L_F(s)}{\gamma_F(s)L_F(s)}$, it follows from (iv) and (v) that $\frac{r'}{r}(s)$ has meromorphic continuation to \mathbb{C} and satisfies $\frac{r'}{r}(s) = -\overline{\frac{r'}{r}(1 - \bar{s})}$. By (i), $\frac{r'}{r}(s)$ is continuous on $\Re(s) \geq \frac{1}{2}$, so it must be entire. Therefore, $r(s)$ is entire and non-vanishing, and satisfies $r(s) = \overline{r(1 - \bar{s})}$. Further, by (ii), there are numbers $d', c'_{-1} \in \mathbb{R}$ and $\mu', c'_0 \in \mathbb{C}$ such that $r(s) = \exp\left(\left(\frac{d'}{2} \log \frac{s}{e} + c'_{-1}\right)(s - \frac{1}{2}) + \frac{\mu'}{2} \log \frac{s}{e} + c'_0\right) (1 + O(|s|^{-1}))$ for $\Re(s) \geq \frac{1}{2}$. The functional equation implies that $r(\frac{1}{2} + it) \in \mathbb{R}$ for all $t \in \mathbb{R}$, and taking $t \rightarrow \infty$ we conclude that $d' = c'_{-1} = \Im(\mu') = 0$. Together with the functional equation, this implies that $r(s) \ll (1 + |s - \frac{1}{2}|)^{\mu'}$, and thus r is a polynomial. Since it does not vanish, it must be a non-zero constant, and invoking the functional equation once more, it is real valued.

2.3. Proof of Theorem 1.6. Clearly (i) \implies (ii) \implies (iii), and we will show that (iii) \implies (iv) \implies (v) \implies (i). Beginning with (iii), suppose that $F = (f, K, m) \in \mathcal{L}$ satisfies $\sum_{n=1}^{\infty} \frac{|a_F(n)|}{\sqrt{n}} < \infty$. Then $L_F(s)$ is holomorphic for $\Re(s) > \frac{1}{2}$ and extends continuously to $\Re(s) \geq \frac{1}{2}$. By Proposition 2.1, $\Lambda_F(\frac{1}{2} + it) = \gamma_F(\frac{1}{2} + it)L_F(\frac{1}{2} + it)$ is well defined and real valued for all sufficiently large $t > 0$. Thus, we have

$$L_F(\frac{1}{2} + it) = \overline{L_F(\frac{1}{2} + it)} e^{-2i \arg \gamma_F(\frac{1}{2} + it)},$$

where $\arg \gamma_F(\frac{1}{2} + it)$ denotes the continuous extension of $\Im \log \gamma_F(s)$ to $\Re(s) = \frac{1}{2}$.

For a fixed positive integer m we multiply both sides of this equation by m^{it} and take the average over $t \in [T, 2T]$ as $T \rightarrow \infty$. On the left-hand side, thanks to the absolute

convergence of $\sum a_F(n)/\sqrt{n}$, this is $a_F(m)/\sqrt{m} + o(1)$, while on the right-hand side we get

$$\frac{1}{T} \int_T^{2T} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} e^{it \log(mn) - 2i \arg \gamma_F(\frac{1}{2} + it)} dt = \frac{1}{T} \int_T^{2T} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} e^{i(t \log(mn) - \varphi(t) + O(1/t))} dt,$$

where, by Proposition 2.1(ii), $\varphi(t) = d_F t \log(t/e) + 2c_{-1}t + \Im(\mu) \log(t/e) + \theta$, for some constants $c_{-1}, \theta \in \mathbb{R}$ and $\mu \in \mathbb{C}$. We may ignore the $O(1/t)$ error term at the cost of $o(1)$. Also, since the sum over n is absolutely convergent, the terms for $n > \log T$ contribute only $o(1)$, so we have

$$\frac{a_F(m)}{\sqrt{m}} = o(1) + \frac{1}{T} \int_T^{2T} \sum_{n \leq \log T} \frac{\overline{a_F(n)}}{\sqrt{n}} e^{i(t \log(mn) - \varphi(t))} dt.$$

If $d_F \neq 0$ then it is easy to see that $t \log(mn) - \varphi(t)$ has no stationary points in $[T, 2T]$ for sufficiently large T , and an application of integration by parts shows that the right-hand side is $o(1)$. To avoid a contradiction for $m = 1$, we must have $d_F = 0$. The main term of $t \log(mn) - \varphi(t)$ is thus $(\log(mn) - 2c_{-1})t$, so there is still no stationary phase unless $mn = e^{2c_{-1}}$. In particular, $e^{2c_{-1}}$ must be a positive integer, say q , and we have $a_F(m) = 0$ unless $m|q$. Thus, $L_F(s)$ is a Dirichlet polynomial, which clearly implies (iv).

Next, assuming (iv), $a_F(n)$ is supported on the y -smooth numbers

$$\{n \in \mathbb{Z}_{>0} : p \text{ prime and } p|n \implies p \leq y\}$$

for some $y > 0$. Taking the logarithm, the same conclusion applies to $f(n)$. By axiom (A1), we have the estimate $f(n) \log^k n \ll_k 1$ for every k , and applying this with $k = \pi(y) + 1$, we see that $\frac{L'_F}{L_F}(s) = -\sum_{n=2}^{\infty} f(n)n^{\frac{1}{2}-s}$ is holomorphic for $\Re(s) > \frac{1}{2}$ and continuous on the boundary. Hence this applies to $\frac{\Lambda'_F}{\Lambda_F}(s) = \frac{\gamma'_F}{\gamma_F}(s) + \frac{L'_F}{L_F}(s)$ as well, and together with the functional equation, this implies that $\Lambda_F(s)$ is entire and non-vanishing. Therefore $m(z) = 0$ identically, which implies (v).

Finally, let us assume (v), and set $N(t) = \sum_{\substack{z \in \text{supp}(m) \\ |\Re(z)| \leq t}} |m(z)|$ for $t \geq 0$. Then by hypothesis, there is a function $\varepsilon(T)$ such that $\lim_{T \rightarrow \infty} \varepsilon(T) = 0$ and $N(t) \leq \varepsilon(T)t$ for all $t \geq T$. We fix a test function g_0 which is non-negative, even, smooth, supported on $[-1, 1]$, satisfies $g_0(0) = 1$, and has Fourier transform $h_0 \in H$. For $\theta \in \mathbb{R}$, $T > 0$ and $x_0 > 0$, we consider axiom (A4) applied to $g(x) = e^{i\theta} g_0(T(x - x_0)) + e^{-i\theta} g_0(T(x + x_0))$, with Fourier transform $h(z) = 2T^{-1} \cos(\theta + x_0 z) h_0(T^{-1}z)$. If we choose $x_0 = \log n$ for some integer $n \geq 2$ and pick θ so that $e^{i\theta} f(n) \in \mathbb{R}_{\geq 0}$, then it is straightforward to see that

$$(2.4) \quad 2\Re \left[\int_0^{\infty} K(x)(g(0) - g(x)) dx - \sum_{n=1}^{\infty} f(n)g(\log n) \right] = -2|f(n)| + o(1)$$

as $T \rightarrow \infty$.

