A SHARPER THRESHOLD FOR RANDOM GROUPS AT DENSITY ONE-HALF

MOON DUCHIN, KASIA JANKIEWICZ, SHELBY C. KILMER,
SAMUEL LELIÈVRE, JOHN M. MACKAY, ANDREW P. SÁNCHEZ

Abstract. In the theory of random groups, we consider presentations with any fixed number m of generators and many random relators of length ℓ, sending $\ell \to \infty$. If d is a "density" parameter measuring the rate of exponential growth of the number of relators compared to the length of relators, then many group-theoretic properties become generically true or generically false at different values of d. The signature theorem for this density model is a phase transition from triviality to hyperbolicity: for $d < 1/2$, random groups are a.a.s. infinite hyperbolic, while for $d > 1/2$, random groups are a.a.s. order one or two. We study random groups at the density threshold $d = 1/2$. Kozma had found that trivial groups are generic for a range of growth rates at $d = 1/2$; we show that infinite hyperbolic groups are generic in a different range. (We include an exposition of Kozma's previously unpublished argument, with slightly improved results, for completeness.)

1. Introduction

We will study random groups on m generators, given by choosing relators of length ℓ through a random process. For a function $\mathsf{num} : \mathbb{N} \to \mathbb{N}$, let $\mathcal{G}(m, \ell, \mathsf{num})$ be the probability space of group presentations with m generators and with $|R| = \mathsf{num}(\ell)$ relators of length ℓ chosen independently and uniformly from the $(2m)(2m-1)^{\ell-1} \approx (2m-1)^{\ell}$ possible freely reduced words of length ℓ. Then the usual *density model of random groups* is the special case $\mathsf{num}(\ell) = (2m-1)^{d\ell}$, and the parameter $0 \le d \le 1$ is called the *density*. We will generalize in a natural way by defining

$$\mathcal{D} := \frac{1}{\ell} \log_{2m-1}(\mathsf{num}(\ell))$$

and saying that the *(generalized) density* is $d = \lim_{\ell\to\infty} \mathcal{D}$.

The foundational theorem in the area of random groups is the result of Gromov and Ollivier [6, Thm 11] that $d = 1/2$ is the threshold for a phase transition between hyperbolicity and triviality. To speak more precisely, the theorem is that for any num with $d > 1/2$, a presentation chosen uniformly at random from $\mathcal{G}(m, \ell, \mathsf{num})$ will be isomorphic to 1 or $\mathbb{Z}/2\mathbb{Z}$ with probability tending to 1 as $\ell \to \infty$; on the other hand, if $d(\mathsf{num}) < 1/2$, a presentation chosen in the same manner will be an infinite, torsion-free, word-hyperbolic group with probability tending to 1 as $\ell \to \infty$. (From now on, we will say that a property of random groups is *asymptotically almost sure* (or a.a.s.) for a certain m and num if its probability tends to 1 as $\ell \to \infty$.)

Here, we study the sharpness of this phase transition. Letting $\mathcal{D} = 1/2 - f(\ell)$ for $f(\ell) = o(1)$ lets us use these functions f to parametrize all cases with generalized density $1/2$. For simplicity of notation, where m is understood to be fixed in advance, let us write $\mathcal{G}_{\frac{1}{2}}(f) = \mathcal{G}\left(m, \ell, (2m-1)^{\ell(\frac{1}{2}-f(\ell))}\right)$. Constant values of $f(\ell)$ change the density, but in the

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$f(\ell) \to 0$ case we show here that the properties of random groups in $\mathcal{G}_{\frac{1}{2}}(f)$ will depend on the rate of vanishing.

Theorem 1. *Consider the density* $1/2$ *model* $\mathcal{G}_{\frac{1}{2}}(f)$ *for various* $f(\ell) = o(1)$.

$$\begin{cases} G \in \mathcal{G}_{\frac{1}{2}}(f) \text{ a.a.s. infinite hyperbolic,} & f(\ell) \geq 10^5 \cdot \log^{1/3}(\ell)/\ell^{1/3}; \\ G \in \mathcal{G}_{\frac{1}{2}}(f) \text{ a.a.s.} \cong 1 \text{ or } \mathbb{Z}/2\mathbb{Z}, & f(\ell) \leq \log(\ell)/4\ell - \log\log(\ell)/\ell. \end{cases}$$

Here and in the rest of the paper logarithms are taken base $2m-1$ and a group isomorphic to 1 or $\mathbb{Z}/2\mathbb{Z}$ is called "trivial." Theorem 1 is illustrated in Figures 1 and 2.

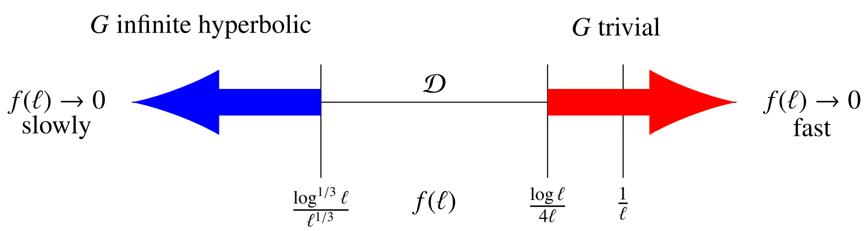

FIGURE 1. We study density $1/2$ by taking $\mathsf{num}(\ell) = (2m-1)^{\ell(\frac{1}{2}-f(\ell))}$ relators for various functions $f(\ell) = o(1)$.

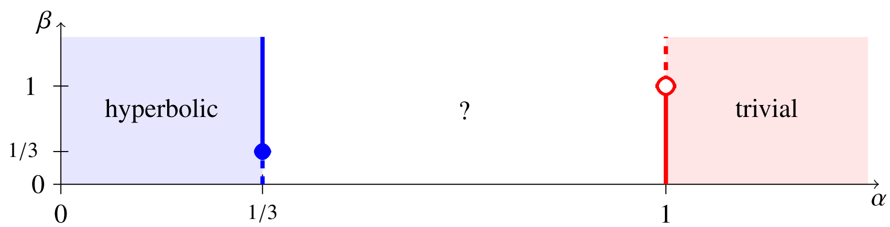

FIGURE 2. A finer view, taking $f(\ell) = f_{\alpha\beta}(\ell) = \frac{\log^{\beta}(\ell)}{\ell^{\alpha}}$.

To interpret Figure 2, note that $\log^{\beta}(\ell) \ll \ell$ for all β. If $G \in \mathcal{G}_{\frac{1}{2}}(f_{\alpha\beta})$ is a.a.s. trivial, then $G' \in \mathcal{G}_{\frac{1}{2}}(f_{\alpha'\beta'})$ is a.a.s. trivial as well whenever $\alpha' > \alpha$ or $\alpha' = \alpha, \beta' < \beta$. Similarly if $G \in \mathcal{G}_{\frac{1}{2}}(f_{\alpha\beta})$ is a.a.s. hyperbolic, then the same is true of $G' \in \mathcal{G}_{\frac{1}{2}}(f_{\alpha'\beta'})$ whenever $\alpha' < \alpha$ or $\alpha' = \alpha, \beta' > \beta$.

