CHAOTIC DYNAMICS IN NONAUTONOMOUS MAPS: APPLICATION TO THE NONAUTONOMOUS HÉNON MAP

FRANCISCO BALIBREA-INIESTA*

Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM,
C/ Nicolás Cabrera 15, Campus de Cantoblanco UAM
Madrid, 28049, Spain
francisco.balibrea@icmat.es

CARLOS LOPESINO†

Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM,
C/ Nicolás Cabrera 15, Campus de Cantoblanco UAM
Madrid, 28049, Spain
carlos.lopesino@icmat.es

STEPHEN WIGGINS‡

School of Mathematics, University of Bristol,
Bristol BS8 1TW, United Kingdom
s.wiggins@bristol.ac.uk

ANA M. MANCHO§

Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM,
C/ Nicolás Cabrera 15, Campus de Cantoblanco UAM
Madrid, 28049, Spain
a.m.mancho@icmat.es

In this paper we analyze chaotic dynamics for two dimensional nonautonomous maps through the use of a nonautonomous version of the Conley-Moser conditions given previously. With this approach we are able to give a precise definition of what is meant by a *chaotic invariant set* for nonautonomous maps. We extend the nonautonomous Conley-Moser conditions by deriving a new sufficient condition for the nonautonomous chaotic invariant set to be hyperbolic. We consider the specific example of a nonautonomous Hénon map and give sufficient conditions, in terms of the parameters defining the map, for the nonautonomous Hénon map to have a hyperbolic chaotic invariant set.

Keywords: chaotic dynamics, invariant set, hyperbolic.

*Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Spain, *francisco.balibrea@icmat.es*.
†Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Spain, *carlos.lopesino@icmat.es*.
‡School of Mathematics, University of Bristol, United Kingdom, *s.wiggins@bristol.ac.uk*.
§Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Spain, *a.m.mancho@icmat.es*.

1. Introduction

Studies of the Hénon map ([Hénon, 1976]) have played a seminal role in the development of our understanding of chaotic dynamics and strange attractors. The map depends on two parameters, A and B, and has the following form:

$$\begin{array}{rccc} H: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2, \\ & (x,y) & \longmapsto & (A + By - x^2, x), \end{array} \tag{1}$$

where we will require $B \neq 0$ in order to endure the existence of the inverse map,

$$\begin{array}{rccc} H^{-1}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2, \\ & (x,y) & \longmapsto & (y, (x - A + y^2)/B). \end{array} \tag{2}$$

The "heart" of chaotic dynamics is exemplified by the so-called "Smale horseshoe map" (see [Smale, 1980] for a general description, with background). The essential feature of the Smale horseshoe map for chaos is that the map contains an invariant Cantor set on which the dynamics are topologically conjugate to a shift map defined on a finite number of symbols (a "chaotic invariant set", sometimes also referred to as a "chaotic saddle"). [Devaney and Nitecki, 1979] gave sufficient conditions, in terms of the parameters A and B, for the Hénon map to have an invariant Cantor set on which it is topologically conjugate to a shift map of two symbols. The proof uses a technique due to Conley and Moser (see [Moser, 1973]) that is referred to as the "Conley-Moser conditions" (but for earlier work in a similar spirit see [Alekseev, 1968a,b, 1969]). [Holmes, 1982] used these conditions to show the existence of a chaotic invariant set in the so-called "bouncing ball map". The Conley-Moser conditions were given a more detailed exposition, along with a slight weakening of the hypotheses, in [Wiggins, 2003]. More recently, the Conley-Moser conditions were used to show the existence of a chaotic invariant set in the Lozi map ([Lopesino et al., 2015]).

The purpose of this paper is to carry out a similar analysis for a nonautonomous version of the Hénon map. The generalization of the Conley-Moser conditions for nonautonomous systems, i.e. in the discrete time setting with dynamics defined by an infinite sequence of maps, was given in [Wiggins, 1999]. We extend the nonautonomous Conley-Moser conditions further by providing an additional condition which is sufficient for the nonautonomous chaotic invariant set to be hyperbolic. Hyperbolicity of nonautonomous invariant sets is discussed in general in [Katok and Hasselblatt, 1995]. Earlier work on chaos in nonautonomous systems is described in [Lerman and Silnikov, 1992; Stoffer, 1988a,b]. Recent interesting work is described in [Lu and Wang, 2010, 2011].

While the development of the "dynamical systems approach to nonautonomous dynamics" is currently a topic of much interest, it is not a topic that is widely known in the applied dynamical systems community (especially the fundamental work that was done in the 1960's). An applied motivation for such work is an understanding of fluid transport from the dynamical systems point of view for aperiodically time dependent flows. [Wiggins and Mancho , 2014] have given a survey of the history of nonautonomous dynamics as well as its application to fluid transport.

This paper is outlined as follows. In Section 2 we develop the required concepts for "building" chaotic invariant sets for two-dimensional nonautonomous maps. In Section 3 we prove the "main theorem" generalizing the Conley-Moser conditions that provide necessary conditions for two-dimensional nonautonomous maps to have a chaotic invariant set. In the course of the proof of the theorem the nature of chaotic invariant sets, and chaos, for nonautonomous maps is developed. This theorem was first given in [Wiggins, 1999], but in Section 3.1 we develop the theory further by providing a more analytical, rather than topological, construction for one of the Conley-Moser conditions that allows us to conclude that the nonautonomous chaotic invariant set is hyperbolic. In Section 4 we develop a version of the nonautonomous Hénon map and use the previously developed results to give sufficient conditions for the map to possess a nonautonomous chaotic invariant set. In Section 5 we discuss directions for future work along these lines.

2. Preliminary concepts

In this section we describe the basic setting and concepts that we will use throughout the remainder of the paper.

Nonautonomous dynamics will be defined by a sequence of maps and domains, $\{f_n, D_n\}_{n=-\infty}^{+\infty}$, acting as follows:

$$f_n : D_n \longrightarrow D_{n+1} \qquad \forall n \in \mathbb{Z} \quad \text{and} \quad f_n^{-1} : D_{n+1} \longrightarrow D_n, \tag{3}$$

where, for our purposes, D_n will be an appropriately chosen domain in $\mathbb{R}^2$, for all n.

Similar to the Smale horseshoe construction ([Wiggins, 2003]), on each domain D_n we must construct a finite collection of vertical strips $V_i^n \subset D_n$ ($\forall n \in \mathbb{Z}$ and $\forall i \in I = \{1, 2, ..., N\}$) which map to a finite collection of horizontal strips H_i^{n+1} located in D_{n+1}:

$$H_i^{n+1} \subset D_{n+1} \quad \text{with} \quad f_n(V_i^n) = H_i^{n+1}\ , \quad \forall n \in \mathbb{Z}\ , \quad i \in I. \tag{4}$$

Associated with these mappings we will need to define a *transition matrix* as follows:

$$A \equiv \{A^n\}_{n=-\infty}^{+\infty} \text{ is a sequence of matrices of dimension } N \times N \text{ such that}$$

$$A_{ij}^n = \begin{cases} 1 & \text{if } f_n(V_i^n) \cap V_j^{n+1} \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \text{or equivalently}$$

$$A_{ij}^n = \begin{cases} 1 & \text{if } H_i^{n+1} \cap V_j^{n+1} \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \forall i, j \in I. \tag{5}$$

However, first we must precisely define the notion of the domains that we will use, horizontal and vertical strips in those domains, and provide a characterization of the intersection of horizontal and vertical strips in the domain appropriate for our purposes.

To begin, let $D \subset \mathbb{R}^2$ denote a closed and bounded set. We consider two associated subsets of $\mathbb{R}$:

$$D_x = \{x \in \mathbb{R} \mid \text{there exists a } y \in \mathbb{R} \text{ with } (x, y) \in D\}$$

$$D_y = \{y \in \mathbb{R} \mid \text{there exists an } x \in \mathbb{R} \text{ with } (x, y) \in D\} \tag{6}$$

Therefore D_x and D_y represent the projections of D onto the x-axis and the y-axis respectively. From this it is easy to see that $D \subset D_x \times D_y$. We consider two closed intervals $I_x \subset D_x$ and $I_y \subset D_y$. We next define μ_h-horizontal and μ_v-vertical curves on these domains.

