A relational view of mathematical concepts

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Many concepts used in mathematics appear self-evidently to be relations. Speed is a relation between distance and time (which are themselves relations); fractions and ratios can both, in different ways, be seen as relations between numbers or lengths or quantities; volume is a relation between one shape and a unit (typically a unit cube); the gradient of a graph is a relation between 'rise' and 'run'. In this chapter, I aim to show that there is considerable pedagogical advantage in viewing *every* mathematical concept as a relation. I aim to show that by introducing mathematical concepts as relations, supposed problems for children in abstraction fall away and learning can become fast, imaginative and engaging. The arguments are primarily pragmatic and empirical ones, based on classroom experiences and experiments. The view of mathematical concepts as relations entails a disruption of the notion that learning proceeds through abstraction from manipulating concrete objects, to visual models and culminating in abstract entities and instead suggests a more circular conception of learning and development.

It is not easy to pin down what kind of a thing a relation *is*; I take examples of relations to be differences, similarities and comparisons and wish to distinguish relations from the objects that are being compared or related. I suggest this difference is one of perspective and sometimes choice – it is perhaps possible to view any object (whether real or imagined) as a relation and any relation as an object. An exercise used at the start of art-college, when faced with a still life, is to move focus away from drawing the things on the table, to the spaces in-between. Instead of focusing on the black text of this page, I invite you to shift attention to the meandering lines of whiteness than run down the page. These are shifts in perception from noticing objects to noticing relations. There is a choice to be made. At moments of insight, we re-see, re-think, re-cognize what we have been attending to. The poet and philosopher Jan Zwicky suggests, 'all genuine understanding is a form of seeing-as' (§3). Similarly, Gregory Bateson described the core of his scientific method as the search for a double description of phenomena, where "[*relationships are*] *always a product of double description*" (1979, p.132, emphasis in the original). When I claim, therefore, that mathematical concepts are relations this is not an ontological statement but rather a suggestion that seeing mathematical concepts as relations has advantages, particularly when this is accompanied by a symbolizing of relations rather than objects. I draw on writers who distinguish between abstract modes of thought and more concrete or empirical modes of thought. What I take this distinction to mean is that abstract thought deals with relations and concrete or empirical modes of thought deal with objects.

The structure of this chapter follows the logic of Walter McCullough's challenge: 'what is a number, that a man (sic) may know it, and a man, that he may know a number?'(1960). To begin, and in order to approach the idea that every mathematical concept is a relation, I take the case of the early learning of number, this leads to a consideration of human cognition more broadly and then suggestions for how other areas of the curriculum can be approached in a relational manner.

What is a number, that a woman may know it?

Bass (2015) suggested there are two competing narratives around the development of number sense and its culmination in an awareness of real numbers and the number line. The predominant narrative in schooling is a 'counting world'. Number is introduced as a label for distinct objects and so, when mapped to an imagined number line, numbers are individual marks. From this point, the number line must be progressively 'filled in', with negative numbers, rational numbers and ultimately real numbers. Such an approach leads to significant difficulties for students in the jump to the existence of the rationals and then to the existence of the reals. If the basic conception of number is to stand for objects, a disruption to this idea is needed to make sense of fractions. Perhaps another way of saying this is that starting with the notion of number as standing for discrete objects, we introduce number to students in an 'empirical' mode of thought (Goutard, 1964) and this way of thinking is not adequate to conceptualise further developments in mathematics.

Bass's alternative approach to number (and the one I advocate in this chapter) is a 'measurement world', or I would prefer to call it a 'relational world'. Bass draws on Davydov's curriculum (1990) as an exemplar of what it might mean to introduce number not as standing for objects, but as measure. In terms of a number line, whole numbers can be seen as scalings of a unit length. The number line is 'full' from the very start in the sense that, intuitively speaking, a scaling can get you anywhere on a line. There is no conceptual stumbling block in moving from integer to fractional scalings. Ma (2015) proposed that the basis for arithmetic is an awareness of quantitative relations. In the counting world, this is undoubtedly the case. However in the measurement world (and for Davydov), before associating specific numbers with specific scalings, awareness can be developed of the broader relations of 'greater than' and 'less than' as the basis for number sense. As well as Davydov, Gattegno (1974) also developed a programme that introduced number not in the context of groups of objects. As far as I am aware their programmes have never been closely compared yet, as I hope to draw out below, there are striking similarities which lead to implications for the whole mathematics curriculum.

