Equivalent Expressions and Performance Analysis of SLNR Precoding Schemes: A Generalisation to Multi-antenna Receivers

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***Abstract*—In this letter, equivalent expressions of transmit precoding solutions based on the maximum signal-to-leakage-plus-noise ratio (SLNR) are derived for multiuser MIMO systems with multi-antenna receivers. The performance of the SLNR precoding scheme is also analysed based on this equivalent form. Further, it is shown that the SLNR scheme can be viewed as a generalised channel regularisation technique and the conditions for an equivalence between the SLNR, the Regularised Block Diagonalisation (RBD) and the Generalised MMSE Channel Inversion (GMI method 2) schemes are given. Consequently, the performance analysis in this letter can be extended to the RBD and GMI schemes. This generalises the equivalence between the SLNR and MMSE schemes and its useful implications, from the case of single-antenna to multi-antenna receivers.**

***Index Terms*—Multiuser MIMO, linear precoding, SLNR, RBD, GMI, equivalent forms, performance analysis**

I. INTRODUCTION

IN multiuser multiple-input multiple-output (MU-MIMO) systems, linear precoding schemes are often considered for practical implementations as they offer near-optimal performance with relatively low computational complexity. Linear precoding techniques, such as Zero-Forcing (ZF) [1] and Minimum Mean Square Error (MMSE) [1], [2], are initially developed for systems with single-antenna receivers. They are later extended to the case of multi-antenna receivers, e.g. ZF is developed into Block Diagonalisation (BD) [3] precoding, and MMSE is extended to Regularised Block Diagonalisation (RBD) [4] and Generalised MMSE Channel Inversion (GMI) [5] schemes. A precoding scheme based on the maximum signal-to-leakage-plus-noise ratio (SLNR) [6] is another attractive technique which provides an alternative approach to the signal-to-interference-plus-noise ratio (SINR) maximisation problem and supports systems with both single-antenna and multi-antenna receivers.

In this letter, equivalent solutions of the SLNR-based precoding scheme with multi-antenna receivers are derived and are shown to be a generalised form of the regularised channel inversion, with regularisation factors for each user inversely proportional to their average signal-to-noise ratios (SNR) per data stream. Conditions for the equivalence between the SLNR, RBD and GMI-2 are also presented. Moreover, the performance of the SLNR scheme is analysed and can be extended to the RBD and GMI-2 schemes, due to their equivalence. This generalises the equivalence between the SLNR and MMSE schemes proven in [7], [8] and establishes the extension of its implications from the case of single-antenna to multi-antenna receivers.

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II. SYSTEM MODEL AND SLNR PRECODING SCHEMES

Consider a single-cell single-carrier downlink MU-MIMO system with M transmit antennas at the base station (BS) and K users, each with N_k receive antennas. Each user k's channel matrix $\mathbf{H}_k \in \mathbb{C}^{N_k \times M}$, assumed to have independent and identically distributed (i.i.d.) entries, is known at the BS. Each user k transmits $B_k \le r_k = rank(\mathbf{H}_k) = \min(N_k, M)$ data streams. The transmitted signal at the BS is expressed as $\mathbf{x} = \mathbf{W}\mathbf{A}\mathbf{s} = \sum_k \mathbf{W}_k\mathbf{A}_k\mathbf{s}_k$. The vector $\mathbf{s} = [\mathbf{s}_1^T, \mathbf{s}_2^T, ..., \mathbf{s}_K^T]^T$ denotes the overall data vector, where $\mathbf{s}_k \in \mathbb{C}^{B_k}$ and $E\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}$. $\mathbf{W} = [\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_K]$ is the transmit precoding matrix, where $\mathbf{W}_k \in \mathbb{C}^{M \times B_k}$. $\mathbf{A}$ is the power normalisation matrix defined by $\mathbf{A} = blkdiag\{\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_K\}$ with $\mathbf{A}_k = diag(\mathbf{a}_k)$ and $\mathbf{a}_k = (a_{k1}, a_{k2}, ..., a_{kB_k})^T \in \mathbb{R}^{B_k}$, such that the total transmission power $\sum \mathrm{Tr}(\mathbf{W}_k\mathbf{A}_k\mathbf{A}_k^H\mathbf{W}_k^H) = \sum P_k = P$. The additive Gaussian noise vector for each user k, denoted as $\mathbf{n}_k$, has zero mean and covariance matrix $E\{\mathbf{n}_k\mathbf{n}_k^H\} = \sigma_k^2\mathbf{I}_{N_k}$. The user k's received signal is given by

