THE SELBERG TRACE FORMULA AS A DIRICHLET SERIES

ANDREW R. BOOKER AND MIN LEE

Abstract. We explore an idea of Conrey and Li of expressing the Selberg trace formula as a Dirichlet series. We describe two applications, including an interpretation of the Selberg eigenvalue conjecture in terms of quadratic twists of certain Dirichlet series, and a formula for an arithmetically weighted sum of the complete symmetric square L-functions associated to cuspidal Maass newforms of squarefree level $N > 1$.

1. Introduction

In this paper, we explore the idea of Conrey and Li [CL01] (later generalized by Li in [Li05]) of presenting the Selberg trace formula for Hecke operators acting on $L^2(\Gamma_0(N)\backslash\mathbb{H})$, N squarefree, as a Dirichlet series. We enhance their work in a few ways:

- We prove the meromorphic continuation of the relevant Dirichlet series to all $s \in \mathbb{C}$ (compared with $\Re(s) > 0$ in [CL01]).
- We give explicit formulas for all terms, without replacing any by estimates. Thus, our formula entails no loss of generality, in the sense that one could reverse the proof to derive the trace formula from it.
- For $N > 1$ we compute the trace over the *newforms* of level N rather than the whole spectrum. The result is a significantly cleaner formula, though again this entails no loss of generality, since one can recover the full formula for level N by summing the formulas for newforms of levels dividing N.
- We treat the Hecke operators T_n for all non-zero n co-prime to the level N, including $n < 0$. When $n < 0$, there are no elliptic terms in the trace formula, and this leads to a simpler result that is useful for applications.
- We base our calculations on a version of the trace formula published by Strömbergsson [Str16], rather than working out each term from first principles. The advantage is that Strömbergsson's formula has been vetted by comparing the two sides numerically, so it is highly robust, and this helps limit the potential for errors in the final formula. For instance, our formula shows that the Dirichlet series we obtain can have poles at the zeros of the scattering determinant (which are in turn related to zeros of the Riemann zeta-function), a fact which seems to have been overlooked in [CL01].

We present two applications of our formula. First, for prime N, we derive a statement equivalent to Selberg's eigenvalue conjecture for $\Gamma_0(N)$, in terms of the analytic properties of twists by the quadratic character (mod N) of the family of Dirichlet series arising from our formula for level 1. A similar criterion was given by Li in [Li08], and in fact Li's formulation is simpler in a way since it involves only a single Dirichlet series. However, our formulation makes plain the fact that the passage from level 1 to level N is essentially a quadratic twist, providing further support for the analogy between exceptional eigenvalues and Siegel zeros.

Both authors were supported by EPSRC Grants `EP/H005188/1`, `EP/L001454/1` and `EP/K034383/1`.

Second, for squarefree $N > 1$, we sum our formula for T_{-n^2} acting on $\Gamma_0(N)\backslash\mathbb{H}$ to obtain an explicit expression for $\sum_{j=1}^{\infty}(-1)^{\epsilon_j} L^*(s, \mathrm{Sym}^2 f_j)$, where $\{f_j\}_{j=1}^{\infty}$ is a complete, arithmetically normalized sequence of Hecke–Maass newforms on $\Gamma_0(N)\backslash\mathbb{H}$, $\epsilon_j \in \{0,1\}$ is the parity of f_j, and $L^*(s, \mathrm{Sym}^2 f_j)$ is the complete symmetric square L-function. When $N = 2$, the answer can be interpreted as the Rankin–Selberg convolution of the weight $\frac{1}{2}$ harmonic weak Maass form defined in [RW11] with a weight 1 Eisenstein series, much like Shimura's integral representation for the symmetric square L-function. Similar formulas have been derived for averages of L-functions over an L^2-normalized basis (see, e.g., [Mot92]); to our knowledge, ours is the first such to be derived from the Selberg trace formula, with arithmetic normalization.

1.1. **Notation and statement of main results.** Let $\mathcal{D}$ denote the set of discriminants, that is

$$\mathcal{D} = \{D \in \mathbb{Z} : D \equiv 0 \text{ or } 1 \pmod 4\}.$$

Any non-zero $D \in \mathcal{D}$ may be expressed uniquely in the form $d\ell^2$, where d is a fundamental discriminant and $\ell > 0$. We define $\psi_D(n) = \left(\frac{d}{n/\gcd(n,\ell)}\right)$, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. Note that ψ_D is periodic modulo D, and if D is fundamental then ψ_D is the usual quadratic character mod D. Set

$$L(s, \psi_D) = \sum_{n=1}^{\infty} \frac{\psi_D(n)}{n^s} \quad \text{for } \Re(s) > 1.$$

Then it is not hard to see that

$$L(s, \psi_D) = L(s, \psi_d) \prod_{p\mid\ell} \left[1 + \big(1 - \psi_d(p)\big) \sum_{j=1}^{\mathrm{ord}_p(\ell)} p^{-js}\right],$$

so that $L(s, \psi_D)$ has analytic continuation to $\mathbb{C}$, apart from a simple pole at $s = 1$ when D is a square. In particular, if D is not a square then we have

$$L(1, \psi_D) = L(1, \psi_d) \cdot \frac{1}{\ell} \prod_{p\mid\ell} \left[1 + \big(p - \psi_d(p)\big) \frac{(\ell, p^\infty) - 1}{p - 1}\right]. \tag{1.1}$$

Our first result is the following:

Theorem 1.1.

(1) *For any positive integer n, the series*

$$\sum_{\substack{t\in\mathbb{Z} \\ \sqrt{t^2+4n}\notin\mathbb{Z}}} \frac{L(1, \psi_{t^2+4n})}{(t^2+4n)^s} \tag{1.2}$$

has meromorphic continuation to $\mathbb{C}$ and is holomorphic for $\Re(s) > 0$, apart from a simple pole of residue $\sigma_{-1}(n)$ at $s = \frac{1}{2}$.

(2) *If n is a positive integer and N is a prime such that $\left(\frac{-4n}{N}\right) = -1$, then the series*

$$\sum_{\substack{t\in\mathbb{Z} \\ \sqrt{t^2+4n}\notin\mathbb{Z}}} \frac{L(1, \psi_{t^2+4n}) \left(\frac{t^2+4n}{N}\right)}{(t^2+4n)^s} \tag{1.3}$$

has meromorphic continuation to $\mathbb{C}$ and is holomorphic for $\Re(s) > \frac{7}{64}$.

(3) *For any prime N, the Selberg eigenvalue conjecture is true for $\Gamma_0(N)$ if and only if (1.3) is holomorphic on $\Re(s) > 0$ for all primes n satisfying $\left(\frac{-4n}{N}\right) = -1$.*

Remarks.

(1) The locations and residues of the poles of (1.2) and (1.3) are related to the trace of T_{-n} over the discrete spectrum of the Laplacian on $L^2(\Gamma_0(1)\backslash\mathbb{H})$ and $L^2(\Gamma_0(N)\backslash\mathbb{H})$, respectively. See Propositions 3.1 and 3.2 for full details.

(2) A simple consequence of (1) is the asymptotic

$$\sum_{\substack{t\in\mathbb{Z}\cap[1,X] \\ \sqrt{t^2+4n}\notin\mathbb{Z}}} L(1,\psi_{t^2+4n}) \sim \sigma_{-1}(n)X \quad \text{as } X\to\infty.$$

In fact, arguing as in the proof of Theorem 1.3 below, one can see that the two sides are equal up to an error of $O_{n,\varepsilon}\big(X^{\frac{3}{5}+\varepsilon}\big)$. Related averages over discriminants of the form $t^{2k}-4$ for fixed k were computed by Sarnak [Sar85] and subsequently generalized by Raulf [Rau09], who obtained averages over arithmetic progressions and also sieved to reach the fundamental discriminants. It would be interesting to see whether our formula for the generating function could be used in conjunction with Raulf's work to obtain sharper error terms. (See also Hashimoto's recent improvement [Has13] of [Sar82] and [Rau09] for the closely related problem of determining the average size of the class number over discriminants ordered by their units.)

Next, we define more general versions of the coefficients $L(1,\psi_D)$ that will turn out to be related to the newforms of a given squarefree level $N>1$. For non-zero $D = d\ell^2 \in \mathcal{D}$, let $m = (N^\infty, \ell)$, and define

$$c_N(D) = \begin{cases} m^{-1}\prod_{p\mid N}(\psi_{D/m^2}(p)-1)\cdot L(1,\psi_{D/m^2}) & \text{if } d\neq 1, \\ \Lambda(N)\big(m^{-1}-\frac{2N}{N+1}\big) & \text{if } d=1, \end{cases} \tag{1.4}$$

where Λ denotes the von Mangoldt function. For notational convenience, we set $c_N(D)=0$ when $D\equiv 2$ or $3 \pmod 4$.

When $N=2$, it was shown in [RW11] that the numbers

$$c^+(n) = \begin{cases} c_2(n) & \text{if } n\equiv 0,1 \pmod 4, \\ 2c_2(4n) & \text{if } n\equiv 2,3 \pmod 4, \end{cases}$$

for $n>0$, are the Fourier coefficients of a weight $\frac{1}{2}$ mock modular form for $\Gamma_0(4)$ with shadow Θ^3, where $\Theta = \sum_{n\in\mathbb{Z}} q^{n^2}$ is the classical theta function.[1]

Now, for a positive integer n, put

$$r(n) = \tfrac{1}{2}\#\big\{(x,y)\in\mathbb{Z}^2 : n = x^2+4y^2\big\} = \big(1+\cos\tfrac{\pi n}{2}\big)\sum_{d\mid n}\psi_{-4}(d).$$

[1]Our definition of $c^+(n)$ differs from that in [RW11] in a few minor ways. First, we have scaled their definition by the constant $\frac{\pi}{6}$. Second, there is a mistake in the formula for $c^+(n)$ given in [RW11] for square values of n, to the effect that their formula should be multiplied by $2-2^{-\operatorname{ord}_2(n)/2}$. Third, the mock modular form is only determined modulo $\mathbb{C}\Theta$ from its defining properties; we add a particular multiple of Θ to make Theorem 1.2 as symmetric as possible.

The factor $\frac{1}{2}$ is chosen to make r multiplicative; in fact, we have

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = (1 - 2^{-s} + 2^{1-2s})\zeta(s)L(s, \psi_{-4}),$$

so that $r(n)$ are the Fourier coefficients of a modular form of weight 1 and level 16.

Theorem 1.2. *Let $N > 1$ be a squarefree integer, and let $\{f_j\}_{j=1}^{\infty}$ be a complete sequence of arithmetically normalized Hecke–Maass newforms on $\Gamma_0(N)\backslash\mathbb{H}$, with parities $\epsilon_j \in \{0, 1\}$, Laplace eigenvalues $\frac{1}{4} + r_j^2$ and Hecke eigenvalues $\lambda_j(n)$. Define*

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right), \quad \zeta^*(s) = \Gamma_{\mathbb{R}}(s)\zeta(s), \quad E_N^*(s) = N^{s/2}\prod_{p|N}(1-p^{-s}), \quad \zeta_N^*(s) = E_N^*(s)\zeta^*(s),$$

$$L^*(s, \mathrm{Sym}^2 f_j) = \frac{\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s - 2ir_j)\Gamma_{\mathbb{R}}(s + 2ir_j)}{\Gamma_{\mathbb{R}}(2s)}\zeta_N^*(2s)\sum_{n=1}^{\infty}\frac{\lambda_j(n^2)}{n^s},$$

$$\mathcal{I}_N(s;\sigma) = \frac{1}{2\pi i}\int_{\Re(u)=-\sigma}\frac{E_N^*(s)E_N^*(1-s)}{E_N^*(u)E_N^*(1-u)}E_N^*(2u)\zeta^*(s-u)\zeta^*(s+u)\,du,$$

and

$$F_N(s) = \zeta_N^*(4s)\Gamma_{\mathbb{R}}(2s)\sum_{n=1}^{\infty}\frac{c_N(n)r(n)}{n^s}.$$

Then, for any $\sigma > 2$ and $s \in \mathbb{C}$ with $\Re(2s) \in (2, \sigma)$, we have

$$\begin{aligned} F_N(s) &= \sum_{j=1}^{\infty}(-1)^{\epsilon_j}L^*(2s, \mathrm{Sym}^2 f_j) \\ &\quad + \zeta^*(2s)\left(\sqrt{N}\zeta_N^*(2s-1)\zeta_N^*(-2s) - \tfrac{N\Lambda(N)}{N+1}\left[\zeta_N^*(4s) + \zeta_N^*(2-4s) + \mathcal{I}_N(2s;\sigma)\right]\right). \end{aligned}$$

In particular, $F_N(s)$ continues to an entire function, apart from at most simple poles at $s \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1\}$, and is symmetric with respect to $s \mapsto \frac{1}{2} - s$.

An analogue of Theorem 1.2 holds for $N = 1$ as well, but the result is more complicated to state. We content ourselves with the following consequence.

Theorem 1.3. *As $X \to \infty$, for any $\varepsilon > 0$,*

$$\sum_{\substack{0<D\le X\\ \sqrt{D}\notin\mathbb{Z}}} L(1, \psi_D)r(D) = \frac{15\zeta(3)}{4\pi}X + O\big(X^{\frac{8}{11}+\varepsilon}\big).$$

1.2. **Outline of the paper.** In Section 2 we present the trace formula for T_n acting on $\Gamma_0(N)\backslash\mathbb{H}$, N squarefree, using a form of the test function that will be convenient for later application; see Propositions 2.1 and 2.2. In Section 3 we specialize the choice of test function as in [CL01], so that the hyperbolic terms become Dirichlet series. Finally, in Section 4 we apply the formula derived in Section 3 to prove Theorems 1.1–1.3.

