



Turner, M. C., Sofrony, J., & Herrmann, G. (2017). An alternative approach to anti-windup in anticipation of actuator saturation. *International Journal of Robust and Nonlinear Control*, 27(6), 963-980. <https://doi.org/10.1002/rnc.3609>

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# An alternative approach to anti-windup in anticipation of actuator saturation

Matthew C. Turner<sup>1\*</sup>, Jorge Sofrony<sup>2</sup>, Guido Herrmann<sup>3</sup>

<sup>1</sup>*Control and Instrumentation Research Group, Department of Engineering, University of Leicester, Leicester, LE1 7RH, UK.*

<sup>2</sup>*Department of Mechatronics, National University of Colombia, Bogota, Colombia*

<sup>3</sup>*Department of Mechanical Engineering, University of Bristol, Bristol, BS8 1TR, UK.*

## SUMMARY

Traditionally, an anti-windup compensator is activated when control signal saturation occurs. An alternative approach is to activate the compensator at a level below that of the physical control constraints: the anti-windup compensator is activated in *anticipation of* actuator saturation. Recent studies have proposed systematic methods for the construction of such anticipatory anti-windup compensators, but a pseudo-LPV representation of the saturated system has been central to these results. This paper approaches the anticipatory anti-windup problem for open-loop stable plants using a “non-square” sector condition which is associated with a combination of deadzone nonlinearities. The advantage of this approach is that it leads to synthesis routines which bear a close resemblance to those associated with traditional immediately activated anti-windup compensators. A by-product of this approach also appears to be that the arising compensators are better numerically conditioned. Some simulation examples illustrating the effectiveness of anticipatory anti-windup compensators and some comments on their wider use complete the paper.

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Received . . .

KEY WORDS: Anti-windup,  $\mathcal{L}_2$  gain, nonlinear systems.

## 1. INTRODUCTION

The anti-windup approach to accommodating actuator saturation in control systems is normally understood to have two distinct design stages. The first stage involves the design of a linear controller which functions satisfactorily in the absence of control limitations; the second involves the design of an anti-windup compensator which is activated upon actuator saturation occurring. Typically, the anti-windup compensator is a linear system and typically it is activated as soon as actuator saturation is encountered (henceforth referred to as “immediate activation”). This type of compensator has been extensively studied in the literature and the reader is referred to the books [17, 6, 9, 27, 18] and papers [19, 5, 1], and references therein, for more detail.

Anti-windup compensators which use immediate activation have been well documented (see references above) to provide improved behaviour in systems which suffer from control saturation

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\*Correspondence to: Control and Instrumentation Research Group, Department of Engineering, University of Leicester, Leicester, LE1 7RH, UK. E-mail: mct6@le.ac.uk

problems, yet for some systems, and some classes of input signal, their behaviour can be disappointing. One reason for this is the linearity of most traditional anti-windup compensators: the response to large saturated control signals is linearly proportional to the response to small saturated control signals. Intuitively, one might expect that compensators with differing behaviour for different signal sizes would perform better. Indeed, this idea was central in the development of early nonlinear anti-windup compensator schemes such as that in [26] (see also [4] and [11]), in which linear anti-windup compensators were scheduled as a function of the proximity of the state to the origin, and in [21], where compensators were combined as a function of the magnitude of the control signal.

More recently the so-called “delayed” or “deferred action” approach to anti-windup has been introduced ([14, 15]) in which the anti-windup compensator is activated when the control signal exceeds a magnitude selected to be *above* that of the physical control signal limits. In effect, this allows the system to operate without the anti-windup compensator when the level of saturation is low, and then only when the saturation level increases beyond a certain point, the anti-windup compensator becomes active. Time-domain simulations using this type of compensator have been extremely encouraging.

In a similar vein, Wu and Lin recently introduced the “anticipatory anti-windup” approach - see [23, 24, 25]. In this approach, the anti-windup compensator is activated at a level *below* that of the physical control limits, meaning that the scheme allows the anti-windup compensator to take action before the real control limits are reached. Simulation results indicated, again, that this approach could be attractive. The anticipatory anti-windup schemes proposed in [24, 25] were developed using the same pseudo-LPV representation first proposed in [14]. Although such an approach does benefit from some intuition, it leads to a rather complicated synthesis routine and, according to the numerical values for the compensators returned in [25], seems to suffer from some numerical issues.

This paper offers an alternative synthesis procedure for the anticipatory anti-windup approach, using a “non-square” sector condition which is associated with an operator consisting of two deadzones, and which arises naturally in the anticipatory anti-windup scenario. The approach is not claimed to out-perform that developed in [25] but it does seem to have better numerical properties and yields synthesis conditions which are much closer to the standard “immediate” approach given in [7]; in fact, the LMI’s obtained have a direct correspondence with those given in [7]. In addition, the order of the (dynamic) anti-windup case is the same as the plant (as in [7]) rather than that of the controller plus the plant (as in [25]). The results developed in this paper are similar to those presented in [20], which treats the deferred-action anti-windup case, but the exact details change due to the differing architectures of the problems. A preliminary version of this paper appeared in [22].

The paper is organised as follows. Section 2 describes the anticipatory anti-windup problem in more detail and introduces the non-square sector condition. Existence conditions for an anticipatory anti-windup compensator, along with an LMI-based synthesis algorithm, are given in Section 3. Section 4 presents two simulation examples and features some comments on the effectiveness of anticipatory anti-windup. Section 5 offers some conclusions.

### 1.1. Notation

$M \in \mathbb{R}_+^{n \times n}$  means that the real  $n \times n$  matrix  $M$  is positive definite;  $M \in \mathbb{D}_+^{n \times n}$  means that it is diagonal and positive definite. He  $M = M + M'$ . Following [23],  $I[1, m]$  denotes the set  $\{1, \dots, m\}$  for some integer  $m > 0$ . The  $\mathcal{L}_2$  norm of a vector valued function  $x(t)$  is defined as  $\|x\|_2 := (\int_0^\infty \|x(t)\|^2 dt)^{1/2}$  where  $\|(\cdot)\|$  denotes the Euclidean norm; any signal whose  $\mathcal{L}_2$  norm is finite is denoted  $x(t) \in \mathcal{L}_2$ . A system,  $\mathcal{T}$ , with input  $u(t)$  and output  $y(t)$  is said to have  $\mathcal{L}_2$  gain less

than  $\gamma$  if there exist scalars  $\gamma, \beta \geq 0$  such that

$$\|y\|_2 < \gamma \|u\|_2 + \beta \quad \forall u \in \mathcal{L}_2$$

The saturation operator,  $\text{Sat}_{\bar{u}}(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^m$ , is defined as

$$\text{Sat}_{\bar{u}}(u) = [ \text{sat}_{\bar{u}_1}(u_1) \quad \dots \quad \text{sat}_{\bar{u}_m}(u_m) ]'$$

where

$$\text{sat}_{\bar{u}_i}(u_i) = \text{sign}(u_i) \min \{|u_i|, \bar{u}_i\}$$

and  $\bar{u} = [ \bar{u}_1 \quad \dots \quad \bar{u}_m ]'$  and  $\bar{u}_i > 0, i \in I[1, m]$ . The deadzone operator,  $\text{Dz}_{\bar{u}}(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^m$  is related to the saturation through the expression

$$\text{Sat}_{\bar{u}}(u) = u - \text{Dz}_{\bar{u}}(u) \quad (1)$$

## 2. PROBLEM FORMULATION

### 2.1. Anticipatory Anti-windup

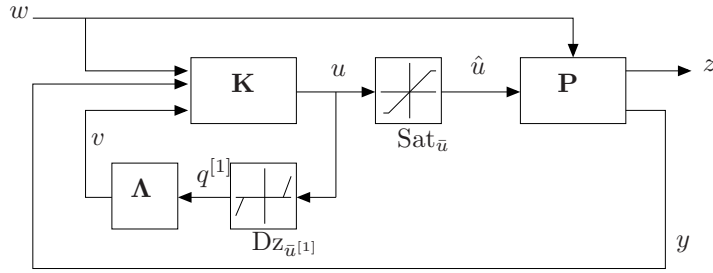


Figure 1. Anticipatory Anti-windup architecture:  $\bar{u}^{[1]} < \bar{u}$  (component-wise). In the anticipatory anti-windup architecture, the deadzone limits,  $\bar{u}^{[1]}$ , are set lower than the physical actuator limits,  $\bar{u}$ , so that the anti-windup compensator is activated before physical saturation occurs.