On the other hand, by axiom (A4) this equals

$$\frac{2}{T} \sum_{z \in \text{supp}(m)} m(z) \cos(\theta + z \log n) h_0(T^{-1}z).$$

Since g_0 is smooth, h_0 decays rapidly in horizontal strips; in particular, for $T \geq 1$,

$$|\cos(\theta + z \log n) h_0(T^{-1}z)| \leq C_n (1 + |\Re(T^{-1}z)|)^{-2} \quad \text{for } |\Im(z)| \leq \frac{1}{2}$$

holds from some $C_n > 0$ depending only on n . Further, since $N(t)$ is non-decreasing, we have $N(t) \leq \varepsilon(T) \max(t, T)$ for all $t \geq 0$. Thus, for $T \geq 1$ we get

$$\begin{aligned} \left| \frac{2}{T} \sum_{z \in \text{supp}(m)} m(z) \cos(\theta + z \log n) h_0(T^{-1}z) \right| &\leq \frac{2C_n}{T} \int_0^\infty \left(1 + \frac{t}{T}\right)^{-2} dN(t) \\ &= \frac{4C_n}{T^2} \int_0^\infty \left(1 + \frac{t}{T}\right)^{-3} N(t) dt \leq \frac{4C_n \varepsilon(T)}{T^2} \int_0^\infty \left(1 + \frac{t}{T}\right)^{-3} \max(t, T) dt \\ &= 3C_n \varepsilon(T) = o(1). \end{aligned}$$

Together with (2.4), this shows that $f(n) = 0$ for all $n \geq 2$. In particular, $L_F(s) = 1$, which implies (iv); in turn, as we saw above, this implies that $m(z) = 0$ identically. Therefore,

$$\Re \left[\int_0^\infty K(x)(g(0) - g(x)) dx - f(1)g(0) \right] = 0.$$

for every suitable test function g . Choosing g as above for an arbitrary $x_0 > 0$ and θ so that $e^{i\theta} K(x_0) \in \mathbb{R}_{\geq 0}$, we see that $K(x_0) = 0$. Finally, $f(1) \in \mathbb{R}$ by axiom (A1), so we have $f(1) = 0$. Thus, (i) holds and this completes the proof.

3. PROOF OF THEOREM 1.7

Let $F \in \mathcal{L}_d^+$ for some $d < 2$. The main object of study in the method initiated by Conrey and Ghosh [7] is the exponential sum

$$S_F(z) = \sum_{n=1}^{\infty} a_F(n) e(nz),$$

defined for $z \in \{x + iy \in \mathbb{C} : y > 0\}$. For $k \in \mathbb{Z}$ we write

$$S_F^{(k)}(z) = \sum_{n=1}^{\infty} a_F(n) (2\pi i n)^k e(nz).$$

Note that this is just the k th derivative for $k \geq 0$. Since $S_F(z)$ is periodic and decays exponentially as $\Im(z) \rightarrow \infty$, it will be enough to consider z in a box

$$B = \{z = -x + iy \in \mathbb{C} : x \in [x_1, x_2], y \in (0, y_1]\},$$

for fixed $x_2 > x_1 > 0$ and $y_1 > 0$ to be specified later.

Lemma 3.1. *For F as above, let $c_{-1} \in \mathbb{R}$ and $\mu \in \mathbb{C}$ be the constants given by Proposition 2.1, and define*

$$G(s) = (2\pi(1 - \frac{d}{2})^{1-\frac{d}{2}} e^{c_{-1}})^{\frac{1}{2}-s} \Gamma\left(\left(1 - \frac{d}{2}\right) \left(s - \frac{1}{2}\right) + \frac{1-\mu}{2}\right).$$

Then for any integer $k \geq 0$ there are constants $c_{kj} \in \mathbb{C}$, $0 \leq j \leq k$, with $c_{kk} \neq 0$, such that for any $\sigma > \max(1, \frac{1}{2} + \frac{\Re(\mu)-1}{2-d})$ and $\varepsilon > 0$,

$$z^k S_F^{(k)}(z) = O_{B,k,\varepsilon}(\Im(z)^{-\varepsilon}) + \sum_{j=0}^k \frac{c_{kj}}{2\pi i} \int_{\Re(s)=\sigma} \Lambda_F(s) G\left(s + \frac{2j}{2-d}\right) (-iz)^{-s} ds,$$

uniformly for $z \in B$.

Proof. By Proposition 2.1(ii) and Stirling's formula, we find that there are constants $\alpha_0, \alpha_1, \dots \in \mathbb{C}$ such that

$$\log \frac{\gamma_F(s)G(s)}{(2\pi)^{-s}\Gamma(s)} = \sum_{j=0}^{n-1} \frac{\alpha_j}{s^j} + O_n(|s|^{-n}),$$

uniformly on $\{s \in \mathbb{C} : \Re(s) \geq \frac{1}{2}\} \cap \{s \in \mathbb{C} : \Re(s) \geq \frac{1}{2} + \frac{\Re(\mu)}{2-d} \text{ or } |\Im(s) - \frac{\Im(\mu)}{2-d}| \geq 1\}$. We take the exponential to get

$$\gamma_F(s)G(s) = (2\pi)^{-s}\Gamma(s) \left(\sum_{j=0}^{n-1} \frac{\beta_j}{s^j} + O_n(|s|^{-n}) \right)$$

for some constants $\beta_j \in \mathbb{C}$, with $\beta_0 \neq 0$.

Next we fix $k \geq 0$, take $n = k + 3$ in the above, and multiply by $G(s + \frac{2k}{2-d})/G(s)$ to get

$$\begin{aligned} \gamma_F(s)G\left(s + \frac{2k}{2-d}\right) &= (2\pi)^{-s}\Gamma(s+k) \frac{G(s + \frac{2k}{2-d})\Gamma(s)}{G(s)\Gamma(s+k)} \left(\sum_{j=0}^{k+2} \frac{\beta_j}{s^j} + O_k(|s|^{-k-3}) \right) \\ &= \sum_{j=-2}^k \gamma_{kj} (2\pi)^{-s}\Gamma(s+j) + O_k(|s|^{-1})(2\pi)^{-s}\Gamma(s-2) \end{aligned}$$

for some $\gamma_{kj} \in \mathbb{C}$ with $\gamma_{kk} \neq 0$, uniformly on the set

$$(3.1) \quad \{s \in \mathbb{C} : \Re(s) \geq \frac{9}{4}\} \cap \{s \in \mathbb{C} : \Re(s) \geq \frac{1}{2} + \frac{\Re(\mu)+2}{2-d} \text{ or } |\Im(s) - \frac{\Im(\mu)}{2-d}| \geq 1\}.$$

Fix $\sigma > \max(1, \frac{1}{2} + \frac{\Re(\mu)-1}{2-d})$, and consider the integral $\frac{1}{2\pi i} \int_{\Re(s)=\sigma} \Lambda_F(s)G(s + \frac{2k}{2-d})(-iz)^{-s} ds$. Shifting the contour to the right if necessary, we may assume without loss of generality that $\sigma \geq \frac{9}{4}$. We write $M_k(s) = \gamma_F(s)^{-1} \sum_{j=-2}^k \gamma_{kj} (2\pi)^{-s}\Gamma(s+j)$, $R_k(s) = G(s + \frac{2k}{2-d}) - M_k(s)$, and split the integral accordingly. For the integral against $R_k(s)$, we shift the contour to the boundary of (3.1), on which we have the estimate $\Lambda_F(s)R_k(s)(-iz)^{-s} \ll_{B,k} |s|^{-5/4}$ for all $z \in B$. Thus, this integral contributes $O_{B,k}(1)$.