Acknowledgements. We thank Dorian Goldfeld and Peter Sarnak for helpful suggestions and corrections.

2. The Selberg trace formula

Let N be a squarefree positive integer, and for any $\lambda \in \mathbb{R}_{\geq 0}$, let $\mathcal{A}(N,\lambda)$ denote the space of automorphic forms $f \in L^2(\Gamma_0(N)\backslash\mathbb{H})$ satisfying $\left(y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\lambda\right)f = 0$. We begin with the trace formula for level 1.

Proposition 2.1. *Let n be a non-zero integer and $q : [0,\infty) \to \mathbb{C}$ a smooth function satisfying $q(v) \ll (1+v)^{-\frac{1}{2}-\delta}$ for some $\delta > 0$. Define*

$$f(y) = q\left(\frac{y^2+2(n-|n|)}{4|n|}\right),$$
$$h(r) = \int_{\mathbb{R}} q\left(\sinh^2\left(\tfrac{u}{2}\right)\right)e^{iru}\,du = 2|n|^{-ir}\int_0^\infty f\left(v-\frac{n}{v}\right)v^{2ir-1}\,dv$$

for $r \in \mathbb{C}$ with $|\Im(r)| < \frac{1}{2}+\delta$,

$$W(D) = \begin{cases} L(1,\psi_D)\frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}}\frac{f(y)}{y^2+|D|}\,dy & \text{if } D<0,\\ L(1,\psi_D)f\left(\sqrt{D}\right) & \text{if } D>0,\ \sqrt{D}\notin\mathbb{Z},\\ \sum_{m\mid\sqrt{D}}\Lambda(m)(1-m^{-1})f(\sqrt{D})+\int_{\sqrt{D}}^\infty\frac{f(y)}{y+\sqrt{D}}\,dy & \text{if } 0\neq\sqrt{D}\in\mathbb{Z},\\ (\gamma-\log 2)f(0)+\frac{1}{2}\int_0^\infty\frac{f(y)+f(y^{-1})-f(0)}{y}\,dy+\frac{1}{3}\int_0^\infty\frac{f(0)-f(y)}{y^2}\,dy & \text{if } D=0 \end{cases}$$

for $D \in \mathcal{D}$, and

$$(2.1)\quad \begin{aligned} F(a) = 2\sum_{m=1}^\infty\frac{\Lambda(m)}{m}f\left(am-\frac{n}{am}\right)+2a\int_a^\infty\frac{f(v-\frac{n}{v})-f(a-\frac{n}{a})}{v^2-a^2}\,dv\\ +\left(\gamma+\log(4\pi)\right)f\left(a-\frac{n}{a}\right)-\frac{h(0)}{4}\end{aligned}$$

for $a \in \mathbb{Z}_{>0}$ with $a \mid n$. Then

$$(2.2)\quad \sum_{\lambda\in\mathbb{R}_{\geq 0}}\operatorname{Tr} T_n|_{\mathcal{A}(1,\lambda)}h\left(\sqrt{\lambda-\tfrac{1}{4}}\right) = \sum_{\substack{a\in\mathbb{Z}_{>0}\\ a\mid n}}F(a)+\sum_{t\in\mathbb{Z}}W(t^2-4n)$$

and

$$(2.3)\quad \sum_{\substack{a\in\mathbb{Z}_{>0}\\ a\mid n}}F(a) = \frac{1}{4\pi}\int_{\mathbb{R}}h(r)\frac{\sigma_{2ir}(|n|)}{|n|^{ir}}\frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)dr+\frac{\sigma_0(|n|)}{4}h(0),$$

where $\phi(s) = \frac{\zeta^*(2(1-s))}{\zeta^*(2s)}$.

Proof. We first derive (2.3). For a sufficiently nice, even Fourier transform pair g, h, it was shown in [Hej83, p. 509] that

$$\frac{1}{4\pi}\int_{\mathbb{R}}h(r)\frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)dr = g(0)\log\pi-\frac{1}{2\pi}\int_{\mathbb{R}}h(r)\psi\left(\tfrac{1}{2}+ir\right)dr-\frac{h(0)}{2}+2\sum_{m=1}^\infty\frac{\Lambda(m)}{m}g(2\log m).$$

Replacing $g(u)$ by $\sum_{\substack{ad=n\\a>0}} g\big(u - \log\big|\frac{a}{d}\big|\big)$ and $h(r)$ by $h(r)\sum_{\substack{ad=n\\a>0}}\big|\frac{a}{d}\big|^{ir}$, we get

$$\begin{aligned}
&\frac{1}{4\pi}\int_{\mathbb{R}} h(r)\frac{\sigma_{2ir}(|n|)}{|n|^{ir}}\frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)dr + \frac{\sigma_0(|n|)}{4}h(0)\\
&\quad = \sum_{\substack{ad=n\\a>0}}\Bigg[g\left(\log\left|\frac{a}{d}\right|\right)\log\pi - \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\left|\frac{a}{d}\right|^{ir}\psi\big(\tfrac{1}{2}+ir\big)\,dr\\
&\qquad - \frac{h(0)}{4} + 2\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m}g\left(\log\left|\frac{a}{d}\right| - 2\log m\right)\Bigg].
\end{aligned}\tag{2.4}$$

Similarly, from the identity

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\psi\big(\tfrac{1}{2}+ir\big)\,dr = g(0)\log\big(4e^{\gamma}\big) + \int_0^{\infty}\frac{g(u)-g(0)}{2\sinh(u/2)}\,du$$

we derive

$$\begin{aligned}
&-\sum_{\substack{ad=n\\a>0}}\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left|\frac{a}{d}\right|^{ir}\psi\big(\tfrac{1}{2}+ir\big)dr\\
&\quad = \sum_{\substack{ad=n\\a>0}}\left[g\left(\log\left|\frac{a}{d}\right|\right)\log\big(4e^{\gamma}\big) + \int_0^{\infty}\frac{g\big(u+\log\big|\frac{a}{d}\big|\big) - g\big(\log\big|\frac{a}{d}\big|\big)}{2\sinh(u/2)}\,du\right],
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{1}{4\pi}\int_{\mathbb{R}} h(r)\frac{\sigma_{2ir}(|n|)}{|n|^{ir}}\frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)dr + \frac{\sigma_0(|n|)}{4}h(0)\\
&\quad = \sum_{\substack{ad=n\\a>0}}\Bigg[g\left(\log\left|\frac{a}{d}\right|\right)\log\big(4\pi e^{\gamma}\big) + \int_0^{\infty}\frac{g\big(u+\log\big|\frac{a}{d}\big|\big) - g\big(\log\big|\frac{a}{d}\big|\big)}{2\sinh(u/2)}\,du\\
&\qquad - \frac{h(0)}{4} + 2\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m}g\left(\log\left|\frac{a}{d}\right| - 2\log m\right)\Bigg].
\end{aligned}$$

Now let $g(u) = q(\sinh^2(\frac{u}{2}))$. Then $g\big(u+\log\big|\frac{a}{d}\big|\big) = f\big(ae^{\frac{u}{2}} - de^{-\frac{u}{2}}\big)$, so on making the substitution $v = ae^{u/2}$, we get $\sum_{\substack{ad=n\\a>0}} F(a)$, as required.

Turning to (2.2), in [Str16, §2.1] we find the following trace formula for level 1:

(2.5)

$$
\begin{aligned}
&\sum_{\lambda\in\mathbb{R}_{\geq 0}} \operatorname{Tr} T_n|_{\mathcal{A}(1,\lambda)} h\Big(\sqrt{\lambda-\tfrac14}\Big)\\
&= \sum_{\substack{t\in\mathbb{Z}\\ \sqrt{t^2-4n}\notin\mathbb{Z}}} \Bigg(\sum_{c|\ell} \mathbf{h}^+(\mathfrak{r}[c])\left[\mathfrak{r}[1]^\times : \mathfrak{r}[c]^\times\right]\Bigg) A(t,n)\\
&+ \begin{cases} \frac{1}{12\sqrt{n}}\int_{\mathbb{R}} r\tanh(\pi r)h(r)\,dr + g(0)\log\frac{\pi\sqrt{n}}{2} + \frac{h(0)}{4} - \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\psi(1+ir)\,dr & \text{if } \sqrt{n}\in\mathbb{Z},\\ 0 & \text{otherwise}\end{cases}\\
&+ \sum_{\substack{a\in\mathbb{Z}_{>0}\\ a|n,\, a^2\neq n}} \Bigg\{\left[\log\pi + \log\left|a-\frac{n}{a}\right| - X\left(\left|a-\frac{n}{a}\right|\right)\right] g\left(\log\left|\frac{a^2}{n}\right|\right)\\
&\qquad\qquad + \frac12\int_{\left|\log\frac{a}{\sqrt{|n|}}\right|}^{\infty} g(u)\frac{e^{\frac{u}{2}}+\operatorname{sgn}(n)e^{-\frac{u}{2}}}{e^{\frac{u}{2}}-\operatorname{sgn}(n)e^{-\frac{u}{2}}+\left|\frac{a}{\sqrt{|n|}}-\operatorname{sgn}(n)\frac{\sqrt{|n|}}{a}\right|}\,du\Bigg\}\\
&+ \sum_{\substack{a\in\mathbb{Z}_{>0}\\ a|n}} \left[2\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m} g\left(\log\left|\frac{a^2}{n}\right| - 2\log m\right) - \frac{h(0)}{4} - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left|\frac{a^2}{n}\right|^{ir}\psi\left(\tfrac12+ir\right)dr\right],
\end{aligned}
$$

where the notation is as follows:

- $t^2-4n = d\ell^2$, where d is a fundamental discriminant and $\ell>0$;
- $\mathfrak{r}[c] = \mathbb{Z}+\mathbb{Z}c\frac{d+\sqrt{d}}{2}$ is the quadratic order of conductor c in $\mathbb{Q}(\sqrt{d})$ and $\mathfrak{r}[c]^\times$ is its unit group;
- $\mathbf{h}^+(\mathfrak{r}[c]) = \frac{\mathbf{h}^+(\mathfrak{r}[1])c\prod_{p|c}(1-\psi_d(p)p^{-1})}{[\mathfrak{r}[1]^\times:\mathfrak{r}[c]^\times]}$ is the narrow class number of $\mathfrak{r}[c]$;
- $\epsilon_d^+>1$ is the smallest unit in $\mathfrak{r}[1]^\times$ with norm 1 (i.e. the fundamental unit when it has norm 1 and its square otherwise);
- $A(t,n) = \begin{cases} \frac{\log\epsilon_d^+}{\sqrt{t^2-4n}} g\left(2\log\frac{|t|+\sqrt{t^2-4n}}{2\sqrt{|n|}}\right) & \text{if } t^2-4n>0,\\ \frac{2}{|\mathfrak{r}[1]^\times|\sqrt{4n-t^2}}\int_{\mathbb{R}}\frac{e^{-2r\arccos(|t|/2\sqrt{|n|})}}{1+e^{-2\pi r}}h(r)\,dr & \text{if } t^2-4n<0;\end{cases}$
- $X(u) = \frac1u\sum_{m \pmod u}\log\gcd(m,u) = \sum_{m|u}\frac{\Lambda(m)}{m}$.

Writing $D = t^2-4n = d\ell^2$, we have

$$
\begin{aligned}
&\mathbf{h}^+(\mathfrak{r}[c])\left[\mathfrak{r}[1]^\times:\mathfrak{r}[c]^\times\right] A(t,n)\\
&= \frac{c}{\ell}\prod_{p|c}\left(1-\frac{\psi_d(p)}{p}\right)\begin{cases} \frac{\mathbf{h}^+(\mathfrak{r}[1])\log\epsilon_d^+}{\sqrt{d}} g\left(2\log\frac{|t|+\sqrt{D}}{2\sqrt{|n|}}\right) & \text{if } D>0,\\ \frac{2\mathbf{h}^+(\mathfrak{r}[1])}{|\mathfrak{r}[1]^\times|\sqrt{|d|}}\int_{\mathbb{R}}\frac{e^{-2r\arccos(|t|/2\sqrt{|n|})}}{1+e^{-2\pi r}}h(r)\,dr & \text{if } D<0.\end{cases}
\end{aligned}
$$

By Dirichlet's class number formula, we have

$$
L(1,\psi_d) = \begin{cases} \frac{\mathbf{h}^+(\mathfrak{r}[1])\log\epsilon_d^+}{\sqrt{d}} & \text{if } d>0,\\ \frac{2\pi\mathbf{h}^+(\mathfrak{r}[1])}{|\mathfrak{r}[1]^\times|\sqrt{|d|}} & \text{if } d<0,\end{cases}
$$

so this becomes

$$L(1,\psi_d)\frac{c}{\ell}\prod_{p\mid c}\left(1-\frac{\psi_d(p)}{p}\right)\begin{cases} g\left(2\log\frac{|t|+\sqrt{D}}{2\sqrt{|n|}}\right) & \text{if } t^2-4n>0,\\ \frac{1}{\pi}\int_{\mathbb{R}}\frac{e^{-2r\arccos(|t|/2\sqrt{|n|})}}{1+e^{-2\pi r}}h(r)\,dr & \text{if } t^2-4n<0.\end{cases}$$

Summing over c and using (1.1), we find by a short computation that

$$\sum_{c\mid\ell}L(1,\psi_d)\frac{c}{\ell}\prod_{p\mid c}\left(1-\frac{\psi_d(p)}{p}\right)=L(1,\psi_D). \tag{2.6}$$