This implies in particular that setting $d = 1/2$ in the classical Gromov model (which corresponds to $f = 0$) gives a.a.s. trivial groups.

In unpublished notes from around 2010, Gady Kozma had given an argument for triviality at density 1/2. We give an expanded exposition here. By tracking through Kozma's argument as sharply as possible, we find triviality at $f(\ell) = \log(\ell)/4\ell - \log\log(\ell)/\ell$, which corresponds to any number of relators greater than

$$\mathsf{num}(\ell) = (2m-1)^{\ell(\frac{1}{2}-f(\ell))} = (2m-1)^{\frac{1}{2}\ell} \cdot \log(\ell) \cdot \ell^{-1/4} < (2m-1)^{\frac{1}{2}\ell} \cdot \ell^{-1/4+\epsilon}$$

for any $\epsilon > 0$. (This is slightly sharper than Kozma's conclusion, and he notes that such a result—with a power of ℓ factor as we have here—would be interesting.)

On the other hand, our hyperbolicity result applies for any number of relators at most

$$\mathsf{num}(\ell) = (2m-1)^{\ell(\frac{1}{2}-10^5\ell^{-1/3}\log^{1/3}\ell)} = (2m-1)^{\frac{1}{2}\ell} \cdot (2m-1)^{-10^5\ell^{2/3}\log^{1/3}(\ell)},$$

i.e., where $(2m-1)^{\frac{1}{2}\ell}$ is divided by a factor that is intermediate between polynomial and exponential. In that case we obtain

Theorem 2. *A random group in $\mathcal{G}_{\frac{1}{2}}(f)$ for $f(\ell) \geq 10^5 \cdot \log^{1/3}(\ell)/\ell^{1/3}$ is a.a.s. δ–hyperbolic with $\delta = c\ell^{5/3}$, for a sufficiently large constant c.*

By contrast, for $d < 1/2$, the best known hyperbolicity constant is $\delta = c_d\ell$, for a coefficient depending on the density.

The proof for triviality given below follows Kozma in using two elementary probabilistic ingredients: a "probabilistic pigeonhole principle" (Lemma 5) and a "decay of influence estimate" (Lemma 4). These may be of independent interest, so they are formulated in §2 in more generality than we need here. The main idea is to find a single short word that is trivial in G and use it to replace the relator set R with an equivalent relator set $\bar{R}$ with higher effective density.

For hyperbolicity, we follow Ollivier [6, Chapter V] in proving a linear isoperimetric inequality, using the local-to-global principle of Gromov as shown by Papasoglu to argue that only a limited number of Van Kampen diagrams need to be checked, then quoting some classic results of Tutte on enumeration of planar graphs to accomplish the necessary estimates.

The sharpest phase transition that one could hope for is to have some precise subexponential function $g(\ell)$ and a pair of constants $c_1 < c_2$ so that $\mathsf{num}(\ell) = c_1(2m-1)^{\frac{1}{2}\ell}g(\ell)^{-1}$ and $\mathsf{num}(\ell) = c_2(2m-1)^{\frac{1}{2}\ell}g(\ell)^{-1}$ yield the hyperbolic and trivial cases, respectively. We hope that in future work we will be able to obtain further refined estimates to "close the gap."

After completing this project we learned of the 2014 preprint [1] which considers very similar threshold sharpness questions for a different model of random groups, called the *triangular model*, in which all relators have length three. They find a one-sided threshold for hyperbolicity and show that triviality admits a very sharp phase transition in a sense similar to our sense above. However, hyperbolicity is not known to have such a sharp threshold in either model, and furthermore there is no guarantee that the hyperbolicity and triviality thresholds would agree, as we conjecture that they do.

1.1. **Conventions.** We will write 1 for the group {1} and will sometimes use the term *trivial* to mean isomorphic to either 1 or $\mathbb{Z}/2\mathbb{Z}$. Throughout the note, when we show that groups are a.a.s. *hyperbolic*, we are proving the same strong isoperimetric inequality as for the $d < 1/2$ case, so the groups in our hyperbolic range are infinite, and furthermore torsion-free, one-ended, with Menger curve boundary.

Since we are concerned with exponential growth with base $(2m-1)$, log will mean $\log_{2m-1}$.

We will use c, c', c'' for constants whose values depend on context and K, k for functions of ℓ. As usual, denote $f/g \to \infty$ by $f \gg g$. Write $[n]$ for $\{1, \ldots, n\}$.

For a word r of length ℓ we denote by $r[i]$ $(1 \leq i \leq \ell)$ the ith letter of r, and write $r[i:j]$ (where $1 \leq i < j \leq \ell$) for the subword $r[i]\,r[i+1]\cdots r[j]$ of r (so that in particular $r = r[1:\ell]$). For words r, r' we write $r = r'$ if r, r' are the same words after free reduction, and $r =_G r'$ if r, r' represent the same element of group G.

As mentioned above, we work with reduced words that need not be cyclically reduced. For models of random groups with cyclically reduced words we expect that the same threshold bounds hold.

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2. Some basic probabilistic facts

2.1. **Distribution of letters in freely reduced words.** Because the relators in these models of random groups are chosen by the uniform distribution on freely reduced words of a given length, it will sometimes be useful to know the conditional probability of seeing a particular letter at a particular position in r, given an earlier letter.

Let $m \geq 2$ be an integer and let $\mathfrak{m} = 1/(2m-1)$.

For any positive integer n let s_n be the partial sum of the alternating geometric series

$$1 - \mathfrak{m} + \mathfrak{m}^2 - \dots,$$

i.e., $s_n = \sum_{k=0}^{n-1}(-\mathfrak{m})^k$, and $s_0 = 0$. Then $\lim_{n\to\infty} s_n = \frac{1}{1+\mathfrak{m}}$.

The following lemma measures the decrease of influence of a letter on its successors.

Lemma 3 (Distribution of letters)**.** *Consider a random freely reduced infinite word $w = x_0x_1x_2\dots$ in m generators. Then for $n > 0$,*

$$\begin{cases} \Pr(x_n = x_0) = \mathfrak{m}\cdot s_{n-1}, & n \text{ even}; \\ \Pr(x_n = y) = \mathfrak{m}\cdot s_n, & n \text{ even}, y \neq x_0; \\ \Pr(x_n = x_0^{-1}) = \mathfrak{m}\cdot s_{n-1}, & n \text{ odd}; \\ \Pr(x_n = y) = \mathfrak{m}\cdot s_n, & n \text{ odd}, y \neq x_0^{-1}. \end{cases}$$

The proof is an easy induction. Note that as $n \to \infty$ the probability of each letter appearing at the nth place tends to $\frac{\mathfrak{m}}{1+\mathfrak{m}} = 1/2m$, recovering the uniform distribution, as one would expect. We immediately deduce bounds on the conditional probability of a later letter given an earlier letter.