Caleb Gattegno (1911-1988), born in Egypt, worked across the world developing a mathematics curriculum based in the use of the Cuisenaire rods[1]. Vasily Davydov (1930-1988) was born and worked in Russia and at around the same time as Gattegno developed a curriculum that was implemented in schools in Russia. Both educators have their advocates today, for example, Gattegno has inspired the Bronx Charter School in New York and Davydov's ideas are behind the 'Measure Up' programme (Dougherty, 2008).

In both programmes for mathematics, children's first lessons revolve around experiences with objects of different lengths. Gattegno (1963) suggests children have some time of 'free play' with the Cuisenaire rods. There is a conviction that children will quite naturally begin making comparisons. Initially the comparisons are 'greater than' and 'less than'. An important step (again in both curricula) is the use of aligning

[1] Cuisenaire rods are cuboids with 1cm^2 cross-sections and ranging in length from 1 to 10cm. Each length is coloured uniquely (eg the cube with 1cm lengths is white, the rod of length 2cm is red).

lengths for comparison. And, equally important, if one rod or length is longer than another, then a third rod/length can be added to make the lengths the same. Madeline Goutard worked closely with Gattegno in developing the use of Cuisenaire rods and wrote a book (1964) in which she detailed her approach. An early activity she would do with children, once they have become familiar with the rods and begun making comparisons between lengths, was to focus on the length 'to be added'. This length can be associated with a subtraction and Goutard would get children working on 'families of subtractions', i.e., pairs or trains of rods where what is 'to be added' is the same.

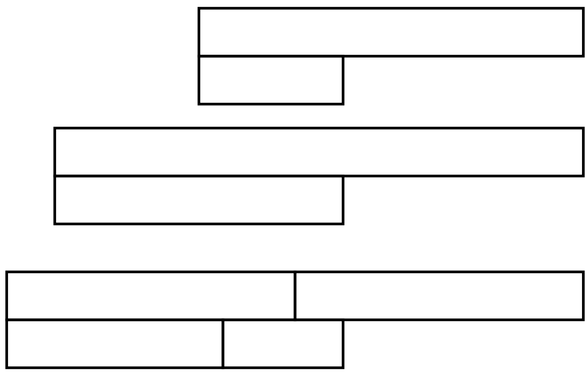

Figure 12.1: 'Families of equivalent subtractions' (adapted from Goutard, 1964)

Davydov worked with a variety of materials (i.e., there is no equivalent of the central place for the Cuisenaire rods) however the focus, as with Gattegno, was on comparison of measures (whether this be a measure of length, area or volume). At some point, there would also be a focus on 'how much' difference there is between two lengths (see Figure 12.2).

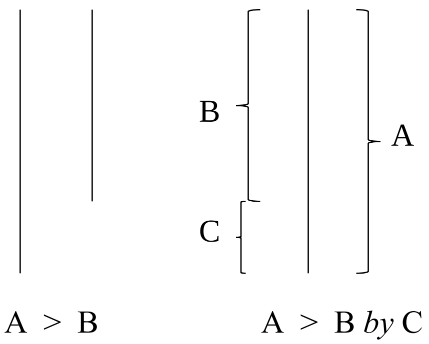

Figure 12.2: Schematic (adapted from Schmittau, 2005, p.19)

Both programmes would get students in the first grade codifying relations such as in Figures 1 and 2, by using letters. Gattegno labels each of the Cuisenaire rods with a letter indicating the colour name, Davydov would use A, B, C to represent the different lengths. The kinds of statements that students would produce are, for Goutard's families of equivalent subtractions:

$$p = b - y = d - r = \cdots$$

and, for Schmittau:

A = B + C
B = A – C

C = A – B

It is important to note that no numbers are used (on either programme) to represent lengths. In both Davydov and Gattegno's curricula, number is first introduced as a *comparison* of measurements, when we have the special case that you can use copies of the same length to make a longer length.