Further, we have $g(u)=f\bigl(v-\frac{n}{v}\bigr)$, where $v=\sqrt{|n|}e^{\frac{u}{2}}$. Hence

$$g\left(2\log\frac{|t|+\sqrt{D}}{2\sqrt{|n|}}\right)=f\bigl(\sqrt{D}\bigr).$$

Next we evaluate the integral $\frac{1}{\pi}\int_{\mathbb{R}}\frac{e^{-2\arccos(|t|/2\sqrt{|n|})r}}{1+e^{-2\pi r}}h(r)\,dr$. It occurs only when $D=t^2-4n<0$, so we may assume that n is positive. Writing $\alpha=\arccos(|t|/2\sqrt{n})$, we have

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{R}}\frac{e^{-2\alpha r}}{1+e^{-2\pi r}}h(r)\,dr&=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{(\pi-2\alpha)r}}{\cosh(\pi r)}h(r)\,dr=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{(\pi-2\alpha)r}}{\cosh(\pi r)}\int_{\mathbb{R}}g(u)e^{iru}\,du\,dr\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}g(u)\int_{\mathbb{R}}\frac{e^{ir(u+i[2\alpha-\pi])r}}{\cosh(\pi r)}\,dr\,du=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{g(u)}{\cosh(\frac{u}{2}+i[\alpha-\frac{\pi}{2}])}\,du\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{g(u)}{\cosh(\frac{u}{2})\sin\alpha-i\sinh(\frac{u}{2})\cos\alpha}\,du\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}g(u)\frac{\cosh(\frac{u}{2})\sin\alpha+i\sinh(\frac{u}{2})\cos\alpha}{\sinh^2(\frac{u}{2})+\sin^2\alpha}\,du=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{g(u)\cosh(\frac{u}{2})\sin\alpha}{\sinh^2(\frac{u}{2})+\sin^2\alpha}\,du,\end{aligned}$$

where in the last line we make use of the fact that g is even. Writing $g(u)=q\bigl(\sinh^2(\frac{u}{2})\bigr)$ and making the substitution $y=2\sqrt{n}\sinh\frac{u}{2}$, this becomes simply

$$\frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}}q\left(\frac{y^2}{4n}\right)\frac{dy}{y^2+|D|}=\frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}}\frac{f(y)}{y^2+|D|}\,dy.$$

Hence, altogether we have

$$\sum_{c\mid\ell}\mathbf{h}^+(\mathfrak{r}[c])\left[\mathfrak{r}[1]^\times:\mathfrak{r}[c]^\times\right]A(t,n)=L(1,\psi_D)\begin{cases} f\bigl(\sqrt{D}\bigr) & \text{if } D>0,\\ \frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}}f(y)\frac{dy}{y^2+|D|} & \text{if } D<0.\end{cases} \tag{2.7}$$

Next, in the penultimate line of (2.5), we write $y=\sqrt{|n|}\bigl(e^{\frac{u}{2}}-(\operatorname{sgn}n)e^{-\frac{u}{2}}\bigr)$, so that $g(u)=f(y)$ and

$$\frac{1}{2}\int_{\left|\log\frac{a}{\sqrt{|n|}}\right|}^{\infty}g(u)\frac{e^{\frac{u}{2}}+\operatorname{sgn}(n)e^{-\frac{u}{2}}}{e^{\frac{u}{2}}-\operatorname{sgn}(n)e^{-\frac{u}{2}}+\left|\frac{a}{\sqrt{|n|}}-\operatorname{sgn}(n)\frac{\sqrt{|n|}}{a}\right|}\,du=\int_{\ell}^{\infty}\frac{f(y)}{\ell+y}\,dy, \tag{2.8}$$

where $\ell=|a-n/a|$, This term contributes whenever $a^2\neq n$, and those a are in one-to-one correspondence with the non-zero square values $D=\ell^2$ in (2.2). Similarly, we get a contribution of

$$\tag{2.9} \big[\log\pi+\log\ell-X(\ell)\big]g\left(\log\left|\frac{a^2}{n}\right|\right)=\left(\log\pi+\sum_{m\mid\ell}\Lambda(m)\big(1-m^{-1}\big)\right)f(\ell)$$

when $D=\ell^2\neq 0$. As for the final line of (2.5), by (2.4) and (2.3), it is

$$\sum_{0<a\mid n}\left[F(a)-g\left(\log\left|\frac{a^2}{n}\right|\right)\log\pi\right]=\sum_{0<a\mid n}\left[F(a)-f\left(a-\frac{n}{a}\right)\log\pi\right],$$

and together with (2.8) and (2.9) we get the contributions from the sum over a and the non-zero square values of D in (2.2).

Finally, the terms of (2.5) with $\sqrt{n}\in\mathbb{Z}$ correspond to $D=0$, and they clearly occur only when n is positive. For any $c>0$, we have

$$\begin{aligned} &g(0)\log\frac{\pi\sqrt{n}}{2}+\frac{h(0)}{4}-\frac{1}{2\pi}\int_{\mathbb{R}}h(r)\psi(1+ir)\,dr\\ &\quad=g(0)\log\frac{\pi e^{\gamma}\sqrt{n}}{2}+\int_0^{\infty}\log(2\sinh(u/2))g'(u)\,du\\ &\quad=g(0)\log\frac{\pi e^{\gamma}\sqrt{n}}{2}+g(0)\log\left(2\sinh\frac{c}{2}\right)+\int_0^c\frac{g(u)-g(0)}{2\tanh\frac{u}{2}}\,du+\int_c^{\infty}\frac{g(u)}{2\tanh\frac{u}{2}}\,du. \end{aligned}$$

Choosing c such that $2\sqrt{n}\sinh\frac{c}{2}=1$ and making the substitution $y=2\sqrt{n}\sinh\frac{u}{2}$, this becomes

$$\begin{aligned} f(0)\log\frac{\pi e^{\gamma}}{2}&+\int_0^1\frac{f(y)-f(0)}{y}\,dy+\int_1^{\infty}\frac{f(y)}{y}\,dy\\ &=f(0)\log\frac{\pi e^{\gamma}}{2}+\frac{1}{2}\int_0^{\infty}\frac{f(y)+f(y^{-1})-f(0)}{y}\,dy. \end{aligned}$$

Similarly, we have

$$\tag{2.10} \begin{aligned} \frac{1}{12\sqrt{n}}\int_{\mathbb{R}}r\tanh(\pi r)h(r)\,dr&=-\frac{1}{12\sqrt{n}}\int_{\mathbb{R}}\frac{g'(u)}{\sinh(u/2)}\,du=\frac{1}{12\sqrt{n}}\int_0^{\infty}\frac{g(0)-g(u)}{\sinh(u/2)\tanh(u/2)}\,du\\ &=\frac{1}{6}\int_{\mathbb{R}}\frac{f(0)-f(y)}{y^2}\,dy. \end{aligned}$$

□

Next, suppose that $N>1$. In this case it is helpful to restrict the trace formula to the newforms of level N. To be precise, if M_1,M_2 are positive integers such that $M_1M_2\mid N$ and $M_1\neq N$, then there is a linear map $L_{M_1,M_2}:\mathcal{A}(M_1,\lambda)\to\mathcal{A}(N,\lambda)$ which sends $f\in\mathcal{A}(M_1,\lambda)$ to the function $z\mapsto f(M_2z)$. Let $\mathcal{A}^{\text{new}}(N,\lambda)\subseteq\mathcal{A}(N,\lambda)$ denote the "new" subspace of forms that are orthogonal (with respect to the Petersson inner product) to the images of L_{M_1,M_2} for all M_1,M_2.

For $D\in\mathbb{Z}$, let

$$\tag{2.11} c_N^{\circ}(D)=\begin{cases}\frac{\varphi(N)}{6} & \text{if } D=0,\\ \frac{\Lambda(N)}{(\ell,N^{\infty})} & \text{if } D=\ell^2\neq 0,\\ c_N(D) & \text{otherwise,}\end{cases}$$

where c_N is as defined in (1.4). Then the trace formula for level N is as follows.

Proposition 2.2. *Let n, f and h be as in Proposition 2.1, and let $N > 1$ be a squarefree integer with $(n, N) = 1$. Then*

$$\sum_{\lambda \in \mathbb{R}_{>0}} \operatorname{Tr} T_n|_{\mathcal{A}^{\mathrm{new}}(N,\lambda)} h\left(\sqrt{\lambda - \tfrac{1}{4}}\right)$$

$$(2.12) \qquad = \sum_{\substack{t \in \mathbb{Z} \\ D = t^2 - 4n}} c_N^\circ(D) \begin{cases} f\left(\sqrt{D}\right) & \text{if } D > 0, \\ \frac{\sqrt{|D|}}{\pi} \int_{\mathbb{R}} \frac{f(y)}{y^2 + |D|}\, dy & \text{if } D < 0, \\ \int_{\mathbb{R}} \frac{f(0) - f(y)}{y^2}\, dy & \text{if } D = 0 \end{cases}$$

$$- \mu(N) \frac{\sigma_1(|n|)}{\sqrt{|n|}} h\left(\frac{i}{2}\right) - 2\Lambda(N) \sum_{\substack{a \in \mathbb{Z}_{>0} \\ a | n}} \sum_{r=0}^{\infty} N^{-r} f\left(aN^r - \frac{n}{aN^r}\right).$$

Proof. Specializing the formula in [Str16, §2.2] to trivial nebentypus character, we have the following trace formula for newforms on $\Gamma_0(N)$, with notation as in (2.5):

$$\frac{\mu(N)\sigma_1(|n|)}{\sqrt{|n|}} h\left(\frac{i}{2}\right) + \sum_{\lambda \in \mathbb{R}_{>0}} \operatorname{Tr} T_n|_{\mathcal{A}^{\mathrm{new}}(N,\lambda)} h\left(\sqrt{\lambda - \tfrac{1}{4}}\right)$$

$$(2.13) \qquad = \sum_{\substack{t \in \mathbb{Z} \\ \sqrt{t^2 - 4n} \notin \mathbb{Z}}} \left(\sum_{\substack{c | \ell \\ (c,N)=1}} \mathbf{h}^+(\mathfrak{r}[c]) \left[\mathfrak{r}[1]^\times : \mathfrak{r}[c]^\times\right] \prod_{p | N} \left[\left(\frac{d}{p}\right) - 1 \right] \right) A(t, n)$$

$$+ \begin{cases} \frac{\varphi(N)}{12\sqrt{n}} \int_{\mathbb{R}} r \tanh(\pi r) h(r)\, dr & \text{if } \sqrt{n} \in \mathbb{Z}, \\ 0 & \text{otherwise} \end{cases}$$

$$+ \Lambda(N) \sum_{\substack{a \in \mathbb{Z}_{>0} \\ a | n,\, a^2 \neq n}} \frac{g\left(\log\left|\frac{a^2}{n}\right|\right)}{(N^\infty, |a - n/a|)} - 2\Lambda(N) \sum_{\substack{a \in \mathbb{Z}_{>0} \\ a | n}} \sum_{r=0}^{\infty} N^{-r} g\left(\log\left|\frac{a^2}{n}\right| - 2r \log N\right).$$

Applying (2.6) with D replaced by $D/(N^\infty, \ell)$ and comparing to the definition (1.4), we find that

$$\sum_{\substack{c | \ell \\ (c,N)=1}} L(1, \psi_d) \frac{c}{\ell} \prod_{p | c} \left(1 - \frac{\psi_d(p)}{p}\right) \cdot \prod_{p | N} \left[\left(\frac{d}{p}\right) - 1 \right] = c_N^\circ(D).$$

Hence, following the derivation of (2.7), we get

$$\sum_{\substack{c | \ell \\ (c,N)=1}} \mathbf{h}^+(\mathfrak{r}[c]) \left[\mathfrak{r}[1]^\times : \mathfrak{r}[c]^\times\right] \prod_{p | N} \left[\left(\frac{d}{p}\right) - 1 \right] A(t, n)$$

$$= c_N^\circ(D) \begin{cases} f\left(\sqrt{D}\right) & \text{if } D > 0, \\ \frac{\sqrt{|D|}}{\pi} \int_{\mathbb{R}} \frac{f(y)}{|D| + y^2}\, dy & \text{if } D < 0. \end{cases}$$

When n is a square, the corresponding term of (2.13) is, by (2.10),

$$\frac{\varphi(N)}{6} \int_{\mathbb{R}} \frac{f(0) - f(y)}{y^2}\, dy,$$

and this matches the contribution to (2.12) from $D = 0$. Similarly, the terms of (2.12) corresponding to $D = \ell^2 \neq 0$ match the first sum on the last line of (2.13). $\square$

3. Specialization of the test function

In this section, we compute the terms of Propositions 2.1 and 2.2 explicitly for $q(v) = [4(v+1)]^{-s}$. We change notation slightly, replacing n by $\pm n$, where $n \in \mathbb{Z}_{>0}$.

