Poncet-Montanges, A. P., Cooper, J. E., Jones, D., Gaitonde, A. L., & Lemmens, Y. (2016). Frequency Domain Approach for Transonic Unsteady Aerodynamic Modelling. In *15th Dynamics Specialists Conference* Article AIAA 2016-1576 American Institute of Aeronautics and Astronautics Inc. (AIAA). https://doi.org/10.2514/6.2016-1576

Frequency Domain Approach for Transonic Unsteady Aerodynamic Modelling

Adrien Poncet-Montanges[1], J.E. Cooper[2], D. Jones[3] and A.L. Gaitonde[4]
University of Bristol, Queens Building, University Walk, Bristol, BS8 1TH, UK

Y. Lemmens[5]
Siemens Industry Software, Leuven, Belgium

This work introduces a method for the construction of a reduced order model in the frequency domain. With input data obtained from the TAU linearized frequency domain solver for a NLR7301 airfoil in the transonic domain, the reduced order model shows a strong ability to reconstruct the full order frequency response. On the other hand, the model built in the frequency domain gives promising results when applied to reconstruct non-periodic motions, as a 1-cos pitching. Compared to a time domain simulation, the lift and the pitching moment obtained are accurate, even with small sizes for the reduced order model, and with a substantial gain of calculation time.

Nomenclature

a	=	angle of attack
a_0	=	amplitude of the pitching motion
a_m	=	mean angle of the pitching motion
C_p	=	pressure coefficient
C_L	=	lift coefficient
C_D	=	Drag coefficient
C_M	=	Pitching moment coefficient
k	=	Reduced frequency
N	=	Number of input points
r	=	size of the reduced order model
U_∞	=	Freestream velocity

I. Introduction

Computational Fluid Dynamics (CFD) now has a wide range of validity where it gives highly accurate results compared to wind tunnel experiments. It is extensively used in industry for steady analysis such as performance studies. However, unsteady aerodynamics is also important for aircraft design and aeroelastic applications such as flutter speed or limit cycle oscillation prediction. Whilst more powerful computers have enabled the application of

[1] Marie Curie Early Stage Research Fellow in Aircraft Loads. Email : adrien.poncet-montanges@bristol.ac.uk
[2] Royal Academy of Engineering Airbus Sir George White Professor of Aerospace Engineering. AFAIAA.
[3] Senior Lecturer, Department of Aerospace Engineering, University of Bristol, AIAA member.
[4] Senior Lecturer, Department of Aerospace Engineering, University of Bristol, AIAA member
[5] Sr Project Leader RTD, Digital Factory Division, Simulation& Test solutions. Siemens Industry Software.

CFD for unsteady loads calculations, in practice the computational cost remains too high for routine use, especially when it comes to viscous flows.

Whilst CFD models are useful to understand phenomena and predict behaviors; a high number of degrees of freedom leads to extended calculation times. Reduced order models (ROMs) can be constructed that aim to decrease the CPU time by capturing the dominant behaviour of the numerical model with a few degrees of freedom, whilst retaining good accuracy and stability. These ROMs enable [7] the system to be studied and simplified to determine the control laws. Model order reduction can be achieved using different methods; these depend on the physics of the system, the accuracy required and the information available. For systems whose model is strongly linked to the physics, order reduction can even be performed by hand, thinking about the independencies between the parameters; interpolation can also be used.

While building a ROM, one technique is to define projection bases and spaces. The idea is to use linear algebra and to construct a subspace orthogonal to the Krylov subspace; this can be performed thanks to the Gram-Schmidt orthonormalization method. Since it can be unstable [8] a modified Gram-Schmidt can be used. In order to achieve this, Arnoldi developed an iterative algorithm [9]. If the system matrix is hermitian, the Lanczos method [10] is much faster. It is based on the Arnoldi method, but as the system matrix is symmetric, the algorithm is much simpler and the recurrence is shorter: each vector U_{j+1} is directly calculated from the two previous ones U_j and U_{j-1}. The Lanczos algorithm can also be combined with a Padé approximation, for a method called Padé via Lanczos (PVL). This method aims to preserve of the stability of the system. In fact, the reduced order modeling techniques using the Padé approximation do not ensure this stability [11]. Other methods such as partial PVL [12] enable the poles and the zeros of the reduced transfer function to be corrected; it leads to an enhanced stability. Antoulas [13] uses the advantages of both Krylov subspaces and balanced truncation approaches. Finally, the Passive Reduced-order Interconnect Macromodelling Algorithm, while using the Arnoldi method guarantees the preservation of passivity and enables an enhanced accuracy [14].

A second stream of scientific analysis uses the system response to different excitations to identify the reduced matrices. Based on Hankel singular values, several algorithms were developed for model reduction such as singular value decomposition (SVD). The idea is to eliminate the states requiring a large amount of energy to be reached, or a large amount of energy to be observed, as both correspond to small eigenvalues [15]. Grammians are introduced since they can be used to quantify these amounts of energy. The reachability grammian quantifies the energy needed to bring a state to a chosen value, whereas the observability grammian quantifies the energy provided by an observed state [16]. The value of these grammians obviously depends on the basis on which they are calculated. In the case of a stable system, a basis in the state space exists in which states that are difficult to reach are also difficult to observe. Normally, the Hankel singular values decrease rapidly. The balanced truncation aims at truncating the modes that are not reachable and observable. They correspond to the smallest Hankel singular values. The singular value decomposition is well-conditioned, stable and can always work, but can be expensive to compute. It solves high-dimensional Lyapunov equations [17] ; the storage required is of the order $O(n^2)$, while the number of operations is of the order $O(n^3)$. Many balancing methods exist, such as stochastic balancing, bounded real balancing, positive real balancing [18]. The frequency weighted balancing [19], can be useful if a good approximation is needed only in a specific frequency range. However, the reduced model is not necessarily stable if both input and output are weighted. These frequency weighted balancing methods have undertaken many improvements: the most recent one guarantees stability and yields to a simple error bound [20]. Based on Markov parameters, the Padé approximation (moment matching method) [21] has then been improved by Arnoldi and Lanczos [10] and is particularly recommended in the case of high dimension systems.