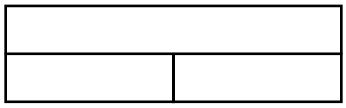

Figure 12.3: The introduction of numbers

For Gattegno, the kind of relationship in Figure 3 is used to introduce the number '2' and immediately (as the second number to appear) the number '½'.

$p = 2r$ and the equivalent, $r = \frac{1}{2}p$ (Gattegno, 1963, p.29)

For Davydov, the role of the 'unit' in measurement is central. From a comparison of lengths, as with Gattegno, the first numbers introduced are in some sense a scaling. Children are encouraged to create their own notation to signify the number of times a unit measure fits into a longer length, for example:

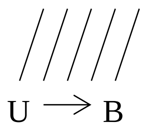

Figure 12.4: A notation of comparison (adapted from Schmittau, 2000, p.66)

As Schmittau explains: 'From this, the idea of the counting sequence is developed, as a tool for labeling how many units make up any particular quantity. Then the concept of number is developed as a relationship of quantities.' (ibid). This relationship is expressed as follows:

$B = 5U$ and the equivalent, $\frac{B}{U} = 5$ (ibid)

As with Gattegno, this scaling is expressed in two different ways, although there is a significant difference in the second expression. Whereas Davydov, in some ways 'defines' number as the ratio of lengths ($\frac{B}{U} = 5$), Gattegno stays with the use of number as an 'operator' (or scale factor) and so introduces fractions in the first grade to capture the inverse relationship ($r = \frac{1}{2}p$).

These parallels in the development and introduction of number are striking. What is also clear from both treatments is that number is not being linked to collections of objects, or at least, not only to collections of objects. Number is first seen as a dynamic relation. Through introducing number as a relation between quantities, whole numbers become, in effect, scale factors. 5U or 2r mean "take five of the unit length" and "take two of the red rod". Numbers are brought into existence through the action of placing objects against each other.

Gattegno makes use of this relational view of number to work with fractions. For anyone skeptical about whether it really is possible to get children aged 6 or 7 working with fractions, it is only necessary to watch a video clip of Gattegno teaching (http://www.calebgattegno.org/mathematics-at-your-fingertips.html) where in one hour children (who have gained familiarity earlier in the year with the rods) move from being introduced to fraction notation, to being able to solve the following questions:

$$\frac{1}{2} \times (36 - 18) =$$

$$\frac{1}{3} \times (18 + 9) =$$

$$\frac{1}{4} \times (9 + 27) =$$

$$\frac{1}{2} \times \left(9 + \tfrac{1}{3} \times 27\right) - \frac{1}{4} \times 36 =$$

These questions would not be out of place on test items for 14 or even 16 year olds in the UK and yet were tackled confidently by 6-7 year olds. Similarly, Davydov develops schematics for working with part-whole relations and students quite quickly are able to solve problems involving proportional reasoning that would usually only be tackled much later on in a typical UK or US mathematics curriculum. Students can find the shift into proportional reasoning a significant barrier in learning mathematics. The remarkable idea of both Gattegno and Davydov is that this problem can be avoided by introducing number, from the very start, as a proportional relation, within a ‘measuring’ or relational world rather than a ‘counting’ world. And, because the symbols are relations it seems, furthermore, that the step to considering symbols in relation to one another, leaving behind the objects from which they initially arose, also occurs swiftly for students. It is evident from the discussion above that when symbolizing (relationally) Figures 3 or 4, there are several descriptions possible – viewing the relation from different perspectives. So, from the very start, students are considering relations *between* symbolic descriptions (e.g., the connection between 2 and ½).

