



Beil, C. (2016). Morita equivalences and Azumaya loci from Higgsing dimer algebras. *Journal of Algebra*, 453, 429-455.
<https://doi.org/10.1016/j.jalgebra.2015.11.052>

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Journal of Algebra

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Morita equivalences and Azumaya loci from Higgsing dimer algebras



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ARTICLE INFO

Article history:

Received 21 October 2015
Available online 29 January 2016
Communicated by J.T. Stafford

MSC:

14A22
16G20
16R20

Keywords:

Dimer model
Dimer algebra
Superpotential algebra
Higgsing
Morita equivalence
Azumaya locus

ABSTRACT

Let $\psi : A \rightarrow A'$ be a cyclic contraction of dimer algebras, with A non-cancellative and A' cancellative. A' is then prime, noetherian, and a finitely generated module over its center. In contrast, A is often not prime, nonnoetherian, and an infinitely generated module over its center. We present certain Morita equivalences that relate the representation theory of A with that of A' .

We then characterize the Azumaya locus of A in terms of the Azumaya locus of A' , and give an explicit classification of the simple A -modules parameterized by the Azumaya locus. Furthermore, we show that if the smooth and Azumaya loci of A' coincide, then the smooth and Azumaya loci of A coincide. This provides the first known class of algebras that are nonnoetherian and infinitely generated modules over their centers, with the property that their smooth and Azumaya loci coincide.

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<http://dx.doi.org/10.1016/j.jalgebra.2015.11.052>

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1. Introduction

1.1. Main results

The main objective of this paper is to characterize Morita equivalences and Azumaya loci of non-cancellative dimer algebras. Dimer algebras were introduced in string theory, and have wide application to many areas of mathematics, such as noncommutative crepant resolutions, Calabi–Yau algebras, the McKay correspondence, mirror symmetry, wall-crossing, and cluster algebras (e.g., [6,7,15,9,14,11,12,10,1]).

A dimer algebra is a quiver algebra whose quiver embeds into a two-torus, with homotopy-like relations derived from a potential (precise definitions will be given in Section 1.2). Cancellative dimer algebras are Calabi–Yau algebras and noncommutative crepant resolutions ([5, Theorem 10.2], [7], [9, Theorem 4.3], [15, Theorem 6.3]). In contrast, non-cancellative dimer algebras are nonnoetherian, infinitely generated modules over their centers, and often not prime or PI [4, Theorems 1.3, 4.17, 4.47]. However, almost all dimer algebras are non-cancellative. Furthermore, non-cancellative dimer algebras correspond to certain unstable quiver gauge theories which may describe the vacuum moduli space at sufficiently high energies. It is therefore of great interest to understand these highly complex algebras, both from a mathematical and a string theoretic perspective.

Our primary tool in studying non-cancellative dimer algebras is a ‘cyclic contraction’, introduced in [4]. Roughly, a cyclic contraction is a k -linear map of dimer algebras $\psi : A = kQ/I \rightarrow A' = kQ'/I'$, where Q' is obtained by contracting a set of arrows of Q to vertices, such that (i) A' is cancellative, and (ii) the commutative algebras generated by the cycles of Q and Q' coincide. This common algebra, denoted S , is isomorphic to the center of A' and is called the ‘cycle algebra’ of A . An example of a cyclic contraction is given in Fig. 1.

Recall that two rings are Morita equivalent if they have equivalent module categories. Our main results are the following.

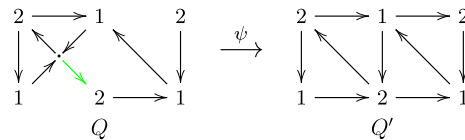


Fig. 1. The non-cancellative dimer algebra $A = kQ/I$ cyclically contracts to the cancellative dimer algebra $A' = kQ'/I'$. Both quivers are drawn on a torus, and the contracted arrow is drawn in green. The cycle algebra of A is $S = k[x^2, y^2, xy, z] \subset B = k[x, y, z]$, and the reduced center of A is $\tilde{Z} = k + (x^2, y^2, xy)S$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Theorem 1.1. (See [Theorems 2.13, 3.4, 3.6](#), and [Corollary 2.14](#).) Let $\psi : A \rightarrow A'$ be a cyclic contraction, and let Z and Z' be the respective centers of A and A' . Denote by $\hat{Z} := Z/\text{nil } Z$ the reduced center of A . Consider the open dense set¹

$$U := \{\mathfrak{n} \in \text{Max } S \mid \hat{Z}_{\mathfrak{n} \cap \hat{Z}} = S_{\mathfrak{n}}\}.$$

(1) Let $\mathfrak{q} \in \text{Spec } S$, and suppose $\mathfrak{p} \in \text{Spec } Z$ satisfies $\mathfrak{p} + \text{nil } Z = \mathfrak{q} \cap \hat{Z}$. Then the following are equivalent:

- The open set U intersects the zero locus $\mathcal{Z}(\mathfrak{q}) \subset \text{Max } S$ of \mathfrak{q} non-trivially.
- The localizations

$$A_{\mathfrak{p}} := A \otimes_Z Z_{\mathfrak{p}} \quad \text{and} \quad A'_{\mathfrak{q}} := A' \otimes_{Z'} Z'_{\mathfrak{q}}$$

are Morita equivalent.

- The localized algebra $A_{\mathfrak{p}}$ is prime, noetherian, and a finitely generated module over its center $Z_{\mathfrak{p}}$ with PI degree $|Q_0|$.

(2) The (noncommutative) function fields are Morita equivalent,

$$A \otimes_Z \text{Frac } Z \sim A' \otimes_{Z'} \text{Frac } Z' \sim \text{Frac } Z \sim \text{Frac } Z'.$$

(3) The Azumaya locus $\mathcal{A} \subset \text{Max } Z$ of A coincides with the intersection of the Azumaya locus $\mathcal{A}' \subset \text{Max } Z'$ of A' and the locus $U \subset \text{Max } Z'$,

$$\mathcal{A} \cong \mathcal{A}' \cap U.$$

From the third statement we obtain the first known class of algebras that are non-noetherian and infinitely generated modules over their centers, with the property that their Azumaya and smooth loci coincide ([Corollary 3.7](#)).

In addition, we give an explicit classification of the simple A -module isoclasses of dimension 1^{Q_0} , or equivalently, the A -modules which sit over the Azumaya locus \mathcal{A} of A ([Proposition 3.10](#) and [Theorem 3.11](#)). Finally, we show that the cycle algebra is isomorphic to the GL-invariant rings

$$S \cong k[\overline{\mathbb{S}(A)}]^{\text{GL}} = k[\overline{\mathbb{S}(A')}]^{\text{GL}},$$

where

$$\mathbb{S}(A) \subset \text{Rep}_{1^{Q_0}}(A) \quad \text{and} \quad \mathbb{S}(A') \subset \text{Rep}_{1^{Q'_0}}(A')$$

are the open subvarieties consisting of simple modules, and $\overline{\mathbb{S}(A)}$ and $\overline{\mathbb{S}(A')}$ are their Zariski closures ([Theorem 3.14](#)).

¹ The locus U was introduced in [\[3\]](#) to construct a theory of geometry for nonnoetherian algebras with finite Krull dimension. U is open dense by [\[3, Proposition 2.4.2\]](#) and [\[4, Theorem 4.63\]](#).

1.2. Preliminary definitions

We begin by recalling the definition of a dimer algebra, which is a type of quiver algebra whose quiver is dual to a dimer model.

Definition 1.2.

- Let Q be a finite quiver whose underlying graph \bar{Q} embeds into a two-dimensional real torus T^2 , such that each connected component of $T^2 \setminus \bar{Q}$ is simply connected and bounded by an oriented cycle of length at least 2, called a *unit cycle*.² The *dimer algebra* of Q is the quiver algebra $A = kQ/I$ with relations

$$I := \langle p - q \mid \exists a \in Q_1 \text{ such that } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ,$$

where p and q are paths.

- Two paths $p, q \in A$ form a *non-cancellative pair* if $p \neq q$, and there is a path $r \in A$ such that

$$rp = rq \neq 0 \quad \text{or} \quad pr = qr \neq 0.$$

A is *non-cancellative* if it contains a non-cancellative pair, and otherwise A is *cancellative*.

To study non-cancellative dimer algebras, the notion of a cyclic contraction was introduced in [4, Section 4.1]. This notion remains our primary tool in this paper. We first define a contraction, which formalizes Higgsing in abelian quiver gauge theories.

Definition 1.3. Let $Q = (Q_0, Q_1, t, h)$ be a dimer quiver, with tail and head maps $t, h : Q_1 \rightarrow Q_0$. Let $Q_1^* \subset Q_1$ be a subset of arrows. Consider the quiver $Q' = (Q'_0, Q'_1, t', h')$ obtained by contracting each arrow in Q_1^* to a vertex. Specifically,

$$Q'_0 := Q_0 / \{h(a) \sim t(a) \mid a \in Q_1^*\}, \quad Q'_1 = Q_1 \setminus Q_1^*,$$

and for each arrow $a \in Q'_1$,

$$h'(a) = h(a) \quad \text{and} \quad t'(a) = t(a).$$

There is a k -linear map of path algebras

$$\psi : kQ \rightarrow kQ'$$

² In contexts such as cluster algebras, it is useful to consider dimer algebras where \bar{Q} embeds into any compact surface; see for example [1].

defined by

$$\psi(a) = \begin{cases} a & \text{if } a \in Q_0 \cup Q_1 \setminus Q_1^* \\ e_{t(a)} & \text{if } a \in Q_1^* \end{cases}$$

and extended multiplicatively to nonzero paths and k -linearly to kQ . We call ψ a *contraction of dimer algebras* if Q' is a dimer quiver, and ψ induces a k -linear map of dimer algebras

$$\psi : A = kQ/I \rightarrow A' = kQ'/I'.$$

We now describe the structure we wish to preserve under a contraction.