The reduced order model developed in this paper falls into the second category of approach and is described in the following sections..

II. Reduced order model

For given flow conditions, the frequency response of the integrated aerodynamic coefficients obtained with a CFD code is directly related to the frequency of the pitching motion. It is therefore appropriate to build a reduced order model of the frequency response in the frequency domain instead of performing a classical reduction in the time domain. After creating a reduced order model in the frequency domain, it can be transformed back into the continuous time domain where it is then possible to reconstruct any motion. The method used in ROM creation means that the the continuous and discrete spaces (and vice versa) are linked via a bilinear transformation and frequencies in the discrete space in the range $[0, \pi]$ are mapped to frequencies in the continuous space in the range $[0,\infty]$. The developed reduced order model gives accurate results when applied to a pitching airfoil in the transonic range, with no shock-induced separation. The method decomposes a Hankel matrix formed using discrete transfer function values to keep the dominant modes of the frequency response. The method could also be used to build a ROM based on experimental without knowing the system matrices. As it needs equispaced input data in the discrete frequency domain, the choice of the sampling spacing is a key element.

The equispaced discrete frequencies are defined by

$$\widehat{\omega}_d(k) = \frac{k\,\pi}{N}, \mathrm{k} \in [0\,,\mathrm{N}] \tag{1}$$

The relationship to continuous frequencies as a result of the linking bilinear transformation is controlled by the sampling time parameter T via

$$\omega(\mathrm{k}) = \frac{2}{T}\tan\frac{\widehat{\omega}_d(k)}{2} \tag{2}$$

T has to be chosen such that the continuous reduced frequencies are in the range of interest for the model input. In aerodynamics it corresponds to continuous reduced frequencies mostly in the interval [0.01,10].

A. Singular value decomposition

The method uses samples of the transfer function G_d impulse response equi-spaced in the interval $[0,\pi]$. To map the whole unit circle, the algorithm extends the domain of the input data to the interval $[\pi,2\pi]$ using the conjugate of G_d:

$$\mathrm{G}_d(k+N) = \mathrm{G}_d^*(N-k) \tag{3}$$

A singular value decomposition of the Hankel matrix defined using the 2N-points inverse discrete Fourier transform (IDFT) is performed [23]. The model reduction is performed by keeping the largest singular values.

B. Calculation of discrete reduced matrices

A discrete-time linear and stable MIMO model of n-th order, with r-inputs and m-outputs can be described using the following state space representation

$$\begin{gathered}\boldsymbol{x}(k+1) = A_d\boldsymbol{x}(k) + B_d\boldsymbol{u}(k)\\ \boldsymbol{y}(k) = C_d\boldsymbol{y}(k) + D_d\boldsymbol{u}(k)\end{gathered} \tag{4}$$

$\boldsymbol{x}(t) \in \mathbb{R}^{\mathrm{n}}$ represents the vector of different degrees of freedom (called state vector in control theory). It contains for example the unknown physical variables, such as velocity, pressure, density. $\boldsymbol{y}(t) \in \mathbb{R}^{\mathrm{p}}$ and $\boldsymbol{u}(t) \in \mathbb{R}^{\mathrm{m}}$ respectively represent the vector of the outputs of interest of the system, and the vector of inputs. Tthis state space model has a transfer function G_d and it is convenient notation is consider that the system can be represented by its transfer function or its state space matrices:

$$G_d : \begin{pmatrix} A_d & B_d \\ C_d & D_d \end{pmatrix} \tag{5}$$

The linked continuous-time state space model is represented in a similar notation by

$$G : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6}$$

The reduced order model in the discrete frequency space has reduced matrices $\hat{A}_d$, $\hat{B}_d$, $\hat{C}_d$ nd $\hat{D}_d$ are calculated [24]. The reduced discrete transfer function $\hat{G}_d$ can be written as

$$\widehat{G_d} = \hat{C}_d(zI - \hat{A}_d)^{-1}\hat{B}_d + \hat{D}_d, z \in \mathbb{C} \quad (7)$$

C. Bilinear transformation

$\hat{G}$ the reduced order transfer function in the continuous time domain is then

$$\hat{G}(z) = \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} \quad (8)$$

and using the bilinear transformation [25] the reduced order system matrices are given by:

$$\hat{A} = \frac{2}{T}(I + \hat{A}_d)^{-1}(\hat{A}_d - I) \quad (9)$$

$$\hat{B} = \frac{2}{\sqrt{T}}(I + \hat{A}_d)\,\hat{B}_d \quad (10)$$

$$\hat{C} = \frac{2}{\sqrt{T}}\hat{C}_d\,(I + \hat{A}_d)^{-1} \quad (11)$$

$$\hat{D} = \hat{D}_d - \hat{C}_d\,(I + \hat{A}_d)^{-1}\,\hat{B}_d \quad (12)$$

III. Reconstruction of pitching motions

The reduced order model is constructed based on a chosen number N_r of equispaced discrete frequencies corresponding to the same number of continuous frequencies. However the transfer function $\hat{G}_d$ can be reconstructed for all frequencies and the response to both periodic and non-periodic inputs in the continuous time-domain can also be simulated.