An obvious question begged by the relational approach to number and its apparent power, is whether the whole curriculum can be treated in similar manner. In thinking about this question, I have found it useful to reflect on what, in these approaches to number, is being symbolized. Clearly, from what has been discussed above, I suggest that numerals denote relations (between lengths). However, what is also highly significant is that number symbols are initially used to stand for an action *performed* on the lengths or rods. The relation captured by ‘2’ or ‘½’ in Figure 3 is performed by students, in the sense that they place the rods in that special configuration and, from their previous work with rods, likely with the awareness that other comparisons of lengths leave something left over. Also important is that different sets of rods are arranged in the same configuration (2 whites and 1 red; 2 yellows and 1 orange, etc) so that ‘2’ is not associated with particular rods but with what is common to a range of instantiations of putting two of the same rod precisely against a third. What is *not* being symbolized with a numeral, at least initially, is any actual length.

Reflecting on the effect of these pedagogical moves it becomes apparent, as I hope to show in the next section, that what happens to learners is not captured by the orthodoxy to a developmental sequence of, say, moving from concrete to visual to abstract. Making this point takes us into questions of what it is to be a human (learning about number).

What is a woman, that she may know a number?

The orthodoxy in the UK in the early years of education and schooling is that children need to work with concrete materials or manipulables to develop their understanding of number. Goutard (1964) pointed to this phenomena:

> It is generally agreed that concrete experience must be the foundation of mathematics learning. When children find it difficult to understand arithmetic it is at once suggested that this is because it is too abstract; for small children the study is then simply reduced to the counting of objects. (1964, p.3)

The importance of the concrete in the learning of mathematics is given a more recent interpretation through, for example, the work of Lakoff and Nunez (2000) who suggested a bodily basis for all mathematical meaning. From this 'embodied' stance, our understanding of number begins with bodily experiences (for example rhythmic clapping, walking up steps) and we progressively abstract from these experiences, re-organising our perception and culminating in symbolic representations of our actions.

However, there is a well-known paradox linked to this familiar story of the movement from the concrete (perhaps via the 'visual') to the abstract. Piaget summed up the paradox as follows: 'that adaptation to the concrete experimental facts is dependent upon the abstract character of the theoretical framework, which allows analysis and apprehension of these facts' (1976, p.353). In order to make sense of our perceptions we need to have some kind of abstract framework within which they fit. Yet we can seemingly only develop these frameworks via abstraction from experience of our perceptions. There is a circularity here: we need abstract structures to make sense of perception and we need perception to build abstract structures (a version of this paradox is the focus of the chapter in this book by Wolff-Michael Roth). Some evidence for the existence of such a circular 'trap' comes from Piaget's experiments with young children. He observed:

> It is obviously not sufficient to make a correct observation of something for it to be accepted, if there appear to be valid reasons for refusing (or repressing) it … if the subject feels that what he (sic) sees happen should not have happened, then this observation is not retained or conceptualized. (1976, p.214)

Piaget observed children ignore results of experiments they had performed accurately (for example using a simple catapult to throw a rubber into a bin) when it came to describing or predicting what to do. A child might have a 'know-how' and yet seemingly lack the abstract structures needed to make sense of this know-how. Piaget embraced a 'circular relationship between subject and object' (ibid, p.343) and arrived at the view that:

> The subject only learns to know himself when acting on the object, and the latter can become known only as a result of progress of the actions carried out on it … this explains the harmony between thought and reality, since action springs from the laws of an organism that is simultaneously one physical object among many and the source of the acting, then thinking, subject. (ibid)

The circularity and entanglement in the emergence of subject and object is a key insight behind the world-view of enactivism (Varela, Thompson, Rosch, 1991), which offers a way through the paradox of how learning and abstraction take place. One difficulty in proposing a viewpoint such as enactivism is that our established patterns of thinking are so firmly embedded in the conviction of the separability of subject and object and, as Piaget notes, humans are very good at refusing or repressing evidence that does not fit an existing way of thinking. One way to catch ourselves out, and allow the possibility of a different perspective, is a consideration of illusions. (For Maturana, a key enactivist thinker, the impossibility of knowing, in the moment, whether a perception is an illusion or not led to an insight about the informational closure of living systems (Maturana and Poerksen, 2004).)