As for the main term, for each $j \geq -2$, Mellin inversion gives

$$\frac{1}{2\pi i} \int_{\Re(s)=\sigma} L_F(s)(-2\pi iz)^{-s}\Gamma(s+j) ds = \sum_{n=1}^{\infty} a_F(n)(-2\pi in z)^j e(nz) = (-z)^j S_F^{(j)}(z).$$

Since $\sum_{n=1}^{\infty} |a_F(n)|n^{-\sigma}$ converges for every $\sigma > 1$, we see that $z^{-2}S_F^{(-2)}(z) \ll_B 1$ and $z^{-1}S_F^{(-1)}(z) \ll_{B,\varepsilon} \Im(z)^{-\varepsilon}$. Thus, altogether we have

$$\frac{1}{2\pi i} \int_{\Re(s)=\sigma} \Lambda_F(s)G\left(s + \frac{2k}{2-d}\right) (-iz)^{-s} ds = O_{B,k,\varepsilon}(\Im(z)^{-\varepsilon}) + \sum_{j=0}^k (-1)^j \gamma_{kj} z^j S_F^{(j)}(z)$$

for $z \in B$. Note that this system of equations is triangular, with non-zero diagonal coefficients $(-1)^k \gamma_{kk}$. The lemma follows on multiplying by the inverse matrix. \square

For integers $k, \ell \geq 0$ we define $\delta_k = \frac{2k}{2-d}$ and $\sigma_\ell = \frac{1}{2} + \frac{2\ell - \Re(\mu)}{2-d}$. Assume that ℓ is such that $\sigma_\ell > \max(1, \frac{1}{2} + \frac{\Re(\mu)-1}{2-d})$; then all poles of $G(s + \delta_k)$ lie to the left of $\Re(s) = \sigma_\ell$ and avoid the line $\Re(s) = 1 - \sigma_\ell$. Since F is positive, there is a number $T \in \mathbb{R}$ such that $\Lambda_F(s)$ is holomorphic for $\Im(s) \geq T$. Fix such a T and let Γ_ℓ be the boundary of the region

$\{s \in \mathbb{C} : \Re(s) \in [1 - \sigma_\ell, \sigma_\ell], \Im(s) \leq T\}$, with counterclockwise orientation. For $z \in B$, $|z| \asymp_B 1$ and $\arg z - \frac{\pi}{2} \gg_B 1$, so it follows from the identity

$$|(-iz)^{-s}| = |z|^{-\Re(s)} \exp\left(\left(\arg z - \frac{\pi}{2}\right)\Im(s)\right),$$

the functional equation $\Lambda_F(s) = \overline{\Lambda_F(1 - \bar{s})}$ and Stirling's formula that

$$\Lambda_F(s)G(s + \delta_k)(-iz)^{-s} \ll_{B,k,\ell} e^{\frac{\pi}{2}\Im(s)} \quad \text{for } s \in \Gamma_\ell,$$

with an implied constant that is independent of z . Thus, we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Re(s)=\sigma_\ell} \Lambda_F(s)G(s + \delta_k)(-iz)^{-s} ds \\ (3.2) \quad &= \frac{1}{2\pi i} \left(\int_{\Gamma_\ell} + \int_{\Re(s)=1-\sigma_\ell} \right) \Lambda_F(s)G(s + \delta_k)(-iz)^{-s} ds \\ &= O_{B,k,\ell}(1) + \frac{1}{2\pi i} \int_{\Re(s)=1-\sigma_\ell} \Lambda_F(s)G(s + \delta_k)(-iz)^{-s} ds \\ &= O_{B,k,\ell}(1) + \frac{1}{2\pi i} \int_{\Re(s)=\sigma_\ell} \overline{\Lambda_F(\bar{s})}G(1 - s + \delta_k)(-iz)^{s-1} ds, \end{aligned}$$

where the last line follows by the functional equation.

3.1. Degree < 1 . Let us first see how to use this to find all elements of \mathcal{L}_d^+ for $d < 1$. Let notation be as in (3.2) above. Then for $s = \sigma_\ell + it$, by the bound $L_F(\bar{s}) \ll_\ell 1$ and Proposition 2.1(ii), we have

$$\overline{\Lambda_F(\bar{s})} \ll_\ell (1 + |t|)^{\frac{d}{2}(\sigma_\ell - \frac{1}{2}) + \frac{\Re(\mu)}{2}} e^{-\frac{\pi}{4}d|t|}.$$

On the other hand, Stirling's formula implies that

$$G(1 - s + \delta_k) \ll_{k,\ell} (1 + |t|)^{\left(\frac{d}{2}-1\right)(\sigma_\ell - \frac{1}{2}) + k - \frac{\Re(\mu)}{2}} e^{-\frac{\pi}{4}(2-d)|t|},$$

and together these estimates yield

$$\overline{\Lambda_F(\bar{s})}G(1 - s + \delta_k)(-iz)^{s-1} \ll_{k,\ell} |z|^{\sigma_\ell-1} (1 + |t|)^{(d-1)(\sigma_\ell - \frac{1}{2}) + k}.$$

Since $d < 1$, taking ℓ (and hence σ_ℓ) sufficiently large, we thus have

$$\frac{1}{2\pi i} \int_{\Re(s)=\sigma_\ell} \Lambda_F(s)G(s + \delta_k)(-iz)^{-s} ds \ll_{B,k} 1.$$

By Lemma 3.1, it follows that $S_F^{(k)}(z) \ll_{B,k,\varepsilon} \Im(z)^{-\varepsilon}$ for any $k \geq 0$. Fixing

$$B = \{-x + iy : x \in [1, 2], y \in (0, 1]\},$$

for any positive integer n and $y \in (0, 1]$, we have

$$(2\pi in)^k a_F(n) e^{-2\pi ny} = \int_{-2}^{-1} S_F^{(k)}(x + iy) e(-nx) dx \ll_{k,\varepsilon} y^{-\varepsilon}.$$

Taking $y = 1/n$, we find $a_F(n) \ll_{k,\varepsilon} n^{-k+\varepsilon}$. In particular, with $k = 1$ we see that $\sum_{n=1}^{\infty} |a_F(n)|/\sqrt{n} < \infty$, and thus Theorem 1.6(iii) implies that $F = (0, 0, 0)$.