Corollary 4 (Decay of influence)**.** *For any letters x, y (not necessarily distinct) and for any $n \geq 1$, $P_n(x,y) = \Pr(x_n = x \mid x_0 = y)$ is bounded between $\mathfrak{m}\cdot s_{n-1}$ and $\mathfrak{m}\cdot s_n$, i.e.,*

$$\begin{array}{lcll} \mathfrak{m} - \mathfrak{m}^2 + \cdots + \mathfrak{m}^{n-1} - \mathfrak{m}^n & \leq P_n(x,y) \leq & \mathfrak{m} - \mathfrak{m}^2 + \cdots + \mathfrak{m}^{n+1} & (n \text{ even}) \\ \mathfrak{m} - \mathfrak{m}^2 + \cdots - \mathfrak{m}^{n+1} & \leq P_n(x,y) \leq & \mathfrak{m} - \mathfrak{m}^2 + \cdots - \mathfrak{m}^{n-1} + \mathfrak{m}^n & (n \text{ odd}). \end{array}$$

In particular, $\frac{2m-2}{(2m-1)^2} \leq \Pr(x_2 = x \mid x_0 = y) \leq \frac{1}{2m-1}$.

2.2. **A generalized "probabilistic pigeonhole principle".** Consider z red balls and z blue balls, and so on for a total of q colors. Each of these qz balls is thrown at random into one of n boxes, giving $[n]$-valued random variables $x_1, \dots, x_{qz}$. We bound the probability that there is some box with balls of all colors.

Lemma 5 (Probabilistic pigeonhole principle on q colors)**.** *Let μ be any probability measure on $[n]$. Fix arbitrary $q, z \in \mathbb{N}$ such that $z \geq 2n^{1-1/q}$. Then if $x_1, \dots, x_{qz}$ are chosen randomly and independently under μ,*

$$\Pr(\exists\, i_1, i_2, \dots, i_q \text{ with } (j-1)z < i_j \leq jz,\ x_{i_1} = x_{i_2} = \cdots = x_{i_q}) \geq 1 - e^{-cz/n^{1-1/q}}$$

for any $c \leq -\frac{1}{4}\ln(1-2^{-q})$, or in particular $c \leq 2^{-q-2}$.

Note that as $n \to \infty$ a q-color coincidence is asymptotically almost sure as long as $z \gg n^{1-1/q}$, and in particular a 2-color coincidence occurs if $z \gg \sqrt{n}$. We further remark that this is equivalent to another probabilistic pigeonhole principle (that for $z \gg n^{1-1/q}$

uncolored balls in n boxes, some box contains at least q balls a.a.s.), in the sense that each implies the other.

Proof. We start by considering the case of a 3-color coincidence ($q = 3$). Let

$$\mathcal{I} := \{(i_1, i_2, i_3) \mid (j-1)z < i_j \le jz\}\ ; \qquad X := \#\{(i_1, i_2, i_3) \in \mathcal{I} \mid x_{i_1} = x_{i_2} = x_{i_3}\}.$$

Since $X \ge 0$ we can bound $\Pr(X > 0)$ using the classical inequality $\Pr(X > 0) > \mathbb{E}^2[X]/\mathbb{E}[X^2]$. We compute expectation by finding the probability of coincidence for some choice of distinct i_1, i_2, i_3 and multiplying by z^3:

$$\mathbb{E}[X] = z^3 \Pr(x_{i_1} = x_{i_2} = x_{i_3}) = z^3 \sum_{p=1}^{n} \mu^3(p).$$

We next write $X = \sum_{i_1} \sum_{i_2} \sum_{i_3} \delta_{x_{i_1} = x_{i_2} = x_{i_3}}$ and reindex as $X = \sum_{i_4} \sum_{i_5} \sum_{i_6} \delta_{x_{i_4} = x_{i_5} = x_{i_6}}$, so by symmetry we get

$$\begin{aligned}\mathbb{E}[X^2] = z^3 \Pr(x_{i_1} = x_{i_2} = x_{i_3}) &+ 3z^3(z-1)\Pr(x_{i_1} = x_{i_2} = x_{i_3} = x_{i_4}) \\ &+ 3z^3(z-1)^2 \Pr(x_{i_1} = x_{i_2} = x_{i_3} = x_{i_4} = x_{i_5}) \\ &+ z^3(z-1)^3 \Pr(x_{i_1} = x_{i_2} = x_{i_3}, x_{i_4} = x_{i_5} = x_{i_6})\end{aligned}$$

with respect to any $(i_1, i_2, i_3), (i_4, i_5, i_6) \in \mathcal{I}$ with the six i_j distinct.

Using $1 < r < s \Longrightarrow \|x\|_r \ge \|x\|_s$, we get

$$\mathbb{E}[X^2] \le z^3 \left(\sum_{p=1}^{n} \mu^3(p)\right)^{3/3} + 3z^4 \left(\sum_{p=1}^{n} \mu^3(p)\right)^{4/3} + 3z^5 \left(\sum_{p=1}^{n} \mu^3(p)\right)^{5/3} + z^6 \left(\sum_{p=1}^{n} \mu^3(p)\right)^{6/3}.$$

These expectation formulas easily generalize from 3 to any number q of colors:

$$\mathbb{E}[X] = z^q \sum_{p=1}^{n} \mu^q(p)\ ; \qquad \mathbb{E}[X^2] \le \sum_{i=0}^{q} \left[\binom{q}{i} \cdot z^{q+i} \cdot \left(\textstyle\sum_{p=1}^{n} \mu^q(p)\right)^{\frac{q+i}{q}}\right].$$

The probability of a coincidence is at least $\mathbb{E}^2[X]/\mathbb{E}[X^2]$. First let us consider a simple case, where the number of balls of each color is chosen to get good cancellation: set $z_0 := \left(\sum_{p=1}^{n} \mu^q(p)\right)^{-1/q}$, so that $1 \le z_0 \le n^{1-1/q}$, where the upper bound follows from Hölder's inequality:

$$1 = \sum_{p=1}^{n} \mu(p) = \sum_{p=1}^{n} \mu(p) \cdot 1 \le \left(\sum_{p=1}^{n} \mu^q(p)\right)^{1/q} \cdot n^{1-1/q}.$$

Then we get $\Pr(X > 0 \mid z \ge z_0) > 1/2^q$.