Consider the illusion in Figure 12.5. Focus on the two lines of grey tiles; do you see the greys as the same shade?

Figure 12.5: The Munker-White illusion

It seems that humans experience these greys as different. Now turn over the page and look at Figure 12.6, where the same tiles are reproduced, but this time without one column of black. In this configuration there seems to be no difficulty in seeing the grey shading as the same (or at least much closer in tone than before), yet it is the identical grey to that used throughout Figure 5. How is it possible to explain the phenomena you have just experienced?

What seems clear from this experiment is that in Figure 12.5, we do not perceive the colours ‘in themselves’. This is an example of a family of illusions where it seems as though the context leads us ‘astray’. The reason for the difference in the way the grey looks in Figure 12.5 has something to do with expectations linked to the pattern of light and dark in the other tiles. In other words, it seems that we do not perceive the grey colour per se, or at least, the context of the grey alters how we perceive it. This has always been known to artists, no doubt; the quilter Kaffe Fassett, famous for his bright designs, talks about the importance of dull and grey colours in his design in terms of the overall colour effect (personal communication).

It might be easy to dismiss this illusion as a gimmick, or to imagine that in this instance we simply need to do some alterations to Figure 12.5 in order to see the colours ‘as they really are’. However, the illusion suggests a more radical view. To put this at its starkest, since we always perceive colours in a context, we can *never* see colours ‘as they really are’ – indeed the entire concept of colour ‘as it really is’ does not even make sense. Therefore, it is not actually the case that we perceive ‘colours’ as such, rather, we perceive *relations* between colours. Even when we know the illusion it seems impossible to 'see' the greys in Figure 12.5 as the same.

Figure 12.6: The illusion unmade

But if we can only make sense of our colour perception of objects through a consideration of context, then it cannot be the case that we really perceive objects at all (what is true of colour will be true of all modalities of perception). Although I may experience the world as a set of discrete objects, if context plays such an important role then the separation of objects from each other (and hence from me) is really a fiction. Again, this is something that has been well known to artists, and connects back to Piaget’s insight about the importance of abstract frameworks in perception:

> to see an object is always to perform an abstraction because seeing consists in the grasping of structural features rather than in the indiscriminate recording of detail (Arnheim, 1969, p.68).

We can of course notice details but when we do, this is preceded by an awareness of more general, abstract and structural features, ‘generalities precede particulars in sensory experience’ (Arnheim, 1969, p.221) – precisely what Piaget concluded from his experiments. The enactivist conviction (drawing on, for example: Bateson, 1972; Maturana and Varela, 1987) is that the core cognitive function is to make distinctions and that even the most basic of organisms operate in the world through acting on relations and differences rather than a representation or awareness of objects, as the colour illusion above illustrates for humans. In other words, there need be no paradox in abstraction coming before particulars, when abstraction is taken to mean attention to relations, if our basic mental function is to attend to relations and differences.

Re-casting the concrete-abstract divide

If humans perceive relations and distinctions then the typical developmental sequence of a move from sensori-motor operations to the mapping onto those sensori-motor experiences of more formal operations is put into question (something also critiqued in deFreitas and Sinclair (this volume, Proposition 2)). The relation ‘double’ (e.g., in

Figure 3) does not exist *in* either the smaller or the larger rods but arises through a human comparison between them. In this sense, relations and differences are *always already* abstracted from the objects that give rise to them. If distinctions are the basic mental function, then whatever problems children have with mathematics it cannot be due to difficulties of abstraction. Rather, it may be that the approach to (for example) early number development, in emphasizing objects, is establishing a pattern of thinking about mathematics that makes it difficult.