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{W}_k\mathbf{A}_k\mathbf{s}_k + \mathbf{H}_k\sum_{j \neq k}\mathbf{W}_j\mathbf{A}_j\mathbf{s}_j + \mathbf{n}_k. \tag{1}$$

At user k, the receive processing can be decomposed as $\mathbf{G}_k = \mathbf{D}_k\bar{\mathbf{G}}_k$, where $\bar{\mathbf{G}}_k \in \mathbb{C}^{B_k \times N_k}$ is the receive filter normalised such that each row has unity norm and $\mathbf{D}_k \in \mathbb{R}^{B_k \times B_k}$ is a diagonal matrix, wherein the diagonal entries represent the norms of the associated rows of $\mathbf{G}_k$. Denoting the received signal at the output of the receive filter as $\hat{\mathbf{y}}_k = \bar{\mathbf{G}}_k\mathbf{y}_k$, the estimated data sequence, $\hat{\mathbf{s}}_k$, can be written as

$$\hat{\mathbf{s}}_k = \mathbf{G}_k\mathbf{y}_k = \mathbf{D}_k\hat{\mathbf{y}}_k. \tag{2}$$

A. The SLNR Precoding Scheme

For the SLNR scheme [6], the power normalisation is usually assumed such that $\mathrm{Tr}\left(\mathbf{W}_k^H\mathbf{W}_k\right) = B_k$ and $\mathbf{A}_k = \sqrt{\frac{P_k}{B_k}}\mathbf{I}_{B_k}$. Here, the SLNR maximisation criterion leads to the following optimisation problem:

$$\mathbf{W}_k^{opt} = arg\max_{\mathbf{W}_k} \frac{\mathrm{Tr}\left[\mathbf{W}_k^H\mathbf{H}_k^H\mathbf{H}_k\mathbf{W}_k\right]}{\mathrm{Tr}\left[\mathbf{W}_k^H\left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k + \alpha_k\mathbf{I}_M\right)\mathbf{W}_k\right]} \quad \text{s.t. } \mathrm{Tr}\left(\mathbf{W}_k^H\mathbf{W}_k\right) = B_k \tag{3}$$

with $\alpha_k = \frac{N_k\sigma_k^2}{P_k}$ and $\tilde{\mathbf{H}}_k = [\mathbf{H}_1^H, ..., \mathbf{H}_{k-1}^H, \mathbf{H}_{k+1}^H, ..., \mathbf{H}_K^H]^H$. The solution to (3) can be given by [6]

$$\mathbf{W}_k = \rho_k\mathbf{T}_k\left[\mathbf{I}_{B_k}; \mathbf{0}_{(M-B_k)\times B_k}\right] \tag{4}$$

where the columns of $\mathbf{T}_k \in \mathbb{C}^{M \times M}$ defines the generalised eigenspace of the pair $\left\{\mathbf{H}_k^H\mathbf{H}_k, \left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k + \alpha_k\mathbf{I}_M\right)\right\}$ and ρ_k is a power normalisation parameter, such that $\mathrm{Tr}\left(\mathbf{W}_k^H\mathbf{W}_k\right) = B_k$. The matched filter, given by

$$\bar{\mathbf{G}}_k = \mathbf{\Psi}_k \mathbf{W}_k^H \mathbf{H}_k^H \quad (5)$$

where $\mathbf{\Psi}_k \in \mathbb{R}^{B_k \times B_k}$ is a diagonal matrix chosen to normalise each row to unity norm, is deployed as the receive filter [6].

III. Equivalent Expressions of SLNR-based Solutions with Multi-antenna Receivers

Lemma 1: Define the following SVD operations

$$\tilde{\mathbf{H}}_k = \tilde{\mathbf{U}}_k \tilde{\mathbf{S}}_k \tilde{\mathbf{V}}_k^H, \quad (6)$$

$$\mathbf{H}_k \tilde{\mathbf{V}}_k \left(\tilde{\mathbf{S}}_k^H \tilde{\mathbf{S}}_k + \alpha_k \mathbf{I}_M\right)^{-\frac{1}{2}} = \bar{\mathbf{U}}_k \bar{\mathbf{S}}_k \bar{\mathbf{V}}_k^H \quad (7)$$