The general case is $z = \gamma z_0$ for some $\gamma \ge 2$. Divide up each of the intervals $((j-1)z, jz]$ into subintervals of length $\lceil z_0 \rceil$, with the last subinterval longer if necessary, and let ρ be the number of subintervals (the hypothesis that $z \ge 2n^{1-1/q}$ ensures that $\gamma/4 \le \rho \le \gamma$). Let X_k count the number of q-color coincidences which occur in the respective kth subintervals. The above calculation tells us that $\Pr(X_k > 0) > 1/2^q$.

By Hölder's inequality again, we have $\gamma \ge \dfrac{z}{n^{1-1/q}}$. It follows that

$$\Pr(X > 0) \ge 1 - \prod_{k=1}^{\rho} (\Pr(X_k = 0)) \ge 1 - (1 - 2^{-q})^{\gamma/4} \ge 1 - (1 - 2^{-q})^{\frac{1}{4} \cdot \frac{z}{n^{1-1/q}}}. \qquad \square$$

We emphasize that this result does not depend on the choice of probability distribution μ.

3. The trivial range

The usual proof that a random group G is trivial at densities $d > 1/2$ uses the probabilistic pigeonhole principle to show that there are pairs of relators r_1, r_2 which have different initial letters $r_1[1] = x, r_2[1] = y$, but with the remainder of the words equal. Consequently $r_1 r_2^{-1} = xy^{-1}$ is trivial. In this way one shows that a.a.s. all generators and their inverses are equal in G.

To show triviality at density $d = 1/2$ is more involved. The overall plan here is to find shorter trivial words than the ones from relator set R; treating these as an alternate relator set will push up the "effective density" of G, then a similar argument as before will show that the group is trivial.

Theorem 6 (Sufficient conditions for triviality)**.** *Given any $f(\ell) = o(1)$, suppose there exists a function $k : \mathbb{N} \to \mathbb{N}$ with $k(\ell) \leq \ell$ for all ℓ and such that*

$$k - 2\ell f \to \infty \tag{$\star$}$$

and

$$\frac{\ell - 2}{(2k+2)(2m-1)^{2k}} \to \infty \tag{$\spadesuit$}$$

as $\ell \to \infty$. Then a.a.s. $G \in \mathcal{G}_{\frac{1}{2}}(f)$ is 1 or $\mathbb{Z}/2\mathbb{Z}$.

Corollary 7. *The functions $k(\ell) = \frac{1}{2}\log(\ell) - \log\log(\ell)$ and $f(\ell) = \frac{\log(\ell)}{4\ell} - \frac{\log\log(\ell)}{\ell}$ satisfy ($\star$), ($\spadesuit$). Thus a random group in $\mathcal{G}_{\frac{1}{2}}\left(\frac{\log(\ell)}{4\ell} - \frac{\log\log(\ell)}{\ell}\right)$ is a.a.s. 1 or $\mathbb{Z}/2\mathbb{Z}$.*

Outline of the proof of Theorem 6.

(Step 1) Using the pigeonhole principle (Lemma 5), we find a freely reduced word w of length $2k$ such that $w =_G 1$. The existence of such a w is guaranteed by ($\star$), and we will use it to reduce other relators.

(Step 2) In each relator r we set aside the first two letters for later use, and then chunk the last $\ell - 2$ letters into b blocks of size $(2k+2)(2m-1)^{2k}$, with the last block possibly smaller. The ($\spadesuit$) condition says that $b \to \infty$. We show that w appears in one of these blocks surrounded by non-canceling letters with probability $> \frac{1}{4}$.

(Step 3) With these reductions, the probability that r reduces to length at most $\ell' = \ell - \frac{bk}{2}$ is more than $1/3$.

(Step 4) Finally we show that for this choice of ℓ', conditions ($\star$) and ($\spadesuit$) ensure that $d\ell - \frac{\ell'}{2} \to \infty$. From this we deduce that for any pair of generators a_i, a_j, we can almost surely find two reduced relators that start with a_i, a_j, respectively, and match after that. Therefore $a_i =_G a_j$ for all pairs of generators (including $a_j = a_i^{-1}$), which establishes the trivality result.

Proof of Theorem 6.

Step 1. *Suppose $k - 2\ell f \to \infty$. Then a.a.s. there exists a reduced word w of length $2k$ such that $w =_G 1$.*

To ensure that the word w we find is independent of the later steps in the proof, we divide the relator set R into to equal subsets R^1 and R^2, and in this step consider only R^1.

For each $r \in R$ the word $r[k+1 : \ell]$ is one of the $2m(2m-1)^{\ell-k-1}$ reduced words of length $\ell - k$. We will find two relators $r_1, r_2 \in R^1$ such that their tails match (i.e., $r_1[k+1 : \ell] = r_2[k+1 : \ell]$) but they differ in the previous letter ($r_1[k] \neq r_2[k]$). We can conclude that $w = r_1 r_2^{-1}$ reduces to a word of length $2k$.

For any word w of length p, we define R_w to be the subset of relators beginning with that word:

$$R_w := \{r \in R \mid r[1 : p] = w\}.$$

To restrict to relators in R^1 (or R^2), we write $R^1_w = R_w \cap R^1$ (or $R^2_w = R_w \cap R^2$).

For letters x, y, z, R_{xz} and R_{yz} are disjoint as long as x and y are distinct and neither one is equal to z^{-1}. Fix such letters x, y, z. There are $2m(2m-1)$ possible two-letter reduced words and since we choose R uniformly, the law of large numbers tells us that a.a.s.

$$|R^1_{xz}| > \frac{1}{2m(2m-1)+1} \cdot |R^1| = \frac{(2m-1)^{\ell(\frac{1}{2}-f(\ell))}}{4m(2m-1)+2}.$$

The same holds for R^1_{yz}.

We will check that

$$\frac{(2m-1)^{\ell(\frac{1}{2}-f(\ell))}}{4m(2m-1)+2} \gg \sqrt{2m(2m-1)^{\ell-k-1}}.$$

Using $2m-1 \geq 3$, we have $4m(2m-1)+2 \leq 4(2m-1)^2$ and $2m(2m-1)^{\ell-k-1} \leq 4(2m-1)^{\ell-k}$, which gives

$$\frac{(2m-1)^{\ell(\frac{1}{2}-f(\ell))}}{4m(2m-1)+2} \cdot \frac{1}{\sqrt{2m(2m-1)^{\ell-k-1}}} \geq c(2m-1)^{\ell(\frac{1}{2}-f(\ell))-\frac{\ell-k}{2}} = c(2m-1)^{\frac{k}{2}-\ell f(\ell)}$$

where $c = 1/8(2m-1)^2 > 0$. The right-hand side goes to infinity precisely when ($\star$) holds.

The purpose of introducing the letter z is to ensure that the tails of words in R^1_{xz} and R^1_{yz} have the same distribution. Hence we can apply Lemma 5 (with $q = 2$) to conclude that a.a.s. there exist $r_1 \in R^1_{xz}$ and $r_2 \in R^1_{yz}$ such that $r_1[k+1 : \ell] = r_2[k+1 : \ell]$. Then setting $w = (r_1[1 : k])^{-1} \cdot r_2[1 : k]$, we have $w =_G 1$.