Goutard (1964) distinguished 'empirical' thought (about objects) from more structural awareness (of relationships). While the objects of mathematics become more and more complex and abstract (for example, we study number patterns that are codified as functions and then treat functions as objects in order to consider their properties, and so on) it does not follow that structural (or relational) thinking is hard, only that the structures about which mathematicians are concerned become nested, one built on another. In the introduction to her book describing her experiences using Cuisenaire rods, Goutard considers this division in ways of thinking:

> It seems to me that there has perhaps been too great-a tendency to make things concrete and that perhaps the difficulties children experience spring from the fact that they are kept too much at the concrete level and are forced to use too empirical a mode of thought. (1964, p.3)

Davydov and Gattegno offer mechanisms for children, from the very beginning, to engage in thinking that is not limited to the concrete or empirical level of thought. It may appear paradoxical that the use of Cuisenaire rods can be talked about, as above, in relation to moving students away from the concrete. And this is where what makes all the difference is what is being symbolized. If, in using Cuisenaire rods, the white rod is always associated with '1', the red rod with '2' and so on, then the use of the rods will remain in an 'empirical mode of thought' and the entire power of Gattegno's approach is lost. By introducing numbers as relations, abstracted from the concrete context at the very start, there is no concrete-abstract divide to cross, for children. Number symbols (initially '2' and '½') denote a particular configuration of material objects but quickly take on properties in relation to each other. In Coles (2014) I described work with children using a visual image (the Gattegno tens chart) where there was precisely this sense of symbols arising for actions and relations within the chart and then children quickly focusing on relations between the symbols themselves.

Some evidence for there being more than one way of thinking about the objects of mathematics comes from recent neuroscientific studies related to early number development. Lyons and Beilock (2013) tested subjects performing basic number tasks such as comparison of size and judgment of whether numerals were in order (4, 5, 6 are in order; 4, 6, 5 are not). What they discovered is that similar patterns of brain activity are present when comparing the sizes of numerals, or collections of dots and even when judging if three sets of dots are in order of size (perhaps all these are examples of Goutard's 'empirical' thinking) and that a different pattern of brain activity is aroused by being asked to make (ordinal) comparisons of whether three numerals are in sequence (perhaps requiring more structural or relational awareness). Not only this, but the kinds of brain activity used in the latter task are similar to those used in more complex arithmetic and, furthermore, speed at this kind of ordinal awareness is the best predictor (compared to speed at any of the other size comparison tasks) of overall mathematical attainment from grades 2 to 6 (Lyons et al., 2014).

The proposal suggested by the considerations above is therefore that the paradox of the concrete-abstract divide in learning number only arises if children are forced to associate numerals too strongly with collections of objects. If, instead, children are introduced to number symbols as relations then they can be inducted, from the start, into the way of thinking about and with number that is needed for success in mathematics. And what is true of number (that it can be introduced relationally), I aim to show in the next section, is true of every mathematical concept.

All mathematical concepts as relations

In order to develop an entire curriculum on the basis of mathematical concepts as relations, it is necessary to devise starting points where what can be symbolized are actions and transformations of objects or images. There is not space to deal with every aspect of the curriculum so in this section I take three concepts and show how they can be thought of as relations and how, in doing this, some re-ordering of the traditional order of curriculum topics may be required. These three concepts are chosen as illustrative and include one example from primary, secondary and post-16 phases of schooling.

Area as a relation

The concept of area can be considered in an empirical manner, with concrete objects or visual images and it is a concept that children can have difficulty with. However, 'area' is an inherently relational concept when considered as a relationship to a unit. In a similar manner to the way that Gattegno introduces the symbol '2' to represent 'double' rather than two objects, an area of '2' can be introduced to represent the situation of one shape fitting into another shape twice (and hence also the first shape has an area ½ the second shape). A relational approach to area would not be so much concerned with attaching numerical values to the sizes of shapes as in comparing them. Which are bigger? Which are smaller? And then considering the case where one shape can fit an exact number of times into another shape. In essence, by considering area as a relation, the concept arises out of the wider mathematical structure of transformation geometry and, in particular, enlargements. It might be, therefore, that we work on enlargements with students before area (a reverse of the typical sequence). It seems, at the least, that there are indeterminate relations between these concepts and ones we typically see as necessary to build on others are not definitively so; any curriculum needs flexibility, therefore, to be responsive to students' awareness. It would be possible to use awarenesses about area to work on enlargements but equally possible to use awarenesses about enlargement to work on area.