Definition 1.4. Let $A = kQ/I$ be a dimer algebra.

- A *perfect matching* $D \subset Q_1$ is a set of arrows such that each unit cycle contains precisely one arrow in D .
- A *simple matching* $D \subset Q_1$ is a perfect matching such that $Q \setminus D$ supports a simple A -module of dimension 1^{Q_0} (that is, $Q \setminus D$ contains a cycle that passes through each vertex of Q). Denote by \mathcal{S} the set of simple matchings of A .

Consider a contraction of dimer algebras $\psi : A \rightarrow A'$, where A is non-cancellative and A' is cancellative. Denote by

$$B := k[x_D \mid D \in \mathcal{S}']$$

the polynomial ring generated by the simple matchings of A' . Denote by $E_{ij} \in M_{|Q'_0|}(B)$ the matrix with a 1 in the ij -th slot and zeros elsewhere. Since A' is cancellative, there is an algebra monomorphism

$$\tau : A' \rightarrow M_{|Q'_0|}(B) \tag{1}$$

defined on $i \in Q'_0$ and $a \in Q'_1$ by

$$\tau(e_i) := E_{ii}, \quad \tau(a) := E_{h(a),t(a)} \prod_{a \in D \in \mathcal{S}'} x_D,$$

and extended multiplicatively and k -linearly to A' [2, Theorem 3.5]. For $p \in e_j A' e_i$, denote by $\bar{\tau}(p) \in B$ the single nonzero matrix entry of $\tau(p)$, that is,

$$\tau(p) = \bar{\tau}(p) E_{ji}.$$

Definition 1.5. (See [2, Definition 4.3].) Let $\psi : A \rightarrow A'$ be a contraction to a cancellative dimer algebra. If

$$S := k [\cup_{i \in Q_0} \bar{\tau}\psi(e_i A e_i)] = k [\cup_{i \in Q'_0} \bar{\tau}(e_i A' e_i)],$$

then we say ψ is *cyclic*, and call S the *cycle algebra* of A .

Throughout, we will consider a cyclic contraction of dimer algebras $\psi : A \rightarrow A'$. Consequently, the respective centers Z and Z' of A and A' both have Krull dimension 3 [4, Theorem 4.66]. Furthermore, Z' is noetherian and isomorphic to the cycle algebra S ,

$$\begin{aligned} Z' &\cong S \\ z &\mapsto \bar{\tau}(ze_i) \end{aligned} \tag{2}$$

where $i \in Q'_0$ is any vertex [4, Theorems 3.3, 3.5]. In contrast, Z is nonnoetherian [4, Theorem 4.45]. Moreover, there is an embedding of algebras

$$\begin{aligned} \hat{Z} := Z/\text{nil } Z &\hookrightarrow R := k [\cap_{i \in Q_0} \bar{\tau}\psi(e_i A e_i)] \subset S \subset B \\ z &\mapsto \bar{\tau}\psi(ze_i) \end{aligned} \tag{3}$$

where again $i \in Q_0$ is any vertex [4, Theorem 4.27]. We will identify the reduced center \hat{Z} of A with its image in R .

Notation 1.6. For each ideal $\mathfrak{p} \in \text{Spec } Z$ (which necessarily contains the nilradical $\text{nil } Z$), set

$$\hat{\mathfrak{p}} := \mathfrak{p} + \text{nil } Z \in \text{Spec } \hat{Z}. \tag{4}$$

Conversely, for $\hat{\mathfrak{p}} \in \text{Spec } \hat{Z}$, denote by $\mathfrak{p} \in \text{Spec } Z$ the prime ideal satisfying (4).

Using the notion of nonnoetherian geometry introduced in [3], we may view the geometry $\text{Max } Z$ of Z as the affine variety $\text{Max } S$, with precisely one ‘smeared-out’ positive dimensional (closed) point $\mathfrak{z}_0 \in \text{Max } Z$ [4, Theorem 4.68]. In particular, the morphism of schemes $\text{Spec } S \rightarrow \text{Spec } \hat{Z}$, $\mathfrak{q} \mapsto \mathfrak{q} \cap \hat{Z}$, induced from the embedding $\hat{Z} \hookrightarrow S$ in (3), is surjective. Furthermore, \hat{Z} is locally noetherian on the complement of $\hat{\mathfrak{z}}_0$:

$$\begin{aligned} U &:= \{\mathfrak{n} \in \text{Max } S \mid \hat{Z}_{\mathfrak{n} \cap \hat{Z}} = S_{\mathfrak{n}}\} \\ &\stackrel{(I)}{=} \{\mathfrak{n} \in \text{Max } S \mid \hat{Z}_{\mathfrak{n} \cap \hat{Z}} \text{ is noetherian}\} \\ &\stackrel{(II)}{=} \{\mathfrak{n} \in \text{Max } S \mid \mathfrak{n} \cap \hat{Z} \neq \hat{\mathfrak{z}}_0\}, \end{aligned} \tag{5}$$

where (I) and (II) hold by [4, Theorem 4.63]. This open set captures the points where \hat{Z} and S look locally the same. The point \mathfrak{z}_0 may be realized explicitly as the common Z -annihilator of the vertex simple A -modules.

Conventions Throughout, all algebras are over an uncountable algebraically closed field k of characteristic zero. For a commutative algebra R , we will denote by $\text{Frac } R$ the ring of fractions of R if prime; by $\text{Max } R$ the set of maximal ideals of R ; by $\text{Spec } R$ either the set of prime ideals of R or the affine k -scheme with global sections R ; by $R_{\mathfrak{p}}$ the localization of R at $\mathfrak{p} \in \text{Spec } R$; by $\text{nil } R$ the nilradical of R ; and by $\mathcal{Z}(\mathfrak{a})$ the closed set $\{\mathfrak{m} \in \text{Max } R \mid \mathfrak{m} \supseteq \mathfrak{a}\}$ of $\text{Max } R$ defined by the subset $\mathfrak{a} \subset R$.

We will denote by $Q = (Q_0, Q_1, t, h)$ a quiver with vertex set Q_0 , arrow set Q_1 , and head and tail maps $h, t : Q_1 \rightarrow Q_0$. We will denote by kQ the path algebra of Q , and by e_i the idempotent corresponding to vertex $i \in Q_0$. Multiplication of paths is read right to left, following the composition of maps. A loop in a quiver is an arrow which is a cycle. By module we mean left module. If $\rho : A \rightarrow \text{End}_k(V)$ is a representation of an algebra A , then we will denote by $V_\rho := V$ the left A -module defined by ρ . If \mathfrak{p} is an ideal of the center of A , then we will denote by A/\mathfrak{p} the quotient $A/\mathfrak{p}A$. By infinitely generated R -module, we mean an R -module that is not finitely generated. A ring is noetherian if it is both left and right noetherian. We will often write rs for $r \otimes s \in R \otimes S$ if the tensor product is clear. Finally, by non-constant monomial, we mean a monomial that is not in k .

2. Morita equivalences

Throughout, A is a non-cancellative dimer algebra; $\psi : A \rightarrow A'$ is a cyclic contraction; and unless stated otherwise, $\mathfrak{q} \in \text{Spec } S$ satisfies

$$\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset.$$

Set $\hat{\mathfrak{p}} := \mathfrak{q} \cap \hat{Z} \in \text{Spec } \hat{Z}$.

Notation 2.1. Let $\pi : \mathbb{R}^2 \rightarrow T^2$ be a covering map such that for some $i \in Q_0$,

$$\pi(\mathbb{Z}^2) = i \in Q_0.$$

Denote by $Q^+ := \pi^{-1}(Q) \subset \mathbb{R}^2$ the covering quiver of Q . For each path p in Q , denote by p^+ the unique path in Q^+ with tail in the unit square $[0, 1) \times [0, 1) \subset \mathbb{R}^2$ satisfying $\pi(p^+) = p$.

Unless stated otherwise, by *path* or *cycle* we mean path or cycle modulo I . By a *cyclic subpath* of a path p , we mean a proper non-trivial closed subpath of p . We denote by $\sigma_i \in A$ the unique unit cycle at vertex $i \in Q_0$, and by σ the $\bar{\tau}\psi$ -image of each unit cycle,

$$\sigma := \bar{\tau}\psi(\sigma_i) = \prod_{D \in S'} x_D \in B.$$

We consider the following sets of cycles in A .

- Let \mathcal{C} be the set of cycles in A (i.e., cycles in Q modulo I).

- For $u \in \mathbb{Z}^2$, let \mathcal{C}^u be the set of cycles $p \in \mathcal{C}$ such that

$$h(p^+) = t(p^+) + u \in Q_0^+.$$

- For $i \in Q_0$, let \mathcal{C}_i be the set of cycles in the vertex corner ring $e_i A e_i$.
- Let $\hat{\mathcal{C}}$ be the set of cycles $p \in \mathcal{C}$ such that $(p^2)^+$ does not have a cyclic subpath; equivalently, the lift of each cyclic permutation of p does not have a cyclic subpath.

We denote the intersection $\hat{\mathcal{C}} \cap \mathcal{C}^u \cap \mathcal{C}_i$, for example, by $\hat{\mathcal{C}}_i^u$.

Notation 2.2. For $p \in e_j A e_i$ and $q \in e_\ell A' e_k$, set

$$\bar{p} := \bar{\tau}\psi(p) \in B \quad \text{and} \quad \bar{q} := \bar{\tau}(q) \in B.$$

Notation 2.3. Set

$$Q_1^S := \{a \in Q_1 \mid a \notin D \text{ for each } D \in \mathcal{S}\} \\ \stackrel{(i)}{=} \{a \in Q_1 \mid \rho(a) \neq 0 \text{ for each simple representation } \rho \text{ of dimension } 1^{Q_0}\},$$

where (i) holds by [4, Lemma 4.39].