A. Periodic input

As a first step, the model has been applied to an airfoil in a pitching motion at different frequencies. The motion is sinusoidal and described by the following equation:

$$\alpha = \alpha_m + \alpha_0 \cdot \sin(\omega \cdot t) \quad (13)$$

where α_m is the mean angle of attack, α_0 the amplitude and ω the frequency of the motion. Let U_∞ be the freestream velocity and c the airfoil chord, the reduced frequency is defined such as

$$k = \frac{\omega \cdot c}{U_\infty} \quad (14)$$

The airfoil chosen is a NLR7301, since it is supercritical it is close to the profiles used in aircraft design and literature provides many results for validation. To be able to use CFD, a mesh is created for inviscid simulations with the TAU CFD code [26]. The Euler equations are discretized using central differences, with a scalar dissipation scheme. Finally, the chosen relaxation solver is Backward Euler. In addition to the classical unsteady Euler computations, as the amplitude of the motion is small and the motion periodic, it is possible to use a linearized frequency domain solver [28].

In order to assess the quality of the reduced order model, three values of T are chosen, and for each of those, N=256 LFD calculations are launched. They enable ROMs of different sizes to be built in a systematic way with different number of input data [29]. N being called the number of training points, different models are created with $N/(2.i)$samples, i ∈ [2,4,8,16]. For each model, the quality is judged by reconstructing each magnitude and phase corresponding to the N frequencies of the training set and by comparing it to the LFD results (Figure 1) .

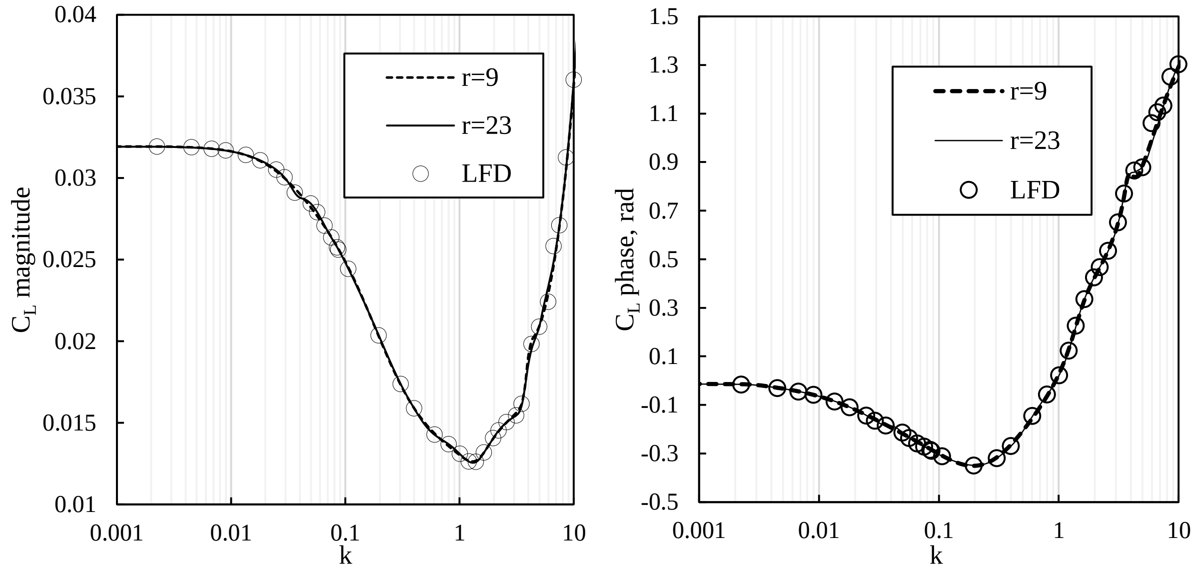


Figure 1: Comparison between the frequency responses of two different ROM, N=64, T=0.05

Even with a very small model (r=9), the model shows a strong ability to reconstruct the frequency response.

The advantage of this method is that it reconstructs the aerodynamic coefficients for any frequency between 0 and infinity. Therefore, for a given frequency, it is straight forward to reconstruct the aerodynamic coefficients during a period using the magnitude and the phase given as an output by the model. Two reduced order model of size Nr=3 and Nr=15 have been built with 32 samples, and the result is compared to the value given by a LFD calculation for the same frequency (Figure 2).

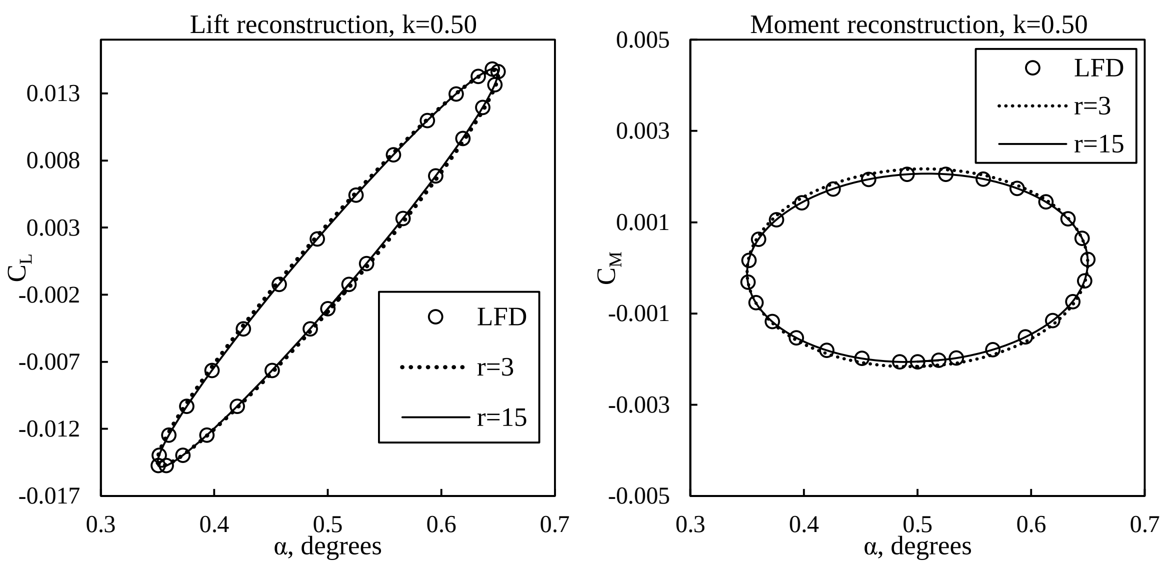


Figure 2: Lift and moment reconstruction, k=0.5, N=32, r=[3,15] vs LFD

B. Non-periodic input

1. Description

The reduced order model built is used to reconstruct a non-periodic motion, here a 1-cosine pitching (Figure 3) . The period of the 1-cosine is proportional to the reduced frequency chosen.