The anticipatory anti-windup architecture is shown in Figure 1:  $w \in \mathbb{R}^{n_w}$  represents the exogenous input (reference/disturbance),  $z \in \mathbb{R}^{n_z}$  the performance output,  $u \in \mathbb{R}^m$  the nominal control signal,  $\hat{u} = \text{Sat}_{\bar{u}}(u)$  the saturated control signal and  $y \in \mathbb{R}^{n_y}$  the measured output. Also shown is the signal  $q^{[1]} = \text{Dz}_{\bar{u}^{[1]}}(u)$  used to drive the anti-windup compensator, and the signal  $v \in \mathbb{R}^{n_c+m}$  which the anti-windup compensator injects into the controller state and output.  $\mathbf{P}$  represents the plant,  $\mathbf{K}$ , the controller and  $\mathbf{\Lambda}$  the anti-windup compensator.

As is standard in the anti-windup literature, it is assumed that in the absence of saturation and anti-windup, the system in Figure 1, is stable and well-posed. In addition, because we seek global finite  $\mathcal{L}_2$  gain guarantees we also stipulate that the linear plant is asymptotically stable i.e.  $\mathbf{P} \in \mathcal{RH}_\infty$

In standard (immediate) anti-windup compensation, the limits of the deadzone which drives the anti-windup compensator,  $\mathbf{\Lambda}$ , are *identical* to those of the physical actuator constraints, viz  $\bar{u}^{[1]} = \bar{u}$ : this ensures that the anti-windup compensator is only activated when saturation of the physical control signal occurs. In the *anticipatory* anti-windup compensation scheme which we consider in this paper, the limits of the deadzone which drives the anti-windup compensator are set to be *lower* than those of the physical actuator constraints, viz  $\bar{u}^{[1]} < \bar{u}$  (component-wise): this means that the signal  $q^{[1]} = \text{Dz}_{\bar{u}^{[1]}}(u)$ , which drives the anti-windup compensator, becomes non-zero *before* the physical actuator constraints are violated and hence the anti-windup compensator is activated *before* saturation is encountered. This mechanism can be considered as *anticipatory* since the anti-windup

compensator is allowed to take pre-emptive action in order to improve the system's behaviour in the event of saturation occurring. Of course, it also means that the compensator may become active if the physical saturation limits are not violated. This architecture is entirely equivalent to the one introduced in [24, 25].

In this paper, consideration is given to the following problem which arises from studying Figure 1.

*Problem 1*

Under the assumptions that

- i)  $\mathbf{P} \in \mathcal{RH}_\infty$
- ii) The closed-loop interconnection of the plant,  $\mathbf{P}$ , and the controller  $\mathbf{K}$ , when  $\hat{u} \equiv u$  and  $\Lambda \equiv 0$ , is asymptotically stable and well-posed
- iii) The physical saturation limits  $\bar{u}$  are given
- iv) The artificial deadzone limits  $\bar{u}^{[1]}$  are given and  $\bar{u}_i^{[1]} < \bar{u}_i \quad \forall i \in \{1, 2, \dots, m\}$

find conditions which ensure the existence of a linear anti-windup compensator  $\Lambda$  which guarantees (a) global asymptotic stability of the origin of the system depicted in Figure 1 when  $w(t) \equiv 0$  and (b) finite  $\mathcal{L}_2$  gain from exogenous inputs  $w$  to performance output  $z$ . Furthermore, determine LMI-based algorithms to enable the construction of such compensators.

In order to transform Problem 1 into a more tractable form, it is necessary to construct a state-space representation of Figure 1. Thus, using the identity (1) and the short-hand  $q^{[2]} := Dz_{\bar{u}}(u)$ , the state-space realisations of the plant, controller and anti-windup compensator, following the notation of [14, 24] can be written as

$$\mathbf{K} \sim \begin{cases} \dot{x}_c &= A_c x_c + B_{cw} w + B_{cy} y + v_1 \\ u &= C_c x_c + D_{cw} w + D_{cy} y + v_2 \end{cases} \quad (2)$$

$$\mathbf{P} \sim \begin{cases} \dot{x}_p &= A_p x_p + B_1 w + B_2 (u - q^{[2]}) \\ z &= C_1 x_p + D_{11} w + D_{12} (u - q^{[2]}) \\ y &= C_2 x_p + D_{21} w + D_{22} (u - q^{[2]}) \end{cases} \quad (3)$$

$$\Lambda \sim \begin{cases} \dot{x}_{aw} &= \Lambda_1 x_{aw} + \Lambda_2 q^{[1]} \\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} \Lambda_{31} \\ \Lambda_{32} \end{bmatrix}}_{\Lambda_3} x_{aw} + \underbrace{\begin{bmatrix} \Lambda_{41} \\ \Lambda_{42} \end{bmatrix}}_{\Lambda_4} q^{[1]} \end{cases} \quad (4)$$

where  $x_p \in \mathbb{R}^{n_p}$ ,  $x_c \in \mathbb{R}^{n_c}$  and  $x_{aw} \in \mathbb{R}^{n_{aw}}$ . The linear dynamics can be grouped to obtain a system in the form

$$\begin{cases} \begin{bmatrix} \dot{x} \\ u \\ z \end{bmatrix} = \begin{bmatrix} A & B_w & \tilde{B}_1 & \tilde{B}_2 \\ C & D_w & \tilde{D}_1 & \tilde{D}_2 \\ C_z & D_{zw} & \tilde{D}_{z1} & \tilde{D}_{z2} \end{bmatrix} \begin{bmatrix} x \\ w \\ q^{[1]} \\ q^{[2]} \end{bmatrix} \end{cases} \quad (5)$$

where  $x \in \mathbb{R}^{n_p+n_c+n_{aw}}$  and explicit expressions for the state-space matrices in terms of the plant/controller/anti-windup compensator parameters can be found in the appendix. Condition ii) of Problem 1 ensures (5) is well-posed.

Equation (5) describes a nonlinear closed-loop system containing two deadzone nonlinearities: one associated with the physical control signal constraints  $Dz_{\bar{u}}(\cdot)$ , and one associated with the anticipatory action of the anti-windup compensator,  $Dz_{\bar{u}^{[1]}}$ . Standard sector-based analysis (e.g. from [10], Chapter 10) could be used directly on the system (5) in order to arrive at anticipatory anti-windup synthesis conditions since the graphs of both deadzones lie within the sector  $[0, I]$ . However

such an analysis *does not* account for the *relationship* between the two deadzones; this relationship can be exploited to obtain further sector-like conditions which can reduce conservatism.

Denoting  $q^{[12]} = q^{[1]} - q^{[2]}$ , an equivalent expression for (5) is given by

$$\Sigma \sim \left\{ \begin{array}{l} \left[ \begin{array}{c} \dot{x} \\ u \\ z \end{array} \right] = \left[ \begin{array}{c|ccc} A & B_w & \tilde{B}_1 + \tilde{B}_2 & -\tilde{B}_2 \\ \hline C & D_w & \tilde{D}_1 + \tilde{D}_2 & -\tilde{D}_2 \\ C_z & D_{zw} & \tilde{D}_{z1} + \tilde{D}_{z2} & -\tilde{D}_{z2} \end{array} \right] \left[ \begin{array}{c} x \\ w \\ q^{[1]} \\ q^{[12]} \end{array} \right] \end{array} \right. \quad (6)$$

where  $\bar{u}^{[2]} := \bar{u}$  and signals  $q^{[1]}$  and  $q^{[12]}$  can be considered as outputs from the nonlinear operator  $\Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^{2m}$  which is defined as

$$\left[ \begin{array}{c} q^{[1]} \\ q^{[12]} \end{array} \right] = \Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u) := \left[ \begin{array}{c} Dz_{\bar{u}^{[1]}}(u) \\ \mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u) \end{array} \right] \quad (7)$$

where the nonlinearity  $\mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}} : \mathbb{R}^m \mapsto \mathbb{R}^m$  is defined as

$$\mathcal{D}_{\bar{u}_1, \bar{u}_2}(u) := Dz_{\bar{u}_1}(u) - Dz_{\bar{u}_2}(u) \quad (8)$$

An illustration of  $\mathcal{D}_{\bar{u}_1, \bar{u}_2}(u)$  can be found in Figure 2

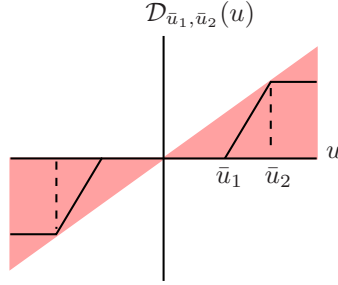


Figure 2. Illustration of a scalar version of  $\mathcal{D}_{\bar{u}_1, \bar{u}_2}(\cdot)$ .