Lemma 2.4. *Let z be a non-constant monomial in Z which is not divisible by σ . Let r be a path whose arrow subpaths are all in Q_1^S . Then there is a path s such that*

$$ze_{t(r)} = sr.$$

Proof. (i) First suppose r is an arrow, $r = \delta \in Q_1^S$. Let $z \in Z$ be a non-constant monomial such that $\sigma \nmid z$. Since $\hat{Z} \subseteq R$ by (3), there are cycles

$$p \in \mathcal{C}_{h(\delta)}^u \quad \text{and} \quad q \in \mathcal{C}_{t(\delta)}^v$$

such that $\bar{p} = \bar{q} = z$. Since $\sigma \nmid z = \bar{p} = \bar{q}$, p and q are in $\hat{\mathcal{C}}$ by [4, contrapositive of Lemma 4.11.2]. In particular, u and v are nonzero. Whence $u = v$ by [4, Lemma 4.13]. Therefore p and q are in $\hat{\mathcal{C}}^u$. Thus the paths $(p\delta)^+$ and $(\delta q)^+$ bound a compact region

$$\mathcal{R}_{p\delta, \delta q} \subset \mathbb{R}^2.$$

Furthermore, since $z \in R$, $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $i \in Q_0$. Thus if a cycle s is formed from subpaths of cycles in $\hat{\mathcal{C}}^u$, then s is in $\hat{\mathcal{C}}^u$ by [4, Proposition 2.20.3]. Therefore we may suppose that the interior of $\mathcal{R}_{\delta p, q\delta}$ does not contain any vertices of Q^+ .

First suppose p^+ and q^+ do not intersect (modulo I). Then δ is contained in a simple matching D of A by [4, Lemma 2.15, second statement]; see Fig. 2.i. But this is a contradiction since $\delta \in Q_1^S$.

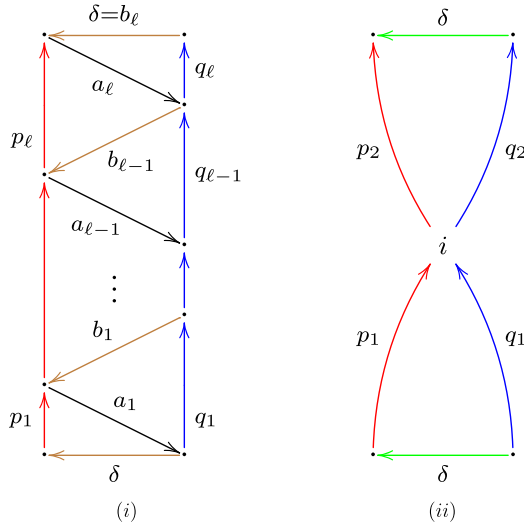


Fig. 2. Cases for Lemma 2.4. In case (i), p and q factor into paths $p = p_\ell \cdots p_2 p_1$ and $q = q_\ell \cdots q_2 q_1$, where $a_1, \dots, a_\ell, b_1, \dots, b_\ell$ are arrows, and the cycles $a_j b_j q_j$ and $b_{j-1} a_j p_j$ are unit cycles. By [4, Lemma 2.16], the b_j arrows, drawn in brown, belong to a simple matching D of A . In case (ii), p and q factor into paths $p = p_2 e_i p_1$ and $q = q_2 e_i q_1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Therefore p^+ and q^+ intersect at a vertex i^+ ; see Fig. 2.ii. By assumption, $\sigma \nmid z$. Thus, since $\bar{\delta} = 1$,

$$\bar{p}_1 = \bar{q}_1 \quad \text{and} \quad \bar{p}_2 = \bar{q}_2,$$

by [4, Lemma 2.3.2]. Whence

$$\overline{q_2 p_1 \delta} = \bar{q}_2 \bar{q}_1 = \bar{q} = z.$$

Therefore, since $z \in R$ and $\sigma \nmid z$,

$$q_2 p_1 \delta \in Z e_{t(\delta)},$$

by [4, Proposition 4.30.1]. In particular, we may take $s = q_2 p_1$.

(ii) Now suppose $r = \delta_\ell \cdots \delta_2 \delta_1 \neq 0$, with each $\delta_i \in Q_1^S$. By Claim (1), for each $1 \leq i \leq \ell$ there is a central element $z_i \in Z$ such that

$$z_i e_{t(\delta_i)} = s_i \delta_i.$$

Set $s := s_1 s_2 \cdots s_\ell$. Then the central element $z := z_\ell \cdots z_2 z_1$ satisfies

$$\begin{aligned} z e_{t(s)} &= z_\ell \cdots z_3 z_2 (s_1 \delta_1) \\ &= z_\ell \cdots z_3 s_1 z_2 \delta_1 \\ &= z_\ell \cdots z_3 s_1 s_2 \delta_2 \delta_1 \end{aligned}$$

$$\begin{aligned} & \vdots \\ & = s_1 s_2 \cdots s_\ell \delta_\ell \cdots \delta_2 \delta_1 \\ & = sr. \quad \square \end{aligned}$$

Denote the origin of $\text{Max } S$ by

$$\mathfrak{n}_0 := (\bar{s} \in S \mid s \text{ a non-trivial cycle in } Q) S \in \text{Max } S.$$

Then the Z -annihilator $\mathfrak{z}_0 \in \text{Max } Z$ of the vertex simple A -modules satisfies

$$\hat{\mathfrak{z}}_0 = \mathfrak{n}_0 \cap \hat{Z} \in \text{Max } \hat{Z}.$$

Recall our standing assumption that $\mathfrak{q} \in \text{Spec } S$ satisfies $\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset$, and $\hat{\mathfrak{p}} := \mathfrak{q} \cap \hat{Z}$.

Lemma 2.5. *Suppose that each non-constant monomial in \hat{Z} is divisible by σ . Then each monomial in \hat{Z} is invertible in the localization $\hat{Z}_{\hat{\mathfrak{p}}}$.*

Proof. By assumption, $\mathfrak{q} \in \text{Spec } S$ satisfies $\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset$. Thus \mathfrak{q} is contained in some maximal ideal $\mathfrak{n} \in \text{Max } S \setminus \{\mathfrak{n}_0\}$.

Assume to the contrary that $\hat{\mathfrak{m}} := \mathfrak{n} \cap \hat{Z}$ contains a monomial. Then $\hat{\mathfrak{m}} = \hat{\mathfrak{z}}_0$ by [4, Lemma 4.56]. Whence

$$\mathfrak{n} \cap \hat{Z} = \hat{\mathfrak{m}} = \hat{\mathfrak{z}}_0 = \mathfrak{n}_0 \cap \hat{Z}.$$

But then $\mathfrak{n} = \mathfrak{n}_0$ by [3, Theorem 2.5.1], a contradiction. Therefore each monomial in \hat{Z} is not in $\hat{\mathfrak{m}}$. In particular, each monomial in \hat{Z} is not in $\hat{\mathfrak{p}} = \mathfrak{q} \cap \hat{Z} \subseteq \hat{\mathfrak{m}}$. \square

Theorem 2.6. *Suppose $\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset$. If $p, q \in A$ is a non-cancellative pair, then $p = q$ in the localization $A_{\mathfrak{p}}$. In particular,*

$$\text{nil } Z \cdot Z_{\mathfrak{p}} = 0. \tag{6}$$

Proof. Fix $\mathfrak{n} \in \mathcal{Z}(\mathfrak{q}) \cap U$; then $\hat{\mathfrak{p}}$ is contained in $\hat{\mathfrak{m}} := \mathfrak{n} \cap \hat{Z} \in \text{Max } \hat{Z}$.

(i) Consider a non-cancellative pair $p, q \in e_j A e_i$. We claim that $p = q$ in $A_{\mathfrak{p}}$. It suffices to suppose that there is no non-cancellative pair s^+, t^+ which bounds a region $\mathcal{R}_{s,t} \subset \mathbb{R}^2$ properly contained in the region $\mathcal{R}_{p,q}$ bounded by p^+, q^+ .³ Let r be a path of minimal length such that $rp = rq \neq 0$. Then by [4, Proposition 4.37], each arrow subpath of r is in Q_1^S .