Algebra as relation

In the book 'Starting Points' (Banwell, Saunders and Tahta, 1986) there is the proposal that whenever functional relationships are discovered by children or being considered in a lesson, that there is always the same way of representing them, in a tabular form:

3	6
5	10
6	12

Table 12.1: A function representation

One way of setting up this representation is through the ‘function game’ (also described in Brown and Waddingham, 1982). The teacher has a rule in mind (‘doubling’ in the case of Table 1) and writes out two ‘inputs’ with arrows to the ‘output’ and then writes a third input and invites the class to guess the output. This game can be powerful when played in silence, with the teachers (and later perhaps a student) indicating if the output is correct with a ☺ or ☹. At some point, when most of the class have figured out the rule, the teacher might put ‘n’ as the input, to invite a sharing of these rules. Even with a rule as simple as ‘doubling’ there may be differences in how students were applying the rule, that can be captured in their algebraic expressions (e.g., $n + n$, or $n \times 2$, or $2n$). The teacher can also put ‘n’ in the ‘output’ column, with the arrow backwards and invite expressions for the inverse of the rule they have just found.

Algebra, in this treatment, represents a relationship between two sets of numbers and working on functions can provide a motivation to consider more routine or technical algebraic techniques (for example, in showing why all the different algebraic rules are the same). In a standard UK curriculum, rather than an introduction to algebra, functions would typically appear much later.

Complex numbers as relations

A complex number can be seen as a relation if it is considered as a function that transforms shapes. Because complex numbers need two dimensions to represent them, to get a sense of how they might transform a shape, we need two sets of (2D) axes. Figure 12.7 shows how a circle is transformed under the operation, $z \rightarrow 2z$

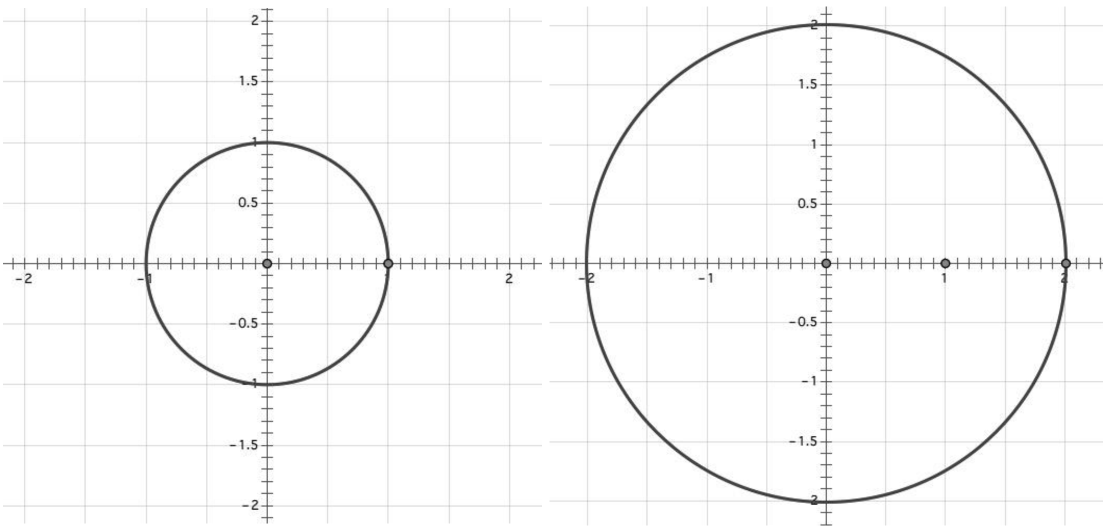

Figure 12.7: A representation of $z \rightarrow 2z$

The pre-requisite to considering complex numbers as relations is to know *i* as the square root of negative 1, and to know how to express complex numbers on an Argand diagram. Starting with the left hand diagram, students can be invited to conjecture what the shape would be transformed to, if both real and imaginary parts were doubled. For example, i, at (0,1) on the diagram above, will be mapped to $2i$, at (0,2) on the right hand diagram.