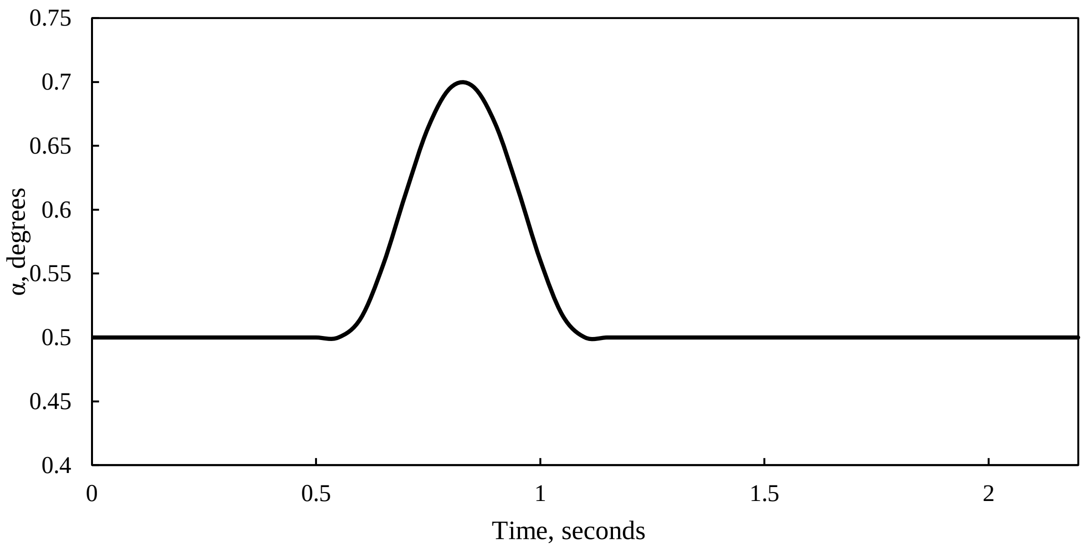


Figure 3: Prescribed motion, k=0.05

The centre of rotation for the pitching motion is at 40% chord. The static lift shows a linear behavior, and the moment is quasi linear (Figure 4) .

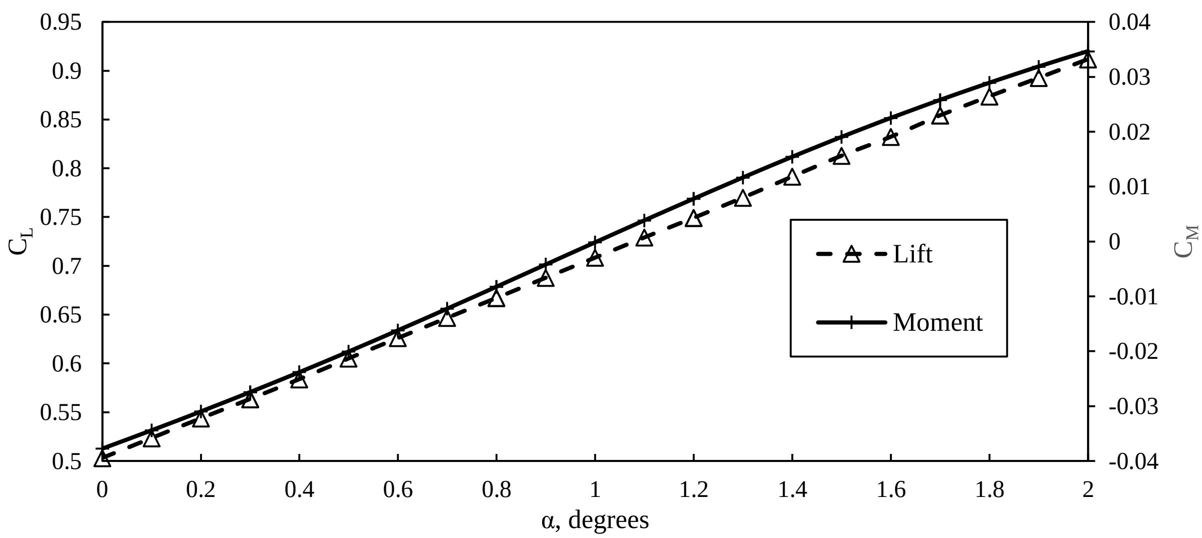
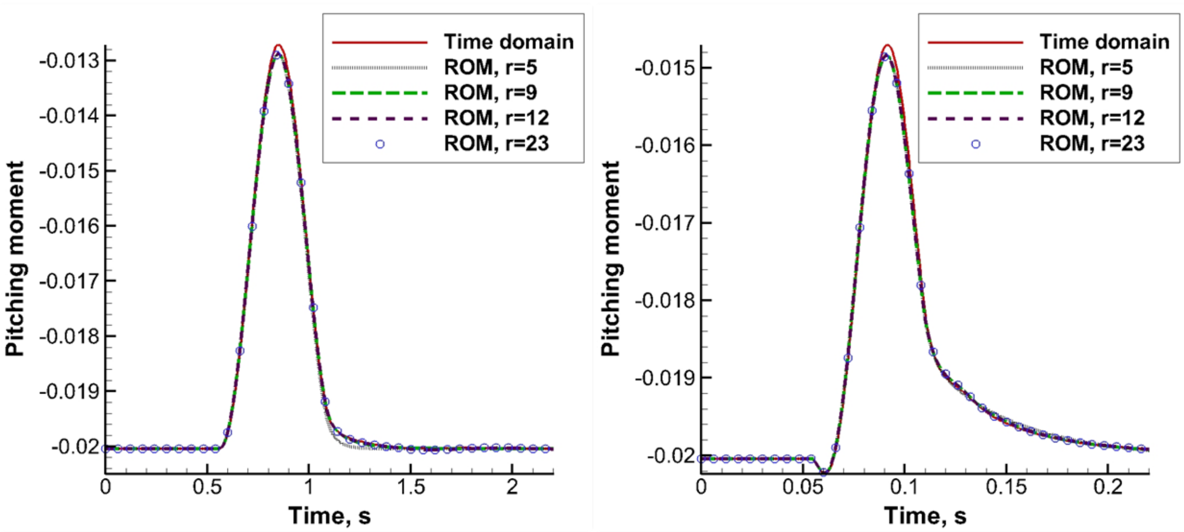