(i.a) First suppose there is a non-constant monomial in \hat{Z} which is not divisible (in B) by σ . Then by [4, Lemma 4.58], there is a non-constant monomial $z \in Z \setminus \mathfrak{m}$ such that $\sigma \nmid z$. Thus by Lemma 2.4, there is a path s such that

³ Such a pair p, q is called *minimal* in [4].

$$ze_{t(r)} = sr.$$

Therefore in $A_{\mathfrak{p}}$,

$$p - q = \frac{z}{z}(p - q) = \frac{sr}{z}(p - q) = \frac{s}{z}r(p - q) = 0.$$

(i.b) Now suppose every non-constant monomial in \hat{Z} is divisible by σ . Let s be a path from $h(r)$ to $t(r)$ that passes through each vertex of Q . Then $\overline{sr} \in R$. Thus there is some $n \geq 1$ such that

$$(sr)^n \in Ze_{t(r)},$$

by [4, Proposition 4.30.3]. Let $z \in Z$ be such that $ze_{t(r)} = (sr)^n$. Then z is invertible in $Z_{\mathfrak{p}}$ by Lemma 2.5. Therefore in $A_{\mathfrak{p}}$,

$$p - q = \frac{z}{z}(p - q) = \frac{(sr)^{n-1}sr}{z}(p - q) = \frac{(sr)^{n-1}s}{z}r(p - q) = 0.$$

(ii) By [4, Theorem 4.24],

$$\text{nil } Z = Z \cap \ker \psi.$$

Therefore by Claim (i),

$$\text{nil } Z \cdot Z_{\mathfrak{p}} = \ker \psi|_Z \cdot Z_{\mathfrak{p}} = 0. \quad \square$$

Lemma 2.7. *There is an algebra isomorphism*

$$A \otimes_Z \hat{Z}_{\mathfrak{p}} \cong A \otimes_Z Z_{\mathfrak{p}}.$$

Proof. (i) We first claim that

$$A_{\mathfrak{p}} \otimes_Z \hat{Z} \cong A \otimes_Z \hat{Z}_{\mathfrak{p}}.$$

It suffices to show that

$$Z_{\mathfrak{p}} \otimes_Z \hat{Z} \cong \hat{Z}_{\mathfrak{p}}.$$

Let $z \in Z$ and set $\hat{z} := z + \text{nil } Z$. If $z^{-1} \in Z_{\mathfrak{p}}$, then $z \notin \mathfrak{p}$. But $z \notin \text{nil } Z$ since $\text{nil } Z \subset \mathfrak{p}$. Thus $\hat{z}^{-1} \in \hat{Z}_{\mathfrak{p}}$. Conversely, if $\hat{z}^{-1} \in \hat{Z}_{\mathfrak{p}}$, then $z \notin \text{nil } Z \cup \mathfrak{p}$. Whence $z \notin \mathfrak{p}$. Therefore $z^{-1} \in Z_{\mathfrak{p}}$.

(ii) Applying the right exact functor $A_{\mathfrak{p}} \otimes_Z -$ to the short exact sequence

$$0 \rightarrow \text{nil } Z \rightarrow Z \rightarrow \hat{Z} \rightarrow 0$$

we obtain the exact sequence

$$A_p \otimes_Z \text{nil } Z \rightarrow A_p \otimes_Z Z \cong A_p \rightarrow A_p \otimes_Z \hat{Z} \stackrel{(i)}{\cong} A \otimes_Z \hat{Z}_p \rightarrow 0,$$

where (i) holds by Claim (i). But the left-most term is zero by Theorem 2.6. Therefore $A \otimes_Z \hat{Z}_p \cong A_p$. \square

Lemma 2.8. *The restriction $\psi : Z \rightarrow A'$ induces an algebra isomorphism*

$$Z_p \cong \hat{Z}_p \xrightarrow{\cong} Z'_q.$$

Proof. There are algebra isomorphisms

$$Z_p \stackrel{(i)}{\cong} \hat{Z}_p \stackrel{(ii)}{\cong} S_q \stackrel{(iii)}{\cong} Z'_q.$$

Indeed, (i) holds by Theorem 2.6. (ii) holds since S is a depiction of \hat{Z} [4, Theorem 4.68.1], and our assumption that $\mathcal{Z}(q) \cap U \neq \emptyset$. Finally, (iii) holds by (2). \square

It follows from Lemma 2.8 that the contraction $\psi : A \rightarrow A'$ extends to a k -linear map

$$\psi_p : A_p = A \otimes_Z Z_p \longrightarrow A'_q = A' \otimes_{Z'} Z'_q,$$

where the restriction of ψ_p to Z_p is the isomorphism $Z_p \rightarrow Z'_q$. We call this extension a *localized contraction*.

Remark 2.9. If a contraction $\psi : A \rightarrow A'$ is cyclic, then

$$Q_1^* \subseteq Q_1^S$$

by [4, Theorem 4.38]. In words, no contracted arrow is represented by zero in any simple representation of A of dimension vector 1^{Q_0} .

Proposition 2.10. *Suppose $\delta \in Q_1^*$, or more generally, $\delta \in Q_1^S$. Then A_p contains an element $\delta^* \in e_{t(\delta)} A_p e_{h(\delta)}$ satisfying*

$$\delta^* \delta = e_{t(\delta)} \quad \text{and} \quad \delta \delta^* = e_{h(\delta)}. \tag{7}$$

Furthermore, if $\delta \in Q_1^*$, then

$$\psi_p(\delta^*) = \psi_p(\delta) \in Q'_0.$$

Proof. Suppose $\delta \in Q_1^S$. Fix $\mathfrak{n} \in \mathcal{Z}(\mathfrak{q}) \cap U$; then $\hat{\mathfrak{p}}$ is contained in $\hat{\mathfrak{m}} := \mathfrak{n} \cap \hat{Z} \in \text{Max } \hat{Z}$.

By Claims (i.a) and (i.b) in the proof of [Theorem 2.6](#) (with $r = \delta$), there is some $z \in Z \setminus \mathfrak{m}$ and a path s such that

$$ze_{t(\delta)} = s\delta.$$

Set

$$\delta^* := \frac{s}{z} \in A_{\mathfrak{p}}.$$

Then in $A_{\mathfrak{p}}$,

$$\delta^* \delta = \frac{s}{z} \delta = \frac{ze_{t(\delta)}}{z} = \frac{z}{z} e_{t(\delta)} = e_{t(\delta)}.$$

Similarly, $\delta \delta^* = e_{h(\delta)}$.

Finally, set $w := \psi(z)$. Then

$$\psi_{\mathfrak{p}}(\delta^*) = \psi_{\mathfrak{p}}(s)w^{-1} = \psi_{\mathfrak{p}}(s\delta)w^{-1} = \psi_{\mathfrak{p}}(e_{t(\delta)}z)w^{-1} = \psi_{\mathfrak{p}}(e_{t(\delta)})ww^{-1} = \psi_{\mathfrak{p}}(\delta). \quad \square$$

For the following, note that cyclic contractions are not surjective in general.

Lemma 2.11. *The localized contraction $\psi_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow A'_{\mathfrak{q}}$ is surjective, but not injective.*

Proof. (i) We first claim that

$$A' \subseteq \psi_{\mathfrak{p}}(A_{\mathfrak{p}}). \tag{8}$$

Indeed, let $q \in A'$ be a path which is not in the ψ -image of A . Then q factors into paths

$$q = q_n \psi(\delta_{n-1}) q_{n-1} \cdots q_2 \psi(\delta_1) q_1,$$

where for each i , there is a path $p_i \in A$ such that $\psi(p_i) = q_i$; $\delta_i \in Q_1^*$; and

$$h(\delta_i) = h(p_i) \quad \text{and} \quad t(\delta_i) = t(p_{i+1}).$$

Furthermore, for each i there is an ‘arrow’ δ_i^* , with opposite orientation to δ_i , satisfying

$$\psi_{\mathfrak{p}}(\delta_i^*) = \psi_{\mathfrak{p}}(\delta_i) \in Q'_0,$$

by [Proposition 2.10](#). Thus q is the $\psi_{\mathfrak{p}}$ -image of the element

$$p = p_n \delta_{n-1}^* p_{n-1} \cdots p_2 \delta_1^* p_1 \in A_{\mathfrak{p}}.$$

Therefore (8) holds.

Furthermore, by the definition of ψ_p ,

$$Z'_q = \psi_p(Z_p) \subseteq \psi_p(A_p).$$

Therefore $A'_q = A' \otimes_{Z'} Z'_q \subseteq \psi_p(A_p)$.

(ii) ψ_p is not injective since $\psi_p(e_t(\delta)) = \psi_p(e_h(\delta))$ whenever $\delta \in Q_1^*$. \square

Set

$$\epsilon_0 := 1_A - \sum_{\delta \in Q_1^*} e_h(\delta).$$

By [4, Remark 4.10], a non-trivial contraction cannot be an algebra homomorphism.

Proposition 2.12.

(1) The map

$$\psi : \epsilon_0 A \epsilon_0 \rightarrow A'$$

is an algebra homomorphism. Furthermore, its localization

$$\psi_p : \epsilon_0 A_p \epsilon_0 \rightarrow A'_q \tag{9}$$

is an algebra isomorphism.

(2) For each $\delta \in Q_0^*$, the k -linear maps

$$\psi_p : \epsilon_0 A_p e_h(\delta) \rightarrow A'_q e_{\psi(\delta)} \quad \text{and} \quad \psi_p : e_h(\delta) A_p \epsilon_0 \rightarrow e_{\psi(\delta)} A'_q \tag{10}$$

are bijective.

(3) For each $i, j \in Q_0$, the k -linear map

$$\psi_p : e_i A_p e_j \rightarrow e_{\psi(i)} A'_q e_{\psi(j)} \tag{11}$$

is bijective. Consequently, if $i = j$, then it is an algebra isomorphism.

Proof. (1.i) We first claim that the restriction $\psi|_{\epsilon_0 A \epsilon_0}$ is an algebra homomorphism. Since ψ is a k -linear map, it suffices to show that the restriction is multiplicative on paths.

Let $p, q \in \epsilon_0 A \epsilon_0$ be paths. First suppose $\psi(q)\psi(p) \neq 0$. Then

$$h(\psi(p)) = t(\psi(q)) \in Q_0'.$$

Thus $h(p) = t(q)$ since $p, q \in \epsilon_0 A \epsilon_0$. Whence $qp \neq 0$. Therefore $\psi(qp) = \psi(q)\psi(p)$.

Now suppose $\psi(q)\psi(p) = 0$. Then $h(\psi(p)) \neq t(\psi(q))$. In particular, $h(p) \neq t(q)$. Therefore $\psi(qp) = \psi(0) = 0 = \psi(q)\psi(p)$.

(1.ii) We now claim that the map (9) is an algebra isomorphism. Indeed, (9) is an algebra homomorphism by Claim (i). Furthermore,

$$\psi(\epsilon_0) = \sum_{i \in Q'_0} e_i = 1_{A'}. \tag{12}$$

Thus (9) is surjective by Lemma 2.11.