3.2. Degree 1. Henceforth we assume that $d \geq 1$. Note that we are free to shift the L -function of F by an imaginary displacement; precisely, for any $t \in \mathbb{R}$ one can see directly from Definition 1.3 that there exists $F_t \in \mathcal{L}_d^+$ with L -function $L_{F_t}(s) = L_F(s + it)$. By the uniqueness of γ -factors we have $\gamma_{F_t}(s) = \gamma_F(s + it)$, and in particular the constant μ given by Proposition 2.1 changes to $\mu_t = \mu + idt$. Hence, replacing F by F_t for a suitable t , we may assume without loss of generality that $\mu \in \mathbb{R}$.

With this convention, after a computation similar to that preceding (3.1), using also the identity $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, we find that

$$(3.3) \quad \overline{\gamma_F(\bar{s})}G(1-s) = A^{s-\frac{1}{2}} \frac{\Gamma((d-1)(s-\frac{1}{2})+\frac{1}{2})}{\cos \frac{\pi}{2}((2-d)(s-\frac{1}{2})+\mu)} \left(\sum_{j=0}^{n-1} \frac{\alpha_j}{s^j} + O_n(|s|^{-n}) \right)$$

for some constants $A \in \mathbb{R}_{>0}$ and $\alpha_j \in \mathbb{C}$ with $\alpha_0 \neq 0$, uniformly on

$$\{s \in \mathbb{C} : \Re(s) \geq \frac{1}{2}\} \cap \{s \in \mathbb{C} : \Re(s) \geq \frac{1}{2} - \frac{\mu}{2-d} \text{ or } |\Im(s)| \geq 1\}.$$

For $d = 1$, we fix any permissible value of ℓ and substitute (3.3) into (3.2) and Lemma 3.1 with $k = 0$, obtaining

$$S_F(z) = O_{B,\varepsilon}(\Im(z)^{-\varepsilon}) + \frac{c}{2\pi i} \int_{\Re(s)=\sigma_\ell} \frac{(-iAz)^{s-1}}{\cos \frac{\pi}{2}(s-\frac{1}{2}+\mu)} \overline{L_F(\bar{s})} (1 + O(|s|^{-1})) ds$$

for some constant $c \in \mathbb{C}^\times$. Using the estimates

$$\frac{(-iAz)^{s-1}}{\cos \frac{\pi}{2}(s-\frac{1}{2}+\mu)} \ll_B e^{-(\pi-\arg z)|\Im(s)|} \quad \text{and} \quad \overline{L_F(\bar{s})} \ll 1$$

for $\Re(s) = \sigma_\ell$, we see that the $O(|s|^{-1})$ error term contributes $\ll_B \log \frac{\pi}{\pi-\arg z} \ll_{B,\varepsilon} \Im(z)^{-\varepsilon}$.

As for the main term, recalling that $\sigma_\ell = \frac{1}{2} + 2\ell - \mu$, we have

$$\begin{aligned} S_F(z) &= O_{B,\varepsilon}(\Im(z)^{-\varepsilon}) - c(-1)^\ell \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(-iAz/n)^{2\ell-\frac{1}{2}-\mu+it}}{\cosh \frac{\pi t}{2}} dt \\ &= O_{B,\varepsilon}(\Im(z)^{-\varepsilon}) - \frac{2ic(-1)^\ell}{\pi} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n} \frac{(-iAz/n)^{2\ell-\frac{1}{2}-\mu}}{\frac{Az}{n} - \frac{n}{Az}}. \end{aligned}$$

We now fix $B = \{-\frac{\alpha-iy}{A} : \alpha \in [1, N+A], y \in (0, 1]\}$ for a large integer $N > 0$, and set $z = -\frac{\alpha-iy}{A}$ in the above. If α is not an integer then we see that the last line above is $O_{\alpha,N,\varepsilon}(y^{-\varepsilon})$, while if $\alpha = n \in \mathbb{Z}_{>0}$ then we get

$$\frac{ce^{i\frac{\pi}{2}(\frac{1}{2}-\mu)}}{\pi y} \overline{a_F(n)} + O_{N,\varepsilon}(y^{-\varepsilon}).$$

Since $S_F(z)$ is periodic and $a_F(1) = 1$, it follows that A is an integer and $a_F(n) = a_F(n+A)$ for all $n \leq N$. Thus, since N is arbitrary, $a_F(n)$ is periodic with period A .

Now, since $L_F(s)$ does not vanish for $\Re(s) > 1$, it follows from [17, Theorem 4] that there is a positive integer $q|A$, a primitive Dirichlet character $\chi \pmod{q}$ and a Dirichlet polynomial $D(s)$ such that $L_F(s) = D(s)L(s, \chi)$. Let $F_\chi \in \mathcal{L}_1^+$ be the L -datum associated to χ , so that $L_{F-F_\chi}(s) = D(s)$. Then Theorem 1.6(iv) implies that $F = F_\chi$, and this completes the proof for $d = 1$.

3.3. Degree > 1 . We assume now that $d \in (1, 2)$ and follow the method of Kaczorowski and Perelli [10]. In what follows we write $\kappa = \frac{1}{d-1}$ and

$$\sigma_k^* = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|a_F(n)|^k}{n^\sigma} < \infty \right\} \quad \text{for } k \in \{1, 2\}.$$

Since $F \neq (0, 0, 0)$, it follows from Theorem 1.6 that $\sum_{n=1}^{\infty} |a_F(n)|/\sqrt{n}$ diverges, so $\sigma_1^* \in [\frac{1}{2}, 1]$. On the other hand, by the Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{|a_F(n)|^2}{n^{2\sigma}} \leq \left(\sum_{n=1}^{\infty} \frac{|a_F(n)|}{n^\sigma} \right)^2 \leq \zeta(1 + \varepsilon) \sum_{n=1}^{\infty} \frac{|a_F(n)|^2}{n^{2\sigma-1-\varepsilon}}$$

for each $\varepsilon > 0$, and thus $2\sigma_1^* \in [\sigma_2^*, \sigma_2^* + 1]$.

Continuing along the same lines as above, we find the following formula for $S_F^{(k)}(z)$ in this case.

Lemma 3.2. *Fix a compact interval $I \subseteq (0, \infty)$. Then for any $k \geq 1$, $\alpha \in I$ and $y > 0$ sufficiently small,*

$$S_F^{(k)}\left(-\frac{\alpha - iy}{A}\right) = O_{I,k,\varepsilon}\left(y^{\frac{1}{2}-k-(d-1)(\sigma_1^*-\frac{1}{2}+\varepsilon)}\right) + \frac{\gamma_k}{y^{k+\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \exp\left(i\left(\frac{n}{\alpha}\right)^\kappa\right) V_k\left(\frac{n}{\alpha^d y^{1-d}}\right),$$

where $\gamma_k \in \mathbb{C}^\times$ is a constant and $V_k(t) = t^{\kappa(k+\frac{1}{2})} \exp(-\kappa(t^\kappa - 1))$.