Figure 4: Static coefficients, Ma=0.68

2. Influence of the size of the reduced order model

For three different reduced frequencies, the pitching moment reconstructed by reduced order models of different sizes r=[5;9;12;23]) are plotted (Figure 5) and compared to the inviscid time domain simulation with TAU.



5.a. k=0.05

5.b. k=0.5

Figure 5: Pitching moment, Euler vs ROM, k=[0.05.0.5], various r

In both cases, the pitching moment given by the ROM seems to be accurate for almost all the ROM sizes. Only r=5 seems to be to low to capture the entire behavior. This is confirmed when displaying the same quantities for k=5 (Figure 6)

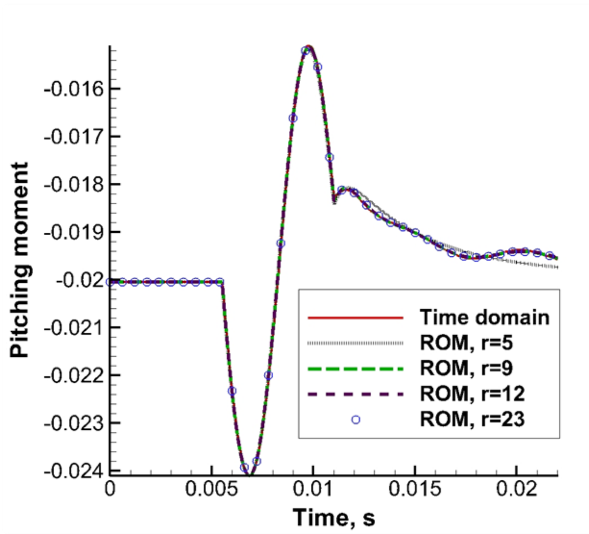


Figure 6: Pitching moment, Euler vs ROM, k=5, various r

The last points being produced with a fixed number of input points, it can be interesting to study more deeply the influence of this parameter.

3. Influence of the number of input points

Different reduced order models are built using a different number of input points, N=[16;32;64;128]. For three different ROM sizes, the effect of changing N is analyzed. Herebelow, the lift and the pitching moment are reconstructed and compared with Euler time domain simulations (Figure 7). The reduced frequency chosen is k=5, since it is the most challenging case, with more dynamics to be captured by the reduced order model.