Step 2. *Let w be as above and $r \in R^2$. Set $s = (2k+2)(2m-1)^{2k}$ and $b = \lfloor \frac{\ell-2}{s} \rfloor$. From the third letter on, divide r into b blocks of length s (with possibly one shorter block at the end). For each such block B, let $\lambda(B)$ be the last letter of r preceding B. Then the conditional probability that w appears in B given any particular value of $\lambda(B)$ is uniformly bounded away from* 0 *as follows:*

$$\forall g, \quad \Pr(w \text{ appears in } B \mid \lambda(B) = g) \geq 1 - e^{-2/3}.$$

Observe that $r \in R^2$ is independent of w, which was found by considering only R^1.

Write $w = w_1 \cdots w_{2k}$, let B be a block of size $(2k+2)(2m-1)^{2k}$, and divide it into $(2m-1)^{2k}$ subblocks $B_1, \ldots, B_{(2m-1)^{2k}}$ of size $2k+2$. Let E_i be the event that the word w appears as $B_i[2 : 2k+1]$. See Figure 3.

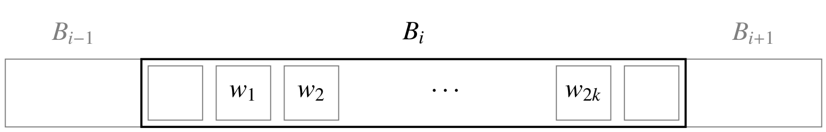


FIGURE 3. A part of block B.

Let us compute the probability of E_i given that none of $E_1, \cdots E_{i-1}$ happens and given any last letter g_0 before B_i. For $1 \le i \le (2m-1)^{2k}$, we have

$$\begin{aligned}
P_i &= \Pr(E_i \mid \neg E_1, \dots, \neg E_{i-1}, \lambda(B_i) = g_0) \\
&\overset{(1)}{=} \Pr(B_i[2] = w_1 \mid \neg E_1, \dots, \neg E_{i-1}, \lambda(B_i) = g_0) \cdot \left(\frac{1}{2m-1}\right)^{2k-1} \\
&\overset{(2)}{\ge} \frac{2m-2}{(2m-1)^2} \cdot \left(\frac{1}{2m-1}\right)^{2k-1} \ge \frac{2}{3}(2m-1)^{-2k}.
\end{aligned}$$

Equality (1) follows from the fact that only $B_i[2]$ could be affected by previous letters in r. Inequality (2) is an application of the decay of influence estimate (Corollary 4), which guarantees that $\Pr(x_2 = x \mid x_0 = y) \ge (2m-2)/(2m-1)^2$ for any x, y. We deduce that

$$\prod_{i=1}^{(2m-1)^{2k}} \Pr(\neg E_i \mid \neg E_0, \dots, \neg E_{i-1}, \lambda(B) = g_0) = \prod_{i=1}^{(2m-1)^{2k}} (1 - P_i) \le \left(1 - \frac{2}{3}(2m-1)^{-2k}\right)^{(2m-1)^{2k}} \le e^{-2/3},$$

and so finally for any g_0,

$$\Pr(w \text{ appears in } B \mid \lambda(B) = g_0) \ge 1 - e^{-\frac{2}{3}} > \frac{1}{4}.$$

Step 2.5. *If there exists a subword w' of B of the form*

$$w' = sdwd^{-1}t$$

for any word d and letters s, t with $s \ne t^{-1}$, then we say that B has a w-reduction. *(In this case $w =_G 1 \implies w' =_G st$. To* perform a w-reduction *we replace w' by st in r and note that the word remains freely reduced.) For k sufficiently large we bound*

$$\Pr(B \textit{ has a } w\textit{-reduction} \mid \lambda(B) = g_0) > \frac{1}{4}.$$

We want to bound from above the conditional probability that w appears in B in the wrong form for a w-reduction. This only happens if B starts or ends with dwd^{-1} for some word $d = d_1 \cdots d_n$. Let us compute the probability that B starts this way. First we bound the probability that w appears in the right place, then conditioning on that we bound the other needed coincidences. We have $\Pr(B[n+1] = w_1) \le \frac{1}{2m-1}$, and

$$\Pr(B[n+1 : n+2k] = w \mid B[n+1] = w_1) = \frac{1}{(2m-1)^{2k-1}}.$$

Next we consider whether $B[n+1-j] = B[n+2k+j]^{-1}$ for each $j = 1, \dots, n$. For $j = 1$, we have

$$\Pr(B[n] = B[n+2k+1]^{-1}) = \frac{1}{2m-1} \quad \text{or} \quad \frac{2m-2}{(2m-1)^2},$$

depending on whether $w_1 = w_{2k}$ or not, but in either case this is $\le 1/(2m-1)$. For $j = 2, \dots, n-1$, the conditional probability is exactly $1/(2m-1)$. For $j = n$, we have the same two possibilities as before, depending on whether $\lambda(B) = d_2$. So all together we find

$$\Pr(B \text{ starts with } dwd^{-1} \mid \lambda(B) = g_0) \le \left(\frac{1}{2m-1}\right)^{2k+n}.$$

The same inequality holds for finding dwd^{-1} at the end of B, so

$$\Pr(w \text{ appears in } B \text{ with no } w\text{-reduction}) \le 2\sum_{n=0}^{\infty} \left(\frac{1}{2m-1}\right)^{2k+n},$$

and the right-hand side goes to 0 as long as $k \to \infty$. So finally for sufficiently large ℓ (and therefore k),

$$\Pr(B \text{ has a } w\text{-reduction} \mid \lambda(B) = g_0) > \frac{1}{4}.$$

Step 3. *For each relator $r \in R^2$ denote by $\bar{r}$ the word obtained by performing the first appearing w-reduction in each block (as described in the previous step). By comparing to an appropriate Bernoulli trial, for k sufficiently large we show that*

$$\Pr\left(\#\{\text{reductions of } w \text{ in } B\} > \tfrac{b}{4} \mid r[1:2] = g_1 g_2\right) > \frac{1}{3},$$

and conclude that

$$\Pr\left(\operatorname{len}(\bar{r}) < \ell - \tfrac{kb}{2} \mid r[1:2] = g_1 g_2\right) > \frac{1}{3}.$$