Definition 2.1. Let $0 \leq \mu_h < +\infty$. A μ_h-horizontal curve $\overline{H}$ is defined to be the graph of a function $h : I_x \to \mathbb{R}$ where h satisfies the following two conditions:

1. The set $\overline{H} = \{(x, h(x)) \in \mathbb{R}^2 \mid x \in I_x\}$ is contained in D.
2. For every $x_1, x_2 \in I_x$ we have the Lipschitz condition

$$|h(x_1) - h(x_2)| \leq \mu_h |x_1 - x_2| \tag{7}$$

Similarly, let $0 \leq \mu_v < +\infty$. A μ_v-vertical curve $\overline{V}$ is defined to be the graph of a function $v : I_y \to \mathbb{R}$ where v satisfies the following two conditions:

1. The set $\overline{V} = \{(v(y), y) \in \mathbb{R}^2 \mid y \in I_y\}$ is contained in D.
2. For every $y_1, y_2 \in I_y$ we have the Lipschitz condition

$$|v(y_1) - v(y_2)| \leq \mu_v |y_1 - y_2| \tag{8}$$

Next we “fatten” these curves into strips.

Definition 2.2. Given two nonintersecting μ_v-vertical curves $v_1(y) < v_2(y)$, $y \in I_y$, we define a μ_v-vertical strip as

$$V = \{(x, y) \in \mathbb{R}^2 \mid x \in [v_1(y), v_2(y)],\ y \in I_y\} \tag{9}$$

Similarly, given two nonintersecting μ_h-horizontal curves $h_1(x) < h_2(x)$, $x \in I_x$, we define a μ_h-horizontal strip as

$$H = \{(x, y) \in \mathbb{R}^2 \mid y \in [h_1(x), h_2(x)],\ x \in I_x\} \tag{10}$$

The width of horizontal and vertical strips is defined as

$$d(H) = \max_{x \in I_x} |h_2(x) - h_1(x)| \quad , \quad d(V) = \max_{y \in I_y} |v_2(y) - v_1(y)| \tag{11}$$

We will need to consider different parts of the boundary of the strips in relation to the domain on which they are defined. The following three definitions provide the necessary concepts.

Definition 2.3. The vertical boundary of a μ_h-horizontal strip H is denoted

$$\partial_v H \equiv \{(x, y) \in H \mid x \in \partial I_x\} \tag{12}$$

The horizontal boundary of a μ_h-horizontal strip H is denoted

$$\partial_h H \equiv \partial H \setminus \partial_v H \tag{13}$$

Definition 2.4. We say that H is a μ_h-horizontal strip contained in a μ_v-vertical strip V if the two μ_h horizontal curves defining the horizontal boundaries of H (denoted by $\partial_h H$) are contained in V, with the remaining boundary components of H (denoted by $\partial_v H$) contained in $\partial_v V$. These two last subsets, $\partial_h H$ and $\partial_v H$ are referred to as the horizontal and vertical boundaries of H, respectively. See Figure 1.

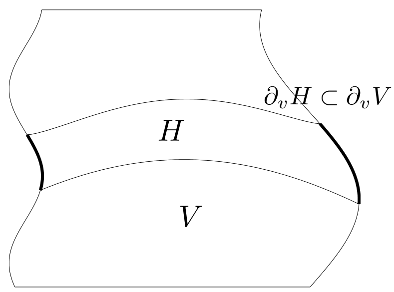

Figure 1. H is bounded by two μ_h-horizontal curves, each of them linking the two μ_v-vertical curves composing $\partial_v V$.

Definition 2.5. Let V and $\widetilde{V}$ be μ_v-vertical strips. $\widetilde{V}$ is said to intersect V fully if $\widetilde{V} \subset V$ and $\partial_h \widetilde{V} \subset \partial_h V$. See Figure 2.

3. The main theorem

In this section we prove the main general theorem which provides sufficient conditions for the existence of a chaotic invariant set for nonautonomous maps. In the course of the proof our meaning of “chaos” for nonautonomous dynamics will be made precise.

Following the original development of the Conley-Moser conditions ([Moser, 1973]), there are three geometrical and analytical conditions that, if satisfied, provide sufficient conditions for an autonomous

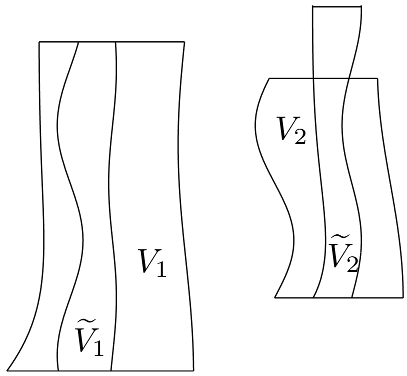

Figure 2. $\widetilde{V}_1$ intersects V_1 fully. This does not happen for $\widetilde{V}_2$ and V_2.

map (in the original formulation) to have a chaotic invariant set. These are referred to as A1, A2, and A3. The conditions A1 and A2 provide sufficient conditions for the existence of a topological chaotic invariant set. The conditions A1 and A3 provide sufficient conditions for a hyperbolic chaotic invariant set. Conditions A1 and A2 were developed for nonautonomous dynamics in [Wiggins, 1999]. In this section we recall A1 and A2, but we also give a new construction of A3 for nonautonomous dynamics[1]. In particular, we will show that A1 and A3 imply that A1 and A2 also hold.

The following two lemmas will play an important role in the proof of the main theorem.

Lemma 1. *i) If $V_1 \supset V_2 \supset \cdots \supset V_k \supset \cdots$ is a nested sequence of μ_v-vertical strips with $d(V_k) \to 0$ as $k \to \infty$, then $\cap_{k=1}^{\infty} V_k \equiv V_\infty$ is a μ_v-vertical curve.*

ii) If $H_1 \supset H_2 \supset \cdots \supset H_k \supset \cdots$ is a nested sequence of μ_h-horizontal strips with $d(H_k) \to 0$ as $k \to \infty$, then $\cap_{k=1}^{\infty} H_k \equiv H_\infty$ is a μ_h-horizontal curve.

Lemma 2. *Suppose $0 \leq \mu_v \mu_h < 1$. Then a μ_v-vertical curve and a μ_h-horizontal curve intersect in a unique point.*

The proof of both these two lemmas can be found in [Wiggins, 2003].

We assume that for each $D_n \subset \mathbb{R}^2$ we have:

$$f_n(D_n) \cap D_{n+1} \neq \emptyset \ , \quad \forall n \in \mathbb{Z} \tag{14}$$

Furthermore, we assume that on each D_n we can find a set of disjoint μ_v vertical strips, $D_V^n \equiv \cup_{i=1}^N V_i^n$, such that each f_n is one-to-one on $D_V^n \equiv \cup_{i=1}^N V_i^n$. We then define

$$H_{ij}^{n+1} \equiv f_n(V_i^n) \cap V_j^{n+1} = H_i^{n+1} \cap V_j^{n+1}, \quad \text{and} \quad V_{ji}^n \equiv f_n^{-1}(V_j^{n+1}) \cap V_i^n \tag{15}$$

with inverse function f_n^{-1} defined on $D_H^{n+1} \equiv \cup_{i=1}^N H_i^{n+1} = f_n\left(\cup_{i=1}^N V_i^n\right)$ for every $n \in \mathbb{Z}$, (see Figure 3). The transition matrix $\{A^n\}_{n=-\infty}^{+\infty}$ is defined as follows:

[1]We point out a minor technical point. In previous development of the Conley-Moser conditions (e.g. [Moser, 1973; Wiggins, 2003]) the set-up considers the mapping of horizontal strips to vertical strips. However, for the Hénon map it is more natural to consider vertical strips mapping to horizontal strips. Of course, the choice of what we refer to as “horizontal” and “vertical” is arbitrary. However, the same choice of coordinate labeling as is used in the previous literature can be used for the Hénon map if we impose a rotation $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the sequence of maps $\{f_n\}_{n=-\infty}^{+\infty}$ or, alternatively, take each map f_n as f_{-n}^{-1} for every $n \in \mathbb{Z}$.

$$A^n_{ij} = \begin{cases} 1 & \text{if } H^{n+1}_{ij} = H^{n+1}_i \cap V^{n+1}_j \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \forall i, j \in I. \tag{16}$$

Now we can state the first two Conley-Moser conditions for a sequence of maps that are sufficient to prove the existence of a chaotic invariant set for nonautonomous systems.