Proof. Using (3.3), we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Re(s)=\sigma_\varepsilon} \overline{\Lambda_F(\bar{s})} G(1-s+\delta_k) (-iz)^{s-1} ds = \\ & \frac{1}{\sqrt{-iz}} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \frac{1}{2\pi i} \int_{\Re(s)=\sigma_\varepsilon} \left(\frac{-iAz}{n}\right)^{s-\frac{1}{2}} \frac{\Gamma\left((d-1)\left(s-\frac{1}{2}\right)+\frac{1}{2}\right)}{\cos\left[\frac{\pi}{2}\left((2-d)\left(s-\frac{1}{2}\right)+\mu\right)\right]} \\ & \quad \cdot \frac{G(1-s+\delta_k)}{G(1-s)} \left(\sum_{j=0}^{n-1} \frac{\alpha_j}{s^j} + O_n(|s|^{-n})\right) ds. \end{aligned}$$

Let Γ be the boundary curve of

$$\{s \in \mathbb{C} : \Re(s) \geq 1 + \varepsilon\} \cap \{s \in \mathbb{C} : \Re(s) \geq \frac{1}{2} + \max(-\frac{\mu}{2-d}, \frac{1}{d-1}) \text{ or } |\Im(s)| \geq 1\}$$

for a sufficiently small $\varepsilon > 0$. Choosing $n = k + 2$, we have

$$\begin{aligned} & \left(\frac{-iAz}{n}\right)^{s-\frac{1}{2}} \frac{\Gamma\left((d-1)\left(s-\frac{1}{2}\right)+\frac{1}{2}\right)}{\cos\left[\frac{\pi}{2}\left((2-d)\left(s-\frac{1}{2}\right)+\mu\right)\right]} \frac{G(1-s+\delta_k)}{G(1-s)} \left(\sum_{j=0}^{n-1} \frac{\alpha_j}{s^j} + O_n(|s|^{-n})\right) \\ & = O_{B,k}(|s|^{-\frac{3}{2}}) + \sum_{j=-1}^k \beta_{kj} \left(\frac{-iAz}{n}\right)^{s-\frac{1}{2}} \frac{\Gamma\left((d-1)\left(s-\frac{1}{2}\right)+j+\frac{1}{2}\right)}{\cos\left[\frac{\pi}{2}\left((2-d)\left(s-\frac{1}{2}\right)+\mu\right)\right]}, \end{aligned}$$

for some constants $\beta_{kj} \in \mathbb{C}$ with $\beta_{kk} \neq 0$, uniformly for $z \in B$ and $s \in \Gamma$. Thus,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Re(s)=\sigma_\ell} \overline{\Lambda_F(\bar{s})} G(1-s+\delta_k)(-iz)^{s-1} ds = \\ O_{B,k}(1) & + \frac{1}{\sqrt{-iz}} \sum_{j=-1}^k \beta_{kj} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{-iAz}{n} \right)^{s-\frac{1}{2}} \frac{\Gamma((d-1)(s-\frac{1}{2})+j+\frac{1}{2})}{\cos[\frac{\pi}{2}((2-d)(s-\frac{1}{2})+\mu)]} ds. \end{aligned}$$

Next, from the identity $\sec s = 2e^{-is} - 2e^{-is} \sec s$ and Stirling's formula, we see that we may replace $1/\cos[\frac{\pi}{2}((2-d)(s-\frac{1}{2})+\mu)]$ by $2 \exp[-\frac{i\pi}{2}((2-d)(s-\frac{1}{2})+\mu)]$ with an error of $O_{B,k}(1)$, uniformly for $z \in B$. Thus, we get

$$(3.4) \quad \begin{aligned} O_{B,k}(1) & + \frac{2e^{-\frac{i\pi}{2}\mu}}{\sqrt{-iz}} \sum_{j=-1}^k \beta_{kj} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{A|z|}{n} \right)^{s-\frac{1}{2}} e^{i[\frac{\pi}{2}(d-1)-(\pi-\arg z)](s-\frac{1}{2})} \\ & \cdot \Gamma((d-1)(s-\frac{1}{2})+j+\frac{1}{2}) ds. \end{aligned}$$

Assuming that $\Im(z)$ is small enough that $\pi - \arg z < \frac{\pi}{2}(d-1)$, we make the change of variables $s \mapsto \kappa s + \frac{1}{2}$, to get

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{A|z|}{n} \right)^{s-\frac{1}{2}} e^{i[\frac{\pi}{2}(d-1)-(\pi-\arg z)](s-\frac{1}{2})} \Gamma((d-1)(s-\frac{1}{2})+j+\frac{1}{2}) ds \\ & = \frac{\kappa}{2\pi i} \int_{\Re(s)=1} \left(-i \left(-\frac{n}{Az} \right)^{\kappa} \right)^{-s} \Gamma(s+j+\frac{1}{2}) ds \\ & = \kappa \left(-i \left(-\frac{n}{Az} \right)^{\kappa} \right)^{j+\frac{1}{2}} \exp\left(i \left(-\frac{n}{Az} \right)^{\kappa} \right), \end{aligned}$$

where all powers are taken with respect to the principal branch of the logarithm.

Let us now fix $B = \{-\frac{\alpha-iy}{A} : \alpha \in I, y \in (0, \delta]\}$ for a sufficiently small $\delta > 0$, and put $z = -\frac{\alpha-iy}{A}$ in the above. Then

$$\left(-\frac{n}{Az} \right)^{\kappa} = \left(\frac{n}{\alpha} \right)^{\kappa} \left(1 - \frac{iy}{\alpha} \right)^{-\kappa},$$

so (3.4) becomes

$$(3.5) \quad \begin{aligned} O_{I,k}(1) & + \sum_{j=-1}^k \frac{2\kappa e^{-\frac{i\pi}{2}(\mu+j+\frac{1}{2})} \beta_{kj}}{\sqrt{-iz}} \left(1 - \frac{iy}{\alpha} \right)^{-\kappa(j+\frac{1}{2})} \\ & \cdot \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \left(\frac{n}{\alpha} \right)^{\kappa(j+\frac{1}{2})} \exp\left(i \left(\frac{n}{\alpha} \right)^{\kappa} \left(1 - \frac{iy}{\alpha} \right)^{-\kappa} \right). \end{aligned}$$

Since $i \left(1 - \frac{iy}{\alpha} \right)^{-\kappa} = i - \frac{\kappa y}{\alpha} + O_I(y^2)$, for sufficiently small y the j th term of (3.5) is bounded above by $O_{I,k,\varepsilon}(1 + y^{-j-\frac{1}{2}-(d-1)(\sigma_1^*-\frac{1}{2}+\varepsilon)})$.