where $\tilde{\mathbf{U}}_k$, $\tilde{\mathbf{V}}_k$, $\bar{\mathbf{U}}_k$, and $\bar{\mathbf{V}}_k$ are unitary matrices; $\tilde{\mathbf{S}}_k$ and $\bar{\mathbf{S}}_k$ are $(\sum_{j\neq k} N_j) \times M$ and $N_k \times M$ (respectively) diagonal matrices, with assumption that the singular values on the diagonal entries of $\bar{\mathbf{S}}_k$ are sorted in decreasing order, i.e. $s_1 \geq s_2 \geq \cdots \geq s_{r_k} > 0$; $r_k = rank(\mathbf{H}_k)$. Let $\breve{\mathbf{S}}_k = diag(s_1, \cdots, s_{B_k})$ and $\breve{\mathbf{U}}_k$, $\breve{\mathbf{V}}_k$ denote submatrices containing B_k ($B_k \leq r_k$) leading columns of $\bar{\mathbf{U}}_k$ and $\bar{\mathbf{V}}_k$ respectively, the solution (4) can be rewritten as:

$$\mathbf{W}_k = \rho_k \tilde{\mathbf{V}}_k \left(\tilde{\mathbf{S}}_k^H \tilde{\mathbf{S}}_k + \alpha_k \mathbf{I}_M\right)^{-\frac{1}{2}} \breve{\mathbf{V}}_k \quad (8)$$

$$= \rho_k \left(\tilde{\mathbf{H}}_k^H \tilde{\mathbf{H}}_k + \alpha_k \mathbf{I}_M\right)^{-1} \mathbf{H}_k^H \breve{\mathbf{U}}_k \breve{\mathbf{S}}_k^{-1}. \quad (9)$$

In addition, the normalised matched filter in (5) can be expressed as

$$\bar{\mathbf{G}}_k = \breve{\mathbf{U}}_k^H. \quad (10)$$

Proof: From the definition of generalised eigenspaces, there exists an invertible matrix $\mathbf{T}_k \in \mathbb{C}^{M\times M}$ such that

$$\mathbf{T}_k^H \mathbf{H}_k^H \mathbf{H}_k \mathbf{T}_k = \mathbf{\Lambda}_k \quad (11)$$

$$\mathbf{T}_k^H \left(\tilde{\mathbf{H}}_k^H \tilde{\mathbf{H}}_k + \alpha_k \mathbf{I}_M\right) \mathbf{T}_k = \mathbf{I}_M \quad (12)$$

where $\mathbf{\Lambda}_k = diag(\lambda_{k,1}, \cdots, \lambda_{k,M})$ with $\lambda_{k,1} \geq \lambda_{k,2} \geq \lambda_{k,r_k} > 0$ and $\lambda_{k,r_k+1} = \cdots = \lambda_{k,M} = 0$. Using the SVD in (6), (12) can be expressed as

$$\mathbf{T}_k^H \tilde{\mathbf{V}}_k \left(\tilde{\mathbf{S}}_k^H \tilde{\mathbf{S}}_k + \alpha_k \mathbf{I}_M\right) \tilde{\mathbf{V}}_k^H \mathbf{T}_k = \mathbf{I}_M. \quad (13)$$

It follows from (13) that $\mathbf{T}_k = \tilde{\mathbf{V}}_k \left(\tilde{\mathbf{S}}_k^H \tilde{\mathbf{S}}_k + \alpha_k \mathbf{I}_M\right)^{-\frac{1}{2}} \mathbf{Q}_k$ where $\mathbf{Q}_k \in \mathbb{C}^{M\times M}$ is a unitary matrix. Let $\tilde{\mathbf{P}}_k = \tilde{\mathbf{V}}_k \left(\tilde{\mathbf{S}}_k^H \tilde{\mathbf{S}}_k + \alpha_k \mathbf{I}_M\right)^{-\frac{1}{2}}$, $\mathbf{Q}_k$ can be determined by substituting $\mathbf{T}_k$ into (11). Hence, (11) can be rewritten as

$$\mathbf{Q}_k^H \left(\tilde{\mathbf{P}}_k^H \mathbf{H}_k^H \mathbf{H}_k \tilde{\mathbf{P}}_k\right) \mathbf{Q}_k = \mathbf{\Lambda}_k. \quad (14)$$

It can be clearly seen from (14) that the unitary matrix $\mathbf{Q}_k$ diagonalises $\left(\tilde{\mathbf{P}}_k^H \mathbf{H}_k^H \mathbf{H}_k \tilde{\mathbf{P}}_k\right)$; thus, $\mathbf{Q}_k$ can be obtained by eigenvalue decomposition of $\left(\tilde{\mathbf{P}}_k^H \mathbf{H}_k^H \mathbf{H}_k \tilde{\mathbf{P}}_k\right)$, i.e. columns of $\mathbf{Q}_k$ contain eigenvectors associated to eigenvalues sorted in decreasing order. Specifically, considering the SVD in (7), it can be chosen such that $\mathbf{Q}_k = \bar{\mathbf{V}}_k$. It also follows that $\mathbf{\Lambda}_k = \bar{\mathbf{S}}_k^H \bar{\mathbf{S}}_k$. Thus, $\mathbf{T}_k$ can be expressed as

$$\mathbf{T}_k = \tilde{\mathbf{V}}_k \left(\tilde{\mathbf{S}}_k^H \tilde{\mathbf{S}}_k + \alpha_k \mathbf{I}_M\right)^{-\frac{1}{2}} \bar{\mathbf{V}}_k. \quad (15)$$