Let X_i, for $i = 1, \ldots, b$, be i.i.d. random variables such that $X_i = 1$ with probability $1/4$ and $X_i = 0$ with probability $3/4$. Then by the central limit theorem,

$$\lim_{b \to \infty} \Pr\left(\sum X_i > \tfrac{b}{4}\right) = \frac{1}{2}.$$

Let $\widetilde{X}_i$ be the indicator random variable for a w-reduction in the ith block $B^{(i)}$ of r. The variables $\widetilde{X}_1, \widetilde{X}_2, \ldots$ are not independent, but each $\widetilde{X}_i$ depends only on $\lambda(B^{(i)})$. By Step 2.5 we know that for any g_0,

$$\Pr(\widetilde{X}_i = 1 \mid \lambda(B^{(i)}) = g_0) > \frac{1}{4} = \Pr(X_i = 1),$$

so

$$\Pr\left(\textstyle\sum_{i=1}^b \widetilde{X}_i > \tfrac{b}{4} \mid r[1:2] = g_1 g_2\right) \geq \Pr\left(\textstyle\sum_{i=1}^b X_i > \tfrac{b}{4}\right) \to \frac{1}{2}.$$

Thus for sufficiently large ℓ,

$$\Pr\left(\textstyle\sum_{i=1}^b \widetilde{X}_i > \tfrac{b}{4} \mid r[1:2] = g_1 g_2\right) > \frac{1}{3},$$

and since each reduction shortens the word by at least $2k$ letters we have

$$\Pr\left(\operatorname{len}(\bar{r}) < \ell - \tfrac{bk}{2} \mid r[1:2] = g_1 g_2\right) > \frac{1}{3}.$$

Step 4. *Let $\bar{R}^2 = \{\bar{r} \mid r \in R^2\}$ be the set of reduced words as above. For each pair of distinct elements x, y chosen from the generators and their inverses, a.a.s. there exists a pair $\bar{r}_1, \bar{r}_2 \in \bar{R}^2$ such that $\bar{r}_1[1] = x$, $\bar{r}_2[1] = y$, and*

$$\bar{r}_1[2 : \operatorname{len}(\bar{r}_1)] = \bar{r}_2[2 : \operatorname{len}(\bar{r}_2)].$$

Consequently, $x =_G y$. Triviality follows.

First, (♠) says that $b \to \infty$, so we have $b \geq 2$ for ℓ sufficiently large, which gives

$$\frac{bk}{4} - \ell f \geq \frac{k}{2} - \ell f,$$

and the right-hand side goes to infinity by ($\star$).

Next, let x, y, z be chosen among the generators and their inverses such that $z^{-1} \neq x, y$ and $x \neq y$. Recall that R^2_w denotes relators in R^2 beginning with subword w. We examine relators $r \in R^2_{xz} \cup R^2_{yz}$ such that $\operatorname{len}(\bar{r}) \leq \ell' = \ell - \frac{bk}{4}$. Note that $|R^2_{xz}|$ is close to $\frac{|R|}{4m(2m-1)}$ a.a.s., and we expect 1/3 of these to have enough reductions so their length is no more than ℓ'. So we get

$$\#\{r \in R^2_{xz} \mid \operatorname{len}(\bar{r}) \le \ell'\} > \frac{(2m-1)^{\ell(\frac{1}{2}-f)}}{3(4m)(2m-1)+1},$$

and the same holds for R^2_{yz}. To apply Lemma 5 to get matching tails, we must compare the number of shortened words to the square root of the number of possible tails. (The two colors are initial 2-letter words and the boxes are final $(\ell'-2)$-letter words; both R^2_{xz} and R^2_{yz} have the same probability distribution from the third letter onwards so the lemma applies.) In order to see that

$$(2m-1)^{\ell(\frac{1}{2}-f)} \gg \sqrt{(2m-1)^{\ell'-2}},$$

note that

$$\frac{(2m-1)^{\ell(\frac{1}{2}-f)}}{\sqrt{(2m-1)^{\ell'-2}}} \ge (2m-1)^{\frac{bk}{4}-\ell f} \to \infty.$$

We may conclude that a.a.s. there exists a pair of words $r_1 \in R^2_{xz}$ and $r_2 \in R^2_{yz}$ such that

$$\bar{r}_1[3 : \operatorname{len}(\bar{r}_1)] = \bar{r}_2[3 : \operatorname{len}(\bar{r}_2)],$$

and since $\bar{r}_1 =_G 1 =_G \bar{r}_2$, we get $xz =_G yz$, so finally $x =_G y$. This means that a.a.s. all generators and their inverses are equal in G. □

Proof of Corollary 7. For (⋆) we compute $k - 2\ell f = \log\log\ell$, which goes to infinity. Condition (♠) is equivalent to $b \to \infty$, and we calculate

$$\begin{aligned}\log b = \log\left(\frac{\ell-2}{(2k+2)(2m-1)^{2k}}\right) &= \log(\ell-2) - \log(2k+2) - 2k \\ &\ge \log\ell - \log\log\ell - \log\ell + 2\log\log\ell - C \\ &= \log\log\ell - C\end{aligned}$$

for a suitable constant C. □

4. The hyperbolic range

To prove hyperbolicity, we establish an isoperimetric inequality on reduced van Kampen diagrams (RVKDs) for a random group, as in Ollivier [6, Chapter 5]. The main difference to our argument is that, rather than aiming to show a linear isoperimetric inequality directly, we show that the random group satisfies a quadratic isoperimetric inequality with a small constant. This in turn implies that the group is hyperbolic by a well-known result of Gromov (see Papasoglu [5] and Bowditch [2]).

Following Ollivier, we write D for a (reduced) van Kampen diagram; $|D|$ for its number of faces, and $|\partial D|$ for the length of its boundary. (Note $|\partial D| \ge$ # boundary edges because of possible "filaments.") A path of contiguous edges so that all interior vertices have valence two is called a *contour*.

The key fact which allows us to check the isoperimetric inequality only on diagrams of certain sizes is the following theorem of Ollivier, which is a variation on Papasoglu's result in [5].

Lemma 8 (Local-global principle [7, Prop 9])**.** *For fixed ℓ and $K \ge 10^{10}$, if*

$$\underbrace{\tfrac{K^2}{4} \le |D| \le 480K^2}_{①} \Rightarrow \underbrace{|\partial D|^2 \ge 2\cdot 10^4\ell^2|D|}_{②},$$

then

$$|D| \geq K^2 \Rightarrow |\partial D| \geq \frac{\ell}{10^4 K}|D|.$$

That is, if RVKDs in a certain size range satisfy a good enough *quadratic* isoperimetric inequality, then all RVKDs satisfy a *linear* isoperimetric inequality. Later, we will let $K = K(\ell)$ to vary the window of diagrams considered.

We will use Ollivier's definitions concerning *abstract diagrams*, which are a device for precise bookkeeping in van Kampen diagrams to control dependencies in probabilities. Roughly speaking, an abstract diagram is a van Kampen diagram where we forget the labelling of edges by generators and the labelling of faces by relators. We do keep track of the orientation and starting point of the boundary of each face, and we also label faces so we know which faces bear the same relator. (Since our relators are reduced but need not be cyclically reduced, each face in an abstract diagram is allowed to have a single "inward spur", see [6, Page 83, footnote 4].)