We substitute (3.5) into (3.2) and Lemma 3.1, and apply the bound noted above to every term with $j < k$, obtaining

$$S_F^{(k)}(z) = O_{I,k,\varepsilon}(y^{\frac{1}{2}-k-(d-1)(\sigma_1^*-\frac{1}{2}+\varepsilon)}) + \frac{\gamma_k e^\kappa}{\alpha^{k+\frac{1}{2}}} \left(1 - \frac{iy}{\alpha}\right)^{-(\kappa+1)(k+\frac{1}{2})} \\ \cdot \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \left(\frac{n}{\alpha}\right)^{\kappa(k+\frac{1}{2})} \exp\left(i \left(\frac{n}{\alpha}\right)^\kappa \left(1 - \frac{iy}{\alpha}\right)^{-\kappa}\right)$$

for any $k > 0$ and some constant $\gamma_k \in \mathbb{C}^\times$. Finally we replace the factor $(1 - \frac{iy}{\alpha})^{-(\kappa+1)(k+\frac{1}{2})}$ by 1 and $\exp\left(i \left(\frac{n}{\alpha}\right)^\kappa \left(1 - \frac{iy}{\alpha}\right)^{-\kappa}\right)$ by $\exp\left(i \left(\frac{n}{\alpha}\right)^\kappa \left(i - \frac{\kappa y}{\alpha}\right)\right)$, both of which contribute an error of at most $O_{I,k,\varepsilon}(y^{\frac{1}{2}-k-(d-1)(\sigma_1^*-\frac{1}{2}+\varepsilon)})$, so that

$$S_F^{(k)}(z) = O_{I,k,\varepsilon}(y^{\frac{1}{2}-k-(d-1)(\sigma_1^*-\frac{1}{2}+\varepsilon)}) + \frac{\gamma_k}{y^{k+\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \exp\left(i \left(\frac{n}{\alpha}\right)^\kappa\right) V_k\left(\frac{n}{\alpha^d y^{1-d}}\right),$$

as desired. \square

Lemma 3.3. *Let $w : \mathbb{R} \rightarrow \mathbb{C}$ be a smooth function supported on a compact subinterval of $(0, \infty)$, and define*

$$\widehat{w}(x) = \int_{\mathbb{R}} w(t) e(xt) dt, \quad W(u) = \begin{cases} u^{\frac{\kappa}{2} + \frac{1}{2d}} (u^{1/d} - 1)^{\frac{\kappa-1}{2}} w((u^{1/d} - 1)^\kappa) & \text{if } u > 1, \\ 0 & \text{if } u \leq 1, \end{cases}$$

and

$$\Sigma(x) = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{ad}} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^{\frac{\kappa+1}{2}}} e\left(a \left(n^{\frac{1}{d}} - x^{\frac{1}{d}}\right)^{\kappa d}\right) W\left(\frac{n}{x}\right),$$

where $a = \frac{1}{2\pi A^\kappa}$. Then for all $x > 0$ sufficiently large,

$$(3.6) \quad \Sigma(x) = \frac{\overline{a_F(n(x))}}{\sqrt{n(x)}} \widehat{w}(-a\kappa n(x)^{\kappa-1}[x - n(x)]) + O_{w,\varepsilon}(x^{\sigma_1^*-\frac{1}{2}-\kappa+\varepsilon}),$$

where $n(x) = \lfloor x + \frac{1}{2} \rfloor$ is the nearest integer to x .

Proof. We apply Lemma 3.2 with $k = 1$. Since $S'_F(z)$ is periodic, the formula is invariant under $\alpha \mapsto \alpha + A$, so that

$$\sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \exp\left(i \left(\frac{n}{\alpha}\right)^\kappa\right) V_1\left(\frac{n}{\alpha^d y^{1-d}}\right) = O_{I,\varepsilon}(y^{1-(d-1)(\sigma_1^*-\frac{1}{2}+\varepsilon)}) \\ + \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \exp\left(i \left(\frac{n}{\alpha+A}\right)^\kappa\right) V_1\left(\frac{n}{(\alpha+A)^d y^{1-d}}\right).$$

Introducing new parameters $x = \alpha^d y^{1-d}$ and $t = (A/\alpha)^\kappa$, this becomes

$$\sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} e(an^\kappa t) V_1\left(\frac{n}{x}\right) = O_{I,\varepsilon}(x^{\sigma_1^*-\frac{1}{2}-\kappa+\varepsilon}) + \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} e\left(\frac{an^\kappa t}{(1+t^{d-1})^\kappa}\right) V_1\left(\frac{n}{x(1+t^{d-1})^d}\right).$$

Now let $w : \mathbb{R} \rightarrow \mathbb{C}$ be as in the hypotheses, and fix the interval I in Lemma 3.2 so that it contains At^{1-d} for every t in the support of w . We multiply both sides of the above by $e(-atx^\kappa)w(t)$ and integrate to get

$$(3.7) \quad \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \widehat{w}(a(n^\kappa - x^\kappa)) V_1\left(\frac{n}{x}\right) = O_{w,\varepsilon}(x^{\sigma_1^* - \frac{1}{2} - \kappa + \varepsilon}) \\ + \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{\sqrt{n}} \int_0^\infty e(x^\kappa \varphi(t, n/x)) \psi(t, n/x) dt,$$

where

$$\varphi(t, u) = at \left[\left(\frac{u}{1+t^{d-1}} \right)^\kappa - 1 \right] \quad \text{and} \quad \psi(t, u) = w(t) V_1\left(\frac{u}{(1+t^{d-1})^d} \right).$$

Suppose that w is supported on $[t_1, t_2] \subseteq (0, \infty)$, and set $\lambda_1 = \frac{1}{2}(1 + (1 + t_1^{d-1})^d)$, $\lambda_2 = 2(1 + t_2^{d-1})^d$. To treat the right-hand side of (3.7), we split the sum over the three ranges $n \leq \lambda_1 x$, $\lambda_1 x < n < \lambda_2 x$ and $n \geq \lambda_2 x$. We compute that

$$\frac{\partial \varphi}{\partial t}(t, u) = a \left[\frac{u^\kappa}{(1+t^{d-1})^{\kappa d}} - 1 \right] \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial t^2}(t, u) = -\frac{adu^\kappa t^{d-2}}{(1+t^{d-1})^{\kappa d+1}},$$

so that $\frac{\partial \varphi}{\partial t}$ vanishes only when $u = (1+t^{d-1})^d$. Hence, for $u \leq \lambda_1$ or $u \geq \lambda_2$, $\frac{\partial \varphi}{\partial t}$ does not vanish for $t \in [t_1, t_2]$, and in fact we have $|\frac{\partial \varphi}{\partial t}| \gg_w 1$ uniformly for $t \in [t_1, t_2]$, $u \notin (\lambda_1, \lambda_2)$. Further, $\frac{\partial^2 \varphi}{\partial t^2}$ never vanishes, so by van der Corput's lemma [20, Chapter VIII, Corollary of Proposition 2], we have

$$\int_0^\infty e(x^\kappa \varphi(t, u)) \psi(t, u) dt \ll_w x^{-\kappa},$$

uniformly for $u \leq \lambda_1$ or $u \geq \lambda_2$. Thus, the ranges $n \leq \lambda_1 x$ and $n \geq \lambda_2 x$ contribute $O_{w,\varepsilon}(x^{\sigma_1^* - \frac{1}{2} - \kappa + \varepsilon})$.