Assumption 1 [**A1**]. *For all $i, j \in I$ such that $A^n_{ij} = 1$, H^{n+1}_{ij} is a μ_h-horizontal strip contained in V^{n+1}_j with $0 \leq \mu_v \mu_h < 1$. Moreover, f_n maps V^n_{ji} homeomorphically onto H^{n+1}_{ij} with $f^{-1}_n(\partial_h H^{n+1}_{ij}) \subset \partial_h V^n_i$.*

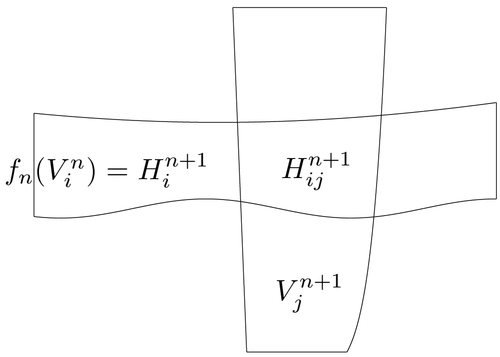

Figure 3. Assuming that A1 is satisfied for a given sequence of maps, this figure illustrates that every non empty $H^{n+1}_{ij} \subset D_{n+1}$ is a μ_h-horizontal strip contained in V^{n+1}_j. This also shows that the two μ_h-horizontal curves which form the boundary $\partial_h f_n(V^n_i) = \partial_h(H^{n+1}_i)$ cut the vertical boundary of V^{n+1}_j in exactly four points.

Also, since f_n maps V^n_{ji} homeomorphically onto H^{n+1}_{ij} with $f^{-1}_n(\partial_h H^{n+1}_{ij}) \subset \partial_h V^n_i$ then f^{-1}_n maps H^{n+1}_{ij} homeomorphically onto V^n_{ji} ($\forall i, j \in I$) with

$$f_n\left(f^{-1}_n(\partial_h H^{n+1}_{ij})\right) = \partial_h H^{n+1}_{ij} \subset f_n(\partial_h V^n_i). \tag{17}$$

Assumption 2 [**A2**]. *Let V^{n+1} be a μ_v-vertical strip which intersects V^{n+1}_j fully. Then $f^{-1}_n(V^{n+1}) \cap V^n_i \equiv \widetilde{V}^n_i$ is a μ_v-vertical strip intersecting V^n_i fully for all $i \in I$ such that $A^n_{ij} = 1$. Moreover,*

$$d(\widetilde{V}^n_i) \leq \nu_v \; d(V^{n+1}) \qquad \textit{for some } 0 < \nu_v < 1 \tag{18}$$

Similarly, let H^n be a μ_h-horizontal strip contained in V^n_j such that also $H^n \subset H^n_{ij}$ for some $i, j \in I$ with $A^{n-1}_{ij} = 1$. Then $f_n(H^n) \cap V^{n+1}_k \equiv \widetilde{H}^{n+1}_k$ is a μ_h-horizontal strip contained in V^{n+1}_k for all $k \in I$ such that $A^n_{jk} = 1$. Moreover,

$$d(\widetilde{H}^{n+1}_k) \leq \nu_h \; d(H^n) \qquad \textit{for some } 0 < \nu_h < 1 \tag{19}$$

Now we develop symbolic dynamics in a form appropriate for nonautonomous dynamics. Let

$$s = (\cdots s_{n-k} \cdots s_{n-2} s_{n-1} . s_n s_{n+1} \cdots s_{n+k} \cdots) \tag{20}$$

denote a bi-infinite sequence with $s_l \in I$ ($\forall l \in \mathbb{Z}$) where adjacent elements of the sequence satisfy the rule $A^n_{s_n s_{n+1}} = 1$, $\forall n \in \mathbb{Z}$.

Similarly to the symbolic dynamics implemented for the Smale horseshoe (see page 575 of [Wiggins, 2003]), here we denote the set of all such symbol sequences by $\Sigma^N_{\{A^n\}}$. If σ denotes the shift map

$$\sigma(s) = \sigma(\cdots s_{n-2}s_{n-1}.s_n s_{n+1}\cdots) = (\cdots s_{n-2}s_{n-1}s_n.s_{n+1}\cdots) \tag{21}$$

on $\Sigma^N_{\{A^n\}}$, we define the "extended shift map" $\tilde{\sigma}$ on $\widetilde{\Sigma} \equiv \Sigma^N_{\{A^n\}} \times \mathbb{Z}$ by

$$\tilde{\sigma}(s,n) = (\sigma(s), n+1). \quad \text{It is also defined } f(x,y;n) = (f_n(x,y), n+1). \tag{22}$$

Now we can state the main theorem.

Theorem 3 [Main theorem]. *Suppose $\{f_n, D_n\}_{n=-\infty}^{+\infty}$ satisfies A1 and A2. There exists a sequence of sets $\Lambda_n \subset D_n$, with $f_n(\Lambda_n) = \Lambda_{n+1}$, such that the following diagram commutes*

$$\begin{array}{ccccc} & \Lambda_n \times \mathbb{Z} & \xrightarrow{f} & \Lambda_{n+1} \times \mathbb{Z} & \\ \phi & \downarrow & & \downarrow & \phi \\ & \Sigma^N_{\{A^n\}} \times \mathbb{Z} & \xrightarrow{\tilde{\sigma}} & \Sigma^N_{\{A^n\}} \times \mathbb{Z} & \end{array} \tag{23}$$

where $\phi(x,y;n) \equiv (\phi_n(x,y), n)$ with $\phi_n(x,y)$ a homeomorphism mapping Λ_n onto $\Sigma^N_{\{A^n\}}$.

Remark 3.1. The sequence of sets $\{\Lambda_n\}_{n=-\infty}^{+\infty}$ is what we mean by a **chaotic set** for nonautonomous dynamics. Consequently our "main theorem" is a theoretical result which gives sufficient conditions for the existence of such a sequence of sets. The original proof can be found in [Wiggins, 1999], keeping in mind the geometrical considerations mentioned before.

Next, we will generalize the third Conley-Moser condition to the nonautonomous case. This will provide an alternative, and more analytical (as opposed to topological) method for proving that the second Conley-Moser condition holds, and it will also provide the additional information that the chaotic invariant set is hyperbolic.

3.1. *Nonautonomous third Conley-Moser condition*

We begin by giving a natural definition of stable and unstable sector bundles for the nonautonomous situation:

$$\mathcal{V}^n \equiv \bigcup_{i,j\in I} V^n_{ji} = \bigcup_{i,j\in I} f_n^{-1}(V_j^{n+1}) \cap V_i^n, \tag{24}$$

$$\mathcal{H}^{n+1} \equiv \bigcup_{i,j\in I} H^{n+1}_{ij} = \bigcup_{i,j\in I} H_i^{n+1} \cap V_j^n, \quad f_n(\mathcal{V}^n) = \mathcal{H}^{n+1} \tag{25}$$

$$S^u_{\mathcal{K}} \equiv \{(\xi_z, \eta_z) \in \mathbb{R}^2 \mid |\eta_z| \leq \mu_h |\xi_z|,\ z \in \mathcal{K}\} \quad \text{(unstable sector bundle)} \tag{26}$$

$$S^s_{\mathcal{K}} \equiv \{(\xi_z, \eta_z) \in \mathbb{R}^2 \mid |\xi_z| \leq \mu_v |\eta_z|,\ z \in \mathcal{K}\} \quad \text{(stable sector bundle)} \tag{27}$$

with $\mathcal{K}$ being either $\mathcal{V}^n$ or $\mathcal{H}^{n+1}$. Then we can state the following assumption: the third Conley-Moser condition for the nonautonomous setting.

Assumption 3 [**A3**]. $Df_n(S^u_{\mathcal{V}^n}) \subset S^u_{\mathcal{H}^{n+1}}$, $Df_n^{-1}(S^s_{\mathcal{H}^{n+1}}) \subset S^s_{\mathcal{V}^n}$.

Moreover, if $(\xi_{f_n(z_0^n)}, \eta_{f_n(z_0^n)}) \equiv Df_n(z_0^n) \cdot (\xi_{z_0^n}, \eta_{z_0^n}) \in S^u_{\mathcal{H}^{n+1}}$ *then*

$$|\xi_{f_n(z_0^n)}| \geq \left(\frac{1}{\mu}\right) |\xi_{z_0^n}| \tag{28}$$

If $(\xi_{f_n^{-1}(z_0^{n+1})}, \eta_{f_n^{-1}(z_0^{n+1})}) \equiv Df_n^{-1}(z_0^{n+1}) \cdot (\xi_{z_0^{n+1}}, \eta_{z_0^{n+1}}) \in S^s_{\mathcal{V}^n}$ *then*

$$|\eta_{f_n^{-1}(z_0^{n+1})}| \geq \left(\frac{1}{\mu}\right) |\eta_{z_0^{n+1}}| \quad \text{for } \mu > 0 \tag{29}$$

Obviously we need to impose an additional condition in order to guarantee the existence of the Jacobian matrices Df_n and Df_n^{-1}. From now we will consider that $f_n, f_n^{-1} \in C^1$ for every $n \in \mathbb{Z}$ on their respective domains. Now we establish an important relationship between assumptions A2 and A3.