Since the solution of the SLNR design only involves the first B_k columns of $\mathbf{Q}_k$, the solution (4) can be rewritten as in (8). In addition, from (7), it can be shown that $\breve{\mathbf{V}}_k = \tilde{\mathbf{P}}_k^H \mathbf{H}_k^H \breve{\mathbf{U}}_k \breve{\mathbf{S}}_k^{-1} = \left(\tilde{\mathbf{S}}_k^H \tilde{\mathbf{S}}_k + \alpha_k \mathbf{I}_M\right)^{-\frac{1}{2}} \tilde{\mathbf{V}}_k^H \mathbf{H}_k^H \breve{\mathbf{U}}_k \breve{\mathbf{S}}_k^{-1}$. It follows that (9) is obtained by substituting $\breve{\mathbf{V}}_k$ into (8).

Furthermore, by considering submatrices of (7) and right multiplication with $\breve{\mathbf{V}}_k$, i.e. $\mathbf{H}_k \tilde{\mathbf{P}}_k \breve{\mathbf{V}}_k = \frac{1}{\rho_k} \mathbf{H}_k \mathbf{W}_k = \breve{\mathbf{U}}_k \breve{\mathbf{S}}_k$, the receive matched filter can be obtained as $\mathbf{W}_k^H \mathbf{H}_k^H = \rho_k \breve{\mathbf{S}}_k^H \breve{\mathbf{U}}_k^H$. Accordingly, the (row) normalised matched filter can be expressed as (10). ■

A. Equivalence between SLNR, RBD and GMI-2 schemes

As clearly seen from (8), (10), the SLNR precoding scheme with multi-antenna receivers can be viewed as a regularised channel inversion technique, similar to RBD [4], with differences in regularisation and power-normalisation parameters. This conforms to the previous observations in [7], [8] for the case of single-antenna receivers. Notice that, for $N_k = 1$, the precoding and decoding matrices in (9) and (10) can be reduced to the solution given in [8]. Analogously, an equivalence between SLNR and RBD schemes with multi-antenna receivers can also be established as given in the following theorem (only the proof outline is provided due to the limited space).

Theorem 1: The precoding and decoding matrices of the SLNR precoding scheme [6], obtained by (4) and (5) respectively, are equivalent to those of RBD [4] and GMI-2 (GMI method 2) [5] schemes under the assumption of the same regularisation parameter and power normalisation procedure.

Proof: This theorem follows by applying [5, *Theorem* 1-3] with the assumption of equal regularisation parameters, i.e. $\alpha_1 = \cdots = \alpha_K = \alpha$, and the same power normalisation, i.e. $\rho_1 = \cdots = \rho_K = \rho$ to the equivalent expressions of precoding and decoding matrices in (9) and (10). ■

For $N_k \leq M$, the above assumptions can be met, for instance, by imposing equal power allocation (EPA) and full-eigenmode transmission constraints, i.e. choosing $\mathbf{A}_k = \sqrt{P_s}\mathbf{I}_{N_k}$ with $B_k = N_k$ and $P_s = P/(\sum_k N_k)$. This is analogous to [8] for the case of single-antenna receivers.

Note that the equivalence between the SLNR and RBD schemes has also been observed in [9] whereby the equivalence has only been verified for the precoding matrices, by back substitution of the RBD solution [e.g. in a form of (8)] into the SLNR precoding design criteria (11) and (12). The equivalence of decoding matrices, e.g. (5) and (10), has not been provided in [9]. This paper, on the other hand, obtains the equivalent expressions of the SLNR solutions (8)-(10) by direct derivation of (11) and (12) and establishes the equivalence of the decoding matrices in addition to that of the precoding matrices as given in Lemma 1.

Theorem 1 implies the possible interchange of existing algorithms and analysis among these schemes. For instance, the matched filter given by (5) can be used instead of (10) in RBD and GMI-2 schemes for reduced complexity. Existing power allocation algorithms in RBD [4] can also be applied to SLNR and GMI-2 schemes. Further, the performance analysis given in the following can be extended to the RBD as well as the GMI-2 schemes.