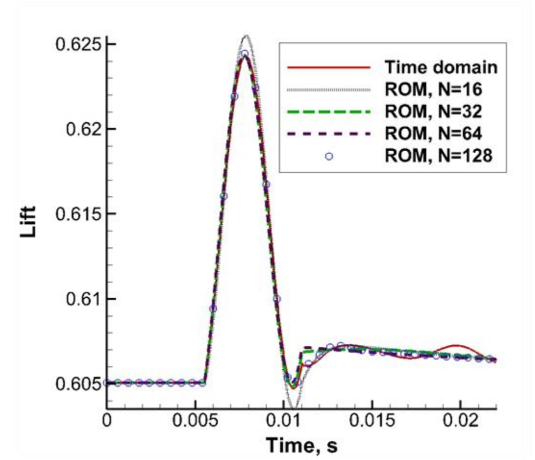
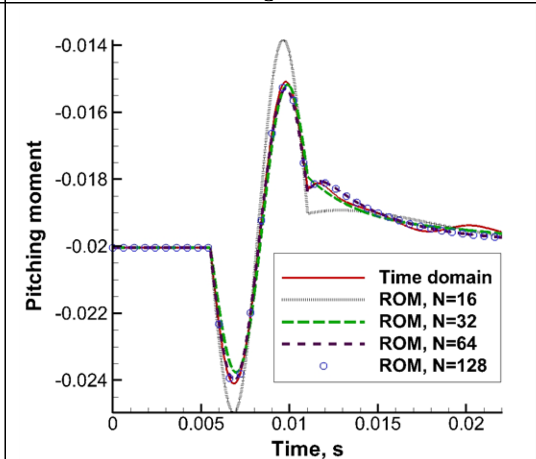
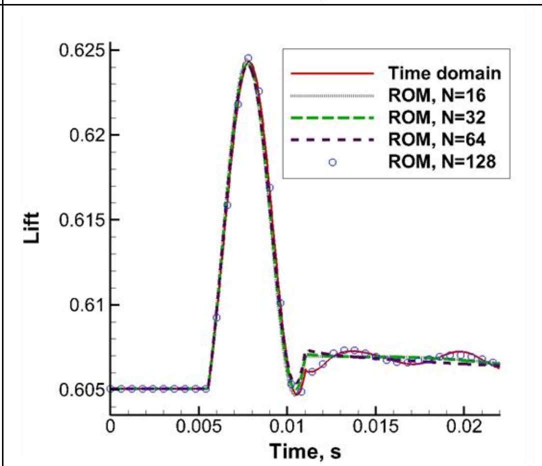
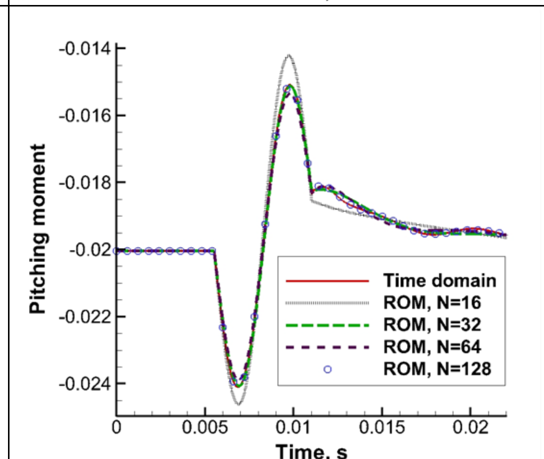
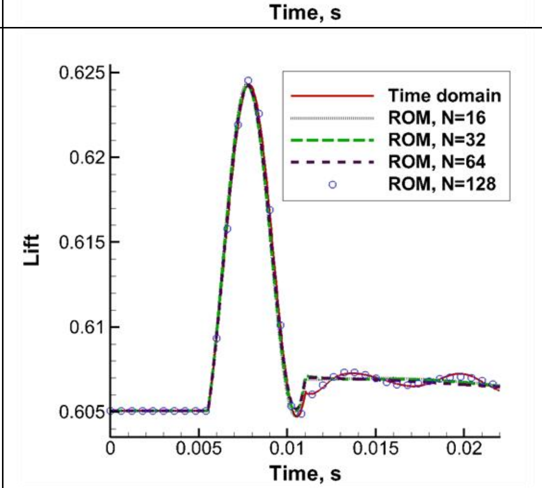
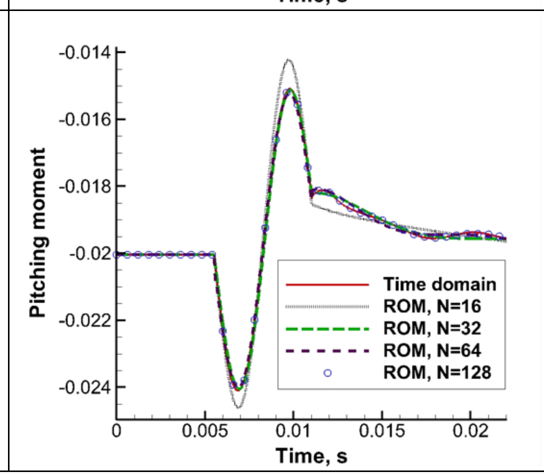
ROM size	Lift	Pitching moment
r=5		
r=9		
r=12		

Figure 7: 1-cos pitching, *k=5*, Euler vs ROM based on LFD

As far as the pitching moment is concerned, N=16 seems to be too small to accurate a good prediction in any case. However, from N=9, the reduced order model gives a satisfactory value.The number of input has an influence on the results, as it can be seen when reconstructing the pitching moment In general, the model of size 3 cannot predict accurately the time domain simulation. However, for both lift and moment, a size of 9 ensures a good reconstruction at both frequencies. Moreover, the peak value is really well represented.

To perform a deeper analysis on the right model size of the bilinear transform, different kinds of error are investigated : relative error in the maximum value predicted by the ROM, the time at which this maximum occurs, and the error in the area between the curves (integral error). The relative errors are given by

$$Relative\ error\ = \frac{|Maximum_{\ ROM} - Maximum_{Euler}|}{Maximum_{Euler}} * 100 \tag{15}$$

First, the error in the maximum value is plotted for k=0.5 for both lift and pitching moment (Figure 8).