Theorem 4. *If nonautonomous A1 and A3 are satisfied for* $0 < \mu < 1 - \mu_h\mu_v$ *then A2 is satisfied.*

Part of the proof of this theorem is based on the following result.

Lemma 5. *Let* $\{f_n, D_n\}_{n=-\infty}^{+\infty}$ *be a sequence of maps satisfying A1 and A3.*
For every $n \in \mathbb{Z}$ *and every pair of indices* $i, j \in I$ *we have that*

i) if $\overline{V}^{n+1} \subset V_j^{n+1}$ *is a* μ_v*-vertical curve, then* $f_n^{-1}(\overline{V}^{n+1}) \cap V_i^n$ *is a* μ_v*-vertical curve in case* $\overline{V}^{n+1} \cap H_i^{n+1} \neq \emptyset$.

ii) if $\overline{H}^n \subset V_{ji}^n$ *is a* μ_h*-horizontal curve, then* $f_n(\overline{H}^n) \cap H_i^{n+1}$ *is a* μ_h*-horizontal curve in case* $\overline{H}^n \cap V_i^n \neq \emptyset$.

Proof. We omit the proof of ii) as it follows the same line of reasoning as i).

We consider a μ_v-vertical curve $\overline{V}^{n+1} \subset V_j^{n+1}$. By definition there exist an interval $T \subset \mathbb{R}$ and a function $v : T \to \mathbb{R}$ such that $\overline{V}^{n+1}$ is the graph of v and also the Lipschitz condition $|v(t_1) - v(t_2)| \leq \mu_v |t_1 - t_2|$ holds for a constant $\mu_v > 0$ and every pair of points $t_1, t_2 \in T$.

It follows from Assumption 1 that (f_n^{-1}) is a homeomorphism over $H_{ij}^{n+1} = H_i^{n+1} \cap V_j^{n+1}$. In particular a homeomorphism over $\overline{V}^{n+1} \cap H_i^{n+1} \neq \emptyset$. This implies that

$$f_n^{-1}(\overline{V}^{n+1} \cap H_i^{n+1}) = f_n^{-1}(\overline{V}^{n+1}) \cap f_n^{-1}(H_i^{n+1}) = f_n^{-1}(\overline{V}^{n+1}) \cap V_i^n \neq \emptyset \tag{30}$$

Since the curve $\overline{V}^{n+1}$ can be parametrized by $(v(t), t)|_{t \in T}$ (take also $v \in C^1$) then this last subset $f_n^{-1}(\overline{V}^{n+1}) \cap V_i^n$ can also have a parametrization but over a smaller domain $T^* \subset T$,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = f_n^{-1}(v(t), t) \quad \text{with} \quad t \in T^* \equiv \{\bar{t} \in T : (v(\bar{t}), \bar{t}) \in H_i^{n+1}\} \tag{31}$$

The image of ant tangent vector of $\overline{V}^{n+1}$ under Df_n^{-1} has the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = Df_n^{-1}(v(t), t) \cdot \begin{pmatrix} \dot{v}(t) \\ 1 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \dot{v}(t) \\ 1 \end{pmatrix} \in S^s_{\mathcal{H}^{n+1}}, \quad \forall t \in T^* \tag{32}$$

This last relation follows directly from the Lipschitz condition:

$$|\dot{v}(t)| \leq \limsup_{\substack{\epsilon \to 0 \\ (t+\epsilon) \in T^*}} \left| \frac{v(t+\epsilon) - v(t)}{\epsilon} \right| \leq \limsup_{\substack{t_1 \in T^* \\ t_1 \neq t}} \frac{|v(t_1) - v(t)|}{|t_1 - t|} \leq \limsup_{\substack{t_1 \in T^* \\ t_1 \neq t}} \frac{\mu_v |t_1 - t|}{|t_1 - t|} = \mu_v \tag{33}$$

By applying Assumption 3 we obtain that the tangent vectors belong to $S^s_{\mathcal{V}^n}$,

$$|\dot{x}(t)| \leq \mu_v \cdot |\dot{y}(t)| \quad , \quad \forall t \in T^* \tag{34}$$

Moreover, as we also assume that $(f_n^{-1}) \in C^1$, any tangent vector

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = Df_n^{-1}(v(t), t) \cdot \begin{pmatrix} \dot{v}(t) \\ 1 \end{pmatrix} \text{ cannot be equal to } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ at any point } t \in T^*.$$

From these two relations it follows that $\dot{y}(t)$ cannot change its sign in the entire domain T^*. Consequently for every pair of points $(x_1, y_1), (x_2, y_2) \in f_n^{-1}(\overline{V}^{n+1}) \cap V_i^n$ there exist $t_1, t_2 \in T^*$ such that $(x_k, y_k) = (x(t_k, y(t_k)), (k = 1, 2)$ and we have the inequality

$$|x_1 - x_2| = |x(t_1) - x(t_2)| = \left| \int_{t_2}^{t_1} \dot{x}(t) dt \right| \leq \int_{t_2}^{t_1} |\dot{x}(t)| dt \leq \mu_v \int_{t_2}^{t_1} |\dot{y}(t)| dt =$$

$$= \mu_v \left| \int_{t_2}^{t_1} \dot{y}(t) dt \right| = \mu_v |y(t_1) - y(t_2)| = \mu_v |y_1 - y_2| \tag{35}$$

and this result implies that $f_n^{-1}(\overline{V}^{n+1}) \cap V_i^n$ is a μ_v-vertical curve. ∎

Proof. [Proof of Theorem 4] The theorem will be proved by verifying the following steps.

Step 1: Let $\overline{V}^{n+1} \subset V_j^{n+1}$ be a μ_v-vertical curve. Then $f_n^{-1}(\overline{V}^{n+1}) \cap V_i^n$ is a μ_v-vertical curve for every $i \in I$ such that $\overline{V}^{n+1} \cap H_i^{n+1} \neq \emptyset$.

Step 2: Let V^{n+1} be a μ_v-vertical strip which intersects V_j^{n+1} fully. Then $f_n^{-1}(V^{n+1}) \cap V_i^n$ is a μ_v-vertical strip that intersects V_i^n fully for every $i \in I$ such that $V^{n+1} \cap H_i^{n+1} \neq \emptyset$.

Step 3: Show that $d(\widetilde{V}_i^n) \leq (\mu/(1 - \mu_h \mu_v)) \cdot d(V^{n+1})$ for $\widetilde{V}_i^n = f_n^{-1}(V^{n+1}) \cap V_i^n$.

We omit the part of the proof dealing with horizontal strips since it follows from the same reasoning used to prove the part concerning vertical strips.

We begin with Step 1. Let $\overline{V}^{n+1} \subset V_j^{n+1}$ be a μ_v-vertical curve. For each $i \in I$ such that $\overline{V}^{n+1} \cap H_i^{n+1} \neq \emptyset$, by applying A1 we obtain that $H_{ij}^{n+1} = V_j^{n+1} \cap H_i^{n+1} \neq \emptyset$ is a μ_h-horizontal strip contained in V_j^{n+1}. Since implicitly we are taking $\overline{V}^{n+1}$ as one of the two components of the vertical boundary of a vertical strip V^{n+1} intersecting V_j^{n+1} fully, the curve $\overline{V}^{n+1}$ intersects $\partial_h H_i^{n+1}$ in exactly two points.

Also because f_n^{-1} maps the horizontal boundaries of each subset $H_{ij}^{n+1} (= H_i^{n+1} \cap V_j^{n+1})$ onto the horizontal boundaries of V_i^n, we have that $f_n^{-1}(\overline{V}^{n+1}) \cap V_i^n$ is a curve linking the two horizontal boundaries of V_i^n. Finally if we apply Lemma 5 to this curve it follows that $f_n^{-1}(\overline{V}^{n+1}) \cap V_i^n$ is also a μ_v-vertical curve.