Proposition 3.1. *Let $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{2}$ and $n \in \mathbb{Z}_{>0}$. Define*

$$\Phi(x,s) = \begin{cases} \frac{1}{s\mathrm{B}(s,\frac{1}{2})} & \text{if } x = 0, \\ x^{-s} I_x\big(s, \frac{1}{2}\big) & \text{if } 0 < x < 1, \\ x^{-s} & \text{if } x \geq 1 \end{cases}$$

and

$$\Psi(x,s) = \int_{\sqrt{x-1}}^{\infty} \frac{(y^2+1)^{-s}}{y + \sqrt{x-1}}\, dy,$$

where $\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ denotes the Euler Beta-function and

$$I_x(a,b) = \frac{1}{\mathrm{B}(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt$$

is the normalized incomplete Beta-function. Then

$$\begin{aligned} &\sum_{\substack{\lambda \in \mathbb{R}_{>0} \\ \lambda = \frac{1}{4}+r^2}} \operatorname{Tr} T_n|_{\mathcal{A}(1,\lambda)} \mathrm{B}(s+ir, s-ir) + \frac{\sigma_1(n)}{\sqrt{n}} \mathrm{B}\left(s - \frac{1}{2}, s + \frac{1}{2}\right) \\ &- \frac{1}{4\pi} \int_{\mathbb{R}} \mathrm{B}(s-ir, s+ir) \frac{\sigma_{2ir}(n)}{n^{ir}} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) dr - \frac{\sigma_0(n)}{4} \mathrm{B}(s,s) \\ &= 4^{-s} \sum_{\substack{t \in \mathbb{Z} \\ D = t^2 - 4n}} \begin{cases} L(1,\psi_D)\Phi\big(\frac{t^2}{4n}, s\big) & \text{if } \sqrt{D} \notin \mathbb{Z}, \\ \sum_{m|\sqrt{D}} \Lambda(m)(1-m^{-1})\Phi\big(\frac{t^2}{4n}, s\big) + \Psi\big(\frac{t^2}{4n}, s\big) & \text{if } 0 \neq \sqrt{D} \in \mathbb{Z}, \\ \frac{1}{2}\big(\psi(s) + \gamma + \log n\big) + \frac{1}{6}\sqrt{\frac{\pi}{n}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)} & \text{if } D = 0, \end{cases} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\sum_{\substack{\lambda \in \mathbb{R}_{>0} \\ \lambda = \frac{1}{4}+r^2}} \operatorname{Tr} T_{-n}|_{\mathcal{A}(1,\lambda)} \mathrm{B}(s+ir, s-ir) + \frac{\sigma_1(n)}{\sqrt{n}} \mathrm{B}\left(s - \frac{1}{2}, s + \frac{1}{2}\right) \\ &- \frac{1}{4\pi} \int_{\mathbb{R}} \mathrm{B}(s-ir, s+ir) \frac{\sigma_{2ir}(n)}{n^{ir}} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) dr - \frac{\sigma_0(n)}{4} \mathrm{B}(s,s) \\ &= \sum_{\substack{t \in \mathbb{Z} \\ D = t^2 + 4n}} \left(\frac{n}{D}\right)^s \begin{cases} L(1,\psi_D) & \text{if } \sqrt{D} \notin \mathbb{Z}, \\ \sum_{m|\sqrt{D}} \Lambda(m)\big(1-m^{-1}\big) + \frac{1}{2}\big(\psi(s+\frac{1}{2}) - \psi(s)\big) & \text{if } \sqrt{D} \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.2}$$

Both (3.1) and (3.2) continue to meromorphic functions on $\mathbb{C}$ and are holomorphic for $\Re(s) > 0$, apart from simple poles of residue $\frac{\sigma_1(n)}{\sqrt{n}}$ at $s = \frac{1}{2}$.

Proof. With $q(v) = \big[4(1+v)\big]^{-s}$ we have

$$\begin{aligned}
h(r) &= \int_{\mathbb{R}} q\big(\sinh^2\big(\tfrac{u}{2}\big)\big)e^{iru}\,du = \int_{\mathbb{R}} \big(2\cosh\tfrac{u}{2}\big)^{-2s}e^{iru}\,du = \int_{\mathbb{R}} e^{u(s+ir)}\big(e^u+1\big)^{-2s}\,du \\
&= \int_0^\infty x^{s+ir}(x+1)^{-2s}\,\frac{dx}{x} = \mathrm{B}(s+ir, s-ir),
\end{aligned} \tag{3.3}$$

by [GR07, 3.194(3)]. By Stirling's formula, for any compact set $K \subset \mathbb{C}$ that omits all poles of $\mathrm{B}(s+ir, s-ir)$, the estimate $|h(r)| = |\mathrm{B}(s+ir, s-ir)| \ll e^{-\pi|r|}$ holds uniformly for $s \in K$. Hence this is a suitable choice of test function for any fixed s with $\Re(s) > \frac{1}{2}$. Further, when combined with the Weyl-type estimate

$$\sum_{\substack{\lambda=\frac{1}{4}+r^2\in\mathbb{R}_{>0}\\ |r|\le T}} \big|\operatorname{Tr} T_{\pm n}|_{\mathcal{A}(1,\lambda)}\big| \ll_n T^2,$$

we see that the sums on the left-hand sides of (3.1) and (3.2) continue to meromorphic functions of $s \in \mathbb{C}$. By (2.3), (3.1) and (3.2) are

$$\sum_{\lambda\in\mathbb{R}_{\ge 0}} \operatorname{Tr} T_{\pm n}|_{\mathcal{A}(1,\lambda)} h\Big(\sqrt{\lambda - \tfrac{1}{4}}\Big) - \sum_{a|n} F(a),$$

and it remains to evaluate $\sum_{t\in\mathbb{Z}} W(t^2 \mp 4n)$.

Let us first consider (3.2). Then $f(y) = n^s|y|^{-2s}$, and we have $D = t^2 + 4n > 0$. Making the substitution $y = \sqrt{D/x}$, we get

$$\int_{\sqrt{D}}^\infty \frac{y^{-2s}}{y+\sqrt{D}}\,dy = \frac{D^{-s}}{2}\int_0^1 \frac{x^{s-1}}{1+\sqrt{x}}\,dx = \frac{D^{-s}}{2}\int_0^1 \frac{x^{s-1}-x^{s-\frac{1}{2}}}{1-x}\,dx = \frac{D^{-s}}{2}\big(\psi(s+\tfrac{1}{2}) - \psi(s)\big),$$

by [GR07, 8.361(4)]. This yields the right-hand side of (3.2).

Next we consider (3.1), in which case $f(y) = n^s(y^2+4n)^{-s}$ and we have $D = t^2 - 4n$. For $D < 0$,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{f(y)}{y^2+|D|}\,dy &= n^s 2\int_0^\infty \frac{(y^2+4n)^{-s}}{y^2+|D|}\,dy = n^s\int_0^\infty (y+4n)^{-s}(y+|D|)^{-1}y^{-\frac{1}{2}}\,dy \\
&= n^s(4n)^{-s}|D|^{-\frac{1}{2}}\mathrm{B}\left(\frac{1}{2}, s+\frac{1}{2}\right){}_2F_1\left(s, \frac{1}{2}; s+1; 1-\frac{|D|}{4n}\right) \\
&= n^s(4n)^{-s}|D|^{-\frac{1}{2}}\mathrm{B}\left(\frac{1}{2}, s+\frac{1}{2}\right) s\mathrm{B}_{\frac{t^2}{4n}}\left(s, \frac{1}{2}\right)\left(\frac{t^2}{4n}\right)^{-s},
\end{aligned}$$

by [GR07, 3.197(1)]. Since

$$s\mathrm{B}\left(\frac{1}{2}, s+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} = \frac{\pi\Gamma(s+\frac{1}{2})}{\Gamma(s)\Gamma(\frac{1}{2})} = \frac{\pi}{\mathrm{B}(s,\frac{1}{2})},$$

we obtain

$$\frac{\sqrt{|D}}{\pi}\int_{\mathbb{R}} \frac{f(y)}{y^2+|D|}\,dy = 4^{-s}I_{\frac{t^2}{4n}}\left(s, \frac{1}{2}\right)\left(\frac{t^2}{4n}\right)^{-s}. \tag{3.4}$$

For a sufficiently small $\varepsilon > 0$, we have

$$f(y) = n^s(y^2+4n)^{-s} = y^{-2s}n^s \frac{1}{2\pi i}\int_{\Re(u)=-\varepsilon} \mathrm{B}(-u, s+u) y^{-2u}(4n)^u \, du, \tag{3.5}$$

by [GR07, 6.422(3)]. Hence, for $0 \neq \sqrt{D} \in \mathbb{Z}$,

$$\int_{\sqrt{D}}^{\infty} \frac{f(y)}{y+\sqrt{D}}\, dy = n^s \frac{1}{2\pi i}\int_{\Re(u)=-\varepsilon} \mathrm{B}(-u, s+u) \int_{\sqrt{D}}^{\infty} \frac{y^{-2u-2s}}{y+\sqrt{D}}\, dy\, (4n)^u \, du$$
$$= 2^{-2s-1}\frac{1}{2\pi i}\int_{\Re(u)=-\varepsilon} \mathrm{B}(-u, s+u)\left(\frac{4n}{D}\right)^{s+u}\left(\psi\left(u+s+\frac{1}{2}\right) - \psi(u+s)\right) du.$$

For $D = 0$,

$$\int_0^{\infty} \frac{f(0)-f(y)}{y^2}\, dy = n^s \int_0^{\infty} \frac{(4n)^{-s} - (y^2+4n)^{-s}}{y^2}\, dy = \frac{4^{-s}}{\sqrt{4n}} \int_0^{\infty} \frac{1-(y^2+1)^{-s}}{y^2}\, dy$$
$$= \frac{4^{-s}}{\sqrt{4n}}\left\{\left[-y^{-1}\left(1-(y^2+1)^{-s}\right)\right]_0^{\infty} + 2s\int_0^{\infty}(y^2+1)^{-s-1}\, dy\right\}.$$

By [GR07, 3.251(2)],

$$\int_0^{\infty} (y^2+1)^{-s-1}\, dy = \frac{1}{2}\mathrm{B}\left(\frac{1}{2}, s+\frac{1}{2}\right),$$

so that

$$\int_0^{\infty} \frac{f(0)-f(y)}{y^2}\, dy = \frac{4^{-s}}{\sqrt{4n}} s\mathrm{B}\left(\frac{1}{2}, s+\frac{1}{2}\right) = \frac{4^{-s}\sqrt{\pi}}{\sqrt{4n}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}. \tag{3.6}$$

Next, we have

$$\int_0^{\infty} \frac{f(y)+f(y^{-1})-f(0)}{y}\, dy = 2\int_0^1 \frac{f(y)+f(y^{-1})-f(0)}{y}\, dy$$
$$= 2\int_0^1\int_0^1 f'(ty)\, dt\, dy + 2\int_0^1 \frac{f(y^{-1})}{y}\, dy.$$

By (3.5), we have

$$f'(ty) = n^s\frac{1}{2\pi i}\int_{\Re(u)=-\varepsilon} \mathrm{B}(-u, s+u)(-2s-2u)(ty)^{-2s-2u-1}(4n)^u\, du$$
$$= n^s\frac{1}{2\pi i}\int_{\Re(u)=-\varepsilon-\Re(s)} \mathrm{B}(-u, s+u)(-2s-2u)(ty)^{-2s-2u-1}(4n)^u\, du,$$

upon moving the line of integration to $\Re(u) = -\varepsilon - \Re(s)$, so that $-\Re(u) - \Re(s) > 0$. Therefore,

$$2\int_0^1\int_0^1 f'(ty)\, dt\, dy = n^s(4n)^{-s}\frac{1}{2\pi i}\int_{\Re(u)=-\varepsilon-\Re(s)} \frac{\mathrm{B}(-u, s+u)}{-u-s}(4n)^{u+s}\, du$$
$$= 4^{-s}\big(\psi(s) - \gamma + \log(4n)\big) - n^s(4n)^{-s}\int_0^{4n}\frac{1}{2\pi i}\int_{\Re(u)=-\varepsilon} \mathrm{B}(-u, s+u)t^{u+s-1}\, du\, dt$$
$$= 4^{-s}\big(\psi(s) - \gamma + \log(4n)\big) - 4^{-s}\int_0^{4n}(t+1)^{-s}t^{s-1}\, dt,$$

by [GR07, 6.422(3)], for $\Re(s) > 0$. On the other hand,

$$2\int_0^1 \frac{f(y^{-1})}{y}\,dy = 2n^s\int_0^1 (1+4ny^2)^{-s}y^{2s-1}\,dy = 4^{-s}\int_0^{4n}(t+1)^{-s}t^{s-1}\,dt,$$

and thus

$$\int_0^\infty \frac{f(y)+f(y^{-1})-f(0)}{y}\,dy = 4^{-s}\big(\psi(s)-\gamma+\log(4n)\big).$$

It remains only to prove that the integral

$$F(s) = \frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{B}(s-ir,s+ir)\frac{\sigma_{2ir}(n)}{n^{ir}}\frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)dr$$

has meromorphic continuation to $s\in\mathbb{C}$. Clearly $F(s)$ is analytic for $\Re(s)>0$. To get meromorphic continuation to $\Re(s)\le 0$, we put $u=ir$ and then deform the contour around $u=0$:

$$F(s) = \frac{1}{4\pi i}\int_{C_0} \mathrm{B}(s-u,s+u)\frac{\sigma_{2u}(n)}{n^u}\left(-\frac{\zeta^{*\prime}}{\zeta^*}(2u)+\frac{\zeta^{*\prime}}{\zeta^*}(1+2u)\right)du,$$

where $C_0 = \{u=it : |t|\ge \frac{1}{2}\}\cup\{u\in\mathbb{C} : |u|=\frac{1}{2},\ \Re(u)\ge 0\}$. Now we replace u by $-u$ in the half of the integral containing $\frac{\zeta^{*\prime}}{\zeta^*}(2u)$ and move the contour back to C_0:

$$F(s) = -\frac{1}{4}\mathrm{B}(s,s)\sigma_0(n) + \frac{1}{2\pi i}\int_{C_0}\mathrm{B}(s-u,s+u)\frac{\sigma_{2u}(n)}{n^u}\frac{\zeta^{*\prime}}{\zeta^*}(1+2u)\,du.$$

Now let $M\in\mathbb{Z}_{\ge 0}$, replace u by $u+s$, and shift the contour to $\Re(u)=M-\frac{1}{2}$. Then we have

$$\begin{aligned}F(s) = &\sum_{m=0}^{M-1}\frac{(-1)^m}{m!}\frac{\Gamma(2s+m)}{\Gamma(2s)}\frac{\sigma_{2s+2m}(n)}{n^{s+m}}\frac{\zeta^{*\prime}}{\zeta^*}(1+2s+2m)\\ &-\frac{1}{4}B(s,s)\sigma_0(n) + \frac{1}{2\pi i}\int_{\Re(u)=M-\frac{1}{2}}\mathrm{B}(-u,2s+u)\frac{\sigma_{2s+2u}(n)}{n^{s+u}}\frac{\zeta^{*\prime}}{\zeta^*}(1+2s+2u)\,du,\end{aligned}$$

and this last line continues meromorphically to $\Re(s) > \frac{1}{4}-\frac{M}{2}$. Taking M arbitarily large, we conclude the meromorphic continuation of $F(s)$ to $\mathbb{C}$.