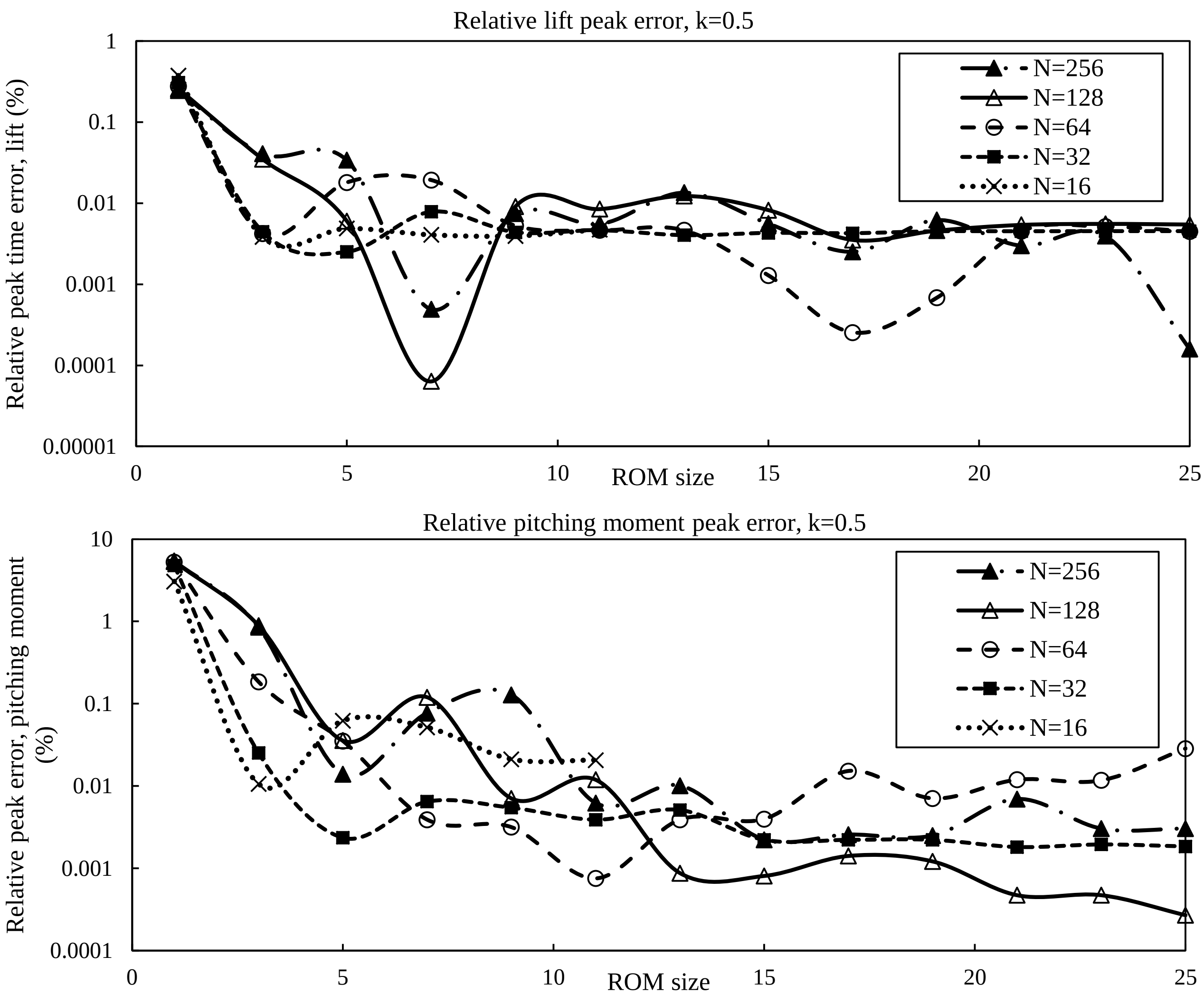


Figure 8: Relative peak errors, lift and pitching moment, *k=0.5*

Up to ROM sizes around 10, the peak error decreases, meaning that the value predicted by the model is closer and closer to the value given by the Euler time domain simulation. For bigger sizes, the error tends to remain constant ; indeed the error between the ROM (built with LFD calculations) and the Euler simulation becomes dominated by the error between LFD and Euler.

As far as the time at which this maximum is reached, the error is plotted for the pitching moment (Figure 9)

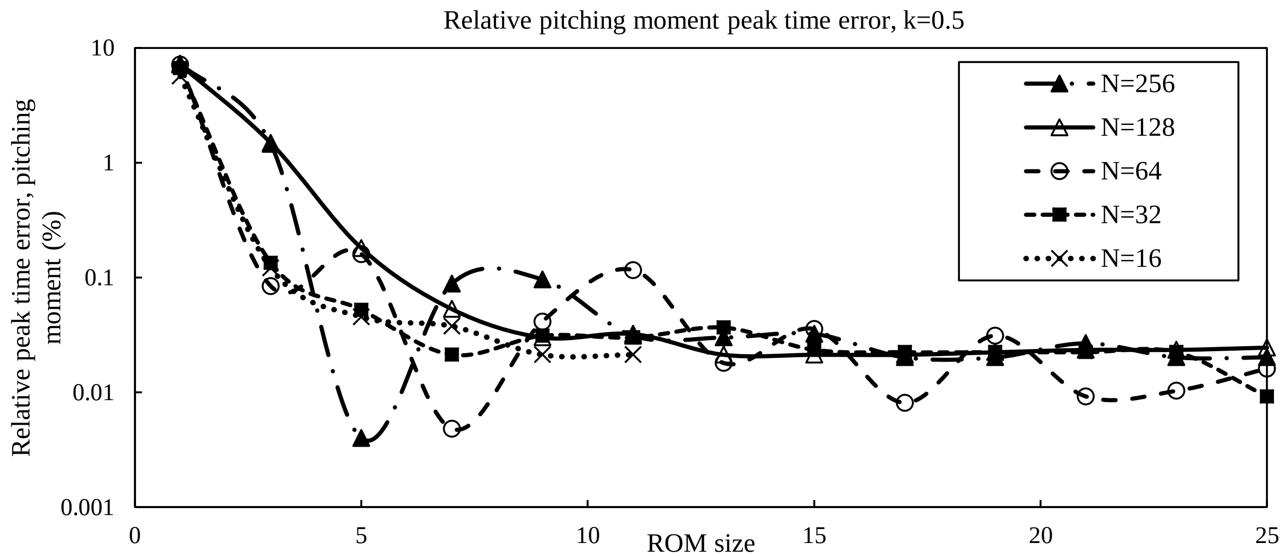


Figure 9: Relative peak time error, k=0.5, pitching moment

Once again, the same trend can be observed, with a fast decrease of the error with the ROM size first. The error reaches a minimum for Rom size of 10. The relative error obtained at this point is low (0.05%).

IV. Conclusions

A reduced order model in the frequency domain has been built and shows a strong ability to reconstruct the frequency response of an airfoil undergoing an harmonic motion. As far as non-periodic motions are concerned, the model has been tested to reconstruct a 1-cos pitching. Even with small reduced model sizes and few input points, it demonstrates to have a good accuracy in the transonic domain, for the whole range of frequencies used in aerodynamics.