To prove Step 2 we apply Step 1 to the μ_v-vertical boundaries of the μ_v-vertical strip V^{n+1} which intersects V_j^{n+1} fully. It then follows that $f_n^{-1}(V^{n+1}) \cap V_i^n$ is also a μ_v-vertical strip for every $i \in I$ such that $V^{n+1} \cap H_i^{n+1} \neq \emptyset$. Moreover this last strip intersects each V_i^n fully because of the geometric considerations in Step 1.

For proving Step 3 first we need to fix an iteration $n \in \mathbb{Z}$ and an index $i \in I$. The width of each μ_v-vertical strip $\widetilde{V}_i^n$ will be the distance between two points $p_0, p_1 \in \widetilde{V}_i^n$ with the same y-component and located in separate vertical boundaries, $d(\widetilde{V}_i^n) = |p_1 - p_0|$.

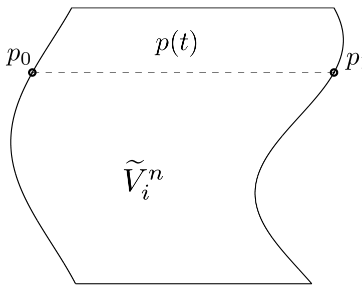

Figure 4. The segment $p(t) = tp_1 + (1-t)p_0$, $t \in [0,1]$, which represents the maximum amplitude of $\widetilde{V}_i^n$ is obviously a μ_h-horizontal curve.

By taking segment $p(t)$ considered in Figure 3.1, $\dot{p}(t) = p_1 - p_0$ is a vector with its y-component equal to zero. Therefore $\dot{p}(t) \in S^u_{\mathcal{V}^n}$, $\forall t \in [0,1]$. Now we have that the curve $f_n(p(t)) \equiv z(t) = (x(t), y(t))$ located in D_{n+1} is a μ_h-horizontal curve because of the second part of Lemma 5,

$$\text{Moreover A3 states that} \quad \dot{z}(t) = D(f_n(p(t))) = Df_n(p(t)) \cdot \dot{p}(t) \in S^u_{\mathcal{H}^{n+1}} \tag{36}$$

Furthermore since the graph of $z(t) = (x(t), y(t))$ is a μ_h-horizontal curve, we obtain that

$$|y(1) - y(0)| \leq \mu_h |x(1) - x(0)| \quad \rightarrow \quad |y_1 - y_0| \leq \mu_h |x_1 - x_0| \tag{37}$$

by denoting $(x_i, y_i) \equiv (x(i), y(i)) = z(i) = f_n(p(i)) = f_n(p_i)$ for $i = 0, 1$.

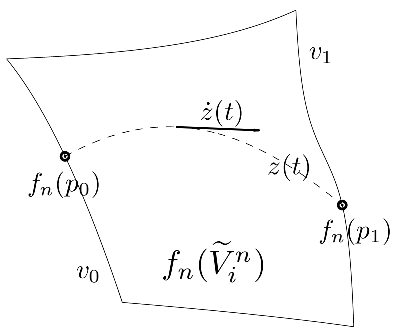

Figure 5. $f_n(p_0)$ and $f_n(p_1)$ are on the graphs of two distinct μ_v-vertical curves, which we denote by v_0 and v_1 respectively.

Using this last fact and also the geometric considerations in Figure 5, it follows that

$$|x_1 - x_0| = |v_1(y_1) - v_0(y_0)| \leq |v_1(y_1) - v_1(y_0)| + |v_1(y_0) - v_0(y_0)| \leq$$

$$\leq \mu_v |y_1 - y_0| + d(V^{n+1}) \leq \mu_v \mu_h |x_1 - x_0| + d(V^{n+1}) \rightarrow$$

$$\rightarrow \quad |x_1 - x_0| \leq \frac{d(V^{n+1})}{(1 - \mu_h \mu_v)} \tag{38}$$

Also as a result of the last part of Assumption 3 there exists a positive constant μ, which we impose to be $\mu < 1 - \mu_h \mu_v$, such that

$$|\dot{x}(t)| \geq \left(\frac{1}{\mu}\right) |\dot{p}(t)| = \left(\frac{1}{\mu}\right) |p_1 - p_0| \quad \text{and then}$$

$$d(\widetilde{V}_i^n) = |p_1 - p_0| \leq \mu \int_0^1 |\dot{x}(t)| dt = \mu \left| \int_0^1 \dot{x}(t) dt \right| = \mu |x_1 - x_0| \tag{39}$$

Note that the two expressions containing the integrals are equal since $\dot{x}(t)$ does not change its sign at any point. This is due to the fact that the graph of $z(t) = (x(t), y(t))$ is a μ_h-horizontal curve.

Finally we arrive at the result,

$$d(\widetilde{V}_i^n) = |p_1 - p_0| \leq \mu |x_1 - x_0| \leq \frac{\mu}{(1 - \mu_h \mu_v)} d(V^{n+1}) \tag{40}$$

$$\text{and} \quad \nu_v = \frac{\mu}{(1 - \mu_h \mu_v)} < 1 \quad \text{will be the required constant for Assumption 2.}$$

■

4. Nonautonomous Hénon map

We have now developed the necessary tools for proving the existence of a chaotic invariant set for the nonautonomous Hénon map. Recall our general notation for nonautonomous dynamics (a sequence of maps defined on a sequence of domains), $\{f_n, D_n\}_{n=-\infty}^{+\infty}$.

We will construct domains D_n for the nonautonomous Hénon map, each of them containing an associated pair of horizontal strips and another of vertical strips. Moreover, the transition matrices will be shown to be identical for each map f_n, with $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ for each iteration n.

Recall that the autonomous Hénon map has the form:

$$H(x, y) = (A + By - x^2, x), \quad \text{with inverse function} \quad H^{-1}(x, y) = (y, (x - A + y^2)/B) \tag{41}$$

Following [Devaney and Nitecki, 1979], sufficient conditions for the existence of a chaotic invariant set in the autonomous context can be proven when the parameters satisfy the following inequalities:

$$A > A_2 = \frac{(5 + 2\sqrt{5})(1 + |B|)^2}{4} \quad , \quad A_2 = 5 + 2\sqrt{5} \approx 9.47 \quad \text{in case} \quad B = \pm 1 \tag{42}$$

Note that when $B = -1$ the map is orientation-preserving and area-preserving. For our version of the nonautonomous Hénon map, we will take $B = -1$ for the sequence of maps $\{f_n, D_n\}_{n=-\infty}^{+\infty}$ in order to retain these properties, but we will allow A to vary for each iteration n. Therefore, we will take:

$$f_n(x,y) = (A(n) - y - x^2, x) \quad , \quad f_n^{-1}(x,y) = (y, A(n) - x - y^2) \tag{43}$$

where

$$A(n) = 9.5 + \epsilon \cdot \cos(n) \quad \text{with} \quad \epsilon = 0.1. \tag{44}$$

This choice is motivated by the fact that $A_2 = 5 + 2\sqrt{5} \approx 9.47$ is the minimum threshold for parameter A for which the autonomous Hénon map satisfies the autonomous versions of Assumptions 1 and 3 of the Conley-Moser conditions.

In the following we will prove that the nonautonomous Hénon map satisfies the conditions described in Theorem 3. In particular, we will prove the following theorem.

Theorem 6. *If $A^* \geq 9.5$ then the nonautonomous Hénon map $f_n = (A(n) - y - x^2, x)$ with $A(n) = A^* + \epsilon \cdot \cos(n)$, $\epsilon = 0.1$ has a nonautonomous chaotic invariant set in $\mathbb{R}^2$.*

Proof. We carry out the proof for the specific case where $A_0 = 9.5$. The case for $A_0 > 9.5$ follows similar reasoning as for the case $A_0 = 9.5$, with the main difference being that some values in the inequalities appearing when checking Assumption 3 must be changed. We begin with the first Conley-Moser condition.

Assumption 1. The domain D_n on which each function f_n will be defined is the square

$$D_n = D = [-R, R] \times [-R, R] \qquad \text{with} \qquad R = \sup_{n \in \mathbb{Z}} R(n) = 1 + \sqrt{1 + A(0)} \approx 4.25 \tag{45}$$

analogously to the domain considered for the autonomous Hénon map.