□

Proposition 3.2. *Let $N>1$ be a squarefree integer, $n\in\mathbb{Z}_{>0}$ with $(n,N)=1$ and $s\in\mathbb{C}$ with $\Re(s)>\frac{1}{2}$. Then*

$$\begin{aligned}(3.7)\quad &\sum_{\substack{\lambda\in\mathbb{R}_{>0}\\ \lambda=\frac{1}{4}+r^2}}\operatorname{Tr} T_n|_{\mathcal{A}^{\mathrm{new}}(N,\lambda)}\mathrm{B}(s+ir,s-ir)+\mu(N)\frac{\sigma_1(n)}{\sqrt{n}}\mathrm{B}\left(s-\frac{1}{2},s+\frac{1}{2}\right)\\ &+2\Lambda(N)\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{B}(s-ir,s+ir)\sigma_{-2ir}(n)n^{ir}\left(1-N^{-1-2ir}\right)^{-1}dr\\ &= 4^{-s}\sum_{\substack{t\in\mathbb{Z}\\ D=t^2-4n}}c_N^\circ(D)\begin{cases}\Phi\left(\frac{t^2}{4n},s\right) & \text{if } D\ne 0,\\ \frac{1}{2}\sqrt{\frac{\pi}{n}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)} & \text{if } D=0\end{cases}\end{aligned}$$

and

$$\text{(3.8)}\qquad \sum_{\substack{\lambda\in\mathbb{R}_{>0}\\ \lambda=\frac{1}{4}+r^2}} \operatorname{Tr} T_{-n}|_{\mathcal{A}^{\mathrm{new}}(N,\lambda)} \mathrm{B}(s+ir,s-ir) + \mu(N)\frac{\sigma_1(n)}{\sqrt{n}}\mathrm{B}\left(s-\frac{1}{2},s+\frac{1}{2}\right)$$

$$+ 2\Lambda(N)\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{B}(s-ir,s+ir)\sigma_{-2ir}(n)n^{ir}\left(1-N^{-1-2ir}\right)^{-1} dr$$

$$= n^s \sum_{\substack{t\in\mathbb{Z}\\ D=t^2+4n}} \frac{c_N^\circ(D)}{D^s},$$

where c_N° *is as defined in* (2.11)*. Both* (3.7) *and* (3.8) *continue to meromorphic functions on* $\mathbb{C}$ *and are holomorphic for* $\Re(s) > \frac{7}{64}$*, apart from simple poles of residue* $\mu(N)\frac{\sigma_1(n)}{\sqrt{n}}$ *at* $s=\frac{1}{2}$.

Proof. With $q(v) = [4(1+v)]^{-s}$, as in the proof of Proposition 3.1, we have $h(r) = \mathrm{B}(s+ir,s-ir)$. Hence, by (2.12), the left-hand sides of (3.8) and (3.7) are

$$\sum_{\lambda\in\mathbb{R}_{>0}} \operatorname{Tr} T_{\pm n}|_{\mathcal{A}^{\mathrm{new}}(N,\lambda)} h\left(\sqrt{\lambda-\tfrac{1}{4}}\right) = \sum_{\substack{\lambda\in\mathbb{R}_{>0}\\ \lambda=\frac{1}{4}+r^2}} \operatorname{Tr} T_n|_{\mathcal{A}^{\mathrm{new}}(N,\lambda)} \mathrm{B}(s+ir,s-ir).$$

Let us first consider (3.8). Then $f(y) = n^s|y|^{-2s}$, and $D = t^2+4n > 0$, so that

$$f(\sqrt{D}) = n^s D^{-s}.$$

Now we consider (3.7), in which case $f(y) = n^2(y^2+4n)^{-s}$ and $D = t^2-4n$. For $D=0$,

$$\int_{\mathbb{R}} \frac{f(0)-f(y)}{y^2}\,dy = \frac{4^{-s}\sqrt{\pi}}{\sqrt{4n}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)},$$

by (3.6). For $D<0$,

$$\frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}} \frac{f(y)}{|D|+y^2}\,dy = 4^{-s} I_{\frac{t^2}{4n}}\left(s,\frac{1}{2}\right)\left(\frac{t^2}{4n}\right)^{-s},$$

by (3.4).

For $D = t^2 \pm 4n$ with $n \geq 1$,

$$f\left(aN^r - \frac{n}{aN^r}\right) = n^s\left(aN^r + \frac{n}{aN^r}\right)^{-2s}.$$

By [GR07, 6.422(1)], for $1-\Re(s) < \sigma < \Re(s)$, we have

$$n^s\left(aN^r + \frac{n}{aN^r}\right)^{-2s} = \frac{1}{2\pi i}\int_{\Re(u)=\sigma} \mathrm{B}(s-u,s+u)\left(\frac{n}{a^2N^{2r}}\right)^u du,$$

so that

$$\sum_{\substack{a\in\mathbb{Z}_{>0}\\ a|n}} \sum_{r=0}^{\infty} N^{-r} f\left(aN^r - \frac{n}{aN^r}\right) = \frac{1}{2\pi i}\int_{\Re(u)=\sigma} \mathrm{B}(s-u,s+u)\sigma_{-2u}(n)n^u\left(1-N^{-1-2u}\right)^{-1} du.$$

This integral and the sum over λ have meromorphic continuation to $s\in\mathbb{C}$, by similar arguments to those of Proposition 3.1. Finally, the fact that the sum over λ is holomorphic for $\Re(s) > \frac{7}{64}$ follows from the best known bound towards the Selberg eigenvalue conjecture, due to Kim and Sarnak [Kim03]. $\square$

4. Proofs of the main results

4.1. **Proof of Theorem 1.1.** By (3.2) in Proposition 3.1, the series given in (1) can be written as

$$\sum_{\substack{t\in\mathbb{Z}\\ \sqrt{t^2+4n}\notin\mathbb{Z}}} \frac{L(1,\psi_{t^2+4n})}{(t^2+4n)^s} = n^{-s}\left(F_1(s)+F_2(s)+F_3(s)\right),$$

where

$$F_1(s) = \sum_{\substack{\lambda\in\mathbb{R}_{>0}\\ \lambda=\frac14+r^2}} \operatorname{Tr} T_{-n}|_{\mathcal{A}(1,\lambda)} \mathrm{B}(s+ir, s-ir) + \frac{\sigma_1(n)}{\sqrt{n}}\mathrm{B}\left(s-\frac12, s+\frac12\right),$$

$$F_2(s) = -\frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{B}(s-ir,s+ir)\frac{\sigma_{2ir}(n)}{n^{ir}}\frac{\phi'}{\phi}\left(\frac12+ir\right)dr - \frac{\sigma_0(n)}{4}\mathrm{B}(s,s)$$

and

$$F_3(s) = -\sum_{\substack{t\in\mathbb{Z}\\ t^2+4n=\ell^2,\ell\in\mathbb{Z}_{>0}}} \frac{\sum_{m|\ell}\Lambda(m)(1-m^{-1}) + \frac12\left(\psi\left(s+\frac12\right)-\psi(s)\right)}{\ell^{2s}}.$$

The meromorphic continuation of $F_1(s)$ and $F_2(s)$ was shown in the proof of Proposition 3.1. In particular, $F_1(s)$ has simple poles at $s=-m+\frac12$ and $s=-m\pm ir$ for $m\in\mathbb{Z}_{\geq 0}$, while $F_2(s)$ has simple poles at $s=-m+\frac{\rho-1}{2}$ for $m\in\mathbb{Z}_{\geq0}$ and ρ a zero or pole of $\zeta^*(s)$. Finally, the series $F_3(s)$ has finitely many terms and is entire apart from simple poles for $s\in-\frac12\mathbb{Z}_{\geq0}$. Moreover, one can check that the poles of $F_2(s)$ and $F_3(s)$ at $s=0$ cancel out, so the only poles of (1.2) for $\Re(s)\geq 0$ are at $s=\frac12$ and $s=\pm ir$, with residues $\sigma_{-1}(n)$ and $n^{\mp ir}\operatorname{Tr} T_{-n}|_{\mathcal{A}(1,\frac14+r^2)}$, respectively. This proves (1).

Let N be a prime and n be a positive integer such that $\left(\frac{-4n}{N}\right)=-1$. Then $N\nmid t^2+4n = d\ell^2 = D$ and

$$c_N^\circ(D) = (\psi_D(N)-1)\,L(1,\psi_D) = \left(\left(\frac{t^2+4n}{N}\right)-1\right)L(1,\psi_{t^2+4n}).$$

Combining (3.8) in Proposition 3.2 and (3.2) in Proposition 3.1, we see that the series

$$\sum_{\substack{t\in\mathbb{Z}\\ \sqrt{t^2+4n}\notin\mathbb{Z}}} \frac{c_N^\circ(t^2+4n)}{(t^2+4n)^s} + \sum_{\substack{t\in\mathbb{Z}\\ \sqrt{t^2+4n}\notin\mathbb{Z}}} \frac{L(1,\psi_{t^2+4n})}{(t^2+4n)^s} = \sum_{\substack{t\in\mathbb{Z}\\ \sqrt{t^2+4n}\notin\mathbb{Z}}} \frac{L(1,\psi_{t^2+4n})\left(\frac{t^2+4n}{N}\right)}{(t^2+4n)^s}$$

has meromorphic continuation to $\mathbb{C}$. This proves (2).

Finally, we turn to (3). If $N=2$ then there are no primes n satisfying $\left(\frac{-4n}{N}\right)=-1$. However, the Selberg eigenvalue conjecture is true for $\Gamma_0(2)$ [Hux85], so (3) is vacuously true in this case. Henceforth we will assume that $N>2$.

If the Selberg conjecture holds for $\Gamma_0(N)$ then, since it also holds for $\Gamma_0(1)$, the first terms of both (3.2) and (3.8) are holomorphic for $\Re(s)>0$, and the second terms cancel. In the other direction, let $\{f_j\}_{j=1}^\infty$ be a complete sequence of arithmetically normalized Hecke–Maass newforms on $\Gamma_0(N)\backslash\mathbb{H}$, with parities $\epsilon_j\in\{0,1\}$, Laplace eigenvalues $\frac14+r_j^2$ and Hecke eigenvalues $\lambda_j(n)$. We need the following lemma.

Lemma 4.1. *Let J be a finite set of positive integers and let $c_j \in \mathbb{C}^\times$ be given for each $j \in J$. Then there is a prime number n such that $\left(\frac{-4n}{N}\right) = -1$ and*

$$\sum_{j\in J} c_j\lambda_j(n) \neq 0.$$

Proof. The main tool is the Rankin–Selberg method, from which it follows that if f and g are normalized Hecke–Maass newforms (of possibly different levels) with Fourier coefficients $\lambda_f(n)$ and $\lambda_g(n)$, respectively, then

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} \lambda_f(p)\overline{\lambda_g(p)} = \begin{cases} 1 & \text{if } f = g, \\ 0 & \text{if } f \neq g, \end{cases} \tag{4.1}$$

where the sum runs through all prime numbers $p \leq x$.

For any prime p, put $S_p = \sum_{j\in J} c_j\lambda_j(p)$. For large $x \geq N$, we consider the sum

$$\sum_{\substack{p\leq x\\ \left(\frac{-4p}{N}\right)=-1}} |S_p|^2 = \frac{1}{2}\sum_{p\leq x} |S_p|^2 - \frac{1}{2}\left(\frac{-4}{N}\right)\sum_{p\leq x}\left(\frac{p}{N}\right)|S_p|^2 - \frac{1}{2}|S_N|^2.$$

Opening up the first sum, we have

$$\sum_{p\leq x} |S_p|^2 = \sum_{j,k\in J} c_j\overline{c_k}\sum_{p\leq x} \lambda_j(p)\overline{\lambda_k(p)}.$$

By (4.1), the inner sum is $o(\pi(x))$ if $j \neq k$ and $(1 + o(1))\pi(x)$ otherwise; thus, in total, the first sum is $\left(\sum_{j\in J} |c_j|^2 + o(1)\right)\pi(x)$.

Expanding the second sum in the same way, we obtain

$$\sum_{j,k\in J} c_j\overline{c_k}\sum_{p\leq x}\left(\frac{p}{N}\right)\lambda_j(p)\overline{\lambda_k(p)}.$$

Since f_j is a newform of prime level N, $\left(\frac{p}{N}\right)\lambda_j(p)$ is the pth Fourier coefficient of a newform of level N^2 (which is therefore distinct from f_k for every k). Thus, (4.1) implies that the second sum is $o(\pi(x))$.