To show injectivity, let $\delta \in Q_1^*$. By [4, Lemma 4.8], no unoriented cycle is contracted to a vertex. Thus, without loss of generality, we may suppose that the head and tail of δ are not the head or tail of another contracted arrow in A . Then the ψ_p -preimage of the vertex $\psi(\delta) \in Q'_0$ consists of the four elements

$$\delta, \delta^*, e_{t(\delta)}, e_{h(\delta)}, \tag{13}$$

by Proposition 2.10. Furthermore, since the tail of δ is not head of another contracted arrow, the only one of these elements in $\epsilon_0 A_p \epsilon_0$ is the idempotent $e_{t(\delta)}$. Therefore (9) is injective by Theorem 2.6 and (7). This proves our claim.

(2) The maps (10) are surjective by (12) and Lemma 2.11. To show injectivity, again suppose that the head and tail of δ are not the head or tail of another contracted arrow in A . Then the ψ_p -preimage of the vertex $\psi(\delta)$ consists of the four elements (13). The only such element in $\epsilon_0 A_p e_{h(\delta)}$ is δ^* , and the only such element in $e_{h(\delta)} A \epsilon_0$ is δ . Therefore the maps (10) are injective by Theorem 2.6 and (7).

(3) The map (11) is surjective by Lemma 2.11, and injective by Theorem 2.6 and (7). Furthermore, the restriction of ψ to the vertex corner ring $e_i A e_i$,

$$\psi : e_i A e_i \rightarrow A',$$

is an algebra homomorphism. \square

Theorem 2.13. *Let $\psi : A \rightarrow A'$ be a cyclic contraction of dimer algebras. Then the localizations A_p and A'_q are Morita equivalent if and only if $\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset$.*

Proof. Set

$$P := A_p = A \otimes_Z Z_p \quad \text{and} \quad Q := A'_q = A' \otimes_{Z'} Z'_q.$$

First suppose $\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset$. Enumerate the contracted arrows,

$$Q_1^* = \{\delta_1, \dots, \delta_n\} \subset Q_1.$$

For $1 \leq i \leq n$, set

$$\epsilon_i := e_{h(\delta_i)} \in Q_0 \quad \text{and} \quad \epsilon'_i := \psi(\delta_i) \in Q'_0. \tag{14}$$

Furthermore, set

$$\epsilon_0 := 1_A - \sum_{i=1}^n \epsilon_i \quad \text{and} \quad \epsilon'_0 := 1_{A'}. \tag{15}$$

Then P is isomorphic to the $(n + 1) \times (n + 1)$ tiled matrix algebra,

$$P \cong [\epsilon_i P \epsilon_j]_{ij}. \tag{16}$$

Consider the k -linear map

$$\zeta : [\epsilon_i P \epsilon_j]_{ij} \longrightarrow [\epsilon'_i Q \epsilon'_j]_{ij}, \tag{17}$$

defined by sending the ij -th entry $\alpha \in \epsilon_i P \epsilon_j$ to the ij -th entry $\psi_p(\alpha) \in \epsilon'_i Q \epsilon'_j$. This map is an algebra isomorphism by [Proposition 2.12](#). Furthermore, using the isomorphism (16), we may view ζ as an algebra isomorphism

$$\zeta : P \xrightarrow{\cong} [\epsilon'_i Q \epsilon'_j]_{ij}.$$

Now consider the bimodules

$${}_P M_Q = \begin{bmatrix} \epsilon'_0 Q \\ \epsilon'_1 Q \\ \vdots \\ \epsilon'_n Q \end{bmatrix} \quad \text{and} \quad {}_Q N_P = \begin{bmatrix} Q \epsilon'_0 & Q \epsilon'_1 & \cdots & Q \epsilon'_n \end{bmatrix}, \tag{18}$$

where P acts via the isomorphism $P \cong \zeta(P)$. The P, P -bimodule homomorphism

$$\theta : M \otimes_Q N \rightarrow \zeta(P) \cong P$$

defined by

$$\begin{bmatrix} \epsilon'_0 s_0 \\ \epsilon'_1 s_1 \\ \vdots \\ \epsilon'_n s_n \end{bmatrix} \otimes \begin{bmatrix} t_0 \epsilon'_0 & t_1 \epsilon'_1 & \cdots & t_n \epsilon'_n \end{bmatrix} \mapsto [\epsilon'_i s_i t_j \epsilon'_j]_{ij}$$

is clearly surjective. Furthermore, the Q, Q -bimodule homomorphism

$$\phi : N \otimes_P M \rightarrow Q \tag{19}$$

defined by

$$\begin{bmatrix} t_0\epsilon'_0 & t_1\epsilon'_1 & \cdots & t_n\epsilon'_n \end{bmatrix} \otimes \begin{bmatrix} \epsilon'_0s_0 \\ \epsilon'_1s_1 \\ \vdots \\ \epsilon'_ns_n \end{bmatrix} \mapsto \sum_{i=0}^n t_i\epsilon'_i\epsilon'_i s_i = \sum_{i=0}^n t_i s_i$$

is also surjective. Thus, since P and Q are unital, θ and ϕ are bimodule isomorphisms [8, Lemma 4.5.2],

$$P \cong M \otimes_Q N \quad \text{and} \quad Q \cong N \otimes_P M.$$

Therefore P and Q are Morita equivalent, with progenerators N and M.

Conversely, suppose $Z(\mathfrak{q}) \cap U = \emptyset$. Then $\hat{Z}_{\mathfrak{p}} \neq S_{\mathfrak{q}}$. Furthermore, $S \cong Z'$ by (2). Whence the centers of P and Q are not isomorphic:

$$Z(P) = \hat{Z}_{\mathfrak{p}} \neq S_{\mathfrak{q}} \cong Z(Q).$$

But P and Q are unital. Therefore P and Q are not Morita equivalent [13, Theorem 5.9.iii]. \square

Although Z may not be reduced, its reduction $\hat{Z} = Z/\text{nil } Z$ is an integral domain [4, Corollary 4.28].

Corollary 2.14.

(1) *The (noncommutative) function fields*

$$A \otimes_Z \text{Frac } \hat{Z}, \quad A' \otimes_{Z'} \text{Frac } Z', \quad \text{Frac } \hat{Z}, \quad \text{Frac } Z',$$

are Morita equivalent.

(2) *If $Z(\mathfrak{q}) \cap U \neq \emptyset$, then the noncommutative residue fields*

$$A_{\mathfrak{p}/\mathfrak{p}} \quad \text{and} \quad A'_{\mathfrak{q}/\mathfrak{q}}$$

are Morita equivalent.

Proof. (1) We have the following Morita equivalences:

$$A \otimes_Z \text{Frac } \hat{Z} \stackrel{(i)}{\sim} A' \otimes_{Z'} \text{Frac } Z' \stackrel{(ii)}{\sim} \text{Frac } Z' \stackrel{(iii)}{=} \text{Frac } S \stackrel{(iv)}{=} \text{Frac } \hat{Z}.$$

Indeed, (i) follows from Theorem 2.13. (ii) holds since A' is a cancellative, whence a non-commutative crepant resolution, and thus an endomorphism ring of a finitely generated

projective Z' -module. (III) holds by (2). Finally, (IV) holds since S is a depiction of \hat{Z} [4, Theorem 4.68.1].

(2) Recall the bimodule isomorphism ϕ defined in (19). Since $\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset$, the localizations $A_{\mathfrak{p}}$ and $A'_{\mathfrak{q}}$ are Morita equivalent by Theorem 2.13. Whence

$$A_{\mathfrak{p}}/\mathfrak{p} \quad \text{and} \quad A'_{\mathfrak{q}}/\phi(N \otimes_{\mathfrak{p}} A_{\mathfrak{p}} M)$$

are Morita equivalent [13, Theorem 5.9.ii]. Furthermore, for $0 \leq i, j \leq n$,

$$\psi_{\mathfrak{p}}(\epsilon_i \mathfrak{p} A_{\mathfrak{p}} \epsilon_j) = \epsilon'_i \mathfrak{q} A'_{\mathfrak{q}} \epsilon'_j, \tag{20}$$

by Proposition 2.12. Thus

$$\begin{aligned} \phi(N \otimes_{\mathfrak{p}} A_{\mathfrak{p}} M) &= N \cdot \zeta(\mathfrak{p} A_{\mathfrak{p}}) \cdot M \\ &= \sum_{0 \leq i, j \leq n} A'_{\mathfrak{q}} \epsilon'_i \cdot \psi_{\mathfrak{p}}(\epsilon_i \mathfrak{p} A_{\mathfrak{p}} \epsilon_j) \cdot \epsilon'_j A'_{\mathfrak{q}} \\ &= \sum_{0 \leq i, j \leq n} A'_{\mathfrak{q}} \epsilon'_i \cdot \epsilon'_i \mathfrak{q} A'_{\mathfrak{q}} \epsilon'_j \cdot \epsilon'_j A'_{\mathfrak{q}} \\ &= \mathfrak{q} \sum_{0 \leq i, j \leq n} A'_{\mathfrak{q}} \epsilon'_i A'_{\mathfrak{q}} \epsilon'_j A'_{\mathfrak{q}} \\ &= \mathfrak{q} A'_{\mathfrak{q}}. \end{aligned}$$

Therefore $A_{\mathfrak{p}}/\mathfrak{p}$ and $A'_{\mathfrak{q}}/\mathfrak{q}$ are Morita equivalent. \square

Remark 2.15. The homotopy algebra of a dimer algebra A is the quotient

$$\tilde{A} := A / \langle p - q \mid p, q \text{ is a non-cancellative pair} \rangle,$$

introduced in [4, Definition 4.33]. The center \tilde{Z} of \tilde{A} is isomorphic to the homotopy center R of A defined in (3) by [4, Theorem 4.35]. Thus $\text{nil } \tilde{Z} = 0$, and in particular (6) holds. It follows that Theorem 2.13 and Corollary 2.14 hold for homotopy dimer algebras as well.