IV. Performance Analysis of the SLNR scheme

A. SLNR analysis when $\sum_{j\neq k} N_j < M$

For $\sum_{j\neq k} N_j < M$, the precoding matrix in (9) can be decomposed into two orthogonal subspaces, i.e.

$$\mathbf{W}_k = \frac{1}{\nu_k}\left[\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{H}_k^H + \alpha_k \mathbf{P}^{//}_{\tilde{\mathbf{H}}_k}\mathbf{H}_k^H\right]\breve{\mathbf{U}}_k\breve{\mathbf{S}}_k^{-1} \quad (16)$$

where $\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{H}_k^H = \left[\mathbf{I} - \tilde{\mathbf{H}}_k^H\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H\right)^{-1}\tilde{\mathbf{H}}_k\right]\mathbf{H}_k^H$ is an orthogonal projection of $\mathbf{H}_k^H$ into the null space of $\tilde{\mathbf{H}}_k$, i.e. aligned with the BD solution, while the other part $\mathbf{P}^{//}_{\tilde{\mathbf{H}}_k}\mathbf{H}_k^H = \tilde{\mathbf{H}}_k^H\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H + \alpha_k\mathbf{I}\right)^{-1}\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H\right)^{-1}\tilde{\mathbf{H}}_k\mathbf{H}_k^H$ leads to signal leakage in the column space of $\tilde{\mathbf{H}}_k$, i.e. inter-user interference. ν_k is the power normalisation parameter. The inter-user interference is well-controlled as $\alpha_k\mathbf{P}^{//}_{\tilde{\mathbf{H}}_k} \to \mathbf{0}$ at high SNR. However, it remains necessary to choose an appropriate number of data streams to avoid a convergence to zero effective gain of some data streams as suggested in the following theorem.

Theorem 2: For $\sum_{j\neq k} N_j < M$, a sufficient condition of the number of data streams that ensures non-zero effective gains at high SNR can be given by

$$B_k \leq \min\{M - \sum_{j\neq k} N_j, N_k\}. \quad (17)$$

Proof: At high SNR, $\alpha_k\mathbf{P}^{//}_{\tilde{\mathbf{H}}_k} \to \mathbf{0}$, the precoding design converges to the BD solution and leakage power converges to zero. The effective channel of user k is thus interference-free and has rank $r = \text{rank}(\mathbf{H}_k\mathbf{W}_k) \to \text{rank}(\mathbf{H}_k\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{H}_k^H) = \min\{M - \sum_{j\neq k} N_j, N_k\}$. Multiplexing excessive data streams over this number involves choosing columns of $\breve{\mathbf{U}}_k$ in the null space of $\mathbf{H}_k\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{H}_k^H$, potentially leading to zero-gain at high SNR. Thus, it suffices to ensure non-zero effective gain for each data stream if (17) is satisfied. ■

Note that substreams with effective gains converging to zero account for irreducible BER and zero throughput at high SNR. This results in an error floor of the average BER. The sum-rate, however, still grows with SNR with a change of slope, i.e. multiplexing gain reduces as substreams with zero-gain no longer contribute to the sum throughput.

Theorem 3: Consider a case wherein the conditions in Theorems 1 and 2 are satisfied, i.e. $\mathbf{A}_k = \sqrt{P_s}\mathbf{I}_{N_k}$, $B_k = N_k$, and $\sum_k N_k = M$ with $N_k \leq \sum_{j\neq k} N_j = \tilde{N} < M$. The SINR of the i^{th} stream of user k can be approximated by

$$\gamma_{k,i}^{SLNR} \approx \text{eig}_i\left\{\left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k + \frac{\sigma_k^2}{P_s}\mathbf{I}_M\right)^{-1}\mathbf{H}_k^H\mathbf{H}_k\right\} \quad (18)$$

$$= \frac{P_s}{\sigma_k^2}\text{eig}_i\left\{\mathbf{B}_k + \mathbf{\Delta}_k\right\} \quad (19)$$

with $\mathbf{B}_k = \mathbf{H}_k\left[\mathbf{I}_M - \tilde{\mathbf{H}}_k^H\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H\right)^{-1}\tilde{\mathbf{H}}_k\right]\mathbf{H}_k^H$ and $\mathbf{\Delta}_k = \mathbf{H}_k\tilde{\mathbf{H}}_k^H\left[\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H\right)^{-1} - \left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H + \frac{\sigma_k^2}{P_s}\mathbf{I}_{\tilde{N}}\right)^{-1}\right]\tilde{\mathbf{H}}_k\mathbf{H}_k^H$, and $\text{eig}_i\{\cdot\}$ denoting the i^{th} largest eigenvalue of the argument.