Finally $\frac{1}{2}|S_N|^2$ is bounded independent of x. Putting these together, we have

$$\sum_{\substack{p\leq x\\ \left(\frac{-4p}{N}\right)=-1}} |S_p|^2 = \left(\frac{1}{2}\sum_{j\in J}|c_j|^2 + o(1)\right)\pi(x) \quad \text{as } x \to \infty.$$

Since $c_j \neq 0$, the last line is positive for large enough x, and thus there exists p satisfying $\left(\frac{-4p}{N}\right) = -1$ and $S_p \neq 0$, as required. □

To conclude the proof, suppose that the Selberg conjecture is false for $\Gamma_0(N)$. Let $\lambda_{\min} \in (0, \frac{1}{4})$ be the smallest non-zero eigenvalue, and put $J = \{j \in \mathbb{Z}_{>0} : \frac{1}{4}+r_j^2 = \lambda_{\min}\}$, $c_j = (-1)^{\epsilon_j}$. Then Lemma 4.1 implies that there exists a prime n such that $\left(\frac{-4n}{N}\right) = -1$ and

$$0 \neq \sum_{j\in J}(-1)^{\epsilon_j}\lambda_j(n) = \operatorname{Tr} T_{-n}|_{\mathcal{A}(N,\lambda_{\min})}.$$

Thus, for this choice of n, (1.3) has a pole at $s = \sqrt{\frac{1}{4} - \lambda_{\min}} > 0$. □

Remark. Using effective estimates in the Rankin–Selberg method, one could give an upper bound for the n produced by Lemma 4.1, so part (3) of Theorem 1.1 could be strengthened to an equivalence with finitely many series. Alternatively, using known results from functoriality would enable equivalences with thinner infinite sequences of n; for instance, if $N \equiv 3 \pmod 4$, then using functoriality of the kth symmetric powers for $k \le 4$, we may take n to be the fourth power of a prime.

4.2. **Proof of Theorem 1.2.** Define $E_N(s) = \prod_{p|N}(1-p^{-s})$. We multiply (3.8) by $n^{-s} \cdot \Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)/E_N(2s+1)$ and sum over square values of n co-prime to N. The result can be expressed in the form $L_1 + L_2 + L_3 = R$, where

$$L_1 = \mu(N)\frac{\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)}{E_N(2s+1)}\frac{\Gamma_{\mathbb{R}}(2s-1)\Gamma_{\mathbb{R}}(2s+1)}{\Gamma_{\mathbb{R}}(4s)}\sum_{\substack{n\in\mathbb{Z}_{>0}\\(n,N)=1}}\frac{\sigma_1(n^2)}{n^{2s+1}},$$

$$L_2 = \frac{\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)}{E_N(2s+1)}\sum_{j=1}^{\infty}(-1)^{\epsilon_j}\frac{\Gamma_{\mathbb{R}}(2s-2ir_j)\Gamma_{\mathbb{R}}(2s+2ir_j)}{\Gamma_{\mathbb{R}}(4s)}\sum_{\substack{n\in\mathbb{Z}_{>0}\\(n,N)=1}}\frac{\lambda_j(n^2)}{n^{2s}},$$

$$L_3 = \frac{\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)}{E_N(2s+1)}2\Lambda(N)\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{B}(s-ir,s+ir)\sum_{\substack{n\in\mathbb{Z}_{>0}\\(n,N)=1}}\frac{\sigma_{-2ir}(n^2)(n^2)^{ir}}{n^{2s}}\left(1-N^{-1-2ir}\right)^{-1}dr,$$

and

$$R = \frac{\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)}{E_N(2s+1)}\sum_{\substack{t\in\mathbb{Z},n\in\mathbb{Z}_{>0}\\(n,N)=1}}\frac{c_N^{\circ}(t^2+4n^2)}{(t^2+4n^2)^s}.$$

By Atkin–Lehner theory, for any $p \mid N$, we have $\lambda_j(p^{2k}) = p^{-k}$. Thus,

$$\sum_{\substack{n\in\mathbb{Z}_{>0}\\(n,N)=1}}\frac{\lambda_j(n^2)}{n^{2s}} = E_N(2s+1)\sum_{n=1}^{\infty}\frac{\lambda_j(n^2)}{n^{2s}},$$

so that

$$L_2 = \sum_{j=1}^{\infty}(-1)^{\epsilon_j}L^*(2s,\mathrm{Sym}^2 f_j).$$

Similarly,

$$\sum_{\substack{n\in\mathbb{Z}_{>0}\\(n,N)=1}}\frac{\sigma_1(n^2)}{n^{2s+1}} = \frac{E_N(2s)E_N(2s+1)E_N(2s-1)}{E_N(4s)}\frac{\zeta(2s)\zeta(2s+1)\zeta(2s-1)}{\zeta(4s)},$$

so that

$$\begin{aligned}L_1 &= \mu(N)N^{2s}E_N(2s)E_N(2s-1)\zeta^*(2s)\zeta^*(2s+1)\zeta^*(2s-1)\\ &= \sqrt{N}\zeta^*(2s)\zeta_N^*(-2s)\zeta_N^*(2s-1).\end{aligned}$$

Turning to the right-hand side, we define

$$r^{(M)}(D) = \tfrac{1}{2}\#\big\{(x,y)\in\mathbb{Z}^2 : D = x^2+4y^2,\ (y,M)=1\big\},$$

for $M \in \mathbb{Z}_{>0}$, so that

$$R = \frac{\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)}{E_N(2s+1)} \sum_{D=1}^{\infty} \frac{c_N^\circ(D) r^{(N)}(D)}{D^s}.$$

Note that $c_N^\circ(m^2 D) = m^{-1} c_N^\circ(D)$ for any $m \mid N^\infty$ and $D \in \mathcal{D}$. Hence, expanding the factor of $E_N(2s+1)^{-1}$ and writing $r^{(N)}(x) = 0$ if x is not a positive integer, we get

$$\begin{aligned}
R &= \Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s) \sum_{m \mid N^\infty} m^{-2s-1} \sum_{D=1}^{\infty} \frac{c_N^\circ(D) r^{(N)}(D)}{D^s} \\
&= \Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s) \sum_{m \mid N^\infty} \sum_{D=1}^{\infty} \frac{c_N^\circ(m^2 D) r^{(N)}(D)}{(m^2 D)^s} \\
&= \Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s) \sum_{m \mid N^\infty} \sum_{D=1}^{\infty} \frac{c_N^\circ(D) r^{(N)}(D m^{-2})}{D^s} \\
&= \Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s) \sum_{D=1}^{\infty} \frac{c_N^\circ(D)}{D^s} \sum_{m \mid N^\infty} r^{(N)}(D m^{-2}).
\end{aligned} \tag{4.2}$$

The inner sum is evaluated by the following three lemmas.

Lemma 4.2. *For positive integers M, ℓ,*

$$r^{(M)}(\ell^2) = \#\big\{(a,n) \in \mathbb{Z}_{>0}^2 : a \mid n^2,\ (n,M) = 1,\ a + \tfrac{n^2}{a} = \ell\big\} + \begin{cases} 1 & \text{if } M = 1, \\ 0 & \text{if } M > 1. \end{cases}$$

Proof. Put $\delta_M = 1$ if $M = 1$ and $\delta_M = 0$ otherwise. Then we have

$$\begin{aligned}
r^{(M)}(\ell^2) &= \tfrac{1}{2}\#\big\{(x,y) \in \mathbb{Z}^2 : \ell^2 = x^2 + 4y^2,\ (y,M) = 1\big\} \\
&= \delta_M + \#\{(x,n) \in \mathbb{Z} \times \mathbb{Z}_{>0} : \ell^2 = x^2 + 4n^2,\ (n,M) = 1\}.
\end{aligned}$$

Note that if $\ell^2 = x^2 + 4n^2$ with $n > 0$ then $x \equiv \ell \pmod 2$ and $|x| < \ell$, so this becomes

$$\begin{aligned}
&\delta_M + \#\big\{(x,n) \in \mathbb{Z} \times \mathbb{Z}_{>0} : \tfrac{\ell+x}{2} \cdot \tfrac{\ell-x}{2} = n^2,\ (n,M) = 1,\ x \equiv \ell \pmod 2, |x| < \ell\big\} \\
&= \delta_M + \#\big\{(a,n) \in \mathbb{Z}_{>0}^2 : a(\ell - a) = n^2,\ (n,M) = 1\big\} \\
&= \delta_M + \#\big\{(a,n) \in \mathbb{Z}_{>0}^2 : a \mid n^2,\ (n,M) = 1,\ a + \tfrac{n^2}{a} = \ell\big\}.
\end{aligned}$$

□

Lemma 4.3. *Let M, D be positive integers with M squarefree, and let p be a prime divisor of M. Then*

$$\sum_{k=0}^{\infty} r^{(M)}(D p^{-2k}) = r^{(M/p)}(D) - \tfrac{1}{2}\#\big\{(x,y) \in \mathbb{Z}^2 : D p^{-2\lfloor \mathrm{ord}_p(D)/2 \rfloor} = x^2 + 4y^2,\ p \mid y\big\}.$$

In particular, if $\psi_d(p) \neq 1$, where d denotes the discriminant of $\mathbb{Q}(\sqrt{D})$, then

$$\sum_{k=0}^{\infty} r^{(M)}(D p^{-2k}) = r^{(M/p)}(D).$$

Proof. If $p^2 \mid n$ then

$$\begin{aligned} r^{(M)}(n) &= \tfrac{1}{2}\#\big\{(x,y)\in\mathbb{Z}^2 : n = x^2+4y^2,\ (y,M)=1\big\} \\ &= r^{(M/p)}(n) - \tfrac{1}{2}\#\big\{(x,y)\in\mathbb{Z}^2 : n = x^2+4y^2,\ (y,M/p)=1,\ p\mid y\big\} \\ &= r^{(M/p)}(n) - r^{(M/p)}(np^{-2}). \end{aligned}$$

Now put $e = \lfloor \operatorname{ord}_p(D)/2\rfloor$. Then for $k<e$ we may apply the above with $n = Dp^{-2k}$:

$$r^{(M)}(Dp^{-2k}) = r^{(M/p)}(Dp^{-2k}) - r^{(M/p)}(Dp^{-2(k+1)}).$$

Hence we have the telescoping sum

$$\sum_{k=0}^{\infty} r^{(M)}(Dp^{-2k}) = \sum_{k=0}^{e} r^{(M)}(Dp^{-2k}) = r^{(M/p)}(D) - r^{(M/p)}(Dp^{-2e}) + r^{(M)}(Dp^{-2e}).$$

The first conclusion follows on noting that

$$r^{(M/p)}(Dp^{-2e}) - r^{(M)}(Dp^{-2e}) = \tfrac{1}{2}\#\big\{(x,y)\in\mathbb{Z}^2 : Dp^{-2e} = x^2+4y^2,\ p\mid y\big\}.$$

As for the second, if $\operatorname{ord}_p(D)$ is odd then $\operatorname{ord}_p(Dp^{-2e}) = 1$. On the other hand, if $p \mid y$ then x^2+4y^2 is either invertible mod p or divisible by p^2. Hence

$$\tfrac{1}{2}\#\big\{(x,y)\in\mathbb{Z}^2 : Dp^{-2e} = x^2+4y^2,\ p\mid y\big\} = 0$$

in this case.

Suppose now that $\operatorname{ord}_p(D)$ is even and $\psi_d(p)\neq 1$. If p is odd then it follows that $\left(\frac{Dp^{-2e}}{p}\right) = -1$, so again $Dp^{-2e} = x^2+4y^2$ is not solvable with $p \mid y$. For $p=2$, we distinguish between even and odd values of d. If d is even then we have $Dp^{-2e} = \frac{1}{4}dm^2$, where m is odd; hence, the equation $Dp^{-2e} = x^2+4y^2$ is not solvable, since $\frac{1}{4}dm^2 \equiv 2$ or $3 \pmod 4$, while $x^2+4y^2\equiv 0$ or $1 \pmod 4$. If d is odd then $d\equiv 5 \pmod 8$ since $\psi_d(p)\neq 1$, so $Dp^{-2e} = dm^2 \equiv 5 \pmod 8$. Again we find that $Dp^{-2e} = x^2+4y^2$ is not solvable with y even. □

Corollary 4.4. *For any positive integer D,*

$$c_N^{\circ}(D)\sum_{m\mid N^\infty} r^{(N)}(Dm^{-2}) = c_N^{\circ}(D)r(D) - \delta_{N,D},$$

where

$$\delta_{N,D} = \begin{cases} \frac{\Lambda(N)}{(\ell,N^\infty)}\big(1+\#\{(a,n)\in\mathbb{Z}^2_{>0} : a\mid n^2,\ N\mid n,\ a+\frac{n^2}{a} = \frac{\ell}{(\ell,N^\infty)}\}\big) & \textit{if } D=\ell^2, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $D\equiv 2$ or $3 \pmod 4$ then both sides vanish. If $D = d\ell^2\in\mathcal{D}$ with $d\neq 1$ then $c_N^{\circ}(D) = 0$ unless $\psi_d(p)\neq 1$ for every $p\mid N$; in that case, we may apply the second conclusion of Lemma 4.3 inductively to see that $\sum_{m\mid N^\infty} r^{(N)}(Dm^{-2}) = r(D)$. Finally, if $D=\ell^2$ is a square then again both sides are 0 unless N is prime, and in that case we apply the first conclusion of Lemma 4.3 with $M=p=N$. The stated formula for $\delta_{N,D}$ follows from Lemma 4.2, since

$$\begin{aligned} &\tfrac{1}{2}\#\big\{(x,y)\in\mathbb{Z}^2 : \ell^2N^{-2\operatorname{ord}_N(\ell)} = x^2+4y^2,\ N\mid y\big\} \\ &\quad = r^{(1)}(\ell^2N^{-2\operatorname{ord}_N(\ell)}) - r^{(N)}(\ell^2N^{-2\operatorname{ord}_N(\ell)}) \\ &\quad = 1+\#\big\{(a,n)\in\mathbb{Z}^2_{>0} : a\mid n^2,\ N\mid n,\ a+\tfrac{n^2}{a} = \tfrac{\ell}{(\ell,N^\infty)}\big\}. \end{aligned}$$