By [4, Theorems 4.17 and 4.45], A is often not prime and nonnoetherian. However, we have the following.

Corollary 2.16. *If $\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset$, then the localization $A_{\mathfrak{p}}$ is prime and noetherian.*

Proof. Since A' is a cancellative dimer algebra, A' is prime and noetherian [4, Theorem 3.3.3 and Proposition 3.11]. Thus $A'_{\mathfrak{q}}$ is prime and noetherian. But $A_{\mathfrak{p}}$ is Morita equivalent to $A'_{\mathfrak{q}}$ by Theorem 2.13. Therefore $A_{\mathfrak{p}}$ is prime and noetherian as well [13, Proposition 5.10].

The fact that $A_{\mathfrak{p}}$ is prime also follows directly from Lemma 3.1 below and the proof of [4, Proposition 3.11, with A and τ replaced by $A_{\mathfrak{p}}$ and $\tilde{\tau}_{\mathfrak{p}}$]. \square

3. Azumaya loci

Throughout, A is a non-cancellative dimer algebra; $\psi : A \rightarrow A'$ is a cyclic contraction; and unless stated otherwise, $\mathfrak{q} \in \text{Spec } S$ satisfies

$$\mathcal{Z}(\mathfrak{q}) \cap U \neq \emptyset.$$

Set $\hat{\mathfrak{p}} := \mathfrak{q} \cap \hat{Z} \in \text{Spec } \hat{Z}$.

3.1. Azumaya and smooth loci

Recall the algebra monomorphism

$$\tau : A' \rightarrow M_{|Q_0|}(B)$$

defined in (1). Similarly, there is an algebra homomorphism

$$\tilde{\tau} : A \rightarrow M_{|Q_0|}(B) \tag{21}$$

defined on $p \in e_j A e_i$ by

$$p \mapsto \tilde{p} E_{ji} = \bar{\tau} \psi(p) E_{ji},$$

and extended k -linearly to A [4, Lemma 4.25]. In contrast to τ , $\tilde{\tau}$ is not injective [4, Lemma 4.12]. However, we have the following.

Lemma 3.1. *There is some $\mathfrak{b} \in \text{Max } B$ such that the algebra homomorphism $\tilde{\tau}$ induces an algebra monomorphism*

$$\tilde{\tau}_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow M_{|Q_0|}(B_{\mathfrak{b}}).$$

Proof. (i) We first claim that $\tilde{\tau}_{\mathfrak{p}}$ is well-defined. Indeed, since A' is cancellative, $Z' \cong S$ by (2). Furthermore, since (τ, B) is an impression of A' , the morphism $\text{Max } B \rightarrow \text{Max } Z' = \text{Max } S$ is surjective [4, Theorem 3.5]. Therefore there is an ideal $\mathfrak{b} \in \text{Spec } B$ such that $\mathfrak{b} \cap S = \mathfrak{q}$ [2, Lemma 2.15]. In particular,

$$\mathfrak{b} \cap \hat{Z} = \hat{\mathfrak{p}}.$$

The claim then follows since $Z_{\mathfrak{p}} = \hat{Z}_{\hat{\mathfrak{p}}}$ by Lemma 2.8.

(ii) We now claim that $\tilde{\tau}_{\mathfrak{p}}$ is injective. The k -linear map

$$\bar{\tau} : e_j A' e_i \rightarrow B$$

is injective for each $i, j \in Q_0$, by [4, Theorem 3.5]. Thus the kernel of $\tilde{\tau}$ is generated by elements of the form $p - q$, where $\psi(p) = \psi(q)$.

If either $t(p) \neq t(q)$ or $h(p) \neq h(q)$, then

$$\tilde{\tau}(p) \propto E_{h(p),t(p)} \quad \text{and} \quad \tilde{\tau}(q) \propto E_{h(q),t(q)}$$

have distinct non-zero matrix entries. Whence $p - q \notin \ker \tilde{\tau}$. Thus if $p - q \in \ker \tilde{\tau}$, then $t(p) = t(q)$ and $h(p) = h(q)$. But then $p = q$ in $A_{\mathfrak{p}}$, by [Theorem 2.6](#). Therefore $\tilde{\tau}_{\mathfrak{p}}$ is injective. \square

Lemma 3.2. *For each $i, j \in Q_0$,*

$$\tilde{\tau}\psi_{\mathfrak{p}}(e_i A_{\mathfrak{p}} e_i) = \tilde{\tau}\psi_{\mathfrak{p}}(e_j A_{\mathfrak{p}} e_j).$$

Proof. We have

$$\tilde{\tau}\psi_{\mathfrak{p}}(e_i A_{\mathfrak{p}} e_i) \stackrel{(i)}{=} \tilde{\tau}(e_{\psi(i)} A'_{\mathfrak{q}} e_{\psi(i)}) \stackrel{(ii)}{=} \tilde{\tau}(e_{\psi(j)} A'_{\mathfrak{q}} e_{\psi(j)}) \stackrel{(iii)}{=} \tilde{\tau}\psi_{\mathfrak{p}}(e_j A_{\mathfrak{p}} e_j).$$

Indeed, (i) and (iii) hold by [Proposition 2.12.2](#), and (ii) holds by [\[4, Theorem 3.3.2\]](#). \square

Proposition 3.3. *$A_{\mathfrak{p}}$ is a finitely generated module over its center $Z_{\mathfrak{p}}$.*

Proof. $A_{\mathfrak{p}}$ is generated as a $Z_{\mathfrak{p}}$ -module by all paths of length at most $|Q_0|$ by [Lemma 3.2](#) and [\[2, second paragraph of proof of Theorem 2.11 with \$e_i A e_i = Z e_i\$ replaced by \$e_i A_{\mathfrak{p}} e_i \subseteq Z_{\mathfrak{p}} e_i\$ \]](#). Therefore $A_{\mathfrak{p}}$ is a finitely generated $Z_{\mathfrak{p}}$ -module. \square

By [\[4, Theorem 4.50\]](#), A is nonnoetherian and an infinitely generated Z -module. In contrast, we have the following.

Theorem 3.4. *Suppose $Z(\mathfrak{q}) \cap U \neq \emptyset$. Then the localized algebra $A_{\mathfrak{p}}$ is prime, noetherian, and a finitely generated module over its center $Z_{\mathfrak{p}}$ with PI degree $|Q_0|$.*

Proof. $A_{\mathfrak{p}}$ is prime, noetherian, and a finitely generated module over its center by [Propositions 2.16 and 3.3](#) respectively. Furthermore, the algebra homomorphism $\tilde{\tau}_{\mathfrak{p}}$ is injective by [Lemma 3.1](#). Thus the PI degree of $A_{\mathfrak{p}}$ is $|Q_0|$ by [\[2, Lemma 2.4, with \$A, U, \tau_{\mathfrak{q}}\$ replaced respectively by \$A_{\mathfrak{p}}, \{\mathfrak{b}\}, \tilde{\tau}_{\mathfrak{p}}\$ \]](#). \square

Lemma 3.5. *Suppose $\mathfrak{n} \notin U$, and set $\hat{\mathfrak{m}} = \mathfrak{n} \cap \hat{Z}$. Then $A_{\mathfrak{m}}$ is not an Azumaya algebra.*

Proof. Suppose the hypotheses hold. Since $\mathfrak{n} \notin U$, we have $\mathfrak{m} = \mathfrak{z}_0$ by [\(5\)](#).

Recall that if $A_{\mathfrak{m}}$ is an Azumaya algebra, then $A_{\mathfrak{m}}/\mathfrak{m}$ is a central simple algebra over k [\[13, Proposition 7.11\]](#) (that is, a simple algebra whose center is k). We claim that $A_{\mathfrak{z}_0}/\mathfrak{z}_0$ is not a central simple algebra. Indeed, since A is non-cancellative and A' is cancellative, the contraction $\psi : A \rightarrow A'$ is non-trivial. Thus there is at least one arrow $\delta \in Q_1$ which is contracted to a vertex. By [\[4, Lemma 4.8.1\]](#), no cycle is contracted to a vertex. In

particular, δ is not a loop. Whence $|Q'_0| < |Q_0|$. Thus $|Q_0| \geq 2$. Therefore there are at least two distinct vertex idempotents $e_1, e_2 \in A$.

Clearly $e_1, e_2 \notin \mathfrak{z}_0 A_{\mathfrak{z}_0}$. Whence the (two-sided) ideal $\langle e_1 \rangle$ of $A_{\mathfrak{z}_0}/\mathfrak{z}_0$ is nonzero. Furthermore, $\langle e_1 \rangle$ is a proper ideal since $e_2 \notin \langle e_1 \rangle$. Therefore $A_{\mathfrak{z}_0}/\mathfrak{z}_0$ is not a simple algebra, and so $A_{\mathfrak{z}_0}$ is not Azumaya. \square

Theorem 3.6. *The Azumaya locus $\mathcal{A} \subset \text{Max } Z$ of A coincides with the intersection of the Azumaya locus $\mathcal{A}' \subset \text{Max } Z'$ of A' and the locus $U \subset \text{Max } Z'$,*

$$\mathcal{A} \cong \mathcal{A}' \cap U.$$

This isomorphism is defined by sending $\mathfrak{n} \in \mathcal{A}' \cap U$ to $\mathfrak{m} \in \mathcal{A}$, where $\hat{\mathfrak{m}} = \mathfrak{n} \cap \hat{Z}$.