Proof: Analogous to [8], it could be argued that the interference-plus-noise covariance matrix can be estimated by a leakage-plus-noise matrix, that is $\sum_{j\neq k}\mathbf{H}_k\mathbf{W}_j\mathbf{W}_j^H\mathbf{H}_k^H + \frac{\sigma_k^2}{P_s}\mathbf{I}_{N_k} \approx \sum_{j\neq k}\mathbf{W}_k^H\mathbf{H}_j^H\mathbf{H}_j\mathbf{W}_k + \frac{\sigma_k^2}{P_s}\mathbf{W}_k^H\mathbf{W}_k = \mathbf{W}_k^H\left(\sum_{j\neq k}\mathbf{H}_j^H\mathbf{H}_j + \frac{\sigma_k^2}{P_s}\mathbf{I}_M\right)\mathbf{W}_k$, which can be shown to be generally tight when the interference power is relatively small compared to the noise power, i.e. at the asymptotic low and high SNR regimes[1]. Using the above approximation, the SINR of the i^{th} stream of user k can be written as

$$\begin{aligned}\gamma_{k,i}^{SLNR} &= \frac{\left[\mathbf{G}_k\mathbf{H}_k\mathbf{W}_k\mathbf{W}_k^H\mathbf{H}_k^H\mathbf{G}_k^H\right]_{ii}}{\left[\mathbf{G}_k\left(\sum_{j\neq k}\mathbf{H}_k\mathbf{W}_j\mathbf{W}_j^H\mathbf{H}_k^H + \frac{\sigma_k^2}{P_s}\mathbf{I}_{N_k}\right)\mathbf{G}_k^H\right]_{ii}} \\ &\approx \frac{\left[\mathbf{W}_k^H\mathbf{H}_k^H\mathbf{H}_k\mathbf{W}_k\mathbf{W}_k^H\mathbf{H}_k^H\mathbf{H}_k\mathbf{W}_k\right]_{ii}}{\left[\mathbf{W}_k^H\mathbf{H}_k^H\mathbf{W}_k^H\left(\tilde{\mathbf{H}}_j^H\tilde{\mathbf{H}}_j + \frac{\sigma_k^2}{P_s}\mathbf{I}_M\right)\mathbf{W}_k\mathbf{H}_k\mathbf{W}_k\right]_{ii}} \\ &\overset{(a)}{=} \lambda_{k,i}\end{aligned} \quad (20)$$

where the equality (a) follows from (11) and (12). Hence, the SINR can be approximated by the corresponding eigenvalue. Note that (20) can be expressed as (18). By using $\text{eig}\{\mathbf{AB}\} = \text{eig}\{\mathbf{BA}\}$, (18) can be rewritten as $\text{eig}_i\left\{\mathbf{H}_k\left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k + \frac{\sigma_k^2}{P_s}\mathbf{I}_M\right)^{-1}\mathbf{H}_k^H\right\}$. (19) follows after applying a few matrix operations. ■

Although the approximation in Theorem 3 could not guarantee the tightness at moderate SNRs, it greatly simplifies the SINR analysis and provides insights into the performance with respect to the BD scheme as given below. Moreover, reasonable accuracy can generally be observed by simulation.

Following [10], the SINR of BD can be expressed as $\gamma_{k,i}^{BD} = \frac{P_s}{\sigma_k^2}\text{eig}_i\{\mathbf{B}_k\}$. Comparing to (19), it follows from [11, 4.3.1] that $\gamma_{k,i}^{SLNR} \approx \lambda_{k,i}^{SLNR} \geq \gamma_{k,i}^{BD} + \lambda_{N_k}(\frac{P_s}{\sigma_k^2}\mathbf{\Delta}_k)$, with the smallest eigenvalue of $\frac{P_s}{\sigma_k^2}\mathbf{\Delta}_k$ denoted as $\lambda_{N_k}(\frac{P_s}{\sigma_k^2}\mathbf{\Delta}_k) = \text{eig}_{N_k}\left\{\mathbf{H}_k\check{\mathbf{V}}_k\left(\check{\mathbf{S}}_k^2 + \frac{\sigma_k^2}{P_s}\mathbf{I}_{\tilde{N}}\right)^{-1}\check{\mathbf{V}}_k^H\mathbf{H}_k^H\right\}$. $\check{\mathbf{V}}_k$ and $\check{\mathbf{S}}_k$ are the submatrices, corresponding to the $\tilde{N}$ non-zero singular values, of $\tilde{\mathbf{V}}_k$ and $\tilde{\mathbf{S}}_k$ respectively. It follows that $\lambda_{N_k}(\frac{P_s}{\sigma_k^2}\mathbf{\Delta}_k) \geq 0$ ($\frac{P_s}{\sigma_k^2}\mathbf{\Delta}_k$ is non-negative definite Hermitian). Further, λ_{N_k} is a non-decreasing function of $\frac{P_s}{\sigma_k^2}$, it converges to $\lambda_{N_k,\infty}(\frac{P_s}{\sigma_k^2}\mathbf{\Delta}_k) = \text{eig}_{N_k}\left\{\mathbf{H}_k\check{\mathbf{V}}_k\check{\mathbf{S}}_k^{-2}\check{\mathbf{V}}_k^H\mathbf{H}_k^H\right\}$ as $\frac{P_s}{\sigma_k^2} \to \infty$. This indicates the superiority of the SLNR scheme over BD. Following [8], for Rayleigh fading channels, it can be shown that the sum-rate of the SLNR scheme converges to that of BD at high SNR, while there remains a non-vanishing gap of BER performance as also shown in Section V.