Acknowledgments

The research leading to these results has received funding from the European Community's Marie Curie Initial Training Network (ITN) on Aircraft Loads Prediction using Enhanced Simulation (ALPES) PEOPLE-ITN-GA-2013-07911. The partners in the ALPES ITN are the University of Bristol, Siemens and Airbus Operations Ltd.

References

[1] P. Eykhoff, "System identification parameter and state estimation," London, Wiley/Interscience, 1974.

[2] G. C. Goodwin, D. J. Hill and M. Palaniswami, "A perspective on convergence of adaptive control algorithms," *Automatica,* vol. 20, no. 5, pp. 519-531, 1984.

[3] L. Ljung, "System identification: theory for the user," *PTR Prentice Hall Information and System Sciences Series,* vol. 198, 1987.

[4] K. J. Åström and B. Wittenmark, Adaptive control, Courier Corporation, 2013.

[5] T. Soderstrom and P. Stoica, System identification, vol. 2, London: Prentice hall, 1989.

[6] M. Kern, "Problèmes inverses," *INRIA, Rocquencourt, BP,* vol. 105, p. 78153, 2002.

[7] W. H. Schilders, H. A. Van der Vorst and J. Rommes, Model order reduction: theory, research aspects and applications, vol. 13, Berlin: Springer, 2008.

[8] Å. Björck, "Solving linear least squares problems by Gram-Schmidt orthogonalization," *BIT Numerical Mathematics,* vol. 7, no. 1, pp. 1-21, 1967.

[9] W. E. Arnoldi, "The principle of minimized iterations in the solution of the matrix eigenvalue problem," *Quarterly of Applied Mathematics,* vol. 9, no. 1, pp. 17-29, 1951.

[10] C. Lanczos, "An iteration method for the solution of the eigenvalue problem of linear differential and integral operators," United States Governm. Press Office, 1950.

[11] Y. Shamash, "Stable reduced-order models using Padé-type approximation," *Automatic Control, IEEE Transactions on,* vol. 19, no. 5, pp. 615-616, 1974.

[12] Z. Bai and R. W. Freund, "A partial Padé-via-Lanczos method for reduced-order modeling," *Linear Algebra and its Applications,* vol. 332, pp. 139-164, 2001.

[13] A. C. Antoulas, "An overview of approximation methods for large-scale dynamical systems," *Annual reviews in Control,* vol. 29, no. 2, pp. 181-190, 2005.

[14] A. Odabasioglu, M. Celik and L. T. Pileggi, "PRIMA : Passive reduced-order interconnect macromodeling algorithm," in *Proceedings of the 1997 IEEE/ACM international conference on Computer-aided design*, IEEE Computer Society, 1997, pp. 58-65.

[15] A. C. Antoulas, Approximation of large-scale dynamical systems, Siam, 2005.

[16] B. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *Automatic control,* vol. 26, no. 1, pp. 17-32, 1981.

[17] A. M. Lyapunov, "The general problem of the stability of motion," *International Journal of Control,* vol. 55, no. 3, pp. 531-534, 1992.

[18] S. Gugercin and A. C. Antoulas, "A survey of model reduction by balanced truncation and some new results," *International Journal of Control,* vol. 77, no. 8, pp. 748-766, 2004.

[19] D. F. Enns, "Model reduction with balanced realizations: An error bound and a frequency weighted generalization.," in *Decision and Control, 1984. The 23rd IEEE Conference on*, IEEE, 1984, pp. 127-132.

[20] G. Wang, V. Sreeram and W. Q. Liu, "A new frequency-weighted balanced truncation method and an error bound," *IEEE Transactions on Automatic Control,* vol. 44, no. 9, pp. 1734-1737, 1999.

[21] P. Feldmann and R. W. Freund, "Efficient linear circuit analysis by Padé approximation via the Lanczos process," *Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on,* vol. 14, no. 5, pp. 639-649, 1995.

[22] J.-N. Juang and R. S. Pappa, "An eigensystem realization algorithm for modal parameter identification and model reduction," *Journal of guidance, control, and dynamics,* vol. 8, no. 5, pp. 620-627, 1985.

[23] G. H. Golub and C. Reinsch, "Singular value decomposition and least squares solutions," *Numerische Mathematik,* vol. 14, no. 5, pp. 403-420, 1970.

[24] T. McKelvey, H. Akçay and L. Ljung, "Subspace-based multivariable system identification from frequency response data," *Automatic Control, IEEE Transactions on,* vol. 41, no. 7, pp. 960-979, 1996.

[25] U. M. Al-Saggaf and G. F. Franklin, "Model reduction via balanced realizations: an extension and frequency weighting techniques," *Automatic Control, IEEE Transactions on,* vol. 33, no. 7, pp. 687-692, 1988.

[26] D. Schwamborn, T. Gerhold and R. Heinrich, "The DLR TAU-code: Recent applications in research and industry," in *ECCOMAS CFD 2006: Proceedings of the European Conference on Computational Fluid Dynamics, Egmond aan Zee, The Netherlands, September 5-8, 2006*, Delft University of Technology, European Community on Computational Methods in Applied Sciences (ECCOMAS), 2006.

[27] P. R. Spalart and S. R. Allmaras, "A one-equation turbulence model for aerodynamic flows, AIAA-92-0439," in *30th aerospace sciences meeting and exhibit*, Reno, NV, 1992.

[28] M. Widhalm, R. P. Dwight, R. Thormann and A. Hübner, "Efficient computation of dynamic stability data with a linearized frequency domain solver," in *European Conference on Computational Fluid Dynamics*, 2010.

[29] A. Poncet-Montanges, D. Jones, A.-L. Gaitonde, J. Cooper and Y. Lemmens, "Frequency-domain approach for transonic unsteady aerodynamic modelling," in *IFASD*, Saint Petersburg, 2015