□

We apply this to (4.2). When N is composite we have $c_N^\circ = c_N$, so that $R = \sum_{D=1}^\infty c_N(D)r(D)D^{-s}$, which completes the proof in that case. Henceforth we assume that N is prime; in particular, $E_N(s) = 1 - N^{-s}$. Thus, by Corollary 4.4,

(4.3)

$$R - \Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)\sum_{D=1}^\infty \frac{c_N^\circ(D)r(D)}{D^s}$$

$$= -\Lambda(N)\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)\sum_{\ell=1}^\infty \frac{1+\#\{(a,n)\in\mathbb{Z}_{>0}^2 : a \mid n^2,\ N \mid n,\ a+\frac{n^2}{a} = \frac{\ell}{(\ell,N^\infty)}\}}{(\ell,N^\infty)\ell^{2s}}$$

$$= -\Lambda(N)\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)\sum_{k=0}^\infty N^{-k(2s+1)}\sum_{\substack{\ell_0\in\mathbb{Z}_{>0}\\(\ell_0,N)=1}} \frac{1+\#\{(a,n)\in\mathbb{Z}_{>0}^2 : a \mid n^2,\ N \mid n,\ a+\frac{n^2}{a} = \ell_0\}}{\ell_0^{2s}}$$

$$= -\frac{\Lambda(N)\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)}{E_N(2s+1)}\left(E_N(2s)\zeta(2s) + \sum_{\substack{n\in\mathbb{Z}_{>0}\\N\mid n}}\sum_{\substack{a\mid n^2\\(a+n^2/a,N)=1}}\left(a+\frac{n^2}{a}\right)^{-2s}\right).$$

We write $n = n_0N^r$ for $r > 0$ and $(n_0,N) = 1$. Then the condition $(a+n^2/a,N) = 1$ is satisfied if and only if $\operatorname{ord}_N(a) \in \{0,2r\}$, so we have

$$\sum_{\substack{n\in\mathbb{Z}_{>0}\\N\mid n}}\sum_{\substack{a\mid n^2\\(a+n^2/a,N)=1}}\left(a+\frac{n^2}{a}\right)^{-2s} = \sum_{r=1}^\infty\sum_{\substack{n_0\in\mathbb{Z}_{>0}\\(n_0,N)=1}}\sum_{a\mid n_0^2}\left(\left(a+\frac{n_0^2N^{2r}}{a}\right)^{-2s} + \left(aN^{2r}+\frac{n_0^2}{a}\right)^{-2s}\right)$$

$$= 2\sum_{r=1}^\infty\sum_{\substack{n_0\in\mathbb{Z}_{>0}\\(n_0,N)=1}}\sum_{a\mid n_0^2}\left(aN^{2r}+\frac{n_0^2}{a}\right)^{-2s} = 2\sum_{\substack{r\in\mathbb{Z}\\r<0}}\sum_{\substack{n_0\in\mathbb{Z}_{>0}\\(n_0,N)=1}} N^{4rs}\sum_{a\mid n_0^2}\left(a+\frac{n_0^2N^{2r}}{a}\right)^{-2s}.$$

Substituting this into (4.3) and combining with R_3, we obtain

(4.4)

$$R - \Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)\sum_{D=1}^\infty \frac{c_N^\circ(D)r(D)}{D^s} = -\Lambda(N)\zeta_N^*(4s)\frac{E_N(2s)\zeta^*(2s)}{E_N(2s+1)}$$

$$- \frac{2\Lambda(N)\Gamma_{\mathbb{R}}(2s)\zeta_N^*(4s)}{E_N(2s+1)}\sum_{\substack{n\in\mathbb{Z}_{>0}\\(n,N)=1}}\sum_{a\mid n^2}\sum_{\substack{r\in\mathbb{Z},\\r<0}}\left(a+\frac{n^2N^{2r}}{a}\right)^{-2s}N^{4rs},$$

which we write as $R_1' + R_2'$.

Next, we convert R_2' into an integral by applying [GR07, 6.422(3)]:

$$\left(a+\frac{n^2N^{2r}}{a}\right)^{-2s} = \frac{1}{2\pi i}\int_{\Re(u)=0}\frac{\Gamma_{\mathbb{R}}(2s+2u)\Gamma_{\mathbb{R}}(2s-2u)}{\Gamma_{\mathbb{R}}(4s)}a^{-2u}(nN^r)^{2(u-s)}\,du. \tag{4.5}$$

We have

$$\sum_{\substack{r\in\mathbb{Z},\\r<0}} N^{2r(u-s)}N^{4rs} = \frac{N^{-2u-2s}}{1-N^{-2u-2s}} = \frac{E_N(2s+1)}{E_N(1-2u)E_N(2u+2s)} - \frac{1}{1-N^{2u-1}},$$

and

$$\sum_{\substack{n\in\mathbb{Z}_{>0}\\(n,N)=1}}\sum_{a\mid n^2} a^{-2u}n^{2u-2s} = \frac{E_N(2s)E_N(2s+2u)E_N(2s-2u)}{E_N(4s)}\frac{\zeta(2s)\zeta(2s+2u)\zeta(2s-2u)}{\zeta(4s)},$$

so that

$$R_2' - L_3 = -2\Lambda(N)N^{2s}E_N(2s)\zeta^*(2s)\int_{\Re(u)=0}\frac{E_N(2s-2u)}{E_N(1-2u)}\zeta^*(2s+2u)\zeta^*(2s-2u)\,\frac{du}{2\pi i}.$$

Now, since

$$\frac{E_N(2s-2u)}{E_N(1-2u)} = N^{1-2s}\left(1-\frac{E_N(1-2s)}{E_N(1-2u)}\right),$$

this equals

$$-2N\Lambda(N)E_N(2s)\zeta^*(2s)\big(A(s)-E_N(1-2s)B_N(s)\big),$$

where

$$A(s) = \frac{1}{2\pi i}\int_{\Re(u)=0}\zeta^*(2s+2u)\zeta^*(2s-2u)\,du$$

and

$$B_N(s) = \frac{1}{2\pi i}\int_{\Re(u)=0}\frac{\zeta^*(2s+2u)\zeta^*(2s-2u)}{E_N(1-2u)}\,du.$$

Next, to compute the contribution from $c_N - c_N^\circ$, we have the following:

Lemma 4.5.

$$\zeta^*(4s)\sum_{n=1}^\infty\frac{r(\ell^2)}{\ell^{2s}} = \zeta(2s)(A(s)+\zeta^*(4s)).$$

Proof. By Lemma 4.2 with $M=1$, we have

$$\sum_{\ell=1}^\infty\frac{r(\ell^2)}{\ell^{2s}} = \zeta(2s)+\sum_{n=1}^\infty\sum_{a\mid n^2}\left(a+\frac{n^2}{a}\right)^{-2s}.$$

On the other hand, by (4.5) with $r=0$,

$$\sum_{n=1}^\infty\sum_{a\mid n^2}\left(a+\frac{n^2}{a}\right)^{-2s} = \frac{1}{2\pi i}\int_{\Re(u)=0}\frac{\Gamma_\mathbb{R}(2s+2u)\Gamma_\mathbb{R}(2s-2u)}{\Gamma_\mathbb{R}(4s)}\sum_{n=1}^\infty\sigma_{-2u}(n^2)n^{2(u-s)}\,du$$

$$= \frac{1}{2\pi i}\int_{\Re(u)=0}\frac{\Gamma_\mathbb{R}(2s+2u)\Gamma_\mathbb{R}(2s-2u)}{\Gamma_\mathbb{R}(4s)}\frac{\zeta(2s)\zeta(2s+2u)\zeta(2s-2u)}{\zeta(4s)}\,du = \frac{\zeta(2s)}{\zeta^*(4s)}A(s).$$

□

Comparing the definitions of c_N and c_N°, we thus have

$$(4.6)\quad R - L_3 - \Gamma_\mathbb{R}(2s)\zeta_N^*(4s)\sum_{D=1}^\infty\frac{c_N(D)r(D)}{D^s}$$

$$= \Lambda(N)\zeta^*(2s)\Bigg[2NA(s)\left(\frac{E_N^*(4s)}{N+1}-E_N(2s)\right)+2NB_N(s)E_N(2s)E_N(1-2s) + \zeta_N^*(4s)\left(\frac{2N}{N+1}-\frac{E_N(2s)}{E_N(2s+1)}\right)\Bigg].$$

Note that, for prime N,

$$\frac{E_N^*(4s)}{N+1} - E_N(2s) = -\frac{N}{N+1} E_N(2s) E_N(1-2s)$$

and

$$\frac{1}{E_N(1-2u)} - \frac{N}{N+1} = \frac{1}{N+1} \frac{E_N^*(4u)}{E_N(2u)E_N(1-2u)},$$

so on making the substitution $u \mapsto \frac{u}{2}$, (4.6) becomes

$$\frac{N\Lambda(N)}{N+1}\zeta^*(2s)\left[\mathcal{I}_N(2s;0) + \zeta_N^*(4s)\left(2 - \frac{(N+1)E_N(2s)}{NE_N(2s+1)}\right)\right]. \tag{4.7}$$

Shifting the contour of $\mathcal{I}_N$ to $\Re(u) = -\sigma$, we pass poles at $u = 1-2s$ and $u = -2s$, with residues

$$E_N^*(2-4s)\zeta^*(4s-1) = \zeta_N^*(2-4s)$$

and

$$-\frac{E_N^*(2s)E_N^*(1-2s)}{E_N^*(-2s)E_N^*(1+2s)} E_N^*(-4s)\zeta^*(4s) = \frac{E_N(2s-1)}{NE_N(2s+1)}\zeta_N^*(4s),$$

respectively. Finally, we have

$$\frac{E_N(2s-1)}{NE_N(2s+1)} + 2 - \frac{(N+1)E_N(2s)}{NE_N(2s+1)} = 1,$$

so that (4.7) is

$$\frac{N\Lambda(N)}{N+1}\zeta^*(2s)\big(\mathcal{I}_N(2s;-\sigma) + \zeta_N^*(4s) + \zeta_N^*(2-4s)\big),$$

as required.

The analytic continuation and functional equation of $L^*(2s, \operatorname{Sym}^2 f_j)$ were proved by Gelbart and Jacquet [GJ78] following ideas of Shimura [Shi75]. By Stirling's formula and the convexity bound, we have

$$L^*(2s, \operatorname{Sym}^2 f_j) \ll_{K,\varepsilon} e^{-(\pi-\varepsilon)|r_j|},$$

uniformly for $s \in K$, for any compact subset $K \subset \mathbb{C}$ and $\varepsilon > 0$. Combining this with the Weyl-type estimate $\#\{j : |r_j| \le T\} \ll T^2$, we see that the series L_2 converges absolutely to an entire function of s. Similarly, $\mathcal{I}_N(2s;\sigma)$ converges absolutely for all s with $\Re(2s) < \sigma$ and satisfies a functional equation. □

4.3. **Proof of Theorem 1.3.** Let

$$F(s) = \sum_{\substack{0<D\in\mathcal{D} \\ \sqrt{D}\notin\mathbb{Z}}} \frac{L(1,\psi_D)r(D)}{D^s}.$$

It is straightforward to show that $L(1,\psi_D)r(D) \ll_\varepsilon D^\varepsilon$, for any $D \in \mathcal{D}$ with $D > 0$ and $\sqrt{D} \notin \mathbb{Z}$. Using this estimate in [Ten15, §II.1, Cor. 2.1], for any $X \ge T \ge 2$, we have

$$\sum_{\substack{0<D\le X \\ \sqrt{D}\notin\mathbb{Z}}} L(1,\psi_D)r(D) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} F(s)X^s\frac{ds}{s} + O_\varepsilon\left(\frac{X^{1+\varepsilon}}{T}\right),$$

where $\kappa = 1 + 1/\log X$.

Our goal now is to shift the contour to $\Re(s) = \frac{1}{4}$. To that end, we will show that $F(s)$ is analytic on $\{s \in \mathbb{C} : \Re(s) \geq \frac{1}{4}\}$, apart from poles at $s \in \{\frac{1}{2}, 1\}$, and satisfies the estimates

$$F(\tfrac{1}{4} + it) \ll_\varepsilon (1 + |t|)^{\frac{9}{4}+\varepsilon} \quad \text{and} \quad \int_{-T}^{T} |F(\tfrac{1}{4} + it)|\, dt \ll_\varepsilon T^{\frac{11}{4}+\varepsilon}. \tag{4.8}$$

Hence, by partial integration, we have

$$\int_{\frac{1}{4}-iT}^{\frac{1}{4}+iT} F(s) X^s \frac{ds}{s} \ll_\varepsilon X^{\frac{1}{4}} T^{\frac{7}{4}+\varepsilon}.$$

By convexity, it follows from the first estimate in (4.8) that

$$F(\sigma \pm iT) \ll_\varepsilon T^{3(1-\sigma)+\varepsilon} \quad \text{for } \sigma \in [\tfrac{1}{4}, \kappa],$$

so that

$$\int_{\frac{1}{4}\pm iT}^{\kappa \pm iT} F(s) X^s \frac{ds}{s} \ll_\varepsilon X^{\frac{1}{4}} T^{\frac{5}{4}+\varepsilon} + X T^{-1+\varepsilon}.$$

The pole at $s = \frac{1}{2}$ contributes a residue of size $O_\varepsilon(X^{\frac{1}{2}+\varepsilon})$, and from the pole at $s = 1$ we get the main term, which turns out to be $\frac{1}{2} \frac{\zeta^*(2)\zeta^*(3)}{\pi^{-1}\zeta^*(4)} X$. Hence, altogether we have

$$\sum_{\substack{D \leq X \\ \sqrt{D} \notin \mathbb{Z}}} L(1, \psi_D) r(D) = \frac{\pi}{2} \frac{\zeta^*(2)\zeta^*(3)}{\zeta^*(4)} X + O_\varepsilon\left(X^{\frac{1}{4}} T^{\frac{7}{4}+\varepsilon} + X^{\frac{1}{2}+\varepsilon} + \frac{X^{1+\varepsilon}}{T} \right)$$

for any $\varepsilon > 0$. Taking $T = X^{\frac{3}{11}}$ yields the desired bound.