To show that a group is hyperbolic, it suffices to have one RVKD for each trivial word that satisfies the linear isoperimetric inequality. Since we get this inequality for *all* RVKDs, following Ollivier's argument on spherical diagrams will give that our group is in addition infinite and torsion-free. We establish that a.a.s. all diagrams satisfy the hypothesis by showing that the probability of a diagram existing that has Ⓘ but not Ⓙ tends to 0. To calculate this, we must first get a bound on how many abstract diagrams have Ⓘ, and the probability that such an abstract diagram is fufillable from our relator set.

4.1. **Probability of fulfillability.** Still following Ollivier, we estimate the probability that some relators exist to fulfill D.

Lemma 9 ([6, Lem 59])**.** *Let R be a random set of relators with $|R| = \mathsf{num}(\ell)$ at length ℓ. Let D be a reduced abstract diagram. Then we have*

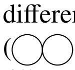
$$\Pr(D \textit{ is fulfillable}) \leq (2m-1)^{\frac{1}{2}\left(\frac{|\partial D|}{|D|} - \ell + 2\log \mathsf{num}\right)} = (2m-1)^{\frac{1}{2}\left(\frac{|\partial D|}{|D|} - \ell(1-2\mathcal{D})\right)}$$

In our case, our choice of $\mathsf{num}(\ell)$ gives $\mathcal{D} = \frac{1}{2} - f(\ell)$. If a diagram satisfies Ⓘ and not Ⓙ, we get

$$\frac{|\partial D|}{|D|} < \frac{\sqrt{2}\cdot 10^2 \ell \sqrt{|D|}}{K^2/4} < \frac{5\cdot 10^3 \ell K}{K^2/4} = \frac{2\cdot 10^4 \ell}{K}.$$

All together, we get

$$\Pr(D \text{ is fulfillable}) \leq (2m-1)^{10^4 \frac{\ell}{K} - \ell\cdot f(\ell)}.$$

4.2. **Counting abstract diagrams.** There is a forgetful map from abstract diagrams Γ to embedded planar graphs Γ' that strips away the data (i.e., subdivision of contours into edges, face labelings, and start points and orientations for reading around each face). Figure 4 shows an example. To see that the planar embedding matters, consider the two different ways of embedding a figure-eight—clearly different as van Kampen diagrams. ( versus ) Adding data to a graph to recover an abstract diagram will be called *filling in*.

In order to find an upper bound on the number of van Kampen diagrams up to a certain size, we will count possible abstract diagrams by enumerating planar graphs and ways of filling in.

Proposition 10 (Diagram count)**.** *Let $N_F(\ell)$ be the number of abstract diagrams with at most F faces, each of boundary length ℓ. Then* $\log N_F(\ell)$ *is asymptotically bounded above by* $6F \log \ell + 2F \log F$.

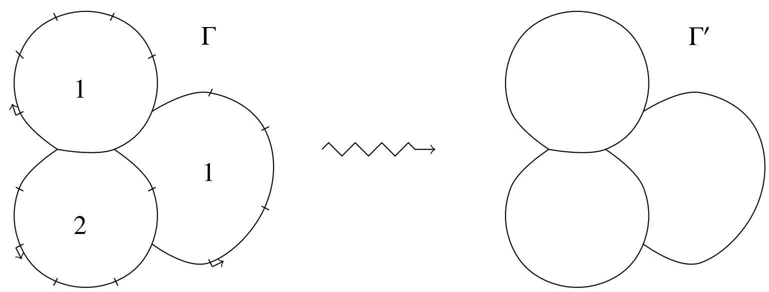


FIGURE 4. Abstract diagram and corresponding embedded planar graph.

Proof. Consider abstract diagrams with no more than F faces. Since there are ℓ edges on the boundary of each face, two orientations, and at most F faces, there are no more than $(2\ell)^F$ choices of oriented start points. Faces can have at most F distinct labels, so there are at most F^F possible labelings.

In order to estimate the number of ways we can subdivide the contours into edges, we first count edges of Γ'. If Γ' has no inward spurs, then every vertex has valence at least three. Since the Euler characteristic is $V - E + F = 1$, we have $2E \geq 3V$, which simplifies to $E \leq 3F - 3 \leq 3F$. Each face of Γ' can have at most one inward spur, which increases the number of edges by ≤ 2 for each face, so the total number of edges in Γ' satisfies $E \leq 5F$.

The number of ways to put ℓ edges around each face can be overcounted by the number of ways to subdivide each contour into exactly ℓ edges, which is ℓ^E and so is bounded above by ℓ^{5F}.

Tutte shows in [8, p. 254] that the number of embedded planar graphs with exactly n edges is $\frac{2(2n)!3^n}{n!(n+2)!}$. Using $E \leq 5F$, and $(n/e)^n \leq n! \leq n^n$ (with lower bound from Stirling's formula), we get

$$\#\,(\Gamma' \text{ with} \leq 5F \text{ edges}) \leq \sum_{n=1}^{5F} \frac{2(2n)!\,3^n}{n!\,(n+2)!} \leq 5F \frac{2(10F)!\,3^{5F}}{(5F)!\,(5F+2)!}$$
$$\leq \frac{(10F)!3^{5F}}{(5F)!(5F)!} \leq \frac{(10F)^{10F}3^{5F}}{(5F/e)^{10F}} = (2e)^{10F}3^{5F} \leq 3^{25F}.$$

Combining the above information, we get

$$N_F(\ell) \leq (2\ell)^F F^F \ell^{5F} 3^{25F},$$

and so

$$\log N_F \leq F \log(2\ell) + F \log F + 5F \log \ell + 25F \log 3.$$

Gathering terms of highest order, we have an upper bound by $6F \log \ell + 2F \log F$, as claimed. □

Corollary 11. *Let $N^I(\ell)$ be the number of reduced van Kampen diagrams with property ① at relator length ℓ. Then $\log N^I(\ell)$ is asymptotically bounded above by $3000K^2 \log(K\ell)$.*

Proof. Considering all diagrams with $|D| \leq 480K^2$ will be an overcount, so we use $F = 480K^2$ in the above estimate, i.e., $N^I(\ell) \leq N_{480K^2}(\ell)$. □

4.3. **Hyperbolicity threshold.**

Theorem 12 (Sufficient condition for hyperbolicity)**.** *Given any $f(\ell) = o(1)$, suppose there exists a function $K : \mathbb{N} \to \mathbb{N}$ such that*

$$(*) \qquad 3000K^2 \log(K\ell) + 10^4 \tfrac{\ell}{K} - \ell \cdot f(\ell) \to -\infty.$$

Then $G \in \mathcal{G}_{\frac{1}{2}}(f)$ is a.a.s. (infinite torsion-free) hyperbolic.