Proof. Let $\mathfrak{n} \in \text{Max } Z' = \text{Max } S$, and set $\hat{\mathfrak{m}} := \mathfrak{n} \cap \hat{Z}$.

If $\mathfrak{n} \notin U$, then $A_{\mathfrak{m}}$ is not an Azumaya algebra by Lemma 3.5. So suppose $\mathfrak{n} \in U$. Then $A_{\mathfrak{m}}/\mathfrak{m}$ and $A'_{\mathfrak{n}}/\mathfrak{n}$ are Morita equivalent by Corollary 2.14.2. In particular, $A_{\mathfrak{m}}/\mathfrak{m}$ is central simple over k if and only if $A'_{\mathfrak{n}}/\mathfrak{n}$ is central simple over k . Since $\mathfrak{n} \in U$, $A_{\mathfrak{m}}$ and $A'_{\mathfrak{n}}$ are both prime, noetherian, and finitely generated modules over their centers with PI degrees $|Q_0|$ and $|Q'_0|$ respectively, by Proposition 3.4. Therefore $A_{\mathfrak{m}}$ is Azumaya if and only if $A'_{\mathfrak{n}}$ is Azumaya by the Artin–Procesi Theorem [13, Theorem 13.7.14]. \square

The following corollary gives the first known class of algebras that are nonnoetherian and infinitely generated modules over their centers, with the property that their Azumaya and smooth loci coincide. The $Y^{p,q}$ dimer algebras are defined in [2, Example 1.3].

Corollary 3.7. *If the Azumaya and smooth loci of A' coincide, then the Azumaya and smooth loci of A coincide. In particular, if A' is a $Y^{p,q}$ algebra then the Azumaya and smooth loci of A coincide.*

Proof. Suppose $\mathfrak{n} \notin U$. Then $\hat{\mathfrak{z}}_0 = \mathfrak{n} \cap \hat{Z}$ by (5). Thus $\hat{Z}_{\hat{\mathfrak{z}}_0}$ is an infinitely generated k -algebra [4, Lemma 4.55]. Whence the residue field $\hat{Z}_{\hat{\mathfrak{z}}_0}/\hat{\mathfrak{z}}_0$ has infinite projective dimension over $\hat{Z}_{\hat{\mathfrak{z}}_0}$. Therefore $\hat{\mathfrak{z}}_0$ is a singular point of \hat{Z} . The corollary then follows from Theorem 3.6. If A' is a $Y^{p,q}$ algebra, then its Azumaya and smooth loci coincide by [2, Theorem 7.3]. \square

Example 3.8. The Azumaya locus \mathcal{A}' and the locus U may be distinct. Indeed, consider the cyclic contraction $\psi : A \rightarrow A'$ given in Fig. 3, with $Q_1^* = \{\delta\}$. Further, consider the cycles $p, q \in A$ drawn in red and blue respectively. Let V_p and $V_{p'}$ be the simple A - and A' -modules of dimensions 1^{Q_0} and $1^{Q'_0}$ defined by

$$\rho(a) := \begin{cases} 1 & \text{if } a \text{ is a subpath of } p \\ 0 & \text{otherwise} \end{cases} \quad \text{for } a \in Q_0 \cup Q_1,$$

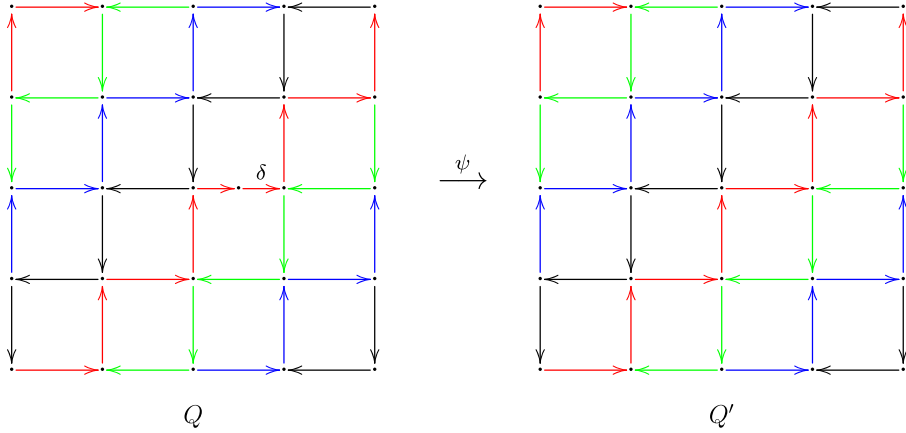


Fig. 3. The quivers Q and Q' in Example 3.8 and Remark 3.13, drawn on a torus. The cycles p, q, s in Q are drawn in red, blue, and green respectively, with suitably chosen tails. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\rho'(a) := \begin{cases} 1 & \text{if } a \text{ is a subpath of } \psi(p) \\ 0 & \text{otherwise} \end{cases} \quad \text{for } a \in Q'_0 \cup Q'_1.$$

Here, we are viewing ρ and ρ' as vector space diagrams on Q and Q' respectively. (In particular, for any path p in Q , $\rho(p)$ is a scalar rather than a $|Q_0| \times |Q_0|$ matrix.)

Set

$$\hat{\mathfrak{m}} := \text{ann}_{\hat{Z}} V_\rho \in \text{Max } \hat{Z} \quad \text{and} \quad \mathfrak{n} := \text{ann}_{Z'} V_{\rho'} \in \text{Max } Z'.$$

Then $\hat{\mathfrak{m}} = \mathfrak{n} \cap \hat{Z}$ under the isomorphism $Z' \cong S$. We claim that \mathfrak{n} is in U , but not in the Azumaya locus of A' .

To show that $\mathfrak{n} \in U$, it suffices to show that if \bar{s} is a monomial in $S \setminus \hat{Z}$, then \bar{s} is also in $\hat{Z}_{\hat{\mathfrak{m}}}$. Since $\sigma \in \hat{Z}$, we may suppose s is a cycle in \mathcal{C}^u with $u \in \mathbb{Z}^2 \setminus 0$.

First note that the only cycle in \mathcal{C}^u which does not share a vertex subpath with p is the cycle q (drawn in blue). But $\bar{q} = \bar{p}$ by [2, Theorem 3.7]. So consider a cycle $s \in \mathcal{C}^u$ which shares a vertex subpath with p , say at vertex $i \in Q_0$. (For example, we may take s to be the green cycle in the figure.) Denote by p_i and s_i the cyclic permutations of p and s with tails at i . Then by the symmetry of Q , it is clear that

$$\bar{s}\bar{p} = \bar{s}_i\bar{p}_i = \overline{s_i p_i} \in R.$$

Furthermore, it is straightforward to verify that there are no central elements in the kernel of ψ in this example. In particular, $\text{nil } Z = 0$ by [4, Theorem 4.24]. Thus $\hat{Z} \cong R$. Consequently, $\bar{s}\bar{p} \in \hat{Z}$. Since p does not annihilate V_ρ , the monomial \bar{p} is in $\hat{Z} \setminus \hat{\mathfrak{m}}$. Thus

$$\bar{s} = \bar{s}\bar{p}\bar{p}^{-1} \in \hat{Z}_{\hat{\mathfrak{m}}}.$$

Therefore $\hat{Z}_{\hat{\mathfrak{m}}} = S_{\mathfrak{n}}$. Whence $\mathfrak{n} \in U$.

Finally, \mathfrak{n} is not in the Azumaya locus of A' since the dimension vector of any simple A' -module of maximal k -dimension is $1^{Q'_0}$ [2, Proposition 2.5, Lemma 2.13].

3.2. Classification of simple modules parameterized by the Azumaya locus

Given a quiver algebra $A = kQ/I$ and dimension vector $d = (d_i)_{i \in Q_0}$, denote by $\text{Rep}_d(A)$ the closed affine variety of d -dimensional representations of A viewed as vector space diagrams on Q ,

$$\text{Rep}_d(A) \subset \bigoplus_{a \in Q_1} M_{d_{h(a)} \times d_{t(a)}}(k) = \mathbb{A}_k^{\sum_{a \in Q_1} d_{h(a)} d_{t(a)}}.$$

Let $\psi : A \rightarrow A'$ be a cyclic contraction of dimer algebras. For the following, consider simple representations $\rho \in \text{Rep}_{1^{Q_0}}(A)$ and $\rho' \in \text{Rep}_{1^{Q'_0}}(A')$. Recall that $\rho(\delta) \neq 0$ for each $\delta \in Q_1^*$, by Proposition 2.10.

Set $\rho_0 := \rho$. For each $n \geq 1$, define the representation $\rho_n \in \text{Rep}_{1^{Q_0}}(A)$ iteratively on $a \in Q_1$ by

$$\rho_{n+1}(a) := \rho_n(a) \prod_{\substack{\delta \in Q_1^* \\ h(\delta)=t(a)}} \rho_n(\delta) \prod_{\substack{\delta' \in Q_1^* \\ h(\delta')=h(a)}} \rho_n(\delta')^{-1}.$$

Since no unoriented cycle is contracted to a vertex by [4, Lemma 4.8], there is an $N \geq 1$ such that for each $n \geq N$,

$$\rho_n = \rho_N \quad \text{and} \quad \rho_N(\delta) = 1 \quad \text{for each } \delta \in Q_1^*.$$

Set $\rho^* := \rho_N$. Clearly ρ^* and ρ are isomorphic representations of A .