The horizontal strips and the vertical strips associated to any iteration $n \in \mathbb{Z}$ will be taken as

$$D_H^{n+1} \equiv f_n(D) \cap D \quad , \quad D_V^n \equiv f_n^{-1}(D) \cap D \tag{46}$$

and since f_n is a homeomorphism we also note that vertical strips "move" to horizontal strips in forward iteration,

$$f_n(D_V^n) = f_n\left(f_n^{-1}(D) \cap D\right) = (f_n \circ f_n^{-1})(D) \cap f_n(D) = D_H^{n+1} \tag{47}$$

Moreover the index I indicating the number of strips in either D_H^n or D_V^n is $I = \{1, 2\}$ and the strips are defined by

$$\begin{aligned} H_1^{n+1} &\equiv f_n(D) \cap ([-R, R] \times [0, R]) \quad , \quad H_2^{n+1} \equiv f_n(D) \cap ([-R, R] \times [-R, 0]) \\ V_1^n &\equiv f_n^{-1}(D) \cap ([0, R] \times [-R, R]) \quad , \quad V_2^n \equiv f_n^{-1}(D) \cap ([-R, 0] \times [-R, R]) \end{aligned} \tag{48}$$

These are determined by the images of D with respect to f_n and f_n^{-1} for every $n \in \mathbb{Z}$. They result easy to compute. Let

$$L_1 = \{(x,y) \in D \mid y = R\} \quad , \quad L_2 = \{(x,y) \in D \mid y = -R\}$$

$$L_3 = \{(x, y) \in D \mid x = R\} \quad , \quad L_4 = \{(x, y) \in D \mid x = -R\}$$

the segments which conform the boundary of D. Their images with respect f_n and f_n^{-1} are either another segment or a parabola, and as f_n is a homeomorphism, both $f_n(D)$ and $f_n^{-1}(D)$ are two strips with a parabolic form.

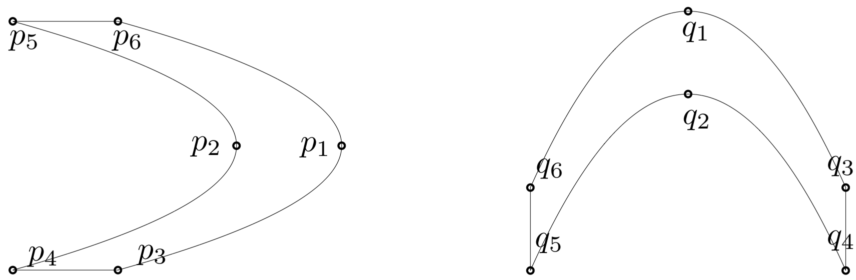

Figure 6. $f_n(D)$ and $f_n^{-1}(D)$ take these two shapes, respectively, for any given $n \in \mathbb{Z}$. The set of points $p_1, p_2, p_3, p_4, p_5, p_6$ and $q_1, q_2, q_3, q_4, q_5, q_6$ determine the height and length of both geometric forms.

The key points of $f_n(D)$ and $f_n^{-1}(D)$ shown in Figure 6 have the following coordinates,

$$p_1 \equiv (A(n) + R, 0) \quad , \quad p_2 \equiv (A(n) - R, 0) \quad , \quad p_3 \equiv (A(n) + R - R^2, -R)$$

$$p_4 \equiv (A(n) - R - R^2, -R) \quad , \quad p_5 \equiv (A(n) - R - R^2, R) \quad , \quad p_6 \equiv (A(n) + R - R^2, R)$$

$$q_1 \equiv (0, A(n) + R) \quad , \quad q_2 \equiv (0, A(n) - R) \quad , \quad q_3 \equiv (R, A(n) + R - R^2)$$

$$q_4 \equiv (R, A(n) - R - R^2) \quad , \quad q_5 \equiv (-R, A(n) - R - R^2) \quad , \quad q_6 \equiv (-R, A(n) + R - R^2)$$

The coordinates of the these points satisfy $A(n) > 2R$, $\forall n \in \mathbb{Z}$ and

$$A(n) + R - R^2 = A(n) + 1 + \sqrt{1 + A(0)} - (1 + \sqrt{1 + A(0)})^2 = (A(n) - A(0)) - R \leq -R \tag{49}$$

with strict inequality when $n \neq 0$. Only in case $n = 0$, the points $p_6, p_3 = q_6, q_3$ are inside the domain D and actually these are three vertices of the square D. In any case, it follows that the points p_1, p_2, p_4, p_5 and q_1, q_2, q_4, q_5 do not belong to D for any n.

We denote the arguments of the parabolic curves by X and Y, so that their equations take the form:

$$Y = \sqrt{A(n) - R - X} \quad \text{in the horizontal case,} \tag{50}$$

$$X = \sqrt{A(n) - R - Y} \quad \text{in the vertical case.} \tag{51}$$

With this notation the absolute value of the derivatives of these functions take the form:

$$\left|\frac{dY}{dX}\right| = \frac{1}{2\sqrt{A(n)-R-X}} \quad , \quad \left|\frac{dX}{dY}\right| = \frac{1}{2\sqrt{A(n)-R-Y}} \tag{52}$$

and their maximum values are assumed when $X = R$ and $Y = R$, respectively. Therefore we have:

$$\frac{1}{2\sqrt{A(n)-2R}} \leq \frac{1}{2\sqrt{9.5+\epsilon\cos(n)-2R}} \approx \frac{1}{2\sqrt{9.5-0.1-8.5}} = \frac{1}{2\sqrt{0.9}} \approx 0.527$$

Due to reasons explained later, for convenience we will choose the thresholds $\mu_h = \mu_v = 0.615$ for the maximum values that the slopes of the horizontal and vertical boundaries can assume, respectively. Using this fact, one can conclude that H_i^{n+1} is a μ_h-horizontal strip and V_i^n a μ_v-vertical strip for every $i \in I$ and $n \in \mathbb{Z}$. Moreover, $\mu_h \cdot \mu_v = (0.615)^2 = 0.378225 < 1$. This proves part of Assumption 1.

Furthermore, for any $i, j \in I$ and $n \in \mathbb{Z}$ we have that the horizontal boundaries of $f_n(V_i^n) = H_i^{n+1}$ are two μ_h-horizontal curves which link the left and right sides of the square D. Since the two μ_v-vertical curves bounding $\partial_v V_j^{n+1}$ link the upper and the bottom sides of D_{n+1}, both boundaries $\partial_h f_n(V_i^n)$ and $\partial_v V_j^{n+1}$ intersect in four different points. From this fact it follows that $H_{ij}^{n+1} = f_n(V_i^n) \cap V_j^{n+1}$ is a μ_h-horizontal strip contained in V_j^{n+1}.

Since f_n is a homeomorphism for every $n \in \mathbb{Z}$,

$$f_n \quad \text{maps} \quad V_{ji}^n = f_n^{-1}(V_j^{n+1}) \cap V_i^n = f_n^{-1}\left(f_n(V_i^n) \cap V_j^{n+1}\right) \quad \text{onto} \quad H_{ij}^{n+1} \quad \text{and}$$

$$f_n^{-1}\left(\partial_h H_{ij}^{n+1}\right) \subset \partial_h V_i^n \quad \text{because by construction} \quad \partial_h H_{ij}^{n+1} \subset \partial_h H_i^{n+1} \tag{53}$$

This can be checked by an easy computation.

Therefore the nonautonomous Hénon map satisfies Assumption 1.

Assumption 3. To begin our verification that A3 is also satisfied, we need to recall the notation for several concepts developed earlier:

$$Df_n(x,y) = \begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad Df_n^{-1}(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & -2y \end{pmatrix} \tag{54}$$

$$S_{\mathcal{K}}^u = \{(\xi_z, \eta_z) \in \mathbb{R}^2 \mid |\eta_z| \leq \mu_h |\xi_z|,\ z \in \mathcal{K}\}$$

$$S_{\mathcal{K}}^s = \{(\xi_z, \eta_z) \in \mathbb{R}^2 \mid |\xi_z| \leq \mu_v |\eta_z|,\ z \in \mathcal{K}\}$$

with $\mathcal{K}$ being either $\mathcal{V}^n$ or $\mathcal{H}^{n+1}$.