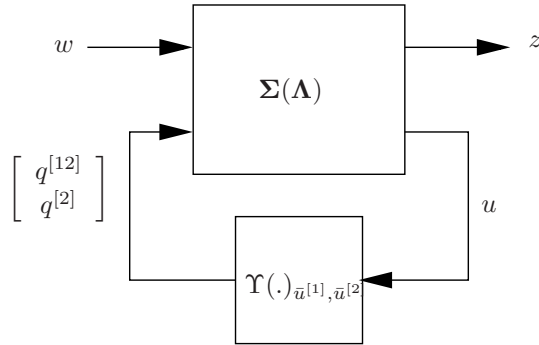


Figure 3. Anticipatory AW problem with operator  $\Upsilon(\cdot)_{\bar{u}^{[1]}, \bar{u}^{[2]}}$

The above analysis means that the anticipatory anti-windup architecture can be represented in the compact form of Figure 3 which admits the following state-space description.

$$\mathcal{T}_{zw} \sim \left\{ \begin{array}{l} \left[ \begin{array}{c} \dot{x} \\ u \\ z \end{array} \right] = \left[ \begin{array}{c|ccc} A & B_w & \tilde{B}_1 + \tilde{B}_2 & -\tilde{B}_2 \\ \hline C & D_w & \tilde{D}_1 + \tilde{D}_2 & -\tilde{D}_2 \\ C_z & D_{zw} & \tilde{D}_{z1} + \tilde{D}_{z2} & -\tilde{D}_{z2} \end{array} \right] \left[ \begin{array}{c} x \\ w \\ q^{[1]} \\ q^{[12]} \end{array} \right] \\ \left[ \begin{array}{c} q^{[1]} \\ q^{[12]} \end{array} \right] = \Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u) \end{array} \right. \quad (9)$$

This now allows Problem 1 to be rephrased as Problem 2 below.

*Problem 2*

Under the assumptions stated in Problem 1, find conditions which guarantee the existence of a linear anti-windup compensator  $\Lambda$  which ensures that

1. when  $w(t) \equiv 0$ ,  $\lim_{t \rightarrow \infty} x(t) = 0$ ; and
2.  $\|\mathcal{T}_{zw}\|_{i,2} < \gamma$ .

Furthermore, determine LMI-based algorithms which allow a compensator achieving these objectives to be computed.

**Remark 1:** The realisations (5) and (6) are mathematically equivalent, but the realisation (6) involves the use of the non-standard nonlinearity  $\Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$  in the feedback loop, rather than the two standard deadzone nonlinearities featured in the realisation (5). The reason for using  $\Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$  is that it enables the relationship between the two deadzone nonlinearities to become clearer, thereby expediting anticipatory anti-windup design.  $\square\square$

*2.2. A non-square sector condition*

The main reason for using the non-square operator  $\Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$  in the expression for  $\mathcal{T}_{zw}$  is that it, in addition to the standard sector conditions, satisfies extra inequalities, which are not normally visible when simply using the deadzone operators individually (as in the realisation (5)). Before, the non-square sector condition is stated, a preliminary lemma from [20] is needed:

*Lemma 1*

Given  $\mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^m$  in (8) and  $\alpha_i$  as

$$\alpha_i := \frac{\bar{u}_i^{[2]} - \bar{u}_i^{[1]}}{\bar{u}_i^{[2]}}, \quad \bar{u}_i^{[2]} > \bar{u}_i^{[1]} \quad (10)$$

then the following properties hold for all  $i \in I[1, m]$ :

$$(a) \quad \text{sign} \{ \alpha_i u_i - \mathcal{D}_i(u_i) \} = \begin{cases} \text{sign}(u_i) & \text{for } |u_i| \neq \bar{u}_i^{[2]} \\ 0 & \text{elsewhere} \end{cases}$$

$$(b) \quad \text{sign} \{ \mathcal{D}_i(u_i) \} = \begin{cases} \text{sign}(u_i) & \text{for } |u_i| > \bar{u}_i^{[1]} \\ 0 & \text{for } |u_i| \leq \bar{u}_i^{[1]} \end{cases}$$

where  $\mathcal{D}_i(u_i)$  is shorthand for  $\mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u_i)$ .

The following fact is also useful

*Fact 1*

For any  $\bar{u} > 0$

$$\text{sign} \{ u - \text{Dz}_{\bar{u}}(u) \} = \text{sign} \{ u \} \quad (11)$$

The following non-square sector condition can now be stated.

*Lemma 2 (Non-square sector condition)*

The operator  $\Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^{2m}$  from (7) satisfies for all  $W_{11}, W_{12}, W_{21}, W_{22} \in \mathbb{D}_+^{m \times m}$  and all  $u \in \mathbb{R}^m$ :

$$\mathcal{S}_1 := \mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u)' W_{11} (\mathcal{A}u - \mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u)) \geq 0 \quad (12)$$

$$\mathcal{S}_2 := \text{Dz}_{\bar{u}^{[1]}}(u)' W_{12} (\mathcal{A}u - \mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u)) \geq 0 \quad (13)$$

$$\mathcal{S}_3 := \mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u)' W_{21} (u - \text{Dz}_{\bar{u}^{[1]}}(u)) \geq 0 \quad (14)$$

$$\mathcal{S}_4 := \text{Dz}_{\bar{u}^{[1]}}(u)' W_{22} (u - \text{Dz}_{\bar{u}^{[1]}}(u)) \geq 0 \quad (15)$$

where  $\mathcal{A} = \text{diag}(\alpha_1, \dots, \alpha_m) \in \mathbb{D}_+^{m \times m}$  and  $\alpha_i$  is defined in (10).

**Proof:** First consider the inequality  $\mathcal{S}_1$ . As  $\mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$  is decentralised<sup>†</sup> and  $W_{11} \in \mathbb{D}_+^{m \times m}$  we have

$$\mathcal{S}_1 = \sum_{i=1}^m \mathcal{D}_i(u_i) W_{11,i} (\alpha_i u_i - \mathcal{D}_i(u_i)) \quad (16)$$

where  $W_{11,i}$  denotes the  $i$ 'th diagonal element of  $W_{11} \in \mathbb{D}_+^{m \times m}$ . Application of Lemma 1, then implies that  $\mathcal{S}_1 \geq 0$ . Fact 1 and Lemma 1 can then be used to prove, in a similar manner, inequalities  $\mathcal{S}_2$  and  $\mathcal{S}_3$ . Inequality  $\mathcal{S}_4$  is simply the standard sector inequality ([10]) for the deadzone.  $\square$

**Remark 2:** Condition (iv) in Problem 1 guarantees that the matrix  $\mathcal{A}$  in Lemma 2 is positive definite. This can also be seen from equation (10) since  $\mathcal{A} = \text{diag}(\alpha_1, \dots, \alpha_m)$ .  $\square\square$

**Remark 3:** Lemma 2 is similar to that proved in [20] except, that the nonlinearity  $\Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$  is defined slightly differently in this paper: it involves  $\mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$  and  $\text{Dz}_{\bar{u}^{[1]}}$ , rather than  $\mathcal{D}_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$  and  $\text{Dz}_{\bar{u}^{[2]}}$  which was used in [20] to define the operator  $\Pi_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$ . The reason we prefer to work with Lemma 2, instead of the result in [20], is that  $\Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot)$  simplifies the algebra significantly. To see, this note that an equivalent realisation of the operator  $\mathcal{T}_{zw}$  can be obtained from (6), and noting that  $\tilde{B}_1$  is an affine function of  $\Lambda$ , as

$$\Sigma \sim \left\{ \begin{array}{l} \left[ \begin{array}{c} \dot{x} \\ u \\ z \end{array} \right] = \left[ \begin{array}{c|ccc} A & B_w & \tilde{B}_1(\Lambda) & (\tilde{B}_2 + \bar{B}_1(\Lambda)) \\ C & D_w & \bar{D}_1(\Lambda) & (\bar{D}_2 + \bar{D}_1(\Lambda)) \\ C_z & D_{zw} & \tilde{D}_{z1}(\Lambda) & (\tilde{D}_{z2} + \bar{D}_{z1}(\Lambda)) \end{array} \right] \left[ \begin{array}{c} x \\ w \\ q^{[12]} \\ q^{[2]} \end{array} \right] \end{array} \right. \quad (17)$$

and

$$\left[ \begin{array}{c} q^{[12]} \\ q^{[2]} \end{array} \right] = \Pi_{\bar{u}^{[1]}, \bar{u}^{[2]}}(u) \quad (18)$$

Clearly this involves the operator  $\Pi_{\bar{u}^{[1]}, \bar{u}^{[2]}} : (\cdot) \mathbb{R}^m \mapsto \mathbb{R}^{2m}$ . However, note that the matrix of anti-windup parameters,  $\Lambda$ , occurs in the *two* far left columns of the realisation (17), whereas it only appears in the penultimate column of the realisation of the realisation (6). This then leads to a substantial increase in algebra when the projection conditions are applied in the next section. For this reason, it is better to work with the alternative non-square operator  $\Upsilon_{\bar{u}^{[1]}, \bar{u}^{[2]}}(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^{2m}$ .  $\square\square$

### 3. MAIN RESULTS

#### 3.1. Existence Conditions

The main results of the paper give existence conditions for an anticipatory anti-windup compensator in a form that is comparable with those associated with the immediate anti-windup compensator construction [7]. Before these conditions are stated, the reader is reminded of the well-known Projection Lemma ([3, 2]).

*Lemma 3* (Projection Lemma)

There exists a matrix  $\Lambda_a$  (of suitable dimensions) solving the matrix inequality

$$\Psi_a + H_a' \Lambda_a G_a + G_a' \Lambda_a H_a < 0 \quad (19)$$

<sup>†</sup>That is  $y_i = \mathcal{D}_{i, u^{[1]}, u^{[2]}}(u_i)$  for all  $i$



if and only if the following inequalities are satisfied

$$W'_{G_a} \Psi_a W_{G_a} < 0 \quad \text{and} \quad W'_{H_a} \Psi_a W_{H_a} < 0 \quad (20)$$

where  $W_{G_a}$  and  $W_{H_a}$  are, respectively, matrices whose columns span the null spaces of  $G_a$  and  $H_a$ .

*Proposition 1*

Consider the interconnection (9) and let the assumptions listed in Problem 1 be satisfied. Let  $\mathcal{A} = \text{diag}(\alpha_1, \dots, \alpha_m) \in \mathbb{D}_+^{m \times m}$  with  $\alpha_i$  defined in (10) and assume there exist positive definite matrices

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \star & \mathbf{R}_{22} \end{bmatrix}, \mathbf{S} \in \mathbb{R}_+^{(n_p+n_c) \times (n_p+n_c)}, \mathbf{R}_{11} \in \mathbb{R}_+^{n_p \times n_p} \quad (21)$$

positive definite diagonal matrices  $\tilde{\mathbf{U}}_1, \mathbf{V}_{11}, \mathbf{V}_{12}, \mathbf{V}_{22} \in \mathbb{D}_+^{m \times m}$ , and a scalar  $\gamma$  such that the following matrix inequalities hold

$$\begin{bmatrix} A_p \mathbf{R}_{11} + \mathbf{R}_{11} A'_p - 2B_2(\tilde{\mathbf{U}}_1 + \mathbf{V}_{11})B'_2 & -B_2(2\mathbf{V}_{11} + \mathbf{V}_{22} + \tilde{\mathbf{U}}_1) & B_1 & \mathbf{R}_{11}C'_1 - 2B_2(\tilde{\mathbf{U}}_1 + \mathbf{V}_{11})D'_{12} \\ \star & -2(\mathbf{V}_{11} + \mathbf{V}_{22} - \mathcal{A}\mathbf{V}_{12}) & 0 & -(2\mathbf{V}_{11} + \mathbf{V}_{22} + \tilde{\mathbf{U}}_1)D'_{12} \\ \star & \star & -\gamma I & D'_{11} \\ \star & \star & \star & -\gamma I - 2D_{12}(\tilde{\mathbf{U}}_1 + \mathbf{V}_{11})D'_{12} \end{bmatrix} < 0 \quad (22)$$

$$\begin{bmatrix} \mathbf{S}A'_{CL} + A_{CL}\mathbf{S} & \bar{B}_2\tilde{\mathbf{U}}_1 + \mathbf{S}C'_{CL} & \bar{B}_w & \mathbf{S}C'_{z,CL} \\ \star & -2\mathbf{V}_{11} - \tilde{\mathbf{U}}_1\tilde{D}'_2 - \tilde{D}_2\tilde{\mathbf{U}}_1 & \bar{D}_w & \tilde{\mathbf{U}}_1\tilde{D}'_{z2} \\ \star & \star & -\gamma I & \tilde{D}'_{zw} \\ \star & \star & \star & -\gamma I \end{bmatrix} < 0 \quad (23)$$

$$\mathbf{R} - \mathbf{S} \geq 0 \quad (24)$$

$$\text{rank}(\mathbf{R} - \mathbf{S}) \leq n_{aw} \quad (25)$$

$$\mathcal{A}\mathbf{V}_{11} - \tilde{\mathbf{U}}_1 < 0 \quad (26)$$

$$\mathcal{A}\mathbf{V}_{12} - \mathbf{V}_{22} < 0 \quad (27)$$

where the constant matrices  $A_{CL}, \bar{B}_2, C_{CL}, \bar{B}_w, C_{z,CL}, \tilde{D}_2, \bar{D}_w, \tilde{D}_{z2}, \bar{D}_{zw}$  are given in the appendix. Then there exists an AW compensator  $\mathbf{\Lambda}$ , (4), of order  $n_{aw}$  which guarantees that the origin of equation (9) is globally asymptotically stable and that the  $\mathcal{L}_2$  gain of the map  $\mathcal{T}_{zw}$  is less than  $\gamma$ .

**Remark 4:** The above inequalities take similar forms to those found in standard *immediate* anti-windup ([7]): there are two inequalities involving open and closed-loop data, (22) and (23) respectively; and two inequalities, (24) and (25), involving the inverse of the Lyapunov matrices. Inequality (23) stipulates that, as global results are sought, the un-saturated closed-loop system must be asymptotically stable; inequality (22) stipulates that the open-loop plant must also be stable - see Remark 4 below. The additional rows/columns in inequalities (23) and (22) and the two additional LMI's, (26) and (27), arise because the anti-windup compensator becomes active before the physical saturation limits are reached. Note that, as  $\mathcal{A}$  becomes "smaller", the conditions in Proposition 1 collapse to those proposed for the immediate anti-windup case [7].  $\square\square$

**Remark 5: Solvability Conditions.** At first glance, it appears that asymptotic stability of the plant is not necessary in order to achieve asymptotic stability of the closed-loop system. In fact, as expected,  $A_p$  indeed has to be Hurwitz for inequality (22) to be solvable. To see this, note that a necessary condition for (22) to be solvable for positive definite matrices,  $\mathbf{R}_{11} \in \mathbb{R}_+^{n_p \times n_p}$  and

$\tilde{\mathbf{U}}_1, \mathbf{V}_{11}, \mathbf{V}_{12}, \mathbf{V}_{22} \in \mathbb{D}_+^{m \times m}$  is for the following matrix to be negative definite

$$\begin{bmatrix} A_p \mathbf{R}_{11} + \mathbf{R}_{11} A'_p - 2B_2(\tilde{\mathbf{U}}_1 + \mathbf{V}_{11})B'_2 & -B_2(2\mathbf{V}_{11} + \mathbf{V}_{22} + \tilde{\mathbf{U}}_1) \\ \star & -2(\mathbf{V}_{11} + \mathbf{V}_{22} - \mathcal{A}\mathbf{V}_{12}) \end{bmatrix} < 0 \quad (28)$$

This inequality can be re-written as

$$\begin{bmatrix} A_p \mathbf{R}_{11} + \mathbf{R}_{11} A'_p & -B_2(\mathbf{V}_{11} + \mathbf{V}_{22}) \\ \star & -2(\mathbf{V}_{11} + \mathbf{V}_{22}) + 2\mathcal{A}\mathbf{V}_{12} \end{bmatrix} + \begin{bmatrix} -2B_2(\mathbf{V}_{11} + \tilde{\mathbf{U}}_1)B'_2 & -B_2(\mathbf{V}_{11} + \tilde{\mathbf{U}}_1) \\ \star & 0 \end{bmatrix} < 0 \quad (29)$$

In turn, this can be expressed as

$$\begin{bmatrix} A_p \mathbf{R}_{11} + \mathbf{R}_{11} A'_p & -B_2(\mathbf{V}_{11} + \mathbf{V}_{22}) \\ \star & -2(\mathbf{V}_{11} + \mathbf{V}_{22}) + 2\mathcal{A}\mathbf{V}_{12} \end{bmatrix} + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} (\tilde{\mathbf{U}}_1 + \mathbf{V}_{11}) \begin{bmatrix} -B'_2 & -I \end{bmatrix} + \begin{bmatrix} -B_2 \\ -I \end{bmatrix} (\tilde{\mathbf{U}}_1 + \mathbf{V}_{11}) \begin{bmatrix} B'_2 & 0 \end{bmatrix} < 0 \quad (30)$$

The projection lemma then implies that a necessary (sufficiency is not proven since the matrix  $\tilde{\mathbf{U}}_1 + \mathbf{V}_{11}$  is structured) condition for this to be solvable is for the following inequality to be solvable for positive definite matrices  $\mathbf{R}_{11}$  and  $\mathbf{V}_{12}$ :

$$A_p \mathbf{R}_{11} + \mathbf{R}_{11} A'_p + 2B_2 \mathcal{A} \mathbf{V}_{12} B'_2 < 0 \quad (31)$$

This clearly requires  $A_p$  to be Hurwitz.  $\square\square$

**Proof of Proposition 1.** The proof is similar to [20, 7], but the details differ significantly due to the architecture of the anticipatory anti-windup problem. The state-space realisation of the linear dynamics (6) can be written ([3]) as

$$\Sigma \sim \left[ \begin{array}{c|ccc} A_0 + H'_1 \Lambda G_1 & B_w & \tilde{B}_2 + H'_1 \Lambda G_2 & -\tilde{B}_2 \\ \hline C_0 + H'_2 \Lambda G_1 & D_w & \tilde{D}_2 + H'_2 \Lambda G_2 & -\tilde{D}_2 \\ C_{z0} + H'_3 \Lambda G_1 & D_{zw} & \tilde{D}_{z2} + H'_3 \Lambda G_2 & -\tilde{D}_{z1} \end{array} \right] \quad (32)$$

where the matrix of the AW compensator matrices is:

$$\Lambda := \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix}.$$

and the remaining matrices are defined in the appendix. For  $\mathcal{T}_{zw}$  to be internally stable with  $\mathcal{L}_2$  gain of  $\gamma > 0$ , it suffices for there to exist a matrix  $P \in \mathbb{R}_+^{(n_p+n_c+n_{aw}) \times (n_p+n_c+n_{aw})}$  ([7, 2]) such that

$$\frac{d}{dt}(x'Px) + \gamma^{-1}\|z\|^2 - \gamma\|w\|^2 < 0 \quad \forall x, w \neq 0 \quad (33)$$

Using Lemma 2, inequality (33) holds if

$$\frac{d}{dt}(x'Px) + \gamma^{-1}\|z\|^2 - \gamma\|w\|^2 + \sum_{j=1}^4 \mathcal{S}_j < 0 \quad \forall x, w \neq 0 \quad (34)$$

for some  $W_{11}, W_{12}, W_{21}$  and  $W_{22} \in \mathbb{R}_+^{m \times m}$ . Using (32), it can be shown, after some algebra, that inequality (34) is equivalent to the matrix inequality:

$$\Psi_0 + H' \Lambda G + G' \Lambda H < 0 \quad (35)$$

where

$$\Psi_0 = \text{He} \begin{bmatrix} PA_0 & P\tilde{B}_2 & -P\tilde{B}_2 & PB_w & 0 \\ \tilde{W}_2 C_0 & -W_{22} + \tilde{W}_2 \tilde{D}_2 & -W_{12} - \tilde{W}_2 \tilde{D}_2 & \tilde{W}_2 D_w & 0 \\ \tilde{W}_1 C_0 & -W_{21} + \tilde{W}_1 \tilde{D}_2 & -2W_{11} - \tilde{W}_1 \tilde{D}_2 & \tilde{W}_1 D_w & 0 \\ 0 & 0 & 0 & -\gamma I & 0 \\ C_{z0} & \tilde{D}_{z2} & -\tilde{D}_{z2} & D_{zw} & -\gamma I \end{bmatrix} \quad (36)$$

$$H = [ H_1 P \quad H_2 \tilde{W}_2 \quad H_2 \tilde{W}_1 \quad 0 \quad H_3 ] \quad (37)$$

$$G = [ G_1 \quad G_2 \quad 0 \quad 0 \quad 0 ] \quad (38)$$

and the nonsingular matrices  $\tilde{W}_1, \tilde{W}_2 \in \mathbb{D}_+^{m \times m}$  are given by

$$\tilde{W}_1 := \mathcal{A}W_{11} + W_{21} =: \tilde{\mathbf{U}}_1^{-1} \quad (39)$$

$$\tilde{W}_2 := \mathcal{A}W_{12} + W_{22} =: \tilde{\mathbf{U}}_2^{-1} \quad (40)$$

and, according to Lemma 2,  $W_{11}, W_{12}, W_{21}, W_{22} \in \mathbb{D}_+^{m \times m}$ .

Invoking the Projection Lemma, (35) holds if and only if

$$W'_G \Psi_0 W_G < 0 \quad \text{and} \quad W'_H \Psi_0 W_H < 0 \quad (41)$$

where  $W_G$  and  $W_H$  are, respectively, full column rank matrices whose columns span the null spaces of  $G$  and  $H$ . The remainder of the proof shows how inequality (41) leads to the inequalities in the statement of the proposition.

Inequality (23). First the matrix  $P$  is partitioned ([7]) as below.

$$P = \left[ \begin{array}{cc|c} P_{11} & P_{12} & P_{13} \\ \star & P_{22} & P_{23} \\ \star & \star & P_{33} \end{array} \right] = \left[ \begin{array}{c|c} \mathbf{S}^{-1} & P_\star \\ \star & P_{33} \end{array} \right] \quad (42)$$

where  $S \in \mathbb{P}_+^{(n_p+n_c) \times (n_p+n_c)}$  and  $P_{33} \in \mathbb{P}_+^{n_{aw} \times n_{aw}}$ . Next, a particular choice of  $W_G$  is made:

$$W'_G = \left[ \begin{array}{c|ccc} I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array} \right] \quad (43)$$

Using the partition given in equation (42) and the choice of  $W'_G$  in equation (43) then allows the left-hand inequality in (41) to be reduced to inequality (23) in the proposition, where

$$\mathbf{V}_{11} := \tilde{\mathbf{U}}_1 W_{11} \tilde{\mathbf{U}}_1 \quad (44)$$

Inequality (22). Let  $Q := P^{-1}$ , where  $Q$  is partitioned as

$$Q = \left[ \begin{array}{cc|c} Q_{11} & Q_{12} & Q_{13} \\ \star & Q_{22} & Q_{23} \\ \star & \star & Q_{33} \end{array} \right] = \left[ \begin{array}{c|c} \mathbf{R} & Q_\star \\ \star & Q_{33} \end{array} \right]$$

Also,  $W_H$  is chosen as:

$$W'_H = \tilde{W}'_H \text{diag}(Q, \tilde{\mathbf{U}}_2, \tilde{\mathbf{U}}_1, I, I) \quad (45)$$

where  $\tilde{W}'_H$  is given by

$$\tilde{W}'_H = \left[ \begin{array}{ccc|ccc} I & 0 & 0 & 0 & -B_2 & 0 & 0 \\ \hline 0 & 0 & 0 & I & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & -D_{12} & 0 & I \end{array} \right] \quad (46)$$

Using this expression for  $W_H$  and the partition of  $Q$ , it can be shown that the right-hand inequality in (41) is equivalent to inequality (22) where

$$\mathbf{V}_{12} := \tilde{U}_2 W_{12} \tilde{U}_2 \quad (47)$$

$$\mathbf{V}_{22} := \tilde{U}_2 - \tilde{U}_2 (W_{12} + W_{21}) \tilde{U}_1 \quad (48)$$

Inequalities (24) and (25). As inequality (23) is expressed in terms of  $\mathbf{S}$  and inequality (22) in terms of  $\mathbf{R}_{11}$ , it is necessary to find conditions which ensure that  $P = Q^{-1}$ , viz

$$\begin{bmatrix} \mathbf{S}^{-1} & P_{\star} \\ \star & P_{33} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{R} & Q_{\star} \\ \star & Q_{33} \end{bmatrix} > 0 \quad (49)$$

According to [13], necessary and sufficient conditions for there to exist matrices  $P_{\star}$ ,  $P_{33}$ ,  $Q_{\star}$  and  $Q_{33}$  satisfying equation (49), is that inequalities (24) and (25) both hold.

Inequalities (26) and (27). Lemma 2 requires the matrices  $W_{11}$ ,  $W_{12}$ ,  $W_{21}$ ,  $W_{22}$  to all be diagonal and positive definite. However, the inequalities (22) and (23) have been stated in new variables  $\tilde{U}_1$ ,  $\mathbf{V}_{11}$ ,  $\mathbf{V}_{12}$  and  $\mathbf{V}_{22}$ . While diagonality follows trivially, to see that  $W_{11}$ ,  $W_{12}$ ,  $W_{21}$ ,  $W_{22}$  are indeed positive definite, note that

- $\mathbf{V}_{11} > 0$  directly implies  $W_{11} > 0$  from equation (44)
- $\mathbf{V}_{12} > 0$  directly implies  $W_{12} > 0$  from equation (47)
- Together inequalities (26) and (39) yield

$$\mathcal{A}\mathbf{V}_{11} - \tilde{U}_1 = \mathcal{A}\tilde{U}_1 W_{11} \tilde{U}_1 - \tilde{U}_1 (\mathcal{A}W_{11} + W_{21}) \tilde{U}_1 < 0 \quad (50)$$

$$\Leftrightarrow -\tilde{U}_1 W_{21} \tilde{U}_1 < 0 \quad (51)$$

$$\Leftrightarrow W_{21} > 0 \quad (52)$$

- From the definition of  $\tilde{U}_1$ ,  $W_{21} = \tilde{U}_1^{-1} - \mathcal{A}W_{11}$ . Using this in equation (48) yields

$$\mathbf{V}_{22} = (\mathcal{A}W_{11} - W_{12}) \tilde{U}_1 \tilde{U}_2 \quad (53)$$

Under the assumption that  $\tilde{U}_2$  is nonsingular, equation (47) gives  $W_{12} = \mathbf{V}_{12} \tilde{U}_2^{-2}$  (It will be shown that  $\tilde{U}_2^{-1}$  exists shortly); using this in equation (53) then becomes

$$\mathbf{V}_{22} = \mathcal{A}W_{11} \tilde{U}_1 \tilde{U}_2 - \mathbf{V}_{12} \tilde{U}_1 \tilde{U}_2^{-1} \quad (54)$$

$$\Leftrightarrow \mathbf{V}_{22} \tilde{U}_2 = \mathcal{A}W_{11} \tilde{U}_1 \tilde{U}_2^2 - \mathbf{V}_{12} \tilde{U}_1 \quad (55)$$

Noting that each matrix is diagonal, the above equation can be expressed as  $m$  quadratic equations in the diagonal elements of  $\tilde{U}_2$ . Denoting  $\tilde{U}_{2,i}$  as the  $i$ 'th diagonal element, the positive root of each quadratic equation is given by

$$\tilde{U}_{2,i} = \frac{\mathbf{V}_{22,i} + \sqrt{\mathbf{V}_{22,i}^2 + 4\alpha_i \mathbf{V}_{11,i} \mathbf{V}_{12,i}}}{2\alpha_i \tilde{U}_{1,i} W_{11,i}} \quad \forall i \in I[1, m] \quad (56)$$

Due to the fact that  $\mathbf{V}_{12}$ ,  $\mathbf{V}_{11}$ ,  $\tilde{U}_1$ ,  $W_{11}$  are all positive definite, and because  $\mathbf{V}_{22}$  is positive definite from equation (27), the above expression implies that  $\tilde{U}_{2,i}$  is also positive definite. Next note that, by definition

$$W_{22} = \tilde{U}_2^{-1} - \mathcal{A}W_{21}$$

so that an equivalent condition for  $W_{22} > 0$  is (pre and post multiplying by  $\tilde{U}_2$ )

$$\tilde{U}_2 - \mathcal{A}\mathbf{V}_{12} > 0 \quad (57)$$

However, because  $\tilde{\mathbf{U}}_1^{-1} = (\mathcal{A}W_{11} + W_{12}) > \mathcal{A}W_{11}$ , from equation (56) it follows that

$$\mathbf{V}_{22} - \mathcal{A}\mathbf{V}_{12} > 0 \Rightarrow \tilde{U}_2 - \mathcal{A}\mathbf{V}_{12} > 0 \quad (58)$$

which is exactly the condition (57) which ensures  $W_{22} > 0$ .

Proposition 1 states non-convex conditions for an AW compensator of arbitrary order ( $n_{aw}$ ) to exist. Convex conditions can be obtained when  $n_{aw} = 0$  (static AW) and  $n_{aw} \geq n_p$  [7]. A useful corollary of Proposition 1 is the full-order case given below.

*Corollary 1*

For a given  $\mathcal{A} = \text{diag}(\alpha_1, \dots, \alpha_m) \in \mathbb{D}_+^{m \times m}$ , there exists an  $n_p$ 'th order AW compensator of the form (4) satisfying the properties of Proposition 1 if inequalities (22), (23), (26) and (27) of Proposition 1 are satisfied and, in addition  $R_{11} - S_{11} > 0$ .

*3.2. Full-order Anti-windup compensator construction*

The construction of the anticipatory AW compensator,  $\Lambda$ , is very similar to [7]. In the general case (Proposition 1) it is difficult to construct an AW compensator of arbitrary order. However, when  $n_{aw} = n_p$ , the conditions become convex, as stated in Corollary 1 and hence the following procedure can be used to construct  $\Lambda$ , and hence  $\mathbf{\Lambda}$  (based on that reported in [7, 3]).

1. Choose  $\mathcal{A}$ .
2. Solve the inequalities indicated in Corollary 1: find the minimum  $\gamma$  and matrices  $\mathbf{S}_{11}$ ,  $\mathbf{R}_{11}$ ,  $\mathbf{V}_{11}$ ,  $\mathbf{V}_{12}$ ,  $\mathbf{V}_{22}$ ,  $\tilde{\mathbf{U}}_1$  and  $\gamma$  for which these inequalities hold.
3. Construct  $\mathbf{R}$  using  $\mathbf{R}_{11}$  from above and with  $R_{12} = \mathbf{S}_{12}$  and  $R_{22} = \mathbf{S}_{22}$ .
4. Construct  $P > 0$  according to equation (42) where

$$P_* P_*' = \mathbf{S}^{-1} \mathbf{R} \mathbf{S}^{-1} - \mathbf{S}^{-1} \quad (59)$$

$$P_{33} = I_{n_{aw}} + P_*' \mathbf{S} P_* \quad (60)$$

5. Determine  $W_{ij}$ 's:

$$W_{11} = \mathbf{V}_{11} \tilde{\mathbf{U}}_1^{-2} \quad (61)$$

$$W_{21} = \tilde{\mathbf{U}}_1^{-1} - \mathcal{A}W_{11} \quad (62)$$

$$\tilde{U}_{2,i} = \frac{\mathbf{V}_{22,i} + \sqrt{\mathbf{V}_{22,i}^2 + 4\alpha_i \mathbf{V}_{11,i} \mathbf{V}_{21,i}}}{2\alpha_i \tilde{\mathbf{U}}_{1,i} W_{11}} \quad (63)$$

$$W_{12} = \mathbf{V}_{12} \tilde{U}_2^{-2} \quad (64)$$

$$W_{22} = \tilde{U}_2^{-1} - \mathcal{A}W_{12} \quad (65)$$

6. Using  $P$ ,  $\gamma$  and  $W_{11}$ ,  $W_{12}$ ,  $W_{21}$ ,  $W_{22}$  solve equation (35) for  $\Lambda$  and construct  $\mathbf{\Lambda}$  according to equation (4).

This design procedure is of a similar complexity to that encountered in standard immediate AW design [7] and is better numerically conditioned than the procedure advocated in [24, 25].

## 4. SIMULATIONS AND COMMENTS

## 4.1. Servo Example

Consider a modified version of the servo example introduced in [27], page 187. The state-space matrices of the plant and the controller are given by

$$\mathbf{P} \sim \left[ \begin{array}{c|c|c} A_p & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{12} & D_{22} \end{array} \right] = \left[ \begin{array}{cc|cc} -0.5000 & -0.0026 & 16 & 16 \\ 0.0020 & 0 & 0 & 0 \\ \hline 0 & -8 & 1 & 0 \\ \hline 0 & 8 & 0 & 0 \end{array} \right] \quad (66)$$

$$\mathbf{K} \sim \left[ \begin{array}{c|c|c} A_c & B_{cy} & B_{cw} \\ \hline C_c & D_{cy} & D_{cw} \end{array} \right] = \left[ \begin{array}{cccc|cc} -49.804 & -30.403 & 55.789 & 0.359 & -120.583 & -1.302 \\ 2.562 & -0.464 & 1.305 & 1.643 & -2.586 & -2.022 \\ -10.699 & -1.910 & 2.523 & -2.491 & -8.363 & 4.248 \\ 0.515 & 0.092 & -0.266 & -2.880 & 0.403 & 6.501 \\ \hline 52.042 & 30.311 & 1.869 & 18.630 & 0 & 0 \end{array} \right] \quad (67)$$

The control signal is constrained to lie in the interval  $[-1, 1]$  units. Two anti-windup compensators were designed for this plant-controller nominal closed-loop: a standard “immediate” anti-windup compensator designed according to the algorithm of [7] for which  $\gamma \approx 4.99 \times 10^4$  (which is large due to the open-loop plant being quite “close” to instability); and an “anticipatory” anti-windup compensator designed using Corollary 1 with  $\mathcal{A} = \alpha = 0.5$ , where again  $\gamma \approx 4.99 \times 10^4$ . This meant that the anticipatory anti-windup compensator would be activated when the control signal magnitude exceeded  $(1 - \alpha)\bar{u} = 0.5$  units. The anticipatory anti-windup compensator is given by

$$\mathbf{\Lambda} \sim \left[ \begin{array}{cc|c} -0.5032 & -0.4159 & -0.0072 \\ -0.5036 & -219.6340 & -8.9295 \\ -641.4410 & -4.2262 & 0.0366 \\ \hline -13.7808 & 1.0596 & -0.0066 \\ -44.5639 & -3.4930 & 0.0244 \\ 1.8875 & 0.1381 & -0.0166 \\ -0.1432 & -63.3882 & -2.2391 \end{array} \right] \quad (68)$$

The left-hand plot in Figure 4 shows the response of the system,  $y(t)$ , due to a square-wave type pulse demand. Without control constraints, the output  $y(t)$  is able to achieve fast, accurate tracking of the reference signal: the rise time is swift although a brief overshoot occurs when the reference demand switches sign. However, the control signal associated with such tracking behaviour, shown to the right of Figure 4, is far beyond the control limits. Thus, when saturation constraints are introduced, without anti-windup compensation, the system loses stability (dotted line). When “immediate” anti-windup compensation is introduced (red dashed line) things improve with stability being restored. However, it is seen that the introduction of “anticipatory” anti-windup leads to a response which is somewhat better than the immediate case: the control signal also appears to be less oscillatory. Thus, in this case, it seems to be preferable to use anticipatory anti-windup to standard immediate anti-windup.

## 4.2. Circuit Example

This example was originally proposed in [8] and was used in [14, 25] to illustrate the merits of deferred action and anticipatory anti-windup, respectively. Here it is used for the same purpose but

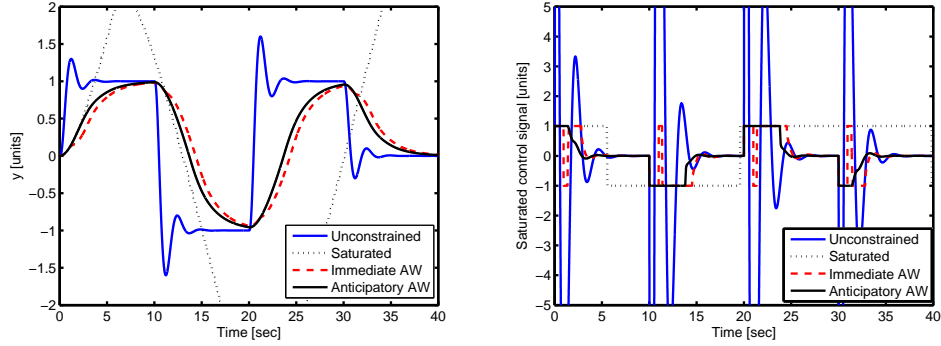


Figure 4. Servo example. Left plot: output response  $y(t)$ . Right plot: control response  $\hat{u}(t)$

the results require some discussion. Therefore, consider the plant and controller with the following state-space realisations.

$$\mathbf{P} \sim \left[ \begin{array}{c|cc} A_p & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{12} & D_{22} \end{array} \right] = \left[ \begin{array}{ccc|cc} -10.6000 & -6.0900 & -0.9000 & 0 & 1 \\ 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ \hline -1 & -11 & -30 & 0 & 0 \\ -1 & -11 & -30 & -1 & 0 \end{array} \right] \quad (69)$$

$$\mathbf{K} \sim \left[ \begin{array}{c|cc} A_c & B_{cy} & B_{cw} \\ \hline C_c & D_{cy} & D_{cw} \end{array} \right] = \left[ \begin{array}{cc|cc} -80 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ \hline 20.25 & 1600 & 80 & -80 \end{array} \right] \quad (70)$$

The control signal is again constrained to lie in the interval  $[-1, 1]$  units and again two anti-windup compensators were designed for this nominal loop. The immediate anti-windup compensator, designed according to the algorithm in [7] had an associated bound on the  $\mathcal{L}_2$  gain of  $\gamma \approx 58.58$ . The second anti-windup compensator was an anticipatory compensator designed using Corollary 1 with  $\mathcal{A} = \alpha = 0.3$ , which meant that the anti-windup compensator would be activated when the control signal magnitude exceeded  $(1 - \alpha) = 0.7$  units. The  $\mathcal{L}_2$  gain bound obtained for this value of  $\alpha$  is  $\gamma \approx 66.63$  and the accompanying anti-windup compensator is given by

$$\mathbf{\Lambda} \sim \left[ \begin{array}{ccc|c} -3.8457 & 5.0120 & 0.0273 & -0.0010 \\ 3.9855 & -6.7417 & 0.0559 & -0.0043 \\ 2.6017 & -3.8043 & -523.6981 & 0.2475 \\ \hline 4.5029 & 7.6318 & -0.1778 & -0.0958 \\ 0.0826 & -0.1060 & 0.0016 & 0.0010 \\ 892.3270 & -68.8792 & 10.1524 & 0.0440 \end{array} \right] \quad (71)$$

The right-hand plot in Figure 5 shows the output response of the circuit example to a pulse like reference signal, similar to that used in [25] and of amplitude 3 units. The saturated response features some overshoot and extensive settling time, but the rise time is relatively fast. Both immediate and anticipatory anti-windup arrest the overshoot and hasten the settling of the system, but the rise time slows in both cases. The anticipatory anti-windup compensator performs a little better than the immediate anti-windup compensator with small rise-time improvements in most of the pulses. However, compared to the deferred-action anti-windup results of [14], which achieves almost linear

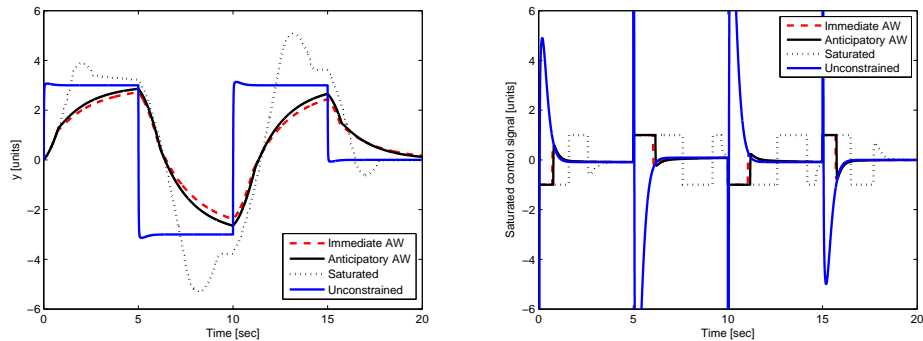


Figure 5. Circuit example. Left plot: output response  $y(t)$ . Right plot: control response  $\hat{u}(t)$

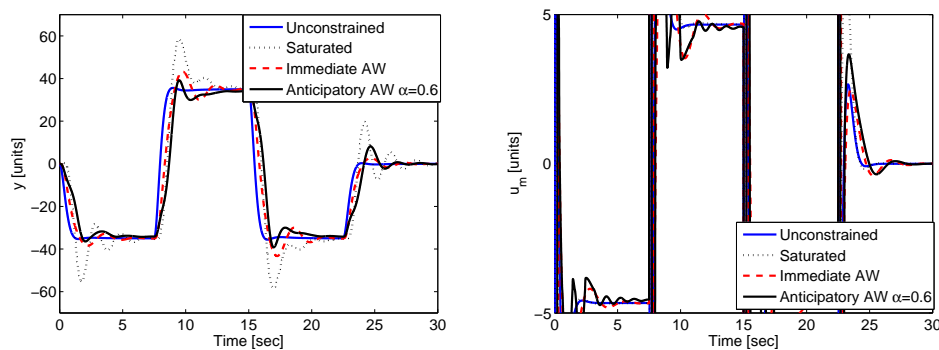


Figure 6. Hydraulic actuator example: output response  $y(t)$  and plant input response  $u_m(t)$ ;  $\bar{u} = 10.5$ .

quality tracking, the performance is much worse. The left-hand plot in Figure 5 shows the saturated control signal response; the anticipatory anti-windup compensator leads to a control signal which stays saturated for longer periods of time, explaining the slightly faster response of this compensator.

From a certain perspective the results in this example agree with the findings of [25]: the use of anticipatory anti-windup can improve upon the results obtained with immediate anti-windup. However, the findings here differ with those in [25] in terms of the magnitude of improvement offered: only slight performance improvement is found here, whereas performance improvement similar to deferred-action anti-windup ([14, 20]) is found in [25]. It is not entirely clear why this difference is apparent, but our current feeling is, due to the combination of poorly conditioned compensators given in [25] and the algebraic loop needed to be solved as part of the simulation, that some overly optimistic simulation results are reported in [25]. It should also be mentioned that, for this example at least, the  $\mathcal{L}_2$  gain bound was fairly insensitive to the particular choice of  $\alpha$  and, indeed, the time domain performance was also relatively insensitive too; the incongruence between our findings and those in [25] is unlikely to be due to differing choices of  $\alpha$ .

### 4.3. Hydraulic Actuator Example

This example is taken from [12] and shows that anticipatory anti-windup must be used with caution; it was also used in [16] to demonstrate scheduled anti-windup. An immediate anti-windup compensator ([7]) and two anticipatory anti-windup compensators using Corollary 1 were synthesised for this system. The first anticipatory compensator was constructed using  $\mathcal{A} = 0.6$



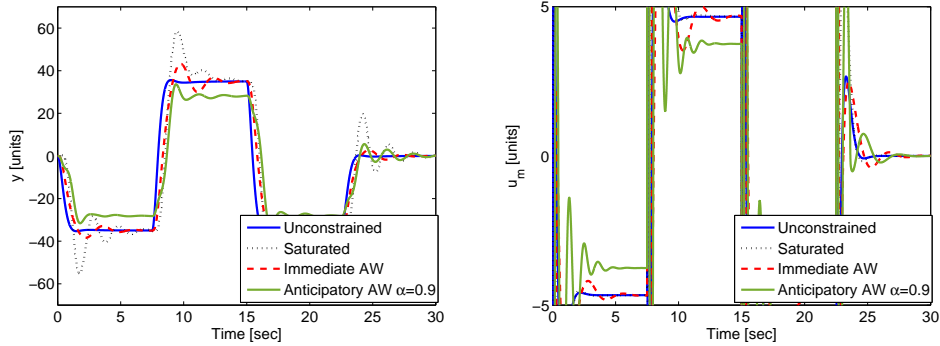


Figure 7. Hydraulic actuator example: output response  $y(t)$  and plant input response  $u_m(t)$ ;  $\bar{u} = 10.5$ .

and provided an  $\mathcal{L}_2$  gain bound of 32.9685; the second anticipatory anti-windup compensator was constructed using  $\mathcal{A} = 0.8$  and provided an  $\mathcal{L}_2$  gain bounds of 48.6219. The immediate anti-windup compensator provided an  $\mathcal{L}_2$  gain bound of 8.85882. Figure 6 shows the responses of the system to a pulse train reference in various circumstances. Note that the anticipatory anti-windup compensator ( $\mathcal{A} = 0.6$ ) arguably leads to the best time-domain performance for this reference despite having a higher  $\mathcal{L}_2$  gain bound. Figure 7 shows a similar comparison to Figure 6, but this time the anticipatory anti-windup compensator ( $\mathcal{A} = 0.9$ ) exhibits some deficiencies in the steady state-response: despite appealing transient behaviour, the controller's steady-state tracking ability is impaired. This is because, the anti-windup compensator is activated when the control signal exceeds  $(1 - \mathcal{A})\bar{u}$  which in this case is  $0.1\bar{u}$  ( $\bar{u} = 10.5$ ) and this is below the value of the steady-state control signal, leading to constant activation of the anti-windup compensation. This phenomenon is *not* present in deferred action anti-windup since the artificial deadzone limits are *above* the physical saturation limits.

#### 4.4. Comments on the use of anticipatory anti-windup

The numerical examples given above, and the papers [23, 24, 25], suggest that in some instances, anticipatory anti-windup may lead to superior time-domain tracking performance to that obtained using immediate anti-windup. However, in our simulations trials, we also found many cases in which anticipatory anti-windup *does not* give superior results to immediate anti-windup, and in fact either gives results close to those corresponding to immediate anti-windup, or actually somewhat *worse*. Therefore, anticipatory anti-windup may not always be the best choice of anti-windup strategy. In fact, we typically found that the deferred action anti-windup provided superior results to anticipatory anti-windup. On the other hand, due to the nonlinear nature of systems containing anti-windup, it seems that anticipatory anti-windup may be a useful tool for the control engineer in some circumstances. As with many nonlinear design techniques, anticipatory AW must be used with care.

## 5. CONCLUSION

This paper has proposed a different approach to synthesising anticipatory anti-windup compensators. Compared to [25], the main difference is that the work here is based on the use of a non-square sector condition which allows similar synthesis inequalities to be proposed to those which appear in standard linear anti-windup, rather than the pseudo-LPV framework used in previous work. In both [25] and in Corollary 1 (above) the anticipatory anti-windup synthesis

conditions are stated in terms of LMIs, but in Corollary 1, the LMI's appear to impose a lower computational burden and the compensators returned tend to have better numerical conditioning. In common with [23, 24, 25], it has also been noted that superior performance can sometimes be obtained with anticipatory anti-windup compensators.

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## A. STATE-SPACE MATRICES

The state-space matrices of  $\Sigma$  are defined in equation (72) and the linear closed-loop matrices are given by equation (73).

$$\left[ \begin{array}{c|c|c|c} A & B_w & \tilde{B}_1 & \tilde{B}_2 \\ \hline C & D_w & \tilde{D}_1 & \tilde{D}_2 \\ \hline C_z & D_{zw} & \tilde{D}_{z1} & \tilde{D}_{z2} \end{array} \right] = \left[ \begin{array}{cc|cc|c} A_{CL} & B_\Lambda \Lambda_3 & \bar{B}_w & B_\Lambda \Lambda_4 & \bar{B}_2 \\ \hline 0 & \Lambda_1 & 0 & \Lambda_2 & 0 \\ \hline C_{CL} & D_\Delta \Lambda_3 & \bar{D}_w & D_\Delta \Lambda_4 & \bar{D}_2 \\ \hline C_{z,CL} & D_\Lambda \Lambda_3 & \bar{D}_{zw} & D_\Lambda \Lambda_4 & \bar{D}_{z2} \end{array} \right] \quad (72)$$

$$\left[ \begin{array}{c|c} A_{CL} & \bar{B}_w \\ \hline C_{CL} & \bar{D}_w \\ \hline C_{z,CL} & \bar{D}_{zw} \end{array} \right] = \left[ \begin{array}{cc|c} A_p + B_2 \Delta D_{cy} C_2 & B_2 \Delta C_c & B_1 + B_2 \Delta (D_{cw} + D_{cy} D_{21}) \\ \hline B_{cy} \tilde{\Delta} C_2 & A_c + B_{cy} \tilde{\Delta} D_{22} C_c & B_{cw} + B_{cy} \tilde{\Delta} (D_{21} + D_{22} D_{cw}) \\ \hline \Delta D_{cy} C_2 & \Delta C_c & \Delta (D_{cw} + D_{cy} D_{21}) \\ \hline C_1 + D_{12} \Delta D_{cy} C_2 & D_{12} \Delta C_c & D_{11} + D_{12} \Delta (D_{cw} + D_{cy} D_{21}) \end{array} \right] \quad (73)$$

where  $\Delta := (I - D_{cy} D_{22})^{-1}$  and  $\tilde{\Delta} := (I - D_{22} D_{cy})^{-1}$ . The existence of these matrices is guaranteed by item (ii) in the definition of Problem 1. The auxiliary matrices are

$$\left[ \begin{array}{c|c} \bar{B}_2 & B_\Lambda \\ \hline \bar{D}_2 & D_\Delta \\ \hline \bar{D}_{z2} & D_\Lambda \end{array} \right] = \left[ \begin{array}{cc|c} -B_2 \Delta & 0 & B_2 \Delta \\ \hline -B_{cy} \tilde{\Delta} D_{22} & I & B_{cy} \tilde{\Delta} D_{22} \\ \hline -\Delta D_{cy} D_{22} & 0 & \Delta \\ \hline -D_{12} \Delta & 0 & D_{12} \Delta \end{array} \right] \quad (74)$$

Also, we have

$$\begin{aligned} A_0 &= \begin{bmatrix} A_{CL} & 0 \\ 0 & 0 \end{bmatrix} & C'_0 &= \begin{bmatrix} C'_{CL} \\ 0 \end{bmatrix} & C'_{z0} &= \begin{bmatrix} C'_{z,CL} \\ 0 \end{bmatrix} & G_2 &= \begin{bmatrix} 0 \\ I \end{bmatrix} \\ G_1 &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} & H_1 &= \begin{bmatrix} 0 & I \\ B'_\Lambda & 0 \end{bmatrix} & H_2 &= \begin{bmatrix} 0 \\ D'_\Delta \end{bmatrix} & H_3 &= \begin{bmatrix} 0 \\ D'_\Lambda \end{bmatrix} \end{aligned}$$