B. SLNR analysis when $\sum_{j\neq k} N_j \geq M$

For $\sum_{j\neq k} N_j \geq M$, i.e. $\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k$ is full rank, leakage power from user k to user j generally increases with P_s at high SNR, as $\|\mathbf{H}_j\mathbf{W}_k\mathbf{A}_k\|_F^2 \propto P_s\|\mathbf{H}_j\left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k\right)^{-1}\mathbf{H}_k^H\breve{\mathbf{U}}_k\breve{\mathbf{S}}_k^{-1}\|_F^2$. In general, this suggests a limit of performance in the high SNR regime. However, from (7), the singular vectors ($\breve{\mathbf{U}}_k$,$\breve{\mathbf{V}}_k$)

[1]This can be shown by first noticing that $\frac{\sigma_k^2}{P_s}\mathbf{W}_k^H\mathbf{W}_k$ converges to $\frac{\sigma_k^2}{P_s}\mathbf{I}_{N_k}$ at low and high SNR. Further, for i.i.d. channels, $E\|\mathbf{H}_k\mathbf{W}_j\mathbf{W}_j^H\mathbf{H}_k^H\|_F = E\|\mathbf{H}_j\mathbf{W}_k\mathbf{W}_k^H\mathbf{H}_j^H\|_F = E\|\mathbf{W}_k^H\mathbf{H}_j^H\mathbf{H}_j\mathbf{W}_k\|_F$. Since $\mathbf{H}_k\mathbf{W}_j\mathbf{W}_j^H\mathbf{H}_k^H$ and $\mathbf{W}_k^H\mathbf{H}_j^H\mathbf{H}_j\mathbf{W}_k$ are presumably small compared to $\frac{\sigma_k^2}{P_s}\mathbf{I}_{N_k}$, it follows that $\sum_{j\neq k}\mathbf{H}_k\mathbf{W}_j\mathbf{W}_j^H\mathbf{H}_k^H + \frac{\sigma_k^2}{P_s}\mathbf{I}_{N_k} \approx \sum_{j\neq k}\mathbf{W}_k^H\mathbf{H}_j^H\mathbf{H}_j\mathbf{W}_k + \frac{\sigma_k^2}{P_s}\mathbf{I}_{N_k} \approx \sum_{j\neq k}\mathbf{W}_k^H\mathbf{H}_j^H\mathbf{H}_j\mathbf{W}_k + \frac{\sigma_k^2}{P_s}\mathbf{W}_k^H\mathbf{W}_k$.

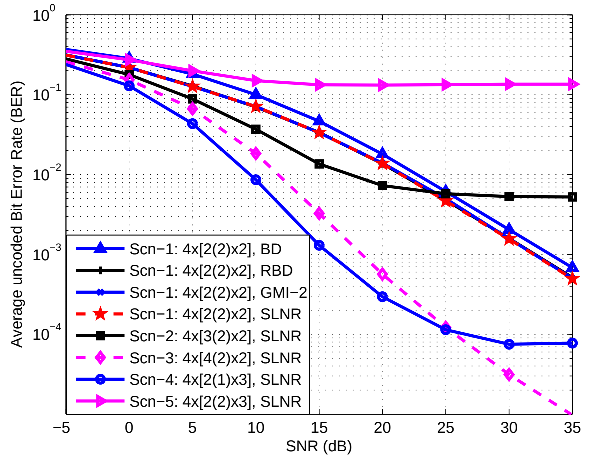

Fig. 1. Average BER (QPSK), with configuration $M \times [N_k(B_k) \times K]$.