Now given any point $z_0 = (x_0, y_0) \in \mathcal{H}^{n+1}$ and any $(\xi_{z_0}, \eta_{z_0}) \in S_{\mathcal{H}^{n+1}}^s$ (which by definition, $|\xi_{z_0}| \leq \mu_v |\eta_{z_0}|$) we have that

$$Df_n^{-1}(z_0) \cdot (\xi_{z_0}, \eta_{z_0}) = \begin{pmatrix} 0 & 1 \\ -1 & -2y_0 \end{pmatrix} \cdot \begin{pmatrix} \xi_{z_0} \\ \eta_{z_0} \end{pmatrix} = \begin{pmatrix} \eta_{z_0} \\ -\xi_{z_0} - 2y_0\eta_{z_0} \end{pmatrix}$$

belongs to $S^s_{\mathcal{V}^n}$ if and only if the inequality

$$|\eta_{z_0}| \leq \mu_v \cdot |\xi_{z_0} + 2y_0\eta_{z_0}| \qquad \text{holds.} \tag{55}$$

Since it is also true that

$$\mu_v \cdot |\xi_{z_0} + 2y_0\eta_{z_0}| \geq \mu_v \cdot [2|y_0||\eta_{z_0}| - |\xi_{z_0}|] \geq$$

$$\geq \mu_v \cdot [2|y_0||\eta_{z_0}| - \mu_v|\eta_{z_0}|] = \left(2|y_0|\mu_v - \mu_v^2\right)|\eta_{z_0}| \tag{56}$$

In case $\left(2|y_0|\mu_v - \mu_v^2\right) \geq 1$, the previous inequality (55) will hold.

To verify this we need to check if

$$|y_0| \geq \frac{1}{2}\left(\mu_v + \frac{1}{\mu_v}\right) = 1.1205 \quad \text{for any } z_0 = (x_0, y_0) \in \mathcal{H}^{n+1}. \tag{57}$$

At this point we give a geometrical argument.

• The horizontal lines $\{Y = \pm 1.1205\}$ cut the parabola $\{X = A(n) - R - Y^2\}$ at two points:

$$(x_1, y_1) = (8.5 + \epsilon\cos(n) - \sqrt{10.6} - 1.2555, 1.1205) \quad \text{and}$$

$$(x_2, y_2) = (8.5 + \epsilon\cos(n) - \sqrt{10.6} - 1.2555, -1.1205)$$

• The horizontal lines $\{Y = \pm 1.1205\}$ cut the parabola $\{Y = A(n+1) + R - X^2\}$ at two points with a positive x-component:

$$(\bar{x}_1, \bar{y}_1) = (\sqrt{10.5 + \epsilon\cos(n+1) + \sqrt{10.6} - 1.1205}, 1.1205) \quad \text{and}$$

$$(\bar{x}_2, \bar{y}_2) = (\sqrt{10.5 + \epsilon\cos(n+1) + \sqrt{10.6} + 1.1205}, -1.1205)$$

From here we have that

$$\bar{x}_1 < \bar{x}_2 = \sqrt{10.5 + \epsilon\cos(n+1) + \sqrt{10.6} + 1.1205} \leq \sqrt{10.6 + \sqrt{10.6} + 1.1205} =$$

$$\sqrt{14.9758} = 3.8699 < 3.8887 = 8.5 - 0.1 - \sqrt{10.6} - 1.2555 \leq$$

$$\leq 8.5 + \epsilon\cos(n) - \sqrt{10.6} - 1.2555 = x_2 = x_1 < 4.25 < R \qquad \forall n \in \mathbb{Z} \tag{58}$$

The inequalities $\bar{x}_1 < \bar{x}_2 < x_2 = x_1$ (note that $\bar{x}_1 < \bar{x}_2$ is trivial due to the definitions) also hold for every parameter $A(n) = A^* + \epsilon\cos(n)$ satisfying $A^* \geq 9.5$ and $\epsilon = 0.1$. The reason comes from comparing the derivatives of $\bar{x}_2$ and x_2 with respect to A^*:

$$\bar{x}_2 = \sqrt{A^* + \epsilon\cos(n+1) + 1 + \sqrt{1 + A^* + \epsilon\cos(n)} + 1.1205} \geq \sqrt{10.4 + \sqrt{10.4} + 1.1205} \approx 3.84 \tag{59}$$

$$x_2 = A^* + \epsilon\cos(n) - 1 - \sqrt{1 + A^* + \epsilon\cos(n)} - 1.1205 \tag{60}$$

$$\frac{d\bar{x}_2}{dA^*} = \frac{1}{2\bar{x}_2}\cdot\left(1 + \frac{1}{2\sqrt{1+A^*+\epsilon\cos(n)}}\right) \leq \frac{1}{2\bar{x}_2}\cdot\left(1+\frac{1}{2\sqrt{1+A^*-\epsilon}}\right) \leq$$
$$\leq \frac{1}{2\bar{x}_2}\cdot\left(1+\frac{1}{2\sqrt{10.4}}\right) \leq \frac{1}{2\cdot 3.84}\cdot\left(1+\frac{1}{2\sqrt{10.4}}\right) \approx 0.1504 \tag{61}$$

$$\frac{dx_2}{dA^*} = 1 - \frac{1}{2\sqrt{1+A^*+\epsilon\cos(n)}} \geq 1 - \frac{1}{2\sqrt{1+A^*-\epsilon}} \geq 1 - \frac{1}{2\sqrt{10.4}} \approx 0.8450 \tag{62}$$

Clearly $\frac{d\bar{x}_2}{dA^*} < \frac{dx_2}{dA^*}$ for every $A^* \geq 9.5$. It follows that $\bar{x}_2 < x_2$ for $A^* \geq 9.5$.

This setup shows that every point $z_0 = (x_0, y_0) \in \mathcal{H}^{n+1}$ satisfies that $|y_0| > 1.1205 = \frac{1}{2}\left(\mu_v + \frac{1}{\mu_v}\right)$, since the four areas composing $\mathcal{H}^{n+1}$ are either beneath the line $\{Y = -1.1205\}$ or above the line $\{Y = 1.1205\}$, as can be observed in Fig. 7.

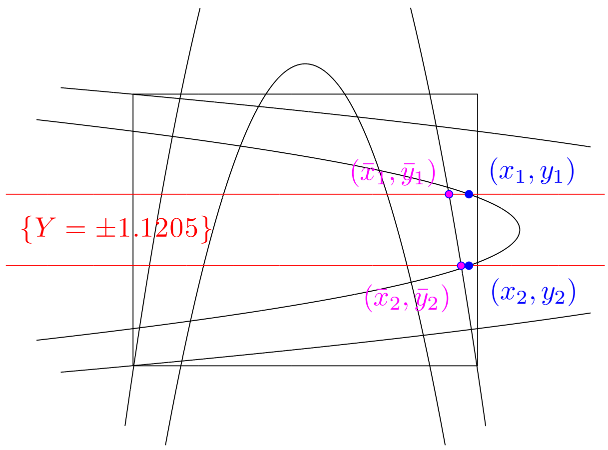

Figure 7. The four areas composing $\mathcal{H}^{n+1}$ are those bounded by the four parabolic strips (two horizontal and two vertical) contained in the square domain D_{n+1}.

Since $z_0 \in \mathcal{H}^{n+1}$ is an arbitrary point, the inclusion $Df_n^{-1}(S^s_{\mathcal{H}^{n+1}}) \subset S^s_{\mathcal{V}^n}$ is proven.

For the second inclusion $Df_n(S^u_{\mathcal{V}^n}) \subset S^u_{\mathcal{H}^{n+1}}$ we focus on the fact that $\mathcal{V}^n = f_n^{-1}(\mathcal{H}^{n+1})$ and since $f_n^{-1}(x,y) = (y, A(n) - x - y^2)$ transforms the y-components of the points of $\mathcal{H}^{n+1}$ into the x-components of the points of $\mathcal{V}^n$, it immediately follows that

$$|x_0| > \frac{1}{2}\left(\mu_v + \frac{1}{\mu_v}\right) = \frac{1}{2}\left(\mu_h + \frac{1}{\mu_h}\right)$$

$$\text{for every point } z_0 = (x_0, y_0) \in \mathcal{V}^n. \tag{63}$$

As in the previous case this inequality allows us to prove the inclusion,

$$Df_n(z_0) \cdot (\xi_{z_0}, \eta_{z_0}) = \begin{pmatrix} -2x_0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \xi_{z_0} \\ \eta_{z_0} \end{pmatrix} = \begin{pmatrix} -2x_0\xi_{z_0} - \eta_{z_0} \\ \xi_{z_0} \end{pmatrix} \in S^u_{\mathcal{H}^{n+1}}$$

$$\text{if and only if} \qquad |\xi_{z_0}| \le \mu_h \cdot |2x_0\xi_{z_0} + \eta_{z_0}| \tag{64}$$

$$\text{(remember that } (\xi_{z_0}, \eta_{z_0}) \in S^u_{\mathcal{V}^n}, \quad |\eta_{z_0}| \le \mu_h |\xi_{z_0}|)$$

$$\text{We see that} \qquad \mu_h \cdot |2x_0\xi_{z_0} + \eta_{z_0}| \ge \mu_h \cdot [2|x_0||\xi_{z_0}| - |\eta_{z_0}|] \ge$$

$$\ge \mu_h \cdot [2|x_0||\xi_{z_0}| - \mu_h|\xi_{z_0}|] = \left[2|x_0|\mu_h - \mu_h^2\right]|\xi_{z_0}| \ge$$

$$\ge \left[2\mu_h \cdot \frac{1}{2}\left(\mu_h + \frac{1}{\mu_h}\right) - \mu_h^2\right]|\xi_{z_0}| = \left[\mu_h^2 + 1 - \mu_h^2\right]|\xi_{z_0}| = |\xi_{z_0}| \tag{65}$$

and then the inclusion $Df_n(S^u_{\mathcal{V}^n}) \subset S^u_{\mathcal{H}^{n+1}}$ is proved.