To prove the meromorphic continuation of $F(s)$ and the estimates (4.8), we compute from (3.2) that

$$\begin{aligned} F(s) = {} & \sum_{j=1}^{\infty} (-1)^{\epsilon_j} \frac{L^*(2s, \operatorname{Sym}^2 f_j)}{\zeta^*(4s)\Gamma_{\mathbb{R}}(2s)} + \frac{\zeta^*(2s)\zeta^*(2s-1)\zeta^*(2s+1)}{\zeta^*(4s)\Gamma_{\mathbb{R}}(2s)} \\ & - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\zeta(2s)\zeta^*(2s-2ir)\zeta^*(2s+2ir)}{\zeta^*(4s)} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) dr - \frac{1}{4} \frac{\zeta^*(2s)^3}{\zeta^*(4s)\Gamma_{\mathbb{R}}(2s)} \\ & - \sum_{\ell=1}^{\infty} \left[\sum_{m \mid \ell} \Lambda(m)(1 - m^{-1}) + \tfrac{1}{2}\big(\psi(s + \tfrac{1}{2}) - \psi(s)\big) \right] \frac{r(\ell^2) - 2}{\ell^{2s}}, \end{aligned} \tag{4.9}$$

where $L^*(2s, \operatorname{Sym}^2 f_j) = \Gamma_{\mathbb{R}}(2s)\Gamma_{\mathbb{R}}(2s - 2ir_j)\Gamma_{\mathbb{R}}(2s + 2ir_j)\zeta(4s) \sum_{n=1}^{\infty} \frac{\lambda_j(n^2)}{n^{2s}}$. We consider each term in turn.

First, by Stirling's formula, for $s = \frac{1}{4} + it$ we have

$$\frac{\Gamma_{\mathbb{R}}(2s - 2ir)\Gamma_{\mathbb{R}}(2s + 2ir)}{\Gamma_{\mathbb{R}}(4s)} \ll (1 + |t^2 - r^2|)^{-\frac{1}{4}} e^{-\pi \max(0, |r| - |t|)},$$

uniformly for $r, t \in \mathbb{R}$. Moreover, $1/\zeta(4s) \ll \log(1 + |t|)$, and by the uniform convexity bound [Har02, Cor. 2], we have

$$L(2s, \operatorname{Sym}^2 f_j) \ll_\varepsilon \big((1 + |t|)(1 + |t^2 - r_j^2|)\big)^{\frac{1}{4}+\varepsilon}.$$

From these estimates and the Weyl bound $\#\{j : |r_j| \leq T\} \ll T^2$, we see that the sum over j in (4.9) is $O_\varepsilon\big((1+|t|)^{\frac{9}{4}+\varepsilon}\big)$, as claimed. Moreover, by Cauchy–Schwarz, we have

$$\begin{aligned}\left(\sum_{j=1}^{\infty}\int_{-T}^{T}\left|\frac{L^*(\frac{1}{2}+2it,\operatorname{Sym}^2 f_j)}{\zeta^*(1+4it)\Gamma_{\mathbb{R}}(\frac{1}{2}+2it)}\right|dt\right)^2 &\ll \log^2 T\sum_{j=1}^{\infty}\int_{-T}^{T}(1+|t^2-r_j^2|)^{-\frac{1}{2}}e^{-\pi\max(0,|r_j|-|t|)}\,dt\\ &\quad\cdot\sum_{j=1}^{\infty}\int_{-T}^{T}|L(\tfrac{1}{2}+2it,\operatorname{Sym}^2 f_j)|^2 e^{-\pi\max(0,|r_j|-|t|)}\,dt\\ &\ll (T\log T)^2\sum_{j=1}^{\infty}\int_{-T}^{T}|L(\tfrac{1}{2}+2it,\operatorname{Sym}^2 f_j)|^2 e^{-\pi\max(0,|r_j|-|t|)}\,dt.\end{aligned}$$

By [KSS06, Thm. 2] and [RW03, Cor. C], we have

$$\int_{-T}^{T}|L(\tfrac{1}{2}+2it,\operatorname{Sym}^2 f_j)|^2\,dt \ll_\varepsilon (1+r_j)^\varepsilon T^{\frac{3}{2}}\log T,$$

and altogether this yields

$$\sum_{j=1}^{\infty}\int_{-T}^{T}\left|\frac{L^*(\frac{1}{2}+2it,\operatorname{Sym}^2 f_j)}{\zeta^*(1+4it)\Gamma_{\mathbb{R}}(\frac{1}{2}+2it)}\right|dt \ll_\varepsilon T^{\frac{11}{4}+\varepsilon}.$$

For all remaining terms we obtain a pointwise bound of at most $O_\varepsilon\big((1+|t|)^{\frac{5}{4}+\varepsilon}\big)$, which suffices for both estimates in (4.8). First, we have

$$\frac{\zeta(2s)\zeta^*(2s-1)\zeta^*(2s+1)}{\zeta^*(4s)} \ll_\varepsilon (1+|t|)^{\frac{3}{4}+\varepsilon}.$$

Next, by a similar analysis to the proof of Proposition 2.1, the second line of (4.9) can be written as

$$\begin{aligned}&\frac{1}{2}\zeta(2s)\frac{\zeta^{*\prime}}{\zeta^*}(2s+1)-\frac{1}{2}\frac{\zeta^*(4s-1)\zeta^{*\prime}(2s)}{\zeta^*(4s)\Gamma_{\mathbb{R}}(2s)}\\ &\qquad-\frac{1}{2\pi i}\int_{\Re(u)=\sigma}\frac{\zeta(2s)\zeta^*(2s-2u)\zeta^*(2s+2u)}{\zeta^*(4s)}\frac{\zeta^{*\prime}}{\zeta^*}(1+2u)\,du,\end{aligned}$$

for any $\sigma > \frac{1}{4}$, and the integral is $\zeta(2s)/\zeta^*(4s)$ times an analytic function for $\Re(s) \in (\frac{1}{2}-\sigma, \sigma)$. The first two terms are analytic for $\Re(s) \geq \frac{1}{4}$ apart from a pole at $s = \frac{1}{2}$, and by the convexity bounds for $\zeta(\frac{1}{2}+2it)$ and $\zeta'(\frac{1}{2}+2it)$, we see that they are $O_\varepsilon\big((1+|t|)^{\frac{1}{4}+\varepsilon}\big)$ for $s = \frac{1}{4}+it$. Taking $\sigma = \frac{1}{4}+\varepsilon$ and writing $s = \frac{1}{4}+it$, $u = \frac{1}{4}+\varepsilon+ir$, we find by a similar analysis to the above that

$$\frac{\zeta(2s)\zeta^*(2s-2u)\zeta^*(2s+2u)}{\zeta^*(4s)}\frac{\zeta^{*\prime}}{\zeta^*}(1+2u) \ll_\varepsilon (1+|t|)^{\frac{1}{4}+\varepsilon}(1+|t^2-r^2|)^\varepsilon e^{-\pi\max(0,|r|-|t|)},$$

so the integral is $O_\varepsilon\big((1+|t|)^{\frac{5}{4}+\varepsilon}\big)$.

Turning to the third line of (4.8), note first that $\sum_{m|\ell}\Lambda(m) = \log\ell$, and

$$-\sum_{\ell=1}^{\infty}\frac{(r(\ell^2)-2)\log\ell}{\ell^{2s}} = \frac{1}{2}\frac{d}{ds}\left(\frac{\zeta^2(2s)L(2s,\psi_{-4})}{\zeta(4s)}-2\zeta(2s)\right).$$

Again using the convexity bound, this is $O_\varepsilon\big((1+|t|)^{\frac{3}{4}+\varepsilon}\big)$ for $s=\frac{1}{4}+it$. Similarly,

$$-\frac{1}{2}\big(\psi(s+\tfrac{1}{2})-\psi(s)\big)\sum_{\ell=1}^{\infty}\frac{r(\ell^2)-2}{\ell^{2s}} = -\frac{1}{2}\big(\psi(s+\tfrac{1}{2})-\psi(s)\big)\left(\frac{\zeta^2(2s)L(2s,\psi_{-4})}{\zeta(4s)}-2\zeta(2s)\right),$$

and this is $O_\varepsilon\big((1+|t|)^{\frac{1}{4}+\varepsilon}\big)$ for $s=\frac{1}{4}+it$. Finally, we have

$$-2\sum_{\ell=1}^{\infty}\ell^{-2s}\sum_{m\mid\ell}\frac{\Lambda(m)}{m} = 2\zeta(2s)\frac{\zeta'}{\zeta}(2s+1),$$

which is $O_\varepsilon\big((1+|t|)^{\frac{1}{4}+\varepsilon}\big)$ for $s=\frac{1}{4}+it$, and

$$\sum_{\ell=1}^{\infty}r(\ell^2)\ell^{-2s}\sum_{m\mid\ell}\frac{\Lambda(m)}{m} = \frac{\zeta^2(2s)L(2s,\psi_{-4})}{\zeta(4s)}\sum_{p}\frac{\log p}{p-1}\left[1-\frac{E_p(p^{-2s-1})}{E_p(p^{-2s})}\right],$$

where $E_p(T)=\frac{1+T}{(1-T)(1-\psi_{-4}(p)T)}$. The sum over p is analytic and bounded for $\Re(s)\geq\frac{1}{4}$, so the last line is $O_\varepsilon\big((1+|t|)^{\frac{3}{4}+\varepsilon}\big)$ for $s=\frac{1}{4}+it$. □

References

[CL01] J. Brian Conrey and Xian-Jin Li. On the trace of Hecke operators for Maass forms for congruence subgroups. *Forum Math.*, 13(4):447–484, 2001.

[GJ78] Stephen Gelbart and Hervé Jacquet. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.

[GR07] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[Har02] Gergely Harcos. Uniform approximate functional equation for principal L-functions. *Int. Math. Res. Not.*, (18):923–932, 2002.

[Has13] Yasufumi Hashimoto. Asymptotic formulas for class number sums of indefinite binary quadratic forms on arithmetic progressions. *Int. J. Number Theory*, 9(1):27–51, 2013.

[Hej83] Dennis A. Hejhal. *The Selberg trace formula for* PSL(2, $\mathbf{R}$). *Vol. 2*, volume 1001 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[Hux85] M. N. Huxley. Introduction to Kloostermania. In *Elementary and analytic theory of numbers (Warsaw, 1982)*, volume 17 of *Banach Center Publ.*, pages 217–306. PWN, Warsaw, 1985.

[Kim03] Henry H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. *J. Amer. Math. Soc.*, 16(1):139–183, 2003. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[KSS06] W. Kohnen, A. Sankaranarayanan, and J. Sengupta. The quadratic mean of automorphic L-functions. In *Automorphic forms and zeta functions*, pages 262–279. World Sci. Publ., Hackensack, NJ, 2006.

[Li05] Xian-Jin Li. On the trace of Hecke operators for Maass forms for congruence subgroups. II. *Forum Math.*, 17(1):1–30, 2005.

[Li08] Xian-Jin Li. On exceptional eigenvalues of the Laplacian for $\Gamma_0(N)$. *Proc. Amer. Math. Soc.*, 136(6):1945–1953, 2008.

[Mot92] Yōichi Motohashi. Spectral mean values of Maass waveform L-functions. *J. Number Theory*, 42(3):258–284, 1992.

[Rau09] Nicole Raulf. Asymptotics of class numbers for progressions and for fundamental discriminants. *Forum Math.*, 21(2):221–257, 2009.

[RW03] Dinakar Ramakrishnan and Song Wang. On the exceptional zeros of Rankin-Selberg L-functions. *Compositio Math.*, 135(2):211–244, 2003.

[RW11] Robert C. Rhoades and Matthias Waldherr. A Maass lifting of Θ^3 and class numbers of real and imaginary quadratic fields. *Math. Res. Lett.*, 18(5):1001–1012, 2011.

[Sar82] Peter Sarnak. Class numbers of indefinite binary quadratic forms. *J. Number Theory*, 15(2):229–247, 1982.

[Sar85] Peter C. Sarnak. Class numbers of indefinite binary quadratic forms. II. *J. Number Theory*, 21(3):333–346, 1985.

[Shi75] Goro Shimura. On the holomorphy of certain Dirichlet series. *Proc. London Math. Soc. (3)*, 31(1):79–98, 1975.

[Str16] Andreas Strömbergsson. Explicit trace formulas for Hecke operators. preprint, 2016.

[Ten15] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015. Translated from the 2008 French edition by Patrick D. F. Ion.

Howard House, University of Bristol, Queens Ave, Bristol, BS8 1SN, United Kingdom
E-mail address: andrew.booker@bristol.ac.uk, min.lee@bristol.ac.uk