From my own experience of teaching, students can quite quickly become independent in testing other complex transformations and generating their own pairs of images. In each case, students can be invited to predict and test what transformation will be described. A potential mathematical appreciation on offer in this activity is around the relations between one entire system (transformation geometry, e.g., translations, rotations, reflections, enlargement) into another system (complex numbers). It is also possible to work on the relations between the transformation entailed in a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and an equivalent complex number (e.g., $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a rotation of -90 degrees around the origin and is therefore equivalent to *i*). Again, the sense of complex numbers as transformations would typically appear late on in any course that introduced complex numbers. The activity above can set up the ideas of the geometrical interpretation of multiplication by complex numbers, leading to deMoivre's theorem, viewed as a statement about transformations.

Discussion

The three examples above and the case of number discussed earlier indicate how a selection of mathematical concepts can be interpreted as relations. It is the proposal of this chapter that *every* concept can be treated in a similar manner. As has been clear in the four examples, the view of mathematical concepts as relations disrupts the typical sequence of the treatment of ideas, generally by familiarizing students with a wider 'whole', out of which particular concepts arise. To generate starting points that allow a relational view of mathematical concepts there is a need to focus on broader (mathematical) structures. There is an important role for visible or tangible objects (Gattegno, 1974) but also a danger, if these are used to symbolize concepts too directly. If a concept is to be approached relationally, then images or materials are needed where what can be symbolized are actions on or relations between the materials, or within the images.

As a mathematician, if I am presented with an expression such as, $2x^2 + 7x - 9$, a whole host of associations arise, I may recognize the statement as an expression that can be factorized, I may picture the graph, or the quadratic formula may come to mind. Depending on what I may be asked, or decide, to do next particular associations will be foregrounded and others will fade. The concepts and associations I invoke primarily gain their use and meaning from their links to other concepts. If concepts have been introduced as relations, then a symbol's links to physical objects or images take their places as just some among a myriad of connections. When introduced as relations, children can access their imaginations when working with symbols, as mathematicians do. It is the perspective of this chapter that difficulties in mathematics are much more likely to arise from children not having access to any imaginative response to symbols (such as my reaction to the equation above) rather than any supposed lack of capacity for abstract thinking (thinking that is amply demonstrated by anyone who has taught themselves their first language). Symbols are meaningful to mathematicians but not in a direct ('this' means 'that') manner. It is in part the ambiguity and flexibility of mathematical symbols that give them power and use; this is the game of mathematics, but it is a game that many students are not let in on.

Gattegno and Davydov achieved extraordinary results when using their curricula. The pedagogical advantages of approaching concepts as relations, as has only been touched on in the examples above, include the following:

- inverse operations can be considered simultaneously (e.g., fractions with integers; fractional enlargements with standard enlargements);
- there is no disruption as ideas become more complex; relational images or representations will not need to be 'unlearnt', for example in the move from integers to fractions;
- there is something the symbols 'mean' and can point to, as well as being abstract from the beginning;
- linked to the point above, symbols can trigger the mathematical *imagination*, i.e., there is something that can be evoked by mathematical symbols if they are introduced as relations;
- symbols can quickly take on relations to each other; because they are introduced in a complex whole there are other symbols they *can* relate to;
- complexity can also be limited in the beginning to allow a gradual development, while still being 'abstract' from the start.

Relations, are at once material (arising from a consideration of objects) and abstract (the relation of, say, 'double' is not linked directly to an object). They exist neither in the objects themselves, nor in any human mind but rather arise from the interaction of humans with the world and each other. Concepts are never fixed: each use adds a different complexion to the web of connections that surround it and changes us as humans, in the kind of circular relationship suggested by Piaget.

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