For the remaining range we apply the method of stationary phase [21, Lemma 2.8]. Note in particular that

$$\frac{\partial^2 \varphi}{\partial t^2}(t_0(u), u) = -\frac{ad}{u^{1/d}(u^{1/d} - 1)^{\kappa-1}} \quad \text{and} \quad \varphi(t_0(u), u) = a(u^{1/d} - 1)^{\kappa d},$$

where $t_0(u) = (u^{1/d} - 1)^\kappa$ is the stationary point of $\varphi(t, u)$. Since $u = n/x$ varies within a compact subset of $(1, \infty)$, we find that

$$\int_0^\infty e(x^\kappa \varphi(t, u)) \psi(t, u) dt = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{ad}} n^{-\frac{\kappa}{2}} e\left(a(n^{1/d} - x^{1/d})^{\kappa d}\right) W\left(\frac{n}{x}\right) + O_w(x^{-\frac{3}{2}\kappa}),$$

with W as defined in statement of the lemma, and with an implied constant that is independent of n . The error term contributes a total of at most $O_{w,\varepsilon}(x^{\sigma_1^* - \frac{1}{2} - \frac{3}{2}\kappa + \varepsilon})$, so altogether the right-hand side of (3.7) is

$$O_{w,\varepsilon}(x^{\sigma_1^* - \frac{1}{2} - \kappa + \varepsilon}) + \frac{e^{-\frac{i\pi}{4}}}{\sqrt{ad}} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^{\frac{\kappa+1}{2}}} e\left(a(n^{1/d} - x^{1/d})^{\kappa d}\right) W\left(\frac{n}{x}\right).$$

Finally, we note that the left-hand side of (3.7) is essentially concentrated at integers. Precisely, for any $n \neq n(x) = \lfloor x + \frac{1}{2} \rfloor$, we have $|n^\kappa - x^\kappa| \gg x^{\kappa-1}$. Since \widehat{w} has very rapid

decay, these terms contribute $O_N(x^{-N})$ to the sum. On the other hand, for $n = n(x)$, writing $\{x\} = x - n(x)$, we have

$$n^\kappa - x^\kappa = n^\kappa - (n + \{x\})^\kappa = -\kappa n^{\kappa-1}\{x\} + O(x^{\kappa-2}\{x\}^2),$$

and it follows that

$$\frac{\overline{a_F(n)}}{\sqrt{n}} \widehat{w}(a(n^\kappa - x^\kappa)) V_1\left(\frac{n}{x}\right) = \frac{\overline{a_F(n)}}{\sqrt{n}} \left(\widehat{w}(-a\kappa n^{\kappa-1}\{x\}) + O_\varepsilon(x^{-\kappa+\varepsilon}) \right).$$

Noting that $a_F(n) \ll_\varepsilon n^{\frac{1}{2}+\varepsilon}$, this concludes the proof. \square

To apply Lemma 3.3, we fix a function $w_0 : \mathbb{R} \rightarrow \mathbb{R}$ which is smooth, even, non-negative, supported on $[-\frac{1}{2}, \frac{1}{2}]$ and L^2 -normalized, and set $w(t) = w_0(t - \frac{3}{2})$. Since the corresponding W is supported away from 0, from the definition of $\Sigma(x)$ we get

$$|\Sigma(x)| \leq \sum_{n=1}^{\infty} \frac{|a_F(n)|}{n^{\frac{\kappa+1}{2}}} \left| W\left(\frac{n}{x}\right) \right| \ll_\varepsilon x^{\sigma_1^* - \frac{\kappa+1}{2} + \varepsilon}.$$

Setting $x = n$, we learn from (3.6) that $a_F(n) \ll_\varepsilon n^{\sigma_1^* - \frac{\kappa}{2} + \varepsilon}$. However, if $\kappa > 2$, this contradicts the definition of σ_1^* , and thus $\mathcal{L}_d^+ = \emptyset$ for $d \in (1, \frac{3}{2})$.

The idea of [10] for going beyond this is to exploit the fact that the right-hand side of (3.6) is concentrated at integers, so that the integral $J(X) = \int_X^{2X} |\Sigma(x)|^2 e(x) dx$ behaves like $\int_X^{2X} |\Sigma(x)|^2 dx$. Since the series defining $\Sigma(x)$ displays no such behavior, we will see that $J(X)$ is small, and this results in a contradiction for $d < \frac{5}{3}$. We assume henceforth that $d \in [\frac{3}{2}, 2)$, so that $\kappa \in (1, 2]$.

Proceeding, we expand the square on the right-hand side of (3.6). The main term is

$$\begin{aligned} & \int_X^{2X} \frac{|a_F(n(x))|^2}{n(x)} \left| \widehat{w}(-a\kappa n(x)^{\kappa-1}[x - n(x)]) \right|^2 e(x) dx \\ &= \sum_{X - \frac{1}{2} \leq n \leq 2X + \frac{1}{2}} \frac{|a_F(n)|^2}{n} \int_{[n - \frac{1}{2}, n + \frac{1}{2}] \cap [X, 2X]} \left| \widehat{w}(-a\kappa n^{\kappa-1}(x - n)) \right|^2 e(x) dx. \end{aligned}$$

For the boundary terms with n near X or $2X$, we use the above estimate

$$|a_F(n)|^2 \ll_\varepsilon n^{2\sigma_1^* - \kappa + \varepsilon} \leq n^{\sigma_2^* + 1 - \kappa + \varepsilon}$$

to see that they contribute at most

$$\frac{|a_F(n)|^2}{n} \int_{\mathbb{R}} \left| \widehat{w}(-a\kappa n^{\kappa-1}x) \right|^2 dx \ll_\varepsilon X^{\sigma_2^* + 1 - 2\kappa + \varepsilon}.$$

For the other terms we translate the integral by n and extend it to \mathbb{R} , which introduces an error of only $O_N(X^{-N})$ thanks to the rapid decay of \widehat{w} . Thus, in total the main term is

$$\begin{aligned} & O_\varepsilon(X^{\sigma_2^* + 1 - 2\kappa + \varepsilon}) + \frac{1}{a\kappa} \sum_{X \leq n \leq 2X} \frac{|a_F(n)|^2}{n^\kappa} w_0 * w_0\left(\frac{1}{a\kappa n^{\kappa-1}}\right) \\ &= O_\varepsilon(X^{\sigma_2^* + 1 - 2\kappa + \varepsilon}) + \frac{1}{a\kappa} \sum_{X \leq n \leq 2X} \frac{|a_F(n)|^2}{n^\kappa}, \end{aligned}$$

where we have used that $w_0 * w_0(t) = 1 + O(|t|)$. Next, the error terms in (3.6) contribute

$$\begin{aligned} &\ll_{\varepsilon} X^{2\sigma_1^* - 2\kappa + \varepsilon} + X^{\sigma_1^* - \frac{1}{2} - \kappa + \varepsilon} \int_X^{2X} \left| \frac{\overline{a_F(n(x))}}{\sqrt{n(x)}} \widehat{w}(-a\kappa n(x)^{\kappa-1}[x - n(x)]) \right| dx \\ &\ll_{\varepsilon} X^{2\sigma_1^* - 2\kappa + \varepsilon} \leq X^{\sigma_2^* + 1 - 2\kappa + \varepsilon}, \end{aligned}$$

so altogether we have

$$(3.8) \quad J(X) = \frac{1}{a\kappa} \sum_{X \leq n \leq 2X} \frac{|a_F(n)|^2}{n^{\kappa}} + O_{\varepsilon}(X^{\sigma_2^* + 1 - 2\kappa + \varepsilon}).$$