Remark 13. *In view of Corollary 11, one intuitive way of choosing a K, f pair is to take $K^2 \log \ell$ and $\ell f(\ell)$ to be of the same order. It turns out that we can do slightly better than that by instead choosing to equalize the orders of $\frac{\ell}{K}$ and $\ell f(\ell)$, which gives the pair below.*

Corollary 14. *For any constants c, c' with $0 < 4000c'^2 + \frac{10^4}{c'} < c$, the functions $f(\ell) = c\frac{\log^{1/3}(\ell)}{\ell^{1/3}}$ and $K(\ell) = c'\frac{\ell^{1/3}}{\log^{1/3}(\ell)}$ satisfy $(*)$.*

In particular, for $c > 10^5$, a random group in $\mathcal{G}_{\frac{1}{2}}\left(c\frac{\log^{1/3}(\ell)}{\ell^{1/3}}\right)$ is a.a.s. (infinite torsion-free) hyperbolic.

Proof of Theorem. Observe that

$$\begin{aligned} P := \Pr\binom{\exists \text{ a van Kampen diagram } D}{\text{that satisfies ① but not ⓙ}} &\le \sum_{\substack{\text{abstract diagrams } D \\ \text{with ① but not ⓙ}}} \Pr(D \text{ is fulfillable}) \\ &\le N^I(\ell) \cdot (2m-1)^{10^4 \frac{\ell}{K} - \ell f(\ell)}, \end{aligned}$$

where $N^I(\ell)$ is as in Corollary 11 and $(2m-1)^{10^4 \frac{\ell}{K} - \ell f(\ell)}$ is the fulfillability bound from Lemma 9. (Note that the last inequality vastly overcounts by replacing [① and not ⓙ] with simply ①.)

We will show that the local-global principle (Lemma 8) holds a.a.s. for all diagrams, by showing that for a K, f pair as in the hypothesis, the above quantities go to zero. In particular, we will show that $\log P \to -\infty$.

We have $\log P \le \log N^I + 10^4 \frac{\ell}{K} - \ell \cdot f(\ell)$. By applying Corollary 11, we have this asymptotically bounded above by

$$3000K^2 \log(K\ell) + 10^4 \tfrac{\ell}{K} - \ell \cdot f(\ell).$$

Requiring that this goes to $-\infty$ is exactly $(*)$. ◻

Proof of Corollary. We calculate each of the four terms of $(*)$ using $K = c'\ell^{1/3} \log^{-1/3} \ell$ and $f = c\ell^{-1/3} \log^{1/3} \ell$. We have

$$\begin{cases} 3000K^2 \log(K\ell) \le 4000c'^2 \ell^{2/3} \log^{1/3} \ell \,; \\ 10^4 \frac{\ell}{K} = \frac{10^4}{c'} \ell^{2/3} \log^{1/3} \ell \,; \\ \ell f = c\ell^{2/3} \log^{1/3} \ell. \end{cases}$$

Provided $4000c'^2 + \frac{10^4}{c'} < c$, the expression goes to $-\infty$ and $(*)$ is verified. For example, we can choose $c' = 1$ and $c = 10^5$. ◻

This completes the proof of Theorem 1.

4.4. **Hyperbolicity constant.** In this section we use the constants in the isoperimetric inequality to estimate the hyperbolicity constant for the density one-half random groups in our hyperbolic range, namely $\mathcal{G}_{\frac{1}{2}}\left(10^5 \log^{1/3}(\ell)/\ell^{1/3}\right)$.

We will use a variation on a result in Bridson–Haefliger [3, III.H.2.9], which differs in two main ways. First, they consider a notion of area for general geodesic metric spaces; however, we are only concerned with simply connected 2-complexes X such as the Cayley complex of a group, so for us the area of an edge loop γ in X is the minimal number of faces in a diagram which fills γ. Second, the statement in Bridson–Haefliger requires a linear isoperimetric inequality for all edge loops, while ours only requires the inequality for loops of sufficiently large area. In both cases, their proof works for our statement.

Theorem 15 (Effective hyperbolicity constant). *Suppose X is a 2–complex that is geometrically finite, i.e., there is some N such that every face has at most N edges. Suppose there is $\kappa > 1/N$ so that X has a linear isoperimetric inequality for large-area loops: if an edge loop γ in X has area $\geq 18\kappa^2 N^2$, then γ can be filled with at most $\kappa|\gamma|$ cells. Then the one-skeleton of X has δ–thin triangles for $\delta = 120\kappa^2 N^3$.*

Proof. We closely follow the proof in [3, p419] from III.H Theorem 2.9 (replacing K by κ to avoid notation clash). If there is a triangle which is not $6k = 18\kappa N^2$–thin, one builds a hexagon $\mathcal{H}$ (or quadrilateral) whose minimal-area filling has area $\geq \kappa(\alpha - 2k) \geq \kappa(6k) = 18\kappa^2 N^2$. So this hexagon satisfies our (large-area-only) linear isoperimetric hypothesis, and thus has area $|\mathcal{H}| \leq \kappa|\partial\mathcal{H}|$. The remainder of the proof shows that the hexagon is δ–thin provided

$$\frac{\delta - 3k}{3N} > 12k\kappa \;\Leftrightarrow\; \delta > 3k + 36kN\kappa = 9\kappa N^2 + 108\kappa^2 N^3.$$

Since $9\kappa N^2 + 108\kappa^2 N^3 \leq 117\kappa^2 N^3$, it suffices to take $\delta = 120\kappa^2 N^3$. □

As a corollary, we obtain Theorem 2: a random group in our hyperbolic range, namely in $\mathcal{G}_{\frac{1}{2}}(f)$ for $f(\ell) \geq 10^5 \cdot \log^{1/3}(\ell)/\ell^{1/3}$, is a.a.s. δ–hyperbolic with $\delta = c\ell^{5/3}$, for a sufficiently large constant c.

By contrast, as noted above, the best known hyperbolicity constant for $d < 1/2$ is proportional to ℓ.

Proof of Theorem 2. The output of the local-to-global principle was the linear isoperimetric inequality $|\partial D| \geq \frac{\ell}{10^4 K}|D|$ and to get the needed case we used $K(\ell) = \frac{\ell^{1/3}}{\log^{2/3}(\ell)}$. This gives $|D| \leq c''\ell^{-2/3}\log^{-2/3}(\ell)\cdot|\partial D| \leq c''\ell^{-2/3}|\partial D|$, so we take $\kappa = c''\ell^{-2/3}$ and $N = \ell$. This linear isoperimetric inequality holds for all diagrams D of size $|D| \geq K^2$; observe that $18\kappa^2 N^2 = 18(c''\ell^{-2/3})^2\ell^2 \geq K^2$ for large K. Therefore, Theorem 15 gives that all triangles are δ–thin for a value of δ proportional to $\ell^{5/3}$. □

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