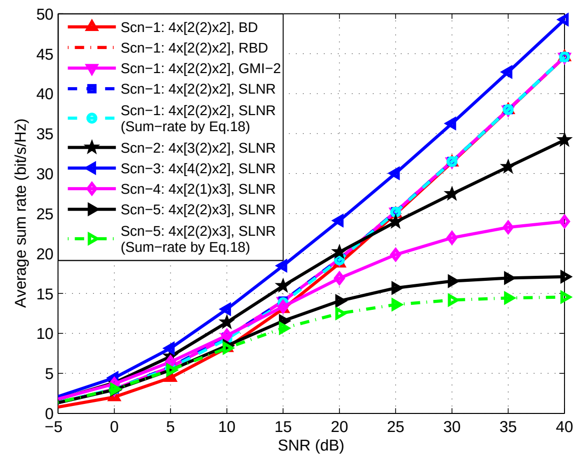

Fig. 2. Average sum rate, with configuration $M \times [N_k(B_k) \times K]$.

are chosen in the directions which have direct and inverse relationships with the singular values of $\mathbf{H}_k$ and those of $\tilde{\mathbf{H}}_k$, respectively. Severe interference can thus be alleviated when M is large (high degree of freedom for transmit beamforming design) and B_k is small (only using data streams with reasonably good designs, i.e. large singular values).

Additionally, with a specific case of $K = 2$, it can be shown that the interference issue at high SNR can be completely avoided as elaborated in the following theorem.

Theorem 4: For $K = 2$ and $N_k \geq M$, a necessary condition for the convergence of inter-user interference to zero at high SNR is given by

$$\sum_{k=1}^{2} B_k \leq M. \tag{21}$$

In fact, at high SNR, the beamforming designs of both users are equivalent with eigenvalues sorted in reverse order.

Proof: For $K = 2$ with $\mathbf{H}_1^H\mathbf{H}_1$ and $\mathbf{H}_2^H\mathbf{H}_2$ full rank, at asymptotically high SNR, (11) and (12) for user 1 can be rewritten as $\left(\mathbf{\Lambda}_1^{-\frac{1}{2}}\mathbf{T}_1^H\right)\mathbf{H}_1^H\mathbf{H}_1\left(\mathbf{T}_1\mathbf{\Lambda}_1^{-\frac{1}{2}}\right) = \mathbf{I}_M$ and $\left(\mathbf{\Lambda}_1^{-\frac{1}{2}}\mathbf{T}_1^H\right)\mathbf{H}_2^H\mathbf{H}_2\left(\mathbf{T}_1\mathbf{\Lambda}_1^{-\frac{1}{2}}\right) = \mathbf{\Lambda}_1^{-1}$, respectively. It follows that $\mathbf{T}_2 = \mathbf{T}_1\mathbf{\Lambda}_1^{-\frac{1}{2}}$ and $\mathbf{\Lambda}_2 = \mathbf{\Lambda}_1^{-1}$. Thus, $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are in reverse order. $\mathbf{T}_1$ and $\mathbf{T}_2$ are equivalent, in the sense that column vectors (in reverse order) are aligned in the same directions, i.e. there are M distinct independent beams available for both users. Hence, zero inter-user interference is plausible if (21) is satisfied. ■

V. Simulation Results and Conclusions

Figs. 1 and 2 show the sum-rate and BER performance for various scenarios (Scn-1 to Scn-5) with EPA and independent Rayleigh fading channels. The SNR is defined as P/σ^2. The equivalence between the SLNR, RBD and GMI-2 schemes is represented by Scn-1. For $\sum_{j \neq k} N_j < M$ with the condition given in Theorem 2 being not satisfied, an error floor and a change of sum-rate slope at high SNR can be observed as shown in Scn-2. For $\sum_{j \neq k} N_j \geq M$, $K = 2$ (Scn-3), a limit of performance at high SNR is shown to be avoided if the condition in Theorem 4 holds. In contrast, for $\sum_{j \neq k} N_j \geq M$, $K > 2$ (Scn-4 and Scn-5), performance floors can be observed. In this case, smaller B_k results in less severe interference as analysed in Section IV-B. The approximation of sum-rate given in Theorem 3 is also provided in Fig. 2. Notice that the approximation is generally tight for the whole SNR range when $\sum_{j \neq k} N_j < M$ (Scn-1), whereas it is slightly loose for high SNR when $\sum_{j \neq k} N_j \geq M$ (Scn-5) as the assumption of small interference is no longer accurate.

In conclusion, this letter derived equivalent expressions of SLNR-based precoding solutions and established the equivalence between the SLNR, RBD, and GMI-2 precoding schemes. With this equivalent form, the performance of the SLNR scheme has been analysed. These analytic results can be extended to the other schemes due to their equivalence. This generalises [8] and its useful implications from the case of single-antenna to multi-antenna receivers.

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