Consider the k -linear map

$$\psi^{-1} : Q'_1 \rightarrow \epsilon_0 Q_1 \epsilon_0 \tag{22}$$

defined by $\psi^{-1}(a) = b$ where $\psi(b) = a$. (Although ψ is not surjective in general, the arrows of Q' are in its image.) We may thus define representations $\psi^{-1}\rho' \in \text{Rep}_{1^{Q_0}}(A)$ and $\psi\rho \in \text{Rep}_{1^{Q'_0}}(A')$ by

$$(\psi^{-1}\rho')(a) := \rho'(\psi(a)) \quad \text{for each } a \in Q_1$$

and

$$(\psi\rho)(a) := \rho^*(\psi^{-1}(a)) \quad \text{for each } a \in Q'_1.$$

Note that $(\psi^{-1}\rho')^* = \psi^{-1}\rho'$.

Lemma 3.9. $\rho^* = \psi^{-1}\rho'$ if and only if $\psi\rho = \rho'$.

Proof. First suppose $\rho^* = \psi^{-1}\rho'$, and let $a \in Q_1$. Then

$$(\psi\rho)(a) = \rho^*(\psi^{-1}(a)) = (\psi^{-1}\rho')(\psi^{-1}(a)) = \rho'(\psi(\psi^{-1}(a))) = \rho'(a).$$

Conversely suppose $\psi\rho = \rho'$, and let $a \in Q'_1$. Then

$$(\psi^{-1}\rho')(a) = \rho'(\psi(a)) = (\psi\rho)(\psi(a)) = \rho^*(\psi^{-1}(\psi(a))) = \rho^*(a). \quad \square$$

Recall that a simple A -module V is said to sit over a point \mathfrak{m} in the Azumaya locus \mathcal{A} if $V_{\mathfrak{m}} := A_{\mathfrak{m}} \otimes_A V$ is the unique simple $A_{\mathfrak{m}}$ -module up to isomorphism. The Azumaya locus then parameterizes a family of simple A -module isoclasses.

Proposition 3.10. $V = V_{\rho}$ is a simple A -module of dimension 1^{Q_0} if and only if V sits over some point $\mathfrak{m} \in \mathcal{A}$.

Proof. (\Leftarrow) First suppose $\mathfrak{m} \in \mathcal{A}$. Let $V_{\mathfrak{m}}$ be the unique simple $A_{\mathfrak{m}}$ -module. We claim that V has dimension 1^{Q_0} .

By [Theorem 3.4](#), the PI degree of $A_{\mathfrak{m}}$ is $|Q_0|$. Thus

$$\dim_k(V_{\mathfrak{m}}) = |Q_0|. \tag{23}$$

Let $\mathfrak{n} \in \text{Max } S$ be such that $\mathfrak{n} \cap Z = \mathfrak{m}$. Since $\mathfrak{m} \in \mathcal{A}$, we have $\mathfrak{n} \in \mathcal{A}' \cap U$ by [Theorem 3.6](#). But $\mathfrak{n} \in U$ implies $\mathfrak{m} \neq \mathfrak{z}_0$ by (5). Thus there is a monomial $z \in Z \setminus \mathfrak{m}$. In particular, $\rho(z) \neq 0$.

Since V is simple and z is central, $\rho(z)$ is a scalar multiple of the identity by Schur’s lemma. Whence $\rho(ze_i) \neq 0$ for each $i \in Q_0$. Thus $\dim_k(e_i V_{\mathfrak{m}}) \geq 1$ for each $i \in Q_0$. Therefore $V_{\mathfrak{m}}$ has dimension 1^{Q_0} by (23).

(\Rightarrow) Now suppose V_{ρ} is simple of dimension 1^{Q_0} . We claim that V sits over a point in \mathcal{A} .

Consider $\rho' := \psi\rho$ as in [Lemma 3.9](#). Set $\mathfrak{m} := \text{ann}_Z V_{\rho}$ and $\mathfrak{n} := \text{ann}_{Z'} V_{\rho'}$. Recall that $V_{\rho'}$ is simple of dimension $1^{Q'_0}$. In particular,

$$\mathfrak{n} \in \mathcal{A}'. \tag{24}$$

Furthermore, since V_{ρ} is simple of dimension 1^{Q_0} , $\mathfrak{m} \neq \mathfrak{z}_0$. Thus by (5),

$$\mathfrak{n} \in U. \tag{25}$$

It then follows from (24), (25), and [Theorem 3.6](#) that $\mathfrak{m} \in \mathcal{A}$. \square

In the following, the algebra homomorphism $\tilde{\tau}$ defined in (21) is used to classify the simple A -modules parameterized by the Azumaya locus. This classification shows that

$\tilde{\tau}$ is very close to being an impression of A even though A may not embed into a matrix ring over a commutative ring; see [2, Proposition 2.5].

Theorem 3.11. *For each A -module V_ρ that sits over a point in the Azumaya locus \mathcal{A} , there is a point $\mathfrak{b} \in \text{Max } B$ such that ρ is isomorphic to the composition*

$$A \xrightarrow{\tilde{\tau}} M_{|Q_0|}(B) \xrightarrow{\epsilon_{\mathfrak{b}}} M_{|Q_0|}(B/\mathfrak{b}).$$

Proof. Suppose V_ρ is an A -module which sits over a point in \mathcal{A} . Then by Proposition 3.10, V_ρ is a simple A -module of dimension 1^{Q_0} . By Lemma 3.9, there is a simple A' -module $V_{\rho'}$ of dimension $1^{Q'_0}$ such that $\psi^{-1}\rho' = \rho^* \cong \rho$. Since (τ, B) is an impression of A , by [2, Proposition 2.5] there is a point $\mathfrak{b} \in \text{Max } B$ such that ρ' is isomorphic to the composition

$$A \xrightarrow{\tau} M_{|Q'_0|}(B) \xrightarrow{\epsilon_{\mathfrak{b}}} M_{|Q'_0|}(B/\mathfrak{b}).$$

But then for each $i, j \in Q_0$,

$$\rho|_{e_j A e_i} \cong (\psi^{-1}\rho')|_{e_j A e_i} = \rho'\psi|_{e_j A e_i} \cong \epsilon_{\mathfrak{b}}\tau\psi|_{e_j A e_i} = \epsilon_{\mathfrak{b}}\tilde{\tau}|_{e_j A e_i}. \quad \square$$

3.3. The cycle algebra is a ring of invariants

Definition 3.12. Let A be a dimer algebra. Denote by $\mathbb{S}(A)$ the open subvariety of $\text{Rep}_{1^{Q_0}}(A)$ of simple representations, and by $\overline{\mathbb{S}(A)}$ its Zariski closure. For an element p in a corner ring $e_j A e_i$, denote by $\mu(p)$ the corresponding function in $k[\text{Rep}_{1^{Q_0}}(A)]$ taking the value

$$\mu(p)(\rho) := \rho(p) \in k$$

on each $\rho \in \text{Rep}_{1^{Q_0}}(A)$.

Remark 3.13. It is possible for the closure $\overline{\mathbb{S}(A)}$ of $\mathbb{S}(A)$ to be properly contained in $\text{Rep}_{1^{Q_0}}(A)$. Indeed, $\overline{\mathbb{S}(A)} \neq \text{Rep}_{1^{Q_0}}(A)$ if there are cycles $p, q \in A$ and a representation $\rho \in \text{Rep}_{1^{Q_0}}(A)$ such that

$$\bar{p} = \bar{q} \quad \text{and} \quad \rho(p) \neq \rho(q) \quad (\text{as scalars}),$$

by Theorem 3.11.

For example, let A be the dimer algebra with quiver Q given in Fig. 3. The paths p and q , drawn in red and blue respectively, satisfy $\bar{p} = \bar{q}$ by [4, Example 3.9]. However, consider the semisimple representation $\rho \in \text{Rep}_{1^{Q_0}}(A)$ where each arrow subpath of p is represented by 1, each arrow subpath of q is represented by 2, and all other arrows are represented by zero. Then $\rho(p) = 1 \neq 2^8 = \rho(q)$.

Recall that the reductive algebraic group

$$GL := \prod_{j \in Q_0} GL_{d_j}(k)$$

acts linearly on $\text{Rep}_d(A)$ by conjugation.

Theorem 3.14. *Suppose $\psi : A \rightarrow A'$ is a cyclic contraction. Then the cycle algebra S is isomorphic to the GL-invariant rings*

$$S = k[\overline{\mathbb{S}(A)}]^{GL} = k[\overline{\mathbb{S}(A')}]^{GL}. \tag{26}$$

Proof. Each arrow $a \in Q_1$ vanishes at some point of $\overline{\mathbb{S}(A)}$, and so $\mu(a)$ is not invertible on $\overline{\mathbb{S}(A)}$ (though if $a \in Q_1^S$, then $\mu(a)$ is invertible on $\mathbb{S}(A)$). Therefore the GL-invariants in $k[\overline{\mathbb{S}(A)}]$ and $k[\overline{\mathbb{S}(A')}]$ are generated by the μ -images of oriented cycles in Q and Q' respectively.

By Proposition 3.10 and Theorem 3.11, for each cycle $p \in A$ we may set

$$\mu(p) = \bar{p}.$$

Therefore (26) holds. \square

Remark 3.15. The ‘mesonic chiral ring’ in an abelian quiver gauge theory is the ring of gauge invariant operators defined on the vacuum moduli space. Morally, the mesonic chiral ring is then the ring of invariants $k[\text{Rep}_{1Q_0}(A)]^{GL}$. However, the mesonic chiral ring may not coincide with the cycle algebra S by Remark 3.13. In the example therein, $\mu(p) \neq \mu(q)$ in $k[\text{Rep}_{1Q_0}(A)]^{GL}$, whereas $\mu(p) = \mu(q)$ in S .

Acknowledgments

Part of this article is based on work supported by the Simons Foundation while the author was a postdoc at the Simons Center for Geometry and Physics at Stony Brook University.

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