Finally for the last part of Assumption 3 we will only prove the inequality

$$|\eta_{f_n^{-1}(z_0)}| \ge \frac{1}{\mu}|\eta_{z_0}| \quad \text{for } 0 < \mu < 1 - \mu_h\mu_v \text{ and } z_0 \in \mathcal{H}^{n+1},\ (\xi_{z_0}, \eta_{z_0}) \in S^s_{\mathcal{H}^{n+1}} \tag{66}$$

since the inequality

$$|\xi_{f_n(z_0)}| \ge \frac{1}{\mu}|\xi_{z_0}|, \quad z_0 \in \mathcal{V}^n,\ (\xi_{z_0}, \eta_{z_0}) \in S^u_{\mathcal{V}^n} \tag{67}$$

is proved by using the same argument.

$$|\eta_{f_n^{-1}(z_0)}| = |2y_0\eta_{z_0} + \xi_{z_0}| \ge 2|y_0||\eta_{z_0}| - |\xi_{z_0}| \ge 2|y_0||\eta_{z_0}| - \mu_v|\eta_{z_0}| =$$

$$[2|y_0| - \mu_v]|\eta_{z_0}| \ge \frac{1}{\mu}|\eta_{z_0}| \qquad \text{if and only if}$$

$$2|y_0| - \mu_v \ge \frac{1}{\mu} \quad \longleftrightarrow \quad |y_0| \ge \frac{1}{2}\left(\mu_v + \frac{1}{\mu}\right) \tag{68}$$

This last inequality is true if we require that $\mu_v < \mu < 1 - \mu_h\mu_v$. This interval exists since $\mu_v = 0.615$ is less than $(1 - \mu_h\mu_v) = 0.621775$. Then we have that

$$|y_0| > \frac{1}{2}\left(\mu_v + \frac{1}{\mu_v}\right) > \frac{1}{2}\left(\mu_v + \frac{1}{\mu}\right) \quad \text{for every } z_0 = (x_0, y_0) \in \mathcal{H}^{n+1} \tag{69}$$

Analogously for any $z_0 = (x_0, y_0) \in \mathcal{V}^n$,

$$|x_0| > \frac{1}{2}\left(\mu_h + \frac{1}{\mu_h}\right) > \frac{1}{2}\left(\mu_h + \frac{1}{\mu}\right) \tag{70}$$

and the proof that the nonautonomous Hénon map satisfies A1 and A3 with $0 < \mu < 1 - \mu_h\mu_v$ is complete. Consequently it also satisfies A2 by using Theorem 4.

By applying the main theorem it follows that there exists a chaotic invariant set $\{\Lambda_n\}_{n=-\infty}^{+\infty}$ (with respect to the nonautonomous Hénon map $\{f_n\}$) contained in $\{D_n\}_{n=-\infty}^{+\infty}$ (let say $\Lambda_n \subset D_n = D$ and $f_n(\Lambda_n) = \Lambda_{n+1}$) which is conjugate to a shift map of two symbols. ■

Remark 4.1. Comparing this result to what happens for the autonomous Hénon map, it is curious that for some $n \in \mathbb{Z}$ the quantity $A(n) = 9.5 + \epsilon \cos(n)$ is less than $A_2 = 5 + 2\sqrt{5} \approx 9.47$, which is the minimum threshold for parameter A for which the autonomous Hénon map satisfies the autonomous Assumption 3.

In other words, this given example shows that although for some $n \in \mathbb{Z}$ the values that parameter A takes imply that f_n does not satisfy the autonomous Assumption 3 separately, this fact does not necessarily mean that the nonautonomous Assumption 3 is not satisfied for the sequence $\{f_n\}_{n=-\infty}^{+\infty}$.

5. Summary and Conclusions

In this paper we have considered a nonautonomous version of the Hénon map and have provided necessary conditions for the map to possess a nonautonomous chaotic invariant set. This is accomplished by using a nonautonomous version of the Conley-Moser conditions given in [Wiggins, 1999]. We sharpen these conditions by providing a more analytical condition that, as a consequence, enables us to show that the chaotic invariant set is hyperbolic. In the course of the proof we provide a precise characterization of what is mean by the phrase “hyperbolic chaotic invariant set” for nonautomous dynamical systems. Currently there is much interest in nonautomous dynamics and a thorough analysis of a specific example might provide a benchmark for further studies, just as the work in [Devaney and Nitecki, 1979] provided a benchmark for studies of chaotic dynamics for autonomous maps. Indeed, our generalization of the Hénon map to the nonautonomous setting provides an approach to generalizing the map to even more general nonautonomous settings, such as a consideration of “noise”. This would be an interesting topic for future studies.

Acknowledgments

The research of FB-I, CL and AMM is supported by the MINECO under grant MTM2014-56392-R. The research of SW is supported by ONR Grant No. N00014-01-1-0769. We acknowledge support from MINECO: ICMAT Severo Ochoa project SEV-2011-0087.

References

Alekseev, V.M. (1968a). Quasirandom dynamical systems, I. Math. USSR-Sb , **5**, 73-128.
Alekseev, V.M. (1968b). Quasirandom dynamical systems, II. Math. USSR-Sb , **6**, 505-560.
Alekseev, V.M. (1969). Quasirandom dynamical systems, III. Math. USSR-Sb , **6**, 1-43.
Devaney, R. L., Nitecki, Z. (1979). Shift automorphisms in the Hénon mapping. *Comm. Math. Phys.*, **67**, 137-179.
Hénon, M. (1976). A Two-Dimensional Mapping with a Strange Attractor. *Comm. Math. Phys.*, **50**, 69-77.
Holmes, P.J. (1982). The dynamics of repeated impacts with a sinusoidally vibrating table. Journal of Sound and Vibration, **84(2)**, 173-189.
Katok, A. and Hasselblatt, B. (1995). *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge.
Lerman, L. and Silnikov, L. (1992). Homoclinical structures in nonautonomous systems: Nonautonomous chaos. *Chaos*, **2**, 447–454.
Lopesino, C., Balibrea-Iniesta, F., Wiggins, S., Mancho, A. (2015). The chaotic saddle in the Lozi map, autonomous and non-autonomous version. (pre-print)

Lu, K. and Wang, Q. (2010). Chaos in differential equations driven by a nonautonomous force. *Nonlinearity*, **23**, 2935–2973.

Lu, K. and Wang, Q. (2011). Chaotic behavior in differential equations driven by a Brownian motion. *J. Diff. Eq.*, **251**, 2853–2895.

Moser, J. (1973). Stable and Random Motions in Dynamical Systems. Annals of mathematical studies, **77**. Princeton University Press.

Smale, S. (1980). *The Mathematics of Time: Essays on Dynamical Systems, Economic Processes and Related Topics* . Springer-Verlag, New York.

Stoffer, D. (1988a). Transversal homoclinic points and hyperbolic sets for non-autonomous maps I. *J. Appl. Math. and Phys. (ZAMP)*, **39**, 518–549.

Stoffer, D. (1988b). Transversal homoclinic points and hyperbolic sets for non-autonomous maps II. *J. Appl. Math. and Phys. (ZAMP)*, **39**, 783–812.

Wiggins, S. (1999). Chaos in the dynamics generated by sequences of maps, with applications to chaotic advection in flows with aperiodic time dependence. Z. angew. Math. Phys. (ZAMP), **50**, 585-616.

Wiggins, S. (2003). Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer Second Edition.

Wiggins, S and Mancho, A. M. (2014). Barriers to transport in aperiodically time-dependent two-dimensional velocity fields: Nekhoroshev's theorem and nearly invariant tori. *Nonlinear Processes in Geophysics*, **21**, 165–185.