Now, for any fixed $\varepsilon > 0$, since $\sum_{n=1}^{\infty} |a_F(n)|^2 n^{-\sigma_2^* + \frac{\varepsilon}{2}}$ diverges, there are arbitrarily large values of X such that $\sum_{X \leq n \leq 2X} |a_F(n)|^2 n^{-\sigma_2^* + \frac{\varepsilon}{2}} \geq \frac{1}{\log X}$, and for these X we have

$$\sum_{X \leq n \leq 2X} \frac{|a_F(n)|^2}{n^{\kappa}} \gg_{\varepsilon} X^{\sigma_2^* - \kappa - \frac{\varepsilon}{2}} \sum_{X \leq n \leq 2X} \frac{|a_F(n)|^2}{n^{\sigma_2^* - \frac{\varepsilon}{2}}} \gg_{\varepsilon} X^{\sigma_2^* - \kappa - \varepsilon}.$$

Hence, since $\kappa > 1$, (3.8) implies that there are arbitrarily large X for which

$$(3.9) \quad |J(X)| \gg_{\varepsilon} X^{\sigma_2^* - \kappa - \varepsilon}.$$

Next we evaluate $J(X)$ using the definition of $\Sigma(x)$. To that end, since our chosen w is real valued, we have

$$|\Sigma(x)|^2 = \frac{1}{ad} \sum_{m, n \geq 1} \frac{a_F(m) \overline{a_F(n)}}{(mn)^{\frac{\kappa+1}{2}}} e\left(a(n^{\frac{1}{d}} - x^{\frac{1}{d}})^{\kappa d} - a(m^{\frac{1}{d}} - x^{\frac{1}{d}})^{\kappa d}\right) W\left(\frac{m}{x}\right) W\left(\frac{n}{x}\right).$$

We are thus faced with the integral

$$J_{m,n}(X) = \int_X^{2X} e(f(x, n) - f(x, m) + x) W\left(\frac{m}{x}\right) W\left(\frac{n}{x}\right) dx,$$

where $f(x, n) = a(n^{\frac{1}{d}} - x^{\frac{1}{d}})^{\kappa d}$, to which we apply van der Corput's method [20, Chapter VIII, Corollary of Proposition 2].

First note that

$$\frac{\partial}{\partial x} (f(x, n) - f(x, m)) = -a\kappa x^{\kappa-1} (t_0(n/x) - t_0(m/x)),$$

where $t_0(u) = (u^{1/d} - 1)^{\kappa}$. For u in the support of W , we have $t'_0(u) = \frac{\kappa}{d}(u^{1/d} - 1)^{\kappa-1} u^{\frac{1}{d}-1} \asymp 1$, so by the mean value theorem, there are positive constants c_1 and c_2 such that

$$c_1 x^{\kappa-2} \leq \frac{\frac{\partial}{\partial x} (f(x, n) - f(x, m))}{m - n} \leq c_2 x^{\kappa-2}$$

for all $m \neq n$ such that $W(m/x)W(n/x) \neq 0$ for some $x \in [X, 2X]$.

Put $I_X = \left[\frac{X^{2-\kappa}}{2c_2}, \frac{2(2X)^{2-\kappa}}{c_1}\right]$. Then for $0 \neq n - m \notin I_X$, it follows that

$$\left| \frac{\partial}{\partial x} (f(x, n) - f(x, m) + x) \right| \geq \frac{c_1}{2} x^{\kappa-2} |m - n| \gg X^{\kappa-2} |m - n|.$$

Further, we compute that

$$(3.10) \quad \begin{aligned} \frac{\partial^2}{\partial x^2} (f(x, n) - f(x, m) + x) &= a\kappa x^{\kappa-2} [ut'_0(u) - (\kappa - 1)t_0(u)] \Big|_{m/x}^{n/x} \\ &= ax^{\kappa-2} (u^{1/d} - 1)^{\kappa-1} \left[\frac{1}{d} u^{1/d} + \kappa(\kappa - 1) \right] \Big|_{m/x}^{n/x}. \end{aligned}$$

Since $(u^{1/d} - 1)^{\kappa-1} [\frac{1}{d} u^{1/d} + \kappa(\kappa - 1)]$ is an increasing function, the last line never vanishes for $m \neq n$, so $\frac{\partial}{\partial x} (f(x, n) - f(x, m) + x)$ is monotonic. Thus, van der Corput's lemma for the first derivative yields the estimate

$$J_{m,n}(X) \ll \frac{X^{2-\kappa}}{|m-n|}$$

for those terms. Similarly, for the diagonal terms $m = n$, we get

$$J_{m,n}(X) = \int_X^{2X} W\left(\frac{n}{x}\right)^2 e(x) dx \ll 1.$$

It remains only to handle the terms with $n - m \in I_X$, to which we apply van der Corput's lemma for the second derivative. From (3.10) and the mean value theorem, we see that

$$\left| \frac{\partial^2}{\partial x^2} (f(x, n) - f(x, m) + x) \right| \gg x^{\kappa-3} |m-n| \gg X^{-1},$$

and thus $J_{m,n}(X) \ll \sqrt{X}$.

Suppose that W is supported on $[u_1, u_2] \subseteq (0, \infty)$. Substituting the above estimates into the definition of $J(X)$, we have

$$\begin{aligned} J(X) \ll & \sum_{u_1 X \leq n \leq 2u_2 X} \frac{|a_F(n)|^2}{n^{\kappa+1}} + \sqrt{X} \sum_{\substack{u_1 X \leq m, n \leq 2u_2 X \\ n-m \in I_X}} \frac{|a_F(n)a_F(m)|}{(mn)^{\frac{\kappa+1}{2}}} \\ & + X^{2-\kappa} \sum_{\substack{u_1 X \leq m, n \leq 2u_2 X \\ n-m \notin I_X \cup \{0\}}} \frac{|a_F(n)a_F(m)|}{|m-n|(mn)^{\frac{\kappa+1}{2}}}. \end{aligned}$$

Using the inequality $|a_F(n)a_F(m)| \leq \frac{1}{2}(|a_F(n)|^2 + |a_F(m)|^2)$ together with the estimates $\#(I_X \cap \mathbb{Z}) \ll X^{2-\kappa}$ and $\sum_{1 \leq n \leq 2u_2 X} \frac{1}{n} \ll \log X \ll_\varepsilon X^\varepsilon$, we see that this is

$$\ll_\varepsilon X^{\sigma_2^* - 1 - \kappa + \varepsilon} + X^{\sigma_2^* + \frac{3}{2} - 2\kappa + \varepsilon} + X^{\sigma_2^* + 1 - 2\kappa + \varepsilon} \ll X^{\sigma_2^* + \frac{3}{2} - 2\kappa + \varepsilon}.$$

Putting this together with the lower bound (3.9), we must have $\sigma_2^* - \kappa - \varepsilon \leq \sigma_2^* + \frac{3}{2} - 2\kappa + \varepsilon$ for all $\varepsilon > 0$, and thus $\kappa \leq \frac{3}{2}$. Hence, $\mathcal{L}_d^+ = \emptyset$ for $d \in [\frac{3}{2}, \frac{5}{3})$, and this concludes the proof.

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