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# STABLE CATEGORIES AND RECONSTRUCTION

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*Dedicated to the memory of Sandy Green*

## 1. INTRODUCTION

The Green correspondence is a fundamental construction in modular representation theory of finite groups. It is expected (Broué's abelian defect group conjecture for example) to be the shadow of a more structural categorical correspondence, yet to be found. In an inductive approach to this, a key case is when the Green correspondence induces a stable equivalence between blocks. This work is an attempt towards a Morita theory for stable equivalences between self-injective algebras. More precisely, given two self-injective algebras  $A$  and  $B$  and an equivalence between their stable categories, consider the set  $\mathcal{S}$  of images of simple  $B$ -modules inside the stable category of  $A$ . That set satisfies some obvious properties of Hom-spaces and it generates the stable category of  $A$ . Keep now only  $\mathcal{S}$  and  $A$ . Can  $B$  be reconstructed? We show how to reconstruct the graded algebra associated to the radical filtration of (an algebra Morita equivalent to)  $B$ . It would be interesting to develop further an obstruction theory for the existence of an algebra  $B$  with that given filtration, starting only with  $\mathcal{S}$  (this might be studied in terms of localization of  $A_\infty$ -algebras). Note that a result of Linckelmann [Li] shows that, if we consider only stable equivalence of Morita type, then  $B$  is characterized by  $\mathcal{S}$  — but this result does not provide a reconstruction of  $B$  from  $\mathcal{S}$ .

We also study a similar problem in the more general setting of a triangulated category  $\mathcal{T}$ . Given a finite set  $\mathcal{S}$  of objects satisfying Hom-properties analogous to those satisfied by the set of simple modules in the derived category of a ring and assuming that the set generates  $\mathcal{T}$ , we construct a  $t$ -structure on  $\mathcal{T}$ . In the case  $\mathcal{T} = D^b(A)$  and  $A$  is a symmetric algebra, the first author has shown [Ri] that there is a symmetric algebra  $B$  with an equivalence  $D^b(B) \xrightarrow{\sim} D^b(A)$  sending the set of simple  $B$ -modules to  $\mathcal{S}$ . The case of a self-injective algebra leads to a slightly more general situation: there is a finite dimensional differential graded algebra  $B$  with  $H^i(B) = 0$  for  $i > 0$  and for  $i \ll 0$  with the same property as above.

## 2. NOTATIONS

Let  $\mathcal{C}$  be an additive category. Given  $S$  a set of objects of  $\mathcal{C}$ , we denote by  $\text{add } S$  the full subcategory of  $\mathcal{C}$  of objects isomorphic to finite direct sums of objects of  $S$ .

Let  $k$  be a field and  $A$  a finite dimensional  $k$ -algebra. We say that  $A$  is split if the endomorphism ring of every simple  $A$ -module is  $k$ . We denote by  $A\text{-mod}$  the category of finitely generated left  $A$ -modules and by  $D^b(A)$  its derived category. For  $A$  self-injective, we denote by  $A\text{-stab}$  the stable category, the quotient of  $A\text{-mod}$  by projective modules. Given  $M$  an  $A$ -module, we denote by  $\Omega M$  the kernel of a projective cover of  $M$  and by  $\Omega^{-1}M$  the cokernel of an injective hull of  $M$ .

## 3. SIMPLE GENERATORS FOR TRIANGULATED CATEGORIES

**3.1. Category of filtered objects.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{S}$  a full subcategory of  $\mathcal{T}$ .

We define a category  $\mathcal{F}$  as follows.

- Its objects are diagrams

$$M = (\cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{\varepsilon_0} N_0)$$

where  $M_i$  is an object of  $\mathcal{T}$ ,  $M_i = 0$  for  $i \gg 0$ , such that

- (i)  $M_1 \xrightarrow{f_1} M_0 \xrightarrow{\varepsilon_0} N_0$  is the beginning of a distinguished triangle
- (ii) for all  $i \geq 1$ , the cone  $N_{i-1}$  of  $f_i$  is in  $\text{add } \mathcal{S}$
- (iii) the canonical map  $\text{Hom}(N_0, S) \rightarrow \text{Hom}(M_0, S)$  is surjective for all  $S \in \mathcal{S}$
- (iv) the canonical map  $\text{Hom}(N_i, S) \rightarrow \text{Hom}(M_i, S)$  is bijective for all  $S \in \mathcal{S}$  and  $i \geq 1$ .

Note that  $\varepsilon_i : M_i \rightarrow N_i = \text{cone}(f_{i+1})$  is well defined up to unique isomorphism for  $i \geq 1$  thanks to property (iv). For  $i \geq 0$ , we define a new object  $M_{\geq i}$  of  $\mathcal{F}$  as  $\cdots \rightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{\varepsilon_i} N_i$ .

- Given another diagram  $M'$ , we define  $\text{Hom}_{\mathcal{F}}(M, M')_0$  as the subspace of  $\text{Hom}(N_0, N'_0)$  consisting of those maps  $g$  such that there is  $h : M_0 \rightarrow M'_0$  with  $\varepsilon'_0 h = g\varepsilon_0$ .

We put  $\text{Hom}_{\mathcal{F}}(M, M')_i = \text{Hom}_{\mathcal{F}}(M, M'_{\geq i})_0$  and  $\text{Hom}_{\mathcal{F}}(M, M') = \bigoplus_{i \geq 0} \text{Hom}_{\mathcal{F}}(M, M')_i$ .

- Consider now  $g_0 \in \text{Hom}_{\mathcal{F}}(M, M')$ . By (iv), there are maps  $h_0, h_1, \dots$  and  $g_1, g_2, \dots$  making the following diagrams commutative

$$\begin{array}{ccccccc} N_i[-1] & \xrightarrow{\rho_i} & M_{i+1} & \xrightarrow{f_{i+1}} & M_i & \xrightarrow{\varepsilon_i} & N_i \\ g_i[-1] \downarrow & & h_{i+1} \downarrow & & h_i \downarrow & & g_i \downarrow \\ N'_i[-1] & \xrightarrow{\rho'_i} & M'_{i+1} & \xrightarrow{f'_{i+1}} & M'_i & \xrightarrow{\varepsilon'_i} & N'_i \end{array}$$

Here,  $\rho_i : N_i[-1] \rightarrow M_{i+1}$  and  $\rho'_i : N'_i[-1] \rightarrow M'_{i+1}$  are the maps making the horizontal rows in the diagram above into distinguished triangles.

**Lemma 3.1.** *The maps  $g_i : N_i \rightarrow N'_i$  (for  $i \geq 1$ ) depend only on  $g_0$ .*

*Proof.* We proceed by induction on  $i$ . We assume  $g_{i-1}$  has been shown to depend only on  $g_0$ . Let us consider the lack of unicity of  $h_i$ . Consider  $h_i, \tilde{h}_i : M_i \rightarrow M'_i$  such that  $h_i \rho_{i-1} = \rho'_{i-1} g_{i-1}[-1] = \tilde{h}_i \rho_{i-1}$ . There is  $p : M_{i-1} \rightarrow M'_i$  such that  $\tilde{h}_i - h_i = pf_i$ .

By (iii) and (iv), there exists  $q : N_{i-1} \rightarrow N_i$  such that  $q\varepsilon_{i-1} = \varepsilon'_i p$ . We have  $\varepsilon'_i p f_i = q\varepsilon_{i-1} f_i = 0$ , hence  $\varepsilon'_i \tilde{h}_i = \varepsilon'_i h_i$ .

By (iv), we deduce that there is a unique map  $g_i : N_i \rightarrow N'_i$  such that  $g_i \varepsilon_i = \varepsilon'_i h_i$  and that map  $g_i$  is the unique one such that  $g_i \varepsilon_i = \varepsilon'_i \tilde{h}_i$ .  $\square$

Let  $g_0 \in \text{Hom}_{\mathcal{F}}(M, M')_i$  and  $g'_0 \in \text{Hom}_{\mathcal{F}}(M', M'')_j$ . We define the product  $g'_0 g_0$  as the composition  $N_0 \xrightarrow{g_0} N'_i \xrightarrow{g'_i} N''_{i+j}$ .

**Lemma 3.2.** *Assume  $\text{Hom}(S, T[n]) = 0$  for all  $S, T \in \mathcal{S}$  and  $n < 0$ . Let  $M$  be an object of  $\mathcal{F}$ . Then, the canonical map  $\text{Hom}(N_0, S) \rightarrow \text{Hom}(M_0, S)$  is an isomorphism.*

*Proof.* By induction on  $-i$ , we see that  $\text{Hom}(M_i, S[n]) = 0$  for  $n < 0$  and  $S \in \mathcal{S}$ . It follows that  $\text{Hom}(M_1[1], S) = 0$ , hence the canonical map  $\text{Hom}(N_0, S) \rightarrow \text{Hom}(M_0, S)$  is injective, as well as being surjective by assumption.  $\square$

**3.2.  $t$ -structures.** Let  $k$  be a field and assume  $\mathcal{T}$  is a  $k$ -linear triangulated category.

We assume from now on the following

**Hypothesis 1.** (1)  $\text{Hom}(S, T) = k^{\delta_{S,T}}$  for  $S, T \in \mathcal{S}$

(2)  $\mathcal{S}$  generates  $\mathcal{T}$  as a triangulated category

(3)  $\text{Hom}(S, T[n]) = 0$  for  $S, T \in \mathcal{S}$  and  $n < 0$ .

3.2.1.

**Lemma 3.3.** *Given  $N \in \mathcal{T}$ , there is a sequence  $0 = M_r \xrightarrow{f_r} \cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$  and  $d : \mathbf{Z}_{>0} \rightarrow \mathbf{Z}$  non increasing such that  $\text{cone}(f_i)[d(i)] \in \mathcal{S}$ .*

*For such a sequence, the maps  $M_{r-1} \rightarrow N$  and  $N \rightarrow \text{cone}(f_1)$  are non zero.*

*Proof.* Since  $\mathcal{T}$  is generated by  $\mathcal{S}$ , there is a sequence  $0 = M_r \rightarrow \cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$  and  $d : \mathbf{Z}_{>0} \rightarrow \mathbf{Z}$  such that  $\text{cone}(f_i)[d(i)] \in \mathcal{S}$ .

We put  $N_i = \text{cone}(f_i) = S_i[-d(i)]$  with  $S_i \in \mathcal{S}$ . Take  $i$  such that  $d(i) > d(i-1)$ . Let  $T$  be the cone of  $f_{i-1}f_i : M_i \rightarrow M_{i-2}$ . The octahedral axiom gives a distinguished triangle  $S_i[-d(i)] \rightarrow T \rightarrow S_{i-1}[-d(i-1)] \rightsquigarrow$ .

Assume the morphism  $S_{i-1}[-d(i-1)] \rightarrow S_i[-d(i)+1]$  is non zero. Then it is an isomorphism and  $d(i) = d(i-1) + 1$ . It follows that  $T = 0$  and  $f_{i-1}f_i$  is an isomorphism. Consequently,

$$0 = M_r \rightarrow \cdots \rightarrow M_{i+1} \xrightarrow{f_{i-1}f_i f_{i+1}} M_{i-2} \rightarrow \cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$$

is a new sequence with successive cones being shifts of objects of  $\mathcal{S}$ .

By induction, we can assume that the morphism  $S_{i-1}[-d(i-1)] \rightarrow S_i[-d(i)+1]$  is zero. Then,  $T \simeq N_i \oplus N_{i-1}$ . There is an object  $M'_{i-1}$  and distinguished triangles  $M_i \rightarrow M'_{i-1} \rightarrow N_{i-1} \rightsquigarrow$  and  $M'_{i-1} \rightarrow M_{i-2} \rightarrow N_i \rightsquigarrow$ . Put  $M'_j = M_j$  for  $j \neq i-1$ . So,

$$0 = M'_r \rightarrow \cdots \rightarrow M'_i \rightarrow M'_{i-1} \rightarrow M'_0 = N$$

is a new sequence with the same cones as in the original sequence except the  $i$  and  $i-1$  ones which have been swapped. By induction, we can reorder the cones in the sequence so that  $d$  is non increasing.

Assume the map  $M_{r-1} \rightarrow N$  is zero. Let  $T$  be its cone. Then  $T \simeq N \oplus M_{r-1}[1]$ . Note that  $T$  is filtered by the  $S_i[-d(i)]$  with  $-d(i) < -d(r) + 1$ , hence  $\text{Hom}(M_{r-1}[1], T) = 0$ . So we have a contradiction. The case of the map  $N \rightarrow N_1$  is similar.  $\square$

Let  $\mathcal{T}^{\leq 0}$  (resp.  $\mathcal{T}^{> 0}$ ) be the full subcategory of objects  $N$  in  $\mathcal{T}$  such that there is a sequence  $0 = M_r \rightarrow \cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$  with  $\text{cone}(f_i)$  a direct sum of objects  $S[r]$  with  $S \in \mathcal{S}$  and  $r \geq 0$  (resp.  $r < 0$ ).

**Proposition 3.4.**  *$(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$  is a bounded  $t$ -structure on  $\mathcal{T}$ .*

*Proof.* By induction, we see there is no non-zero map from an object of  $\mathcal{T}^{\leq 0}$  to an object of  $\mathcal{T}^{> 0}$ . Furthermore, we have  $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$  and  $\mathcal{T}^{> 0} \subseteq \mathcal{T}^{> 0}[1]$ .

Let  $N \in \mathcal{T}$ . Pick a sequence as in Lemma 3.3. Take  $s$  such that  $d(s) > 0$  and  $d(s+1) \leq 0$ . Let  $L$  be the cone of  $f_1 \cdots f_s : M_s \rightarrow N$ . We have a distinguished triangle

$$M_s \rightarrow N \rightarrow L \rightsquigarrow$$

with  $M_s \in \mathcal{T}^{\leq 0}$  and  $L \in \mathcal{T}^{> 0}$ . □

We have a characterization of  $\mathcal{T}^{\geq 0}$  and  $\mathcal{T}^{\leq 0}$  :

**Proposition 3.5.** *Let  $N \in \mathcal{T}$ . Then,  $N \in \mathcal{T}^{\leq 0}$  if and only if  $\text{Hom}(N, S[i]) = 0$  for  $S \in \mathcal{S}$  and  $i < 0$ .*

*Similarly,  $N \in \mathcal{T}^{\geq 0}$  if and only if  $\text{Hom}(S[i], N) = 0$  for  $S \in \mathcal{S}$  and  $i > 0$ .*

*Proof.* We have  $\text{Hom}(N, S[i]) = 0$  for  $S \in \mathcal{S}$  and  $i < 0$ , if  $N \in \mathcal{S}[r]$  with  $r \geq 0$ . By induction, it follows that if  $N \in \mathcal{T}^{\leq 0}$ , then  $\text{Hom}(N, S[i]) = 0$  for  $S \in \mathcal{S}$  and  $i < 0$ .

Assume now  $\text{Hom}(N, S[i]) = 0$  for  $S \in \mathcal{S}$  and  $i < 0$ . Pick a filtration of  $N$  as in Lemma 3.3. Then,  $d(1) \leq 0$ , hence  $d(i) \leq 0$  for all  $i$  and  $N \in \mathcal{T}^{\leq 0}$ .

The other case is similar. □

Note that the heart  $\mathcal{A}$  of the  $t$ -structure is artinian and noetherian. Its set of simple objects is  $\mathcal{S}$ .

**Remark 3.6.** Assume  $\mathcal{T}$  can be generated by a finite set of objects. Then, there is a finite subcategory  $\mathcal{S}'$  of  $\mathcal{S}$  generating  $\mathcal{T}$ . It follows immediately from condition (i) that  $\mathcal{S} = \mathcal{S}'$ . So,  $\mathcal{S}$  has only finitely many objects.

3.2.2. In §3.2.2, we assume  $\mathcal{T} = D^b(A)$  where  $A$  is a finite dimensional  $k$ -algebra. By Remark 3.6,  $\mathcal{S}$  is finite (note that  $\mathcal{T}$  is generated by the simple  $A$ -modules, up to isomorphism).

**Proposition 3.7.** *Let  $S \in \mathcal{S}$ . There is a bounded complex of finitely generated injective  $A$ -modules  $I_S(S) \in \mathcal{T}^{\geq 0}$  such that, given  $T \in \mathcal{S}$  and  $i \in \mathbf{Z}$ , we have*

$$\text{Hom}_{D^b(A)}(T, I_S(S)[i]) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases}$$

*Similarly, there is a bounded complex of finitely generated projective  $A$ -modules  $P_S(S) \in \mathcal{T}^{\leq 0}$  such that, given  $T \in \mathcal{S}$  and  $i \in \mathbf{Z}$ , we have*

$$\text{Hom}_{D^b(A)}(P_S(S)[i], T) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The construction of a complex  $I_S(S)$  of  $A$ -modules with the Hom property is [Ri, §5] (note that the proof of [Ri, Lemma 5.4] is valid for non-symmetric algebras). It is in  $\mathcal{T}^{\geq 0}$  by Proposition 3.5. Since  $\bigoplus_{i \in \mathbf{Z}} \dim \text{Hom}_{D^b(A)}(V, I_S(S)[i]) = 0$  for all simple  $A$ -modules  $V$ , we deduce that  $I_S(S)$  is isomorphic to a bounded complex of finitely generated injective  $A$ -modules.

The second case follows from the first one by passing to  $A^{\text{opp}}$  and taking the  $k$ -duals of elements of  $\mathcal{S}$ . □

We denote by  $\tau^{> 0}$ , etc... the truncation functors and  ${}^t H^0$  the  $H^0$ -functor associated to the  $t$ -structure constructed in §3.2.1.

**Lemma 3.8.** *The object  ${}^t H^0(I_S(S))$  of  $\mathcal{A}$  is an injective hull of  $S$  and  ${}^t H^0(P_S(S))$  is a projective cover of  $S$ .*

*Proof.* We have a distinguished triangle

$${}^tH^0(I_{\mathcal{S}}(S)) \rightarrow I_{\mathcal{S}}(S) \rightarrow \tau^{>0}I_{\mathcal{S}}(S) \rightsquigarrow .$$

Let  $N \in \mathcal{A}$ . We have  $\text{Hom}(N, \tau^{>0}I_{\mathcal{S}}(S)) = 0$  and  $\text{Hom}(N, I_{\mathcal{S}}(S)[1]) = 0$ , so we deduce that  $\text{Hom}(N, {}^tH^0(I_{\mathcal{S}}(S))[1]) = 0$ . It follows that  $\text{Ext}_{\mathcal{A}}^1(N, {}^tH^0(I_{\mathcal{S}}(S))) = 0$ , hence  ${}^tH^0(I_{\mathcal{S}}(S))$  is injective. Since  $\text{Hom}(T, (\tau^{>0}I_{\mathcal{S}}(S))[-1]) = 0$ , we have  $\text{Hom}(T, {}^tH^0(I_{\mathcal{S}}(S))) \xrightarrow{\sim} \text{Hom}(T, I_{\mathcal{S}}(S)) = k^{\delta_{ST}}$  for  $T \in \mathcal{S}$ . So  ${}^tH^0(I_{\mathcal{S}}(S))$  is an injective hull of  $S$ . The projective case is similar.  $\square$

Let us consider the finite dimensional differential graded algebra

$$B = \text{End}_{\mathcal{A}}^{\bullet}(\bigoplus_S P_{\mathcal{S}}(S)) = \bigoplus_i \text{Hom}_{\mathcal{A}}(\bigoplus_S P_{\mathcal{S}}(S), \bigoplus_S P_{\mathcal{S}}(S)[i]).$$

Denote by  $D^b(B)$  the derived category of finite dimensional differential graded  $B$ -modules.

**Theorem 3.9.** *We have  $H^i(B) = 0$  for  $i > 0$  and for  $i \ll 0$ . We have  $H^0(B)\text{-mod} \simeq \mathcal{A}$  and  $D^b(B) \simeq D^b(A)$ .*

*Proof.* Let  $N \in \mathcal{T}$  and consider a filtration of  $N$  as in Lemma 3.3. Take  $S \in \mathcal{S}$  such that  $S[i]$  is isomorphic to the cone of  $M_d \rightarrow M_{d-1}$ . Then,  $\text{Hom}(P_{\mathcal{S}}(S)[i], N) \neq 0$ . It follows that the right orthogonal category of  $\{P_{\mathcal{S}}(S)[i]\}_{S \in \mathcal{S}, i \in \mathbf{Z}}$  is zero. Since the  $P_{\mathcal{S}}(S)$  are perfect, it follows that  $\bigoplus_S P_{\mathcal{S}}(S)$  generates the category of perfect complexes of  $A$ -modules as a triangulated category closed under taking direct summands [Nee, Lemma 2.2]. The functor  $\text{Hom}_{\mathcal{A}}^{\bullet}(\bigoplus_S P_{\mathcal{S}}(S), -)$  gives an equivalence  $D^b(A) \xrightarrow{\sim} D^b(B)$  [Ke, Theorem 4.3].

Let  $C = \bigoplus_{S \in \mathcal{S}} P_{\mathcal{S}}(S)$  and  $N = {}^tH^0(C)$ . We have a distinguished triangle  $\tau^{<0}C \rightarrow C \rightarrow N \rightsquigarrow$ . We have  $\text{Hom}(\tau^{<0}C, N[i]) = 0$  for  $i \leq 0$ . We deduce that the canonical morphism  $\text{Hom}(N, N) \rightarrow \text{Hom}(C, N)$  is an isomorphism. We have  $\text{Hom}(C, (\tau^{<0}C)[i]) = 0$  for  $i \geq 0$  since  $\tau^{<0}C$  is filtered by objects in  $\mathcal{S}[d]$ ,  $d > 0$  (cf Proposition 3.7). It follows that the canonical morphism  $\text{Hom}(C, C) \rightarrow \text{Hom}(C, N)$  is an isomorphism.

This shows that the canonical morphism  $\text{End}(C) \rightarrow \text{End}({}^tH^0(C))$  is an isomorphism. By Lemma 3.8,  ${}^tH^0(C)$  is a progenerator for  $\mathcal{A}$ . So  $H^0(B)\text{-mod} \simeq \mathcal{A}$ .

Note that  $H^i(B) = 0$  for  $i \ll 0$  because  $\bigoplus_S P_{\mathcal{S}}(S)$  is bounded. Since  $P_{\mathcal{S}}(S)$  is filtered by objects in  $\mathcal{S}[d]$  with  $d \geq 0$ , it follows from Proposition 3.7 that  $\text{Hom}(P_{\mathcal{S}}(T), P_{\mathcal{S}}(S)[i]) = 0$  for  $i > 0$ . So,  $H^i(B) = 0$  for  $i > 0$ .  $\square$

The following proposition is clear.

**Proposition 3.10.** *Let  $B$  be a dg-algebra with  $H^i(B) = 0$  for  $i > 0$  and for  $i \ll 0$ . Let  $C$  be the sub-dg-algebra of  $B$  given by  $C^i = B^i$  for  $i < 0$ ,  $C^0 = \ker d^0$  and  $C^i = 0$  for  $i > 0$ . Then the restriction  $D(B) \rightarrow D(C)$  is an equivalence.*

*Let  $\mathcal{S}$  be a complete set of representatives of isomorphism classes of simple  $H^0(B)$ -modules (viewed as dg- $C$ -modules). Then  $\mathcal{S}$  satisfies Hypothesis 1. Furthermore,  $\mathcal{A} \simeq H^0(B)\text{-mod}$ .*

So we have a bijection between

- the sets  $\mathcal{S}$  (up to isomorphism) satisfying Hypothesis 1
- the equivalences  $D^b(B) \xrightarrow{\sim} D^b(A)$  where  $B$  is a dg-algebra with  $H^i(B) = 0$  for  $i > 0$  and for  $i \ll 0$  and where  $B$  is well-defined up to quasi-isomorphism and the equivalence is taken modulo self-equivalences of  $D^b(B)$  that fix the isomorphism classes of simple  $H^0(B)$ -modules.

We recover a result of Al-Nofayee [Al, Theorem 4] :

**Proposition 3.11.** *Assume  $A$  is self-injective with Nakayama functor  $\nu$ . The following are equivalent*

- $H^i(B) = 0$  for  $i \neq 0$
- $\nu(\mathcal{S}) = \mathcal{S}$  (up to isomorphism).

*Proof.* Note that  $\mathcal{S}$  is stable under  $\nu$  if and only if  $\{P_S(S)\}_{S \in \mathcal{S}}$  is stable under  $\nu$  (up to isomorphism). Given  $S, T \in \mathcal{S}$  and  $i \in \mathbf{Z}$ , we have

$$\mathrm{Hom}_{D^b(A)}(P_S(S), P_S(T)[i])^* \simeq \mathrm{Hom}_{D^b(A)}(P_S(T), \nu(P_S(S))[-i]).$$

If  $\mathcal{S}$  is stable under  $\nu$ , then  $\mathrm{Hom}_{D^b(A)}(P_S(T), \nu(P_S(S))[-i]) = 0$  for  $i > 0$ , hence  $H^{<0}(B) = 0$ .

Assume now  $H^{<0}(B) = 0$ . Then, viewed as an object of  $D^b(B)$ ,  $\nu(P_S(S))$  is concentrated in degree 0. Since it is perfect, it is isomorphic to a projective indecomposable module, hence to  $P_S(S')$  for some  $S' \in \mathcal{S}$ . So,  $\mathcal{S}$  is stable under  $\nu$ .  $\square$

We recover now the main result of [AlRi]:

**Corollary 3.12.** *Let  $A$  be a self-injective algebra and  $B$  an algebra derived equivalent to  $A$ . Then  $B$  is self-injective.*

From Proposition 3.11, we recover [Ri, Theorem 5.1] :

**Theorem 3.13.** *If  $A$  is symmetric then  $H^i(B) = 0$  for  $i \neq 0$ , i.e., there is an equivalence  $D^b(\mathcal{A}) \xrightarrow{\sim} D^b(A)$  where  $\mathcal{S}$  is the set of images of the simple objects of  $\mathcal{A}$ .*

**Remark 3.14.** Theorem 3.13 does not hold in general for a self-injective algebra. Take  $A = k[\varepsilon]/(\varepsilon^2) \rtimes \mu_2$ , where  $\mu_2 = \{\pm 1\}$  acts on  $k[\varepsilon]/(\varepsilon^2)$  by multiplication on  $\varepsilon$ . Assume  $k$  does not have characteristic 2. This is a self-injective algebra which is not symmetric. The Nakayama functor swaps the two simple  $A$ -modules  $U$  and  $V$ .

Let  $P_U$  (resp.  $P_V$ ) be a projective cover of  $U$  (resp.  $V$ ). Take  $S = U$  and  $T = P_U[1]$ . Then, the set  $\mathcal{S} = \{S, T\}$  satisfies Hypothesis 1. We have  $I_S(T) \simeq T$  and  $I_S(S) \simeq 0 \rightarrow P_U \rightarrow P_V \rightarrow 0$ , a complex with homology  $V$  in degree 0 and  $-1$ .

The dg-algebra  $B$  has homology  $H^0(B)$  isomorphic to the path algebra of the quiver  $\bullet \longrightarrow \bullet$ ,  $H^{-1}(B) = k$  and  $H^i(B) = 0$  for  $i \neq 0, -1$ .

The derived category of the hereditary algebra  $H^0(B)$  is not equivalent to  $D^b(A)$ .

**3.3. Graded of an abelian category.** Let  $\mathcal{A}$  be an abelian  $k$ -linear artinian and noetherian category with finitely many simple objects up to isomorphism and  $\mathcal{S}$  a complete set of representatives of isomorphism classes of simple objects. We assume  $\mathcal{A}$  is split, i.e., endomorphism rings of simple objects are isomorphic to  $k$ . Let  $\mathcal{T} = D^b(\mathcal{A})$ .

Let  $\mathrm{gr}\mathcal{A}$  be the category with objects the objects of  $\mathcal{A}$  and where  $\mathrm{Hom}_{\mathrm{gr}\mathcal{A}}(M, N)$  is the graded vector space associated to the filtration of  $\mathrm{Hom}_{\mathcal{A}}(M, N)$  given by  $\mathrm{Hom}_{\mathcal{A}}(M, N)^i = \{f \mid \mathrm{im} f \subseteq \mathrm{rad}^i N\}$ .

Given  $M$  in  $\mathcal{A}$ , let  $M_i = \mathrm{rad}^i M$ ,  $f_i : M_i \rightarrow M_{i-1}$  the inclusion,  $N_0 = M/M_1$  and  $\varepsilon_0 : M \rightarrow M/M_1$  the projection. This defines an object of  $\mathcal{F}$ .

We obtain a functor  $\mathrm{gr}\mathcal{A} \rightarrow \mathcal{F}$ .

**Proposition 3.15.** *The canonical functor  $\mathrm{gr}\mathcal{A} \rightarrow \mathcal{F}$  is an equivalence.*

*Proof.* The image of  $\text{Hom}_{\mathcal{A}}(N, N')$  in  $\text{Hom}_{\mathcal{A}}(N, N'_0)$  is isomorphic to the quotient of  $\text{Hom}_{\mathcal{A}}(N, N')$  by  $\text{Hom}_{\mathcal{A}}(N, \text{rad } N')$  and it follows that the functor is fully faithful.

Let us show that it is essentially surjective. Let  $M \in \mathcal{F}$ . Let  $r \geq 0$  such that  $M_{r+1} = 0$ . Then,  $M_r \xrightarrow{\sim} N_r$  has homology concentrated in degree 0 and is semi-simple. By induction on  $-i$ , it follows from the distinguished triangle  $M_{i+1} \rightarrow M_i \rightarrow N_i \rightsquigarrow$  that  $M_i$  has homology concentrated in degree 0.

Note that we have an exact sequence  $0 \rightarrow H^0 M_{i+1} \rightarrow H^0 M_i \rightarrow H^0 N_i \rightarrow 0$ . Since the canonical map  $\text{Hom}(H^0 N_i, S) \rightarrow \text{Hom}(H^0 M_i, S)$  is bijective for any simple  $S$ , it follows that  $H^0 N_i$  is the largest semi-simple quotient of  $H^0 M_i$ . So,  $M_i \xrightarrow{\sim} \text{rad}^i M_0$  and  $M$  comes from an object of  $\mathcal{A}$ .  $\square$

#### 4. SIMPLE GENERATORS FOR STABLE CATEGORIES

**4.1. From equivalences.** Let  $k$  be a field and  $A$  a split self-injective  $k$ -algebra with no projective simple module.

Let  $B$  be another split self-injective  $k$ -algebra with no projective simple module, and let  $F : B\text{-stab} \xrightarrow{\sim} A\text{-stab}$  be an equivalence of triangulated categories. Let  $\mathcal{S}'$  be a complete set of representatives of isomorphism classes of simple  $B$ -modules. For  $L \in \mathcal{S}'$ , let  $L'$  be an indecomposable  $A$ -module isomorphic to  $F(L)$  in  $A\text{-stab}$ . Let  $\mathcal{S} = \{L'\}_{L \in \mathcal{S}'}$ . Then,

- (i)  $\text{Hom}_{A\text{-stab}}(S, T) = k^{\delta_{S,T}}$  for  $S, T \in \mathcal{S}$
- (ii) Every object  $M$  of  $A\text{-stab}$  has a filtration  $0 = M_r \rightarrow M_{r-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 = M$  such that the cone of  $M_i \rightarrow M_{i-1}$  is isomorphic to an object of  $\mathcal{S}$ .

Note that (ii) is equivalent to

- (ii') Given  $M$  in  $A\text{-mod}$ , there is a projective module  $P$  such that  $M \oplus P$  has a filtration  $0 = N_r \subset N_{r-1} \subset \cdots \subset N_1 \subset N_0 = M \oplus P$  with the property that  $N_i/N_{i-1}$  is isomorphic (in  $A\text{-mod}$ ) to an object of  $\mathcal{S}$ .

Linckelmann has shown the following [Li, Theorem 2.1 (iii)] :

**Proposition 4.1.** *Assume that  $F$  is induced by an exact functor  $B\text{-mod} \rightarrow A\text{-mod}$ . If  $\mathcal{S}$  consists of simple modules, then there is a direct summand of  $F$  that is an equivalence  $B\text{-mod} \xrightarrow{\sim} A\text{-mod}$ .*

We deduce :

**Corollary 4.2.** *Let  $B_1, B_2$  be split self-injective algebras with no projective simple modules and  $G_i : B_i\text{-mod} \rightarrow A\text{-mod}$  exact functors inducing stable equivalences. Assume  $\mathcal{S}_1 = \mathcal{S}_2$  (up to isomorphism). Then,  $B_1$  and  $B_2$  are Morita equivalent.*

So, if we assume in addition that  $F$  comes from an exact functor  $G$  between module categories, then  $B$  is determined by  $\mathcal{S}$ , up to Morita equivalence.

The functor  $G$  is isomorphic to  $X \otimes_B -$  where  $X$  is an  $(A, B)$ -bimodule. We can (and will) choose  $G$  so that  $X$  has no non-zero projective direct summand. Then,  $G(L)$  is indecomposable for  $L$  simple [Li, Theorem 2.1 (ii)], so  $\mathcal{S} = \{G(L)\}_{L \in \mathcal{S}'}$ , up to isomorphism.

**Proposition 4.3.** *An  $A$ -module  $M$  is in the image of  $G$  if and only if there is a filtration  $0 = M_r \subset M_{r-1} \subset \cdots \subset M_1 \subset M_0 = M$  such that  $M_i/M_{i-1}$  is isomorphic to an object of  $\mathcal{S}$ .*



*Proof.* Take  $L$  a  $B$ -module. Then the image by  $G$  of a filtration of  $L$  whose successive quotients are simple provides a filtration as required.

Conversely, we proceed by induction on  $r$ . We have an exact sequence  $0 \rightarrow G(N) \rightarrow M \rightarrow G(L) \rightarrow 0$  and a corresponding element  $\zeta \in \text{Ext}_A^1(G(L), G(N))$ . We have an isomorphism  $\text{Ext}_B^1(L, N) \xrightarrow{\sim} \text{Ext}_A^1(G(L), G(N))$  and we take  $\zeta'$  to be the inverse image of  $\zeta$  under this isomorphism. This gives an exact sequence  $0 \rightarrow N \rightarrow M' \rightarrow L \rightarrow 0$ , and hence an exact sequence  $0 \rightarrow G(N) \rightarrow G(M') \rightarrow G(L) \rightarrow 0$  with class  $\zeta$ . It follows that  $M \simeq G(M')$  and we are done.  $\square$

## 4.2. Filtrable objects.

4.2.1. Given two  $A$ -modules  $M$  and  $N$ , we write  $M \sim N$  to denote the existence of an isomorphism between  $M$  and  $N$  in  $A\text{-stab}$ . Given  $f, g \in \text{Hom}_A(M, N)$ , we write  $f \sim g$  if  $f - g$  is a projective map.

**Lemma 4.4.** *Let  $f, f' : M \rightarrow N$  be two surjective maps with  $f \sim g$ . Then there is  $\sigma \in \text{Aut}_A(M)$  with  $f' = f\sigma$  and  $\sigma \sim \text{id}_M$ .*

*Similarly, let  $f, f' : N \rightarrow M$  be two injective maps with  $f \sim g$ . Then there is  $\sigma \in \text{Aut}_A(M)$  with  $f' = \sigma f$  and  $\sigma \sim \text{id}_M$ .*

*Proof.* Let  $L = \ker f$  and  $L' = \ker f'$ . Let  $L = L_0 \oplus P$  and  $L' = L'_0 \oplus P'$  with  $P, P'$  projective and  $L_0, L'_0$  without non-zero projective direct summands. We have an isomorphism  $\bar{\alpha}_0 \in \text{Hom}_{A\text{-stab}}(L_0, L'_0)$  in  $A\text{-stab}$  giving rise to an isomorphism of distinguished triangles in  $A\text{-stab}$

$$\begin{array}{ccccccc} L_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Omega^{-1}L_0 \\ \bar{\alpha}_0 \downarrow \sim & & \parallel & & \parallel & & \Omega^{-1}(\bar{\alpha}_0) \downarrow \sim \\ L'_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Omega^{-1}L'_0 \end{array}$$

Let  $\alpha_0 \in \text{Hom}_A(L_0, L'_0)$  lifting  $\bar{\alpha}_0$ . This is an isomorphism. There is now a commutative diagram of  $A$ -modules, where the exact rows come from the elements of  $\text{Ext}_A^1(N, L_0)$  and  $\text{Ext}_A^1(N, L'_0)$  defined above :

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_0 & \longrightarrow & M_0 & \longrightarrow & N \longrightarrow 0 \\ & & \alpha_0 \downarrow \sim & & \sigma_0 \downarrow \sim & & \parallel \\ 0 & \longrightarrow & L'_0 & \longrightarrow & M'_0 & \longrightarrow & N \longrightarrow 0 \end{array}$$

We have  $M \simeq M_0 \oplus P \simeq M'_0 \oplus P'$ , hence  $P \simeq P'$ . Let  $\alpha : L \xrightarrow{\sim} L'$  extending  $\alpha_0$ . Then there is  $\sigma : M \xrightarrow{\sim} M$  making the following diagram commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\ & & \alpha \downarrow \sim & & \sigma \downarrow \sim & & \parallel \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \end{array}$$

and we are done.

The second part of the lemma has a similar proof — it can also be deduced from the first part by duality.  $\square$

4.2.2.

**Hypothesis 2.** Let  $\mathcal{S}$  be a finite set of indecomposable finitely generated  $A$ -modules such that  $\text{Hom}_{A\text{-stab}}(S, T) = k^{\delta_{S,T}}$  for  $S, T \in \mathcal{S}$ .

An  $\mathcal{S}$ -filtration for an  $A$ -module  $M$  is a filtration  $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$  such that  $\bar{M}_i = M_i/M_{i+1}$  is in  $\text{add}(\mathcal{S})$  for  $0 \leq i \leq r-1$ .

We say that  $M$  is *filtrable* if it admits an  $\mathcal{S}$ -filtration.

**Lemma 4.5.** *Let  $M$  be a non-projective filtrable  $A$ -module. Then there is  $S \in \mathcal{S}$  such that  $\text{Hom}_{A\text{-stab}}(M, S) \neq 0$  (resp. such that  $\text{Hom}_{A\text{-stab}}(S, M) \neq 0$ ).*

*Proof.* Assume  $\text{Hom}_{A\text{-stab}}(M, S) = 0$  for all  $S \in \mathcal{S}$ . Since  $M$  is filtrable, it follows that  $\text{End}_{A\text{-stab}}(M) = 0$ , and hence  $M$  is projective, which is not true. The second case is similar.  $\square$

**Lemma 4.6.** *Let  $M$  be a filtrable module and  $S \in \mathcal{S}$ . Given  $f : M \rightarrow S$  non-projective, there is  $g : M \rightarrow S$  surjective with filtrable kernel such that  $f \sim g$ . Similarly, given  $f : S \rightarrow M$  non-projective, there is  $g : S \rightarrow M$  injective with filtrable cokernel such that  $f \sim g$ .*

*Proof.* We proceed by induction on the number of terms in a filtration of  $M$ . The result is clear if  $M \in \mathcal{S}$ .

Let  $0 \rightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} T \rightarrow 0$  be an exact sequence with  $T \in \mathcal{S}$  and  $N$  filtrable.

Assume first  $f\alpha : N \rightarrow S$  is projective. Then there is  $p : M \rightarrow S$  projective and  $g : T \rightarrow S$  with  $f - p = g\beta$ . Since  $g$  is not projective, it is an isomorphism. Consequently,  $f - p$  is surjective and its kernel is isomorphic to  $N$  by Lemma 4.4, so we are done.

Assume now  $f\alpha : N \rightarrow S$  is not projective. By induction, there is  $q : N \rightarrow S$  projective such that  $f\alpha + q$  is surjective with filtrable kernel  $N'$ . Since  $\alpha : N \rightarrow M$  is injective, there is a projective map  $p : M \rightarrow S$  with  $q = p\alpha$ . Now, we have an exact sequence  $0 \rightarrow N/N' \xrightarrow{\bar{\alpha}} M/\alpha(N') \rightarrow T \rightarrow 0$  and a non-projective surjection  $f + p : M/\alpha(N') \rightarrow S$ . Since  $(f + p)\bar{\alpha} : N/N' \rightarrow S$  is an isomorphism, it follows that the kernel of the map  $M/\alpha(N') \rightarrow S$  is isomorphic to  $T$ . Since  $N'$  is filtrable, it follows that  $\ker(f + p)$  is filtrable and we are done. The second assertion follows by duality.  $\square$

From Lemmas 4.4 and 4.6, we deduce :

**Lemma 4.7.** *Let  $S \in \mathcal{S}$  and let  $M$  be a filtrable module.*

*If  $f : M \rightarrow S$  be a surjective and non-projective map, then  $\ker f$  is filtrable.*

*Similarly, if  $g : S \rightarrow M$  is injective and non-projective, then  $\text{coker } g$  is filtrable.*

From Lemmas 4.5 and 4.6, we deduce :

**Lemma 4.8.** *Let  $M$  be filtrable non-projective. Then there is a submodule  $S$  of  $M$ , with  $S \in \mathcal{S}$ , such that  $M/S$  is filtrable and the inclusion  $S \rightarrow M$  is not projective. Similarly, there is a filtrable submodule  $N$  of  $M$  such that  $M/N \in \mathcal{S}$  and  $M \rightarrow M/N$  is not projective.*

**Proposition 4.9.** *Let  $M$  be an  $A$ -module with a decomposition  $M \sim M'_1 \oplus M'_2$  in the stable category. If  $M$  is filtrable then there is a decomposition  $M = M_1 \oplus M_2$  such that  $M_i$  is filtrable and  $M_i \sim M'_i$*

*Proof.* We can assume  $M$  is not projective, for otherwise the proposition is trivial. We prove the proposition by induction on the dimension of  $M$ .

Let  $M = T_1 \oplus T_2 \oplus P$  with  $P$  projective,  $T_i$  without non-zero projective direct summand and  $T_i \sim M'_i$ . Denote by  $\pi : M \rightarrow T_1$  the projection.

By Lemma 4.5, there is  $S \in \mathcal{S}$  such that  $\text{Hom}_{A\text{-stab}}(M, S) \neq 0$ . Hence,  $\text{Hom}_{A\text{-stab}}(T_i, S) \neq 0$  for  $i = 1$  or  $i = 2$ . Assume for instance  $i = 1$ . Pick a non-projective map  $\alpha : T_1 \rightarrow S$ . So,  $\alpha\pi : M \rightarrow S$  is not projective. By Lemma 4.6, there is a surjective map  $\beta : M \rightarrow S$  with  $\beta \sim \alpha\pi$  and  $N = \ker \beta$  filtrable. Then  $N \sim L \oplus T_2$ , where  $L$  is the kernel of  $\alpha + p : T_1 \oplus P_S \rightarrow S$  and  $p : P_S \rightarrow S$  is a projective cover of  $S$ . By induction, we have  $N = N_1 \oplus N_2$  with  $N_i$  filtrable and  $N_1 \sim L$ ,  $N_2 \sim T_2$ . Now, the map  $S \rightarrow L[1]$  gives a map  $S \rightarrow N_1[1]$  (in  $A\text{-stab}$ ). Let  $M_1$  be the extension of  $S$  by  $N_1$  corresponding to that map. Then  $M \simeq M_1 \oplus N_2$ , the modules  $M_1$  and  $N_2$  are filtrable,  $M_1 \sim M'_1$ , and  $N_2 \sim M'_2$ .  $\square$

Let  $M$  be a filtrable module. We say that  $M$  has no projective remainder if there is no direct sum decomposition  $M = N \oplus P$  with  $P \neq 0$  projective and  $N$  filtrable.

**Lemma 4.10.** *Let  $M$  be a filtrable module with no projective remainder and let  $S \in \mathcal{S}$ .*

*For  $f : M \rightarrow S$  surjective,  $\ker f$  is filtrable if and only if  $f$  is non-projective.*

*For  $f : S \rightarrow M$  injective,  $\text{coker } f$  is filtrable if and only if  $f$  is non-projective.*

*Proof.* Assume  $f$  is projective. Then there is a decomposition  $M = N \oplus P$  and  $f = (0, g)$  with  $P$  projective. Now,  $\ker f = N \oplus \ker g$ . If  $\ker f$  is filtrable, then it follows from Lemma 4.9 that  $M$  has a non-zero projective submodule whose quotient is filtrable.

The converse is given by Lemma 4.7. The second part of the Lemma has a similar proof.  $\square$

**Lemma 4.11.** *Let  $M = M_0 \oplus M_1$  with  $M$  and  $M_0$  filtrable and such that  $M_0$  has no projective remainder. Then  $M_1$  is filtrable.*

*Proof.* We proceed by induction on  $\dim M_0$  — the result is clear for  $M_0 = 0$ . Assume  $M_0 \neq 0$ . Let  $f : M_0 \rightarrow S$  be a surjection with  $S \in \mathcal{S}$  and  $\ker f$  filtrable. By Lemma 4.10,  $f$  is not projective. Then  $f' : M \xrightarrow{\text{can}} M_0 \xrightarrow{f} S$  is a non-projective surjection. By Lemma 4.7,  $\ker f'$  is filtrable. We have  $\ker f' = \ker f \oplus M_1$  and we are done.  $\square$

4.2.3. We now turn to filtrations by objects in  $\text{add}(\mathcal{S})$ .

**Lemma 4.12.** *Let  $M$  be a filtrable module and  $N$  a filtrable submodule of  $M$  such that  $M/N \in \text{add } \mathcal{S}$ . Then,  $N$  is minimal with these properties if and only if  $N$  has no projective remainder and the canonical map  $\text{Hom}_{A\text{-stab}}(M/N, S) \rightarrow \text{Hom}_{A\text{-stab}}(M, S)$  is surjective for every  $S \in \mathcal{S}$ .*

*Proof.* Let  $N$  be a minimal filtrable submodule of  $M$  such that  $M/N \in \text{add } \mathcal{S}$ . Denote by  $i : N \rightarrow M$  the injection and  $p : M \rightarrow M/N$  the quotient map.

Let  $S \in \mathcal{S}$ . Fix  $f_1, \dots, f_r : M/N \rightarrow S$  such that  $\sum_i f_i : M/N \rightarrow S^r$  is surjective and  $\ker \sum_i f_i$  has no direct summand isomorphic to  $S$ . Let  $T$  be the subspace of  $\text{Hom}_{A\text{-stab}}(M, S)$  generated by  $f_1 p, \dots, f_r p$ . Assume this is a proper subspace, so there is  $f' : M \rightarrow S$  whose image in  $\text{Hom}_{A\text{-stab}}(M, S)$  is not in  $T$ . Then  $f' i : N \rightarrow S$  is not projective, hence there is a projective map  $q : N \rightarrow S$  such that  $f' i + q$  is surjective and has filtrable kernel  $N'$  (Lemma 4.6). There is  $q' : M \rightarrow S$  projective such that  $q = q' i$ . Now,  $M/N' \simeq M/N \oplus S$  and this contradicts the minimality of  $N$ . It follows that the canonical map  $\text{Hom}_{A\text{-stab}}(M/N, S) \rightarrow \text{Hom}_{A\text{-stab}}(M, S)$  is surjective. Assume  $N = N' \oplus P$  with  $N'$  filtrable with no projective remainder and  $P$  projective.

By Lemma 4.11,  $P$  is filtrable. We have  $M/N' \simeq M/N \oplus P$ . Since  $M/N$  is a maximal quotient of  $M$  in  $\text{add}(\mathcal{S})$  and  $P$  is filtrable, it follows that  $P = 0$ .

Conversely, take  $f : N \rightarrow S$  surjective with filtrable kernel such that the extension of  $M/N$  by  $S$  splits. Then  $f$  lifts to  $M \rightarrow S$  and it is not projective by Lemma 4.10. This contradicts the surjectivity of  $\text{Hom}_{A\text{-stab}}(M/N, S) \rightarrow \text{Hom}_{A\text{-stab}}(M, S)$ . Consequently,  $N$  is minimal.  $\square$

**Lemma 4.13.** *Let  $M$  be a filtrable  $A$ -module with no projective remainder.*

*Let  $f : M \rightarrow L$  be a surjection with  $L \in \text{add } \mathcal{S}$ . Then  $\ker f$  is filtrable if and only if the canonical map  $\text{Hom}_{A\text{-stab}}(L, S) \rightarrow \text{Hom}_{A\text{-stab}}(M, S)$  is injective for all  $S \in \mathcal{S}$ .*

*Proof.* Note that the canonical map  $\text{Hom}_{A\text{-stab}}(L, S) \rightarrow \text{Hom}_{A\text{-stab}}(M, S)$  is injective if and only if, given  $p : L \rightarrow S$  surjective with  $S \in \mathcal{S}$ ,  $pf$  is not projective.

Assume  $\ker f$  is filtrable. Let  $p : L \rightarrow S$  be a surjective map with  $S \in \mathcal{S}$ . Then  $\ker pf$  is filtrable, hence  $pf$  is not projective (Lemma 4.10).

Let us now prove the converse by induction on the dimension of  $M$ . Assume that given  $p : L \rightarrow S$  surjective with  $S \in \mathcal{S}$ , then  $pf$  is not projective. Pick  $p : L \rightarrow S$  surjective and let  $L' = \ker p$ . Let  $M' = \ker pf$ . Then  $f$  induces a surjection  $f' : M' \rightarrow L'$  and we have  $L' \in \text{add } \mathcal{S}$  (since  $p$  is split). Let  $p' : L' \rightarrow T$  be a surjective map with  $T \in \mathcal{S}$ . Fix a left inverse  $\sigma : L \rightarrow L'$  to the inclusion  $L' \rightarrow L$ .

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & T \\
 & & \downarrow & & \downarrow & \nearrow p' & \\
 0 & \longrightarrow & \ker f' & \longrightarrow & M' & \xrightarrow{f'} & L' & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow \sigma & & \\
 0 & \longrightarrow & \ker f & \longrightarrow & M & \xrightarrow{f} & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow p & & \\
 & & & & S & \xlongequal{\quad} & S & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 
 \end{array}$$

If  $S \neq T$ , then  $\text{Hom}_{A\text{-stab}}(S, T) = 0$ , and hence  $p'\sigma f$  doesn't factor through  $S$  in the stable category. On the other hand, if  $S = T$  then  $pf$  and  $p'\sigma f$  define linearly independent elements of  $\text{Hom}_{A\text{-stab}}(M, S)$ . Consequently,  $p'\sigma f$  doesn't factor through  $S$  in the stable category. It follows that  $p'f'$  is not projective. By Lemma 4.7,  $M'$  is filtrable. By induction, it follows that  $\ker f'$  is filtrable and we are done.  $\square$

**Proposition 4.14.** *Let  $M$  be a filtrable  $A$ -module with no projective remainder.*

*Let  $N$  be a minimal filtrable submodule of  $M$  such that  $M/N \in \text{add } \mathcal{S}$ . Then there is an isomorphism*

$$M/N \xrightarrow{\sim} \bigoplus_{S \in \mathcal{S}} S \otimes \text{Hom}_{A\text{-stab}}(M, S)$$

*that induces the canonical map  $M \rightarrow \bigoplus_{S \in \mathcal{S}} S \otimes \text{Hom}_{A\text{-stab}}(M, S)$  in the stable category.*

*Given  $\tau \in \text{Aut}(N)$  such that  $\tau \sim \text{id}_N$ , there is  $\sigma \in \text{Aut}(M)$  with  $\sigma \sim \text{id}_M$  and  $\sigma|_N = \tau$ .*

*Let  $N'$  be a minimal filtrable submodule of  $M$  such that  $M/N' \in \text{add } \mathcal{S}$ . Then there is  $\sigma \in \text{Aut}(M)$  such that  $N' = \sigma(N)$  and  $\sigma \sim \text{id}_M$ .*

*Proof.* The first part of the proposition follows from Lemmas 4.12 and 4.13.

Let  $\tau \in \text{Aut}(N)$  such that  $\tau = \text{id}_N + p$  with  $p : N \rightarrow N$  projective. Then there is a projective map  $q : M \rightarrow N$  with  $p = qi$ . Let  $\sigma = \text{id}_M + q$ . Then  $\sigma|_N = \tau$ . Now, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N \longrightarrow 0 \\ & & \tau \downarrow \sim & & \sigma \downarrow & & \text{id} \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N \longrightarrow 0 \end{array}$$

and hence  $\sigma$  is an automorphism of  $M$ .

Let  $N'$  be a minimal filtrable submodule of  $M$  such that  $M/N' \in \text{add } \mathcal{S}$ . Then we have shown that  $M/N \xrightarrow{\sim} M/N'$  and that via such an isomorphism, the maps  $M \rightarrow M/N$  and  $M \rightarrow M/N'$  are stably equal. Now, Lemma 4.4 shows there is  $\sigma \in \text{Aut}(M)$  with  $N' = \sigma(N)$  and  $\sigma \sim \text{id}_M$ .  $\square$

Let  $M$  be filtrable. An  $\mathcal{S}$ -radical filtration of  $M$  is a filtration  $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$  such that  $M_i$  is a minimal filtrable submodule of  $M_{i-1}$  with  $M_{i-1}/M_i \in \text{add } \mathcal{S}$ .

**Proposition 4.15.** *Let  $M$  be a filtrable  $A$ -module with no projective remainder. Let  $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$  and  $0 = M'_{r'} \subseteq M'_{r'-1} \subseteq \cdots \subseteq M'_0 = M$  be two  $\mathcal{S}$ -radical filtrations of  $M$ . Then,  $r = r'$  and there is an automorphism of  $M$  that swaps the two filtrations and that is stably the identity.*

*Proof.* We prove this lemma by induction on the dimension of  $M$ . By Proposition 4.14, there is  $\sigma \in \text{Aut}(M)$  such that  $\sigma(M'_1) = M_1$  and  $\sigma \sim \text{id}_M$ . Now, by induction, we have  $r = r'$  and there is  $\tau \in \text{Aut}(M_1)$  such that  $\tau\sigma(M'_i) = M_i$  for  $i > 0$  and  $\tau \sim \text{id}_{M_1}$ . By Proposition 4.14, there is  $\tau' \in \text{Aut}(M)$  such that  $\tau'_{|M_1} = \tau$  and  $\tau' \sim \text{id}_M$ . Now,  $\tau'\sigma$  sends  $M'_i$  onto  $M_i$ .  $\square$

**Remark 4.16.** A filtrable projective module can have two  $\mathcal{S}$ -radical filtrations with non-isomorphic layers.

Consider  $A = k\mathfrak{A}_4$ , the group algebra of the alternating group of degree 4 and assume  $k$  has characteristic 2 and contains a cubic root of 1. Let  $B$  be the principal block of  $k\mathfrak{A}_5$ . Then, the restriction functor is a stable equivalence between  $B$  and  $A$ . Let  $\mathcal{S}$  be the set of images of the simple  $B$ -modules. Denote by  $k$  the trivial  $A$ -module and by  $k_+, k_-$  the non-trivial simple  $A$ -modules. Then  $\mathcal{S} = \{k, S_+, S_-\}$  where  $S_\varepsilon$  is a non-trivial extension of  $k_\varepsilon$  by  $k_{-\varepsilon}$ . Let  $P$  and  $P'$  be the two projective indecomposable  $B$ -modules that don't have  $k$  as a quotient. Then  $\text{Res}_{\mathfrak{A}_4} P \simeq \text{Res}_{\mathfrak{A}_4} P'$ . This projective module has two  $\mathcal{S}$ -radical filtrations with non-isomorphic layers : one coming from the radical filtration of  $P$  and one coming from the radical filtration of  $P'$ .

While  $\mathcal{S}$ -radical filtrations are not unique in general for filtrable modules with a projective remainder, there are some cases where uniqueness still holds :

**Proposition 4.17.** *Assume  $A$  is a symmetric algebra. Let  $0 \rightarrow S \rightarrow M \rightarrow T \rightarrow 0$  and  $0 \rightarrow S' \rightarrow M \rightarrow T' \rightarrow 0$  be two exact sequences with  $S, S', T, T' \in \mathcal{S}$ . Assume that the sequences don't both split. Then there is an automorphism of  $M$  swapping the two exact sequences.*

*Proof.* If  $M$  is non-projective, then this is a consequence of Proposition 4.14.

Assume  $M$  is projective. Since  $A$  is symmetric, we have a non-projective map  $T \simeq \Omega^{-1}S \rightarrow S$ . It follows that  $S = T$ . Similarly,  $T' = S'$ . We have exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}(S', S) \rightarrow \text{Hom}(S', M) \rightarrow \text{Hom}(S', S) \rightarrow \text{Ext}^1(S', S) \rightarrow 0 \\ 0 \rightarrow \text{Hom}(S', S') \rightarrow \text{Hom}(S', M) \rightarrow \text{Hom}(S', S') \rightarrow \text{Ext}^1(S', S') \rightarrow 0 \end{aligned}$$

We have  $\Omega^{-1}S' \simeq S'$ , and hence  $\dim \text{Ext}^1(S', S') = 1$ . Consequently,  $\dim \text{Hom}(S', M)$  is an odd integer. It follows that  $\text{Ext}^1(S', S) \neq 0$ , hence  $\text{Hom}_{A\text{-stab}}(S', S) \neq 0$ , so  $S' = S$  and we are done by Lemma 4.4.  $\square$

**Lemma 4.18.** *Let  $0 = M_r \subset M_{r-1} \subset \cdots \subset M_0 = M$  be a filtration of  $M$  with  $M_{i-1}/M_i \in \text{add } \mathcal{S}$ .*

- (i) *If  $M$  has no projective remainder, then  $M_i$  has no projective remainder, for all  $i$ .*
- (ii) *If the filtration is an  $\mathcal{S}$ -radical filtration, then  $M_i$  has no projective remainder for  $i \geq 1$ .*

*Proof.* Consider an exact sequence  $0 \rightarrow N \oplus P \rightarrow M \rightarrow L \rightarrow 0$  of filtrable modules with  $P$  projective and  $N$  filtrable. Then there is an extension  $M'$  of  $L$  by  $N$  such that  $M = M' \oplus P$  and  $M'$  is filtrable. The first part of the lemma follows.

Assume now the filtration is an  $\mathcal{S}$ -radical filtration. Assume for some  $i \geq 1$ , we have  $M_i = N \oplus P$  with  $N$  filtrable with no projective remainder and  $P$  projective and filtrable (Lemma 4.11). Then,  $M = M' \oplus P$  with  $P$  filtrable by (i). There is an exact sequence  $0 \rightarrow L \rightarrow P \rightarrow S \rightarrow 0$  with  $S \in \mathcal{S}$  and  $L$  filtrable. Now, the canonical surjection  $M' \oplus P \rightarrow M/M_1 \oplus S$  has filtrable kernel and this contradicts the minimality of  $M_1$ .  $\square$

**Proposition 4.19.** *Let  $M_1$  and  $M_2$  be two filtrable  $A$ -modules with no projective remainder. If  $M_1 \sim M_2$ , then  $M_1 \simeq M_2$ .*

*Proof.* We prove the proposition by induction on  $\min(\dim M_1, \dim M_2)$ . Fix an isomorphism  $\phi$  from  $M_2$  to  $M_1$  in the stable category. Let  $X = \bigoplus_{S \in \mathcal{S}} S \otimes \text{Hom}_{A\text{-stab}}(M_1, S)$  and  $g_1 \in \text{Hom}_{A\text{-stab}}(M_1, X)$  be the canonical map. Let  $g_2 = g_1\phi$ . By Propositions 4.14 and 4.15, there are exact sequences

$$0 \rightarrow N_1 \rightarrow M_1 \xrightarrow{f_1} X \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N_2 \rightarrow M_2 \xrightarrow{f_2} X \rightarrow 0$$

with the image of  $f_i$  in the stable category equal to  $g_i$ . So, there is an isomorphism from  $N_2$  to  $N_1$  in the stable category compatible with  $\phi$ . By Lemma 4.18,  $N_1$  and  $N_2$  have no projective remainder. By induction, we deduce that there is an isomorphism  $N_2 \xrightarrow{\sim} N_1$  lifting the stable isomorphism. So,  $M_1$  and  $M_2$  are extensions of isomorphic modules, with the same class in  $\text{Ext}^1$ , hence are isomorphic.  $\square$

### 4.3. Generators and reconstruction.

4.3.1. We assume from now on that

**Hypothesis 3.**  $\mathcal{S}$  satisfies Hypothesis 2 and given  $M \in A\text{-mod}$ , there is a projective  $A$ -module  $P$  such that  $M \oplus P$  is filtrable.

**Proposition 4.20.** *Let  $S \in \mathcal{S}$ . Let  $P_S \rightarrow S$  be a projective cover of  $S$  and  $P$  minimal projective such that  $\Omega S \oplus P$  is filtrable. Let  $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = \Omega S \oplus P$  be an  $\mathcal{S}$ -radical filtration.*

*Then  $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 \subseteq P_S \oplus P$  is an  $\mathcal{S}$ -radical filtration.*

*If  $A$  is symmetric, then  $M_{r-1} \simeq S$ .*

*Proof.* Let  $f_1 : P_S \rightarrow S$  be a surjective map and  $f = (f_1, 0) : P_S \oplus P \rightarrow S$ . Let  $T \in \mathcal{S}$  and  $g : P_S \oplus P \rightarrow T$  such that we have an exact sequence  $0 \rightarrow L \rightarrow P_S \oplus P \xrightarrow{f+g} S \oplus T \rightarrow 0$  with  $L$  filtrable.

We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & P_S \oplus P & \xrightarrow{f+g} & S \oplus T \longrightarrow 0 \\ & & \parallel & & \uparrow & & \downarrow (0, \text{id}) \\ 0 & \longrightarrow & L & \longrightarrow & \Omega S \oplus P & \longrightarrow & T \longrightarrow 0 \end{array}$$

The surjection  $\Omega S \oplus P \rightarrow T$  is projective and has filtrable kernel. From Lemma 4.10, we get a contradiction to the minimality of  $P$ . It follows that  $\Omega S \oplus P$  is a minimal submodule of  $P_S \oplus P$  such that the quotient is in  $\text{add } \mathcal{S}$ .

We have  $\text{Hom}_{A\text{-stab}}(T, \Omega S) \simeq \text{Hom}_{A\text{-stab}}(S, T)^*$ , since  $A$  is symmetric. Now,  $\text{Hom}_{A\text{-stab}}(M_{r-1}, \Omega S \oplus P) \neq 0$  by Lemma 4.10. The second part of the proposition follows.  $\square$

Let  $M$  and  $N$  be two  $A$ -modules with filtrations  $0 = M_r \subseteq M_{r-1} \subseteq \dots \subseteq M_0 = M$  and  $0 = N_s \subseteq N_{s-1} \subseteq \dots \subseteq N_0 = N$ . Let  $\text{Hom}_A^f(M, N)$  be the subspace of  $\text{Hom}_A(M, N)$  of filtered maps (i.e., those  $g$  such that  $g(M_i) \subseteq N_i$ ). We put  $\bar{M}_i = M_i/M_{i+1}$ . We denote by  $\phi_i$  the composition of canonical maps  $\phi_i : \text{Hom}_A^f(M, N) \rightarrow \text{Hom}_A(\bar{M}_i, \bar{N}_i) \rightarrow \text{Hom}_{A\text{-stab}}(\bar{M}_i, \bar{N}_i)$ .

We view  $N' = N_i$  as a filtered module with the induced filtration  $0 = N'_{s-i} \subseteq N'_{s-i-1} \subseteq \dots \subseteq N'_1 = N_{i+1} \subseteq N'_0 = N'$ .

**Lemma 4.21.** *Let  $M$  be a filtrable  $A$ -module with an  $\mathcal{S}$ -radical filtration and  $N$  be a filtrable  $A$ -module with an  $\mathcal{S}$ -filtration. Let  $f \in \text{Hom}_A^f(M, N)$  with  $\phi_0(f) = 0$ . Then  $\phi_i(f) = 0$  for all  $i$ .*

*Proof.* The map  $\bar{f}_0 : \bar{M}_0 \rightarrow \bar{N}_0$  induced by  $f$  is projective. So there is a projective module  $P$  and a commutative diagram

$$\begin{array}{ccc} M & & N \\ \downarrow & \nearrow & \downarrow \\ \bar{M}_0 & \xrightarrow{\bar{f}_0} & \bar{N}_0 \\ & \nearrow P & \searrow P \end{array}$$

Let  $p$  be the composition  $p : M \rightarrow \bar{M}_0 \rightarrow P \rightarrow N$ . Then  $f - p \sim f$ ,  $f - p$  and  $f$  have the same restriction to  $M_1$ , and  $(f - p)_0 = 0$ . Consequently it is enough to prove the lemma in the case where  $\bar{f}_0 = 0$ .

From now on, we assume  $\bar{f}_0 = 0$ . Assume the map  $\bar{f}_1 : \bar{M}_1 \rightarrow \bar{N}_1$  is not projective. So there is  $S \in \mathcal{S}$  and a (split) surjection  $g : \bar{N}_1 \rightarrow S$  such that  $g\bar{f}_1 : \bar{M}_1 \rightarrow S$  is not projective. Let  $s : S \rightarrow \bar{M}_1$  be a right inverse to  $g$ , and let  $L$  be the kernel of  $g\bar{f}_1$ .

We have an exact sequence  $0 \rightarrow L \rightarrow M/M_2 \xrightarrow{(\text{can}, g\bar{f}_1)} \bar{M}_0 \oplus S \rightarrow 0$ . So the inverse image of  $L$  in  $M_1$  is a filtrable submodule of  $M$  with quotient isomorphic to  $\bar{M}_0 \oplus S$ . This contradicts the fact that  $M_1$  is a minimal filtrable submodule of  $M$  such that  $M/M_1 \in \text{add } \mathcal{S}$ . So  $\bar{f}_1$  is projective; i.e.,  $\phi_1(f) = 0$ .

We now prove by induction that  $\phi_i(f) = 0$  for all  $i$ . Assume  $\phi_d(f) = 0$ . Then, we apply the result above to the filtered modules  $M_d$  and  $N_d$  to get  $\phi_{d+1}(f) = 0$ .  $\square$

4.3.2. We define a category  $\mathcal{G}$  as follows.

- Its objects are  $A$ -modules together with a fixed  $\mathcal{S}$ -radical filtration.
- We define  $\text{Hom}_{\mathcal{G}}(M, N)_i$  as the image of  $\text{Hom}_A^f(M, N_i)$  in  $\text{Hom}_{A\text{-stab}}(\bar{M}_0, \bar{N}_i)$ . We put  $\text{Hom}_{\mathcal{G}}(M, N) = \bigoplus_i \text{Hom}_{\mathcal{G}}(M, N)_i$ .
- Let  $f \in \text{Hom}_{\mathcal{G}}(M, N)_i$  and  $g \in \text{Hom}_{\mathcal{G}}(L, M)_j$ . Let  $\tilde{f} : M \rightarrow N_i$  be a filtered map lifting  $f$ . It induces a map  $\phi_j(\tilde{f}) \in \text{Hom}_{A\text{-stab}}(\bar{M}_j, \bar{N}_{i+j})$  independent of the choice of  $\tilde{f}$  (Lemma 4.21). We define the product  $fg$  to be  $\phi_j(\tilde{f}) \circ \phi_0(g)$ .

Given  $S \in \mathcal{S}$ , let  $P_S \rightarrow S$  be a projective cover of  $S$  and  $Q_S$  projective minimal such that  $\Omega S \oplus Q_S$  is filtrable. Fix a radical filtration of  $P_S \oplus Q_S$  with first term  $\Omega S \oplus Q_S$ .

Let  $M = \bigoplus_{S \in \mathcal{S}} (P_S \oplus Q_S)$ . This comes with an  $\mathcal{S}$ -radical filtration. We have constructed a  $\mathbf{Z}_{\geq 0}$ -graded  $k$ -algebra  $\text{End}_{\mathcal{G}}(M)$ .

The following Lemma is clear.

**Lemma 4.22.** *Let  $\mathcal{S}$  be a complete set of representatives of isomorphism classes of simple  $A$ -modules. Then we have an equivalence  $\text{gr}(A\text{-mod}) \xrightarrow{\sim} \mathcal{G}$ . If  $A$  is basic, then  $\text{End}_{\mathcal{G}}(M)$  is isomorphic to the graded algebra associated with the radical filtration of  $A$ .*

We have now obtained our partial reconstruction result :

**Theorem 4.23.** *Let  $B$  be a selfinjective algebra with no simple projective module. Let  $M$  be an  $(A, B)$ -bimodule inducing a stable equivalence and having no projective direct summand. Let  $\mathcal{S} = \{M \otimes_B L\}$  where  $L$  runs over a complete set of representatives of isomorphism classes of simple  $B$ -modules.*

*Then, there is an equivalence  $\text{gr}(B\text{-mod}) \xrightarrow{\sim} \mathcal{G}$ . If  $B$  is basic, there is an isomorphism between the graded algebra associated with the radical filtration of  $B$  and  $\text{End}_{\mathcal{G}}(M)$ .*

4.3.3. The category  $\mathcal{G}$  can be constructed directly as in §3.1, using only the stable category with its triangulated structure.

**Proposition 4.24.** *Let  $M$  be a module with an  $\mathcal{S}$ -filtration  $0 = M_r \subseteq M_{r-1} \subseteq \dots \subseteq M_0 = M$ . This is an  $\mathcal{S}$ -radical filtration if and only if*

- $\text{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_i, S)$  is an isomorphism for all  $S \in \mathcal{S}$  and  $i > 0$ ,
- $\text{Hom}_{A\text{-stab}}(M_0/M_1, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_0, S)$  is surjective for all  $S \in \mathcal{S}$ , and
- $M_i$  has no projective remainder for  $i > 0$ .

*Assume the filtration is an  $\mathcal{S}$ -radical filtration. Then  $M$  has no projective remainder if and only if  $\text{Hom}_{A\text{-stab}}(M_0/M_1, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_0, S)$  is an isomorphism.*

*Proof.* Let  $M$  be a module with an  $\mathcal{S}$ -radical filtration  $0 = M_r \subseteq M_{r-1} \subseteq \dots \subseteq M_0 = M$ . The canonical map  $\text{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_i, S)$  is surjective for all  $S \in \mathcal{S}$ , by Lemma 4.12. Note that  $M_i$  has no projective remainder for  $i > 0$ , by Lemma 4.18. It follows that the canonical map  $\text{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_i, S)$  is an isomorphism for all  $S \in \mathcal{S}$  (Lemma 4.13).

Let us now prove the other implication. Since  $M_i$  has no projective remainder for  $i > 0$ , it follows from Lemma 4.12 that  $0 = M_r \subseteq M_{r-1} \subseteq \dots \subseteq M_1$  is an  $\mathcal{S}$ -radical filtration of  $M_1$ .



Assume the filtration is an  $\mathcal{S}$ -radical filtration. If  $M$  has no projective remainder, then  $\mathrm{Hom}_{A\text{-stab}}(M_0/M_1, S) \rightarrow \mathrm{Hom}_{A\text{-stab}}(M_0, S)$  is injective by Lemma 4.13.

Assume now that  $\mathrm{Hom}_{A\text{-stab}}(M/M_1, S) \rightarrow \mathrm{Hom}_{A\text{-stab}}(M, S)$  is bijective. Assume  $M = M' \oplus P$  with  $M'$  filtrable and  $P$  projective. We have  $\mathrm{Hom}_{A\text{-stab}}(M/M_1, S) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-stab}}(M, S) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-stab}}(M', S)$ . There is a surjective map  $g : M' \rightarrow M/M_1$  with filtrable kernel such that the composition  $M \xrightarrow{\mathrm{can}} M' \xrightarrow{g} M/M_1$  is equal to the canonical map  $M \rightarrow M/M_1$  in the stable category, by Proposition 4.14. By Lemma 4.4, we have  $M_1 \simeq \ker g \oplus P$ . Since  $M_1$  has no projective remainder by the first part of the proposition, we get  $P = 0$ , hence  $M$  has no projective remainder.  $\square$

Let  $\mathcal{T} = A\text{-stab}$ . Note that  $\mathcal{S}$  is determined by its image in  $\mathcal{T}$  and it satisfies Hypothesis 3 if and only if  $\mathrm{Hom}_{\mathcal{T}}(S, T) = k^{\delta_{ST}}$  for all  $S, T \in \mathcal{S}$  and every object of  $\mathcal{T}$  is an iterated extension of objects of  $\mathcal{S}$ .

We have a functor  $\mathcal{G} \rightarrow \mathcal{F}$  : it sends a module  $M$  with an  $\mathcal{S}$ -radical filtration  $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$  to  $\cdots \rightarrow 0 \rightarrow M_{r-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M \rightarrow M/M_1$  (cf Proposition 4.24).

**Proposition 4.25.** *The canonical functor  $\mathcal{G} \xrightarrow{\sim} \mathcal{F}$  is an equivalence.*

*Proof.* The functor is clearly fully faithful.

Start with  $0 = N_r \xrightarrow{f_r} N_{r-1} \rightarrow \cdots \rightarrow N_1 \xrightarrow{f_1} N_0 \xrightarrow{\varepsilon_0} M_0$ . Adding a projective direct summand to the  $N_i$ 's, we can lift the maps  $f_i$  to maps that are injective in the module category and such that the successive quotients have no projective direct summands. So we have a filtration  $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_1 \subseteq M'_0$  such that  $M'_i/M'_{i+1}$  is stably isomorphic to a direct sum of objects of  $\mathcal{S}$ . Since it has no projective summand, it is actually isomorphic to a sum of objects of  $\mathcal{S}$ ; *i.e.*, we have an  $\mathcal{S}$ -filtration. Consider  $i$  maximal such that  $M'_i$  has a projective remainder. Then  $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_i$  is an  $\mathcal{S}$ -radical filtration by Proposition 4.24 (first part). The second part of Proposition 4.24 shows now that  $M'_i$  has no projective remainder, a contradiction. So the filtration is an  $\mathcal{S}$ -filtration